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# Consensus and Flocking in Cooperative Systems with Random Communication Failures

Benoît Bonnet\*, Émilien Flayac† and Francesco Rossi‡

**Abstract**—We study sufficient conditions for convergence to consensus and flocking of non-linear multi-agent systems subject to random-in-time communication failures. Our approach is based on Lyapunov methods adapted to this non-stationary setting, under a persistence of excitation condition. This assumption has an interpretation in terms of average connectedness of the interaction graph.

## I. INTRODUCTION

The study of emerging patterns in dynamical systems describing collective behaviour has been the object of an increasing attention in the last decades. There is now a large literature devoted to the analysis of *consensus formation* in the class of so-called *cooperative systems*, see e.g. [1]. These systems are widely used, for example, to study crowd motion [2], robot swarms [3], [4] and animal groups [5], [6] such as bird flocks or fish schools.

Since the seminal paper [7] by Cucker and Smale, a great deal of interest has been manifested towards the analysis of the so-called *flocking behaviour* (see Definition 4 below), which describes the appearance of alignment patterns in second-order cooperative multi-agent systems. In [8], the authors proposed an alternative proof of the emergence of asymptotic flocking, based on Lyapunov methods. This method was then extended both to finite and infinite dimension [9], and to specific time and state-dependent interaction topologies [10]. It also allowed to control key models towards consensus and flocking [11–14].

When communications between agents are subject to random failure, it is clearly crucial to verify whether convergence is still guaranteed. For discrete-time first and second order systems, opinion formation models have been thoroughly investigated in a graph theoretic framework, see for instance the seminal paper [15]. Further results allowed to incorporate asymmetric communication rates and random communication failures e.g. in [16],

[17]. However, to the best of our knowledge, there is no general proof of convergence for general time-continuous systems subject to random communication failures.

In this paper, we investigate sufficient conditions for both asymptotic consensus and flocking formation based on Lyapunov methods. The main ingredient is the introduction of a condition of *persistence of excitation* (see Definition 3 below). This type of condition appeared quite recently in stability theory, and has proven its adaptability to build strict Lyapunov functions, see [18–20]. Besides the interest of having a strict Lyapunov function, e.g. for studying input-to-state stability, persistence of excitation has a both deep and simple signification in cooperative dynamics. Indeed, it transcribes the fact that, on average on a given time window, the interaction graph describing a multi-agent system is connected with a prescribed lower bound on the intensity of this averaged interaction. This type of average connectedness assumption is standard when studying general time-varying interaction topologies (see e.g. [15], [21]), and it is even proven to be necessary for consensus in a large number of cases in [15]. In the way we formulate it, this condition further encodes the idea that one only requires the system to be persistently exciting with respect to the agents which have not yet reached consensus.

The structure of the paper is the following. In Section II, we prove the asymptotic consensus for persistently excited first-order dynamics. We then extend this result in Section III to asymptotic flocking for Cucker-Smale type systems with strongly interacting kernels in the sense of (13), which is the main result of this paper. We provide numerical examples in Section IV and conclude with open perspectives in Section V.

## II. CONSENSUS UNDER PERSISTENT EXCITATION FOR FIRST-ORDER DYNAMICS

In this section, we introduce the tools used in the article, in the particular case of consensus formation. We study first-order cooperative systems of the form

$$\begin{cases} \dot{x}_i(t) = \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i(t) - x_j(t)|) (x_j(t) - x_i(t)), \\ x_i(0) = x_i^0, \end{cases} \quad (\text{CS}_1)$$

where  $(x_1^0, \dots, x_N^0) \in \mathbb{R}^{dN}$  is a given initial datum. We assume that the interaction kernel  $\phi \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$  is **strictly positive**.

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The functions  $\xi_{ij} \in L^\infty(\mathbb{R}_+, [0, 1])$  represent **communication rates**, taking into account potential communication failures that can occur in the system (when  $\xi_{ij}(t) < 1$ ). We require them to be symmetric, i.e.  $\xi_{ij}(\cdot) = \xi_{ji}(\cdot)$ . One of the main motivations for this choice of communication rates is to study consensus and flocking when *random interaction failures* occur. This article is the first step towards a more general theory for such systems, in which the  $\xi_{ij}(\cdot)$  will be realizations of stochastic processes.

From now on, we use the notation  $\mathbf{x} = (x_1, \dots, x_N)$  for the state in  $\mathbb{R}^{dN}$  and  $\bar{\mathbf{x}} = \frac{1}{N} \sum_{i=1}^N x_i$  for its mean value. For systems of the form (CS<sub>1</sub>), we aim to study the formation of *asymptotic consensus*, defined as follows.

*Definition 1:* A solution  $\mathbf{x}(t)$  of (CS<sub>1</sub>) *asymptotically converges to consensus* if  $\lim_{t \rightarrow +\infty} |x_i(t) - \bar{\mathbf{x}}(t)| = 0$  for all  $i \in \{1, \dots, N\}$ .

As a consequence of the symmetry of the rates  $\xi_{ij}(\cdot)$ , (CS<sub>1</sub>) can be rewritten as

$$\dot{\mathbf{x}}(t) = -\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (\text{CSM}_1)$$

where  $\mathbf{L} : \mathbb{R}_+ \times \mathbb{R}^{dN} \rightarrow \mathcal{L}(\mathbb{R}^{dN})$  is the so-called *graph Laplacian*, defined by

$$(\mathbf{L}(t, \mathbf{x})\mathbf{y})_i := \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i - x_j|) (y_i - y_j). \quad (1)$$

In the following, we will also use  $\mathbf{L}_\xi(\cdot)$  defined by

$$(\mathbf{L}_\xi(t)\mathbf{y})_i := \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) (y_i - y_j). \quad (2)$$

Observe that both  $\mathbf{L}(\cdot, \cdot)$  and  $\mathbf{L}_\xi(\cdot)$  depend on the time-dependent communication rates  $\xi_{ij}(\cdot)$ , that are  $L^\infty$  functions, thus defined for almost every  $t \in [0, +\infty)$ . For simplicity, we will drop this 'almost everywhere' definition from now on.

The structure displayed in (1) is fairly general and allows for a comprehensive study of both consensus and flocking problems in a unified way via Lyapunov methods. With this goal in mind, we introduce the following bilinear form in the spirit of [11], [12].

*Definition 2:* The variance bilinear form  $B(\cdot, \cdot)$  is

$$B(\mathbf{x}, \mathbf{y}) := \frac{1}{N} \sum_{i=1}^N \langle x_i, y_i \rangle - \langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle. \quad (3)$$

It is symmetric and positive semi-definite.

Such bilinear form is the squared distance of a given  $\mathbf{x} \in \mathbb{R}^{dN}$  from the so-called *consensus manifold*  $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^{dN} \text{ s.t. } x_1 = \dots = x_N\}$ . As a consequence,  $B(\mathbf{x}, \mathbf{x}) = 0$  if and only if  $x_i = \bar{\mathbf{x}}$  for any index  $i \in \{1, \dots, N\}$ , i.e. if  $\mathbf{x}$  is a consensus.

We now list useful properties linking  $B(\cdot, \cdot)$  and  $\mathbf{L}(\cdot, \cdot)$ .

*Proposition 1:* The graph Laplacian  $\mathbf{L}(t, \mathbf{x})$  is positive-semi definite with respect to  $B(\cdot, \cdot)$ . Moreover, vectors of the form  $\mathbf{L}(t, \mathbf{x})\mathbf{y}$  have zero mean.

*Proof:* By summing over  $i \in \{1, \dots, N\}$  the components in (1), the mean of  $\mathbf{L}(t, \mathbf{x})\mathbf{y}$  is zero. As a consequence, and by symmetry of the communication

rates  $\xi_{ij}(\cdot)$ , it holds

$$\begin{aligned} B(\mathbf{L}(t, \mathbf{x})\mathbf{y}, \mathbf{y}) &= \frac{1}{N^2} \sum_{i,j=1}^N \xi_{ij}(t) \phi(|x_i - x_j|) \langle y_i, y_i - y_j \rangle \\ &= \frac{1}{2N^2} \sum_{i,j=1}^N \xi_{ij}(t) \phi(|x_i - x_j|) |y_i - y_j|^2 \geq 0. \end{aligned}$$

We now introduce a definition of *persistence of excitation* adapted to our study.

*Definition 3:* Let  $\tau, \mu > 0$  be given parameters. We say that the **persistence of excitation condition** (PE <sub>$\tau, \mu$</sub> ) holds for (CSM<sub>1</sub>) if

$$B\left(\left(\frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}_\xi(s) ds\right) \mathbf{x}, \mathbf{x}\right) \geq \mu B(\mathbf{x}, \mathbf{x}). \quad (\text{PE}_{\tau, \mu})$$

*Remark 1:* Condition (PE <sub>$\tau, \mu$</sub> ) only involves the communication weights  $\xi_{ij}(\cdot)$  through  $\mathbf{L}_\xi(\cdot)$  and not the state of the system. Moreover, it is formulated using the bilinear form  $B(\cdot, \cdot)$ , representing the fact that one only needs the persistence to hold along directions orthogonal to the consensus manifold  $\mathcal{C}$ . Finally, (PE <sub>$\tau, \mu$</sub> ) can be interpreted as a connectivity condition of the time-averaged interaction graph of the system, which is fairly common in the literature, see [15] and [21, Chapter 2].

In the following theorem, we state a general result of convergence for (CSM<sub>1</sub>), when (PE <sub>$\tau, \mu$</sub> ) holds.

*Theorem 1 (Main result - Consensus):* Let  $\phi(\cdot)$  be positive and non-increasing. Let  $\xi_{ij} \in L^\infty(\mathbb{R}_+, [0, 1])$  satisfy (PE <sub>$\tau, \mu$</sub> ) for some  $\tau, \mu > 0$ . Then, any solution  $\mathbf{x}(\cdot)$  of (CSM<sub>1</sub>) asymptotically converges to consensus.

*Proof:* Let  $c^2$  be the operator norm of  $\mathbf{L}(\cdot, \cdot)$  with respect to  $B(\cdot, \cdot)$ , i.e.

$$\begin{aligned} c^2 &:= \sup_{(t, \mathbf{x})} \|\mathbf{L}(\cdot, \cdot)\|_B \\ &= \sup_{(t, \mathbf{x}, \mathbf{y})} \left\{ \sqrt{B(\mathbf{L}(t, \mathbf{x})\mathbf{y}, \mathbf{L}(t, \mathbf{x})\mathbf{y})} \text{ s.t. } B(\mathbf{y}, \mathbf{y}) = 1 \right\}. \end{aligned}$$

Denote by  $X(\cdot)$  the *standard deviation* of  $\mathbf{x}(\cdot)$ , given by

$$X(t) := \sqrt{B(\mathbf{x}(t), \mathbf{x}(t))},$$

By definition of  $B(\cdot, \cdot)$ ,  $\mathbf{x}(\cdot)$  asymptotically converges to consensus if and only if  $\lim_{t \rightarrow +\infty} X(t) = 0$ .

Define the time-dependent family of linear operators  $\psi_\tau : \mathbb{R}_+ \rightarrow \mathcal{L}(\mathbb{R}^{dN})$  along the trajectory  $\mathbf{x}(\cdot)$  by

$$\psi_\tau(t) := (1 + c^2)\tau I_{dN} - \frac{1}{\tau} \int_t^{t+\tau} \int_t^s \mathbf{L}(\sigma, \mathbf{x}(\sigma)) d\sigma ds. \quad (4)$$

Then,  $\psi_\tau(\cdot)$  is Lipschitz with pointwise derivative

$$\dot{\psi}_\tau(t) = \mathbf{L}(t, \mathbf{x}(t)) - \frac{1}{\tau} \int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s)) ds. \quad (5)$$

By definition, it further holds

$$\sqrt{\tau} X(t) \leq \sqrt{B(\psi_\tau(t)\mathbf{x}, \mathbf{x})} \leq \sqrt{(1 + c^2)\tau} X(t). \quad (6)$$

Define the candidate Lyapunov function

$$\mathcal{X}_\tau(t) := \lambda X(t) + \sqrt{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))} \quad (7)$$

where  $\lambda > 0$  is a tuning parameter. This type of construction is rather recent in the design of strict Lyapunov

function under persistent excitation assumptions: see e.g. [18–20]. By (6), it holds that

$$(\lambda + \sqrt{\tau})X(t) \leq \mathcal{X}_\tau(t) \leq (\lambda + \sqrt{(1+c^2)\tau})X(t). \quad (8)$$

By Proposition 1, any solution  $\mathbf{x}(\cdot)$  of (CSM<sub>1</sub>) has constant mean, i.e.  $\bar{\mathbf{x}}(\cdot) \equiv \bar{\mathbf{x}}_0$ . By invariance with respect to translation of (CSM<sub>1</sub>), we assume without loss of generality  $\bar{\mathbf{x}}(\cdot) \equiv 0$  from now on. We now aim to prove a strict-dissipation inequality of the form

$$\dot{\mathcal{X}}_\tau(t) \leq -\alpha \mathcal{X}_\tau(t), \quad (9)$$

for some  $\alpha > 0$ . With this goal, we first compute

$$\begin{aligned} \dot{\mathcal{X}}_\tau(t) = & -\frac{\lambda}{X(t)}B(\mathbf{L}(t, \mathbf{x}(t)), \mathbf{x}(t)) + \frac{B(\dot{\psi}_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}{2\sqrt{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}} \\ & - \frac{B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \psi_\tau(t)\mathbf{x}(t))}{\sqrt{B(\psi_\tau(t)\mathbf{x}(t), \mathbf{x}(t))}}. \end{aligned}$$

By (5)-(6), it holds

$$\begin{aligned} \dot{\mathcal{X}}_\tau(t) \leq & -\frac{1}{2\sqrt{(1+c^2)\tau}X(t)}B\left(\left(\frac{1}{\tau}\int_t^{t+\tau}\mathbf{L}(s, \mathbf{x}(s))ds\right)\mathbf{y}, \mathbf{y}\right) \\ & + \frac{1}{\sqrt{\tau}X(t)}B\left(\left(\frac{1}{\tau}\int_t^{t+\tau}\int_t^s\mathbf{L}(\sigma, \mathbf{x}(\sigma))d\sigma ds\right)\mathbf{y}, \mathbf{L}(t, \mathbf{y})\mathbf{y}\right) \\ & + \frac{1}{\sqrt{\tau}X(t)}\left(\frac{1}{2} - \sqrt{(1+c^2)\tau} - \sqrt{\tau}\lambda\right)B(\mathbf{L}(t, \mathbf{y})\mathbf{y}, \mathbf{y}), \end{aligned} \quad (10)$$

where we wrote  $\mathbf{y} \equiv \mathbf{x}(t)$  for conciseness.

To estimate the first line of (10), recall that first-order cooperative systems have uniformly compactly supported trajectories, see e.g. [14, Lemma 1]. Since  $\phi(\cdot)$  is positive and continuous, there exists a positive constant  $C(\mathbf{x}_0)$  such that

$$\min_{i,j} \phi(|x_i(t) - x_j(t)|) \geq C(\mathbf{x}_0)$$

for all times  $t \geq 0$ . By definition of  $\mathbf{L}(\cdot, \cdot)$ , this implies

$$\begin{aligned} B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{y}, \mathbf{y}) & \geq \frac{C(\mathbf{x}_0)}{2N^2} \sum_{i,j=1}^N \xi_{ij}(t) |y_i - y_j|^2 \\ & = C(\mathbf{x}_0)B(\mathbf{L}_\xi(t)\mathbf{y}, \mathbf{y}), \end{aligned}$$

for any  $\mathbf{y} \in \mathbb{R}^{dN}$ . By using (PE $_{\tau, \mu}$ ), it holds

$$B\left(\left(\frac{1}{\tau}\int_t^{t+\tau}\mathbf{L}(s, \mathbf{x}(s))ds\right)\mathbf{x}(t), \mathbf{x}(t)\right) \geq C(\mathbf{x}_0)\mu X^2(t). \quad (11)$$

For the second line of (10), one has that

$$\begin{aligned} & B\left(\frac{1}{\tau}\left(\int_t^{t+\tau}\int_t^s\mathbf{L}(\sigma, \mathbf{x}(\sigma))d\sigma ds\right)\mathbf{x}(s), \mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t)\right) \\ & \leq \tau c^2 X(t) \sqrt{B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t))} \\ & \leq \tau c^2 X(t) \|\mathbf{L}(t, \mathbf{x}(t))\|_B^{1/2} \sqrt{B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t))} \\ & \leq \tau c^3 \left(\frac{\epsilon}{2} X(t)^2 + \frac{1}{2\epsilon} B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t))\right), \end{aligned} \quad (12)$$

for any  $\epsilon > 0$ , by definition of  $\|\cdot\|_B$  and Young's inequality. Merge (10)-(11)-(12) and recall that  $\mathbf{L}(\cdot, \cdot)$  is positive semi-definite to obtain

$$\begin{aligned} \dot{\mathcal{X}}_\tau(t) \leq & -\left(\frac{C(\mathbf{x}_0)\mu}{2\sqrt{(1+c^2)\tau}} - \frac{c^3\sqrt{\tau}}{2}\epsilon\right) X(t) \\ & + \frac{1}{X(t)}\left(\frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon} - \lambda\right) B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{x}(t), \mathbf{x}(t)). \end{aligned}$$

Choose

$$\epsilon = \frac{C(\mathbf{x}_0)\mu}{2c^3\tau\sqrt{(1+c^2)}}, \quad \lambda = \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon}.$$

Using (8), we recover (9) for a positive constant  $\alpha(\mu, c, \tau, C(\mathbf{x}_0))$ . Then, it holds  $\lim_{t \rightarrow +\infty} \mathcal{X}_\tau(t) = 0$ , thus  $\lim_{t \rightarrow +\infty} X(t) = 0$  by (8). By definition of  $X(\cdot)$ , this implies that  $\mathbf{x}(\cdot)$  converges to consensus. ■

### III. FLOCKING FOR CUCKER-SMALE TYPE SYSTEMS WITH STRONG INTERACTIONS

In this section, we derive sufficient conditions for the asymptotic convergence to flocking of general Cucker-Smale type dynamics subject to random communication failures. These systems are of the form

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N \xi_{ij}(t) \phi(|x_i(t) - x_j(t)|) (v_j(t) - v_i(t)). \end{cases} \quad (\text{CS}_2)$$

Similarly to Section II, (CS<sub>2</sub>) can be rewritten in matrix form using the graph Laplacian defined in (1):

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{v}(t), & \mathbf{x}(0) = \mathbf{x}_0, \\ \dot{\mathbf{v}}(t) = -\mathbf{L}(t, \mathbf{x}(t))\mathbf{v}(t), & \mathbf{v}(0) = \mathbf{v}_0. \end{cases} \quad (\text{CSM}_2)$$

We now recall the definition of flocking.

*Definition 4:* A solution  $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$  of (CSM<sub>2</sub>) converges to flocking if for any  $i \in \{1, \dots, N\}$  it holds

$$\sup_{t \geq 0} |x_i(t) - \bar{\mathbf{x}}(t)| < +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} |v_i(t) - \bar{\mathbf{v}}(t)| = 0.$$

For this problem, we always assume to have  $\phi(\cdot) \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$  **non-increasing** and such that

$$\int_0^{+\infty} \phi(r) dr = +\infty. \quad (13)$$

Equation (13) is known as a **strong interaction** condition, since it describes the fact that the interaction between agents does not decrease too fast when their distance goes to infinity.

*Remark 2:* When  $\phi(\cdot)$  is uniformly bounded from below by a positive constant, then flocking in the full-communication setting occurs, see e.g. [7], [8], [14]. In our framework, this result is a simple consequence of Theorem 2. For positive kernels not satisfying (13), one can easily construct examples of initial conditions  $(\mathbf{x}_0, \mathbf{v}_0)$  for which flocking does not occur, see [7].

Solutions of (CSM<sub>2</sub>) satisfy

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{v}}(t), \quad \dot{\bar{\mathbf{v}}}(t) = 0.$$

By invariance properties, we assume  $\bar{\mathbf{x}}(\cdot) = \bar{\mathbf{v}}(\cdot) \equiv 0$  from now on, with no loss of generality. Define

$$X(t) := \sqrt{B(\mathbf{x}(t), \mathbf{x}(t))}, \quad V(t) := \sqrt{B(\mathbf{v}(t), \mathbf{v}(t))}$$

As a consequence of symmetry of  $\xi_{ij}(\cdot)$ , system (CSM<sub>2</sub>) is *weakly dissipative* in the sense that

$$\dot{X}(t) \leq V(t), \quad \dot{V}(t) \leq 0, \quad (14)$$

along any solution  $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$ .

In their seminal paper [8], Ha and Liu produced a concise proof of the Cucker-Smale flocking based on the analysis of a system of *strictly dissipative inequalities*: it holds that

$$\dot{X}(t) \leq V(t), \quad \dot{V}(t) \leq -\phi(2\sqrt{N}X(t))V(t), \quad (15)$$

where  $\phi(\cdot)$  satisfies the strong interaction condition (13), then the system converges to flocking. Our aim is to adapt their strategy while taking into account possible communication failures. We prove the following, main result of this paper.

*Theorem 2 (Main result - Flocking)*: Let  $(\text{PE}_{\tau, \mu})$  hold and  $\phi(\cdot)$  be positive, non-increasing, and satisfying (13). Then, any solution of  $(\text{CSM}_2)$  converges to flocking.

The proof of this result relies on the construction of a strict Lyapunov function for  $(\text{CSM}_2)$ , for which a system of inequalities akin to (15) holds **only on a bounded time interval**. This finite-time strict dissipation allows us to recover the asymptotic flocking of the system as a consequence of the weak dissipativity (14) of  $(\text{CSM}_2)$ . To the best of our knowledge, this combination of strict Lyapunov design and flocking analysis via systems of dissipative inequalities has not been covered by the existing literature.

We first prove a technical lemma.

*Lemma 1*: Let  $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$  be a solution of  $(\text{CSM}_2)$ . If  $(\text{PE}_{\tau, \mu})$  holds, then one has

$$\begin{aligned} & B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s))ds\right) \mathbf{w}, \mathbf{w}\right) \\ & \geq \mu\phi(2\sqrt{N}(X(t) + \tau V(0)))B(\mathbf{w}, \mathbf{w}) \end{aligned} \quad (16)$$

for any  $\mathbf{w} \in \mathbb{R}^{dN}$ .

*Proof*: By definition of  $\mathbf{L}(\cdot, \cdot)$ , it holds

$$\begin{aligned} & B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s))ds\right) \mathbf{w}, \mathbf{w}\right) \\ & \frac{1}{2N^2} \sum_{i,j=1}^N \left(\frac{1}{\tau}\int_t^{t+\tau} \xi_{ij}(s)\phi(|x_i(s) - x_j(s)|)ds\right) |v_i - v_j|^2 \\ & \geq \frac{1}{2N^2} \sum_{i,j=1}^N \left(\frac{1}{\tau}\int_t^{t+\tau} \xi_{ij}(s)\phi(2\sqrt{N}X(s))ds\right) |v_i - v_j|^2, \end{aligned} \quad (17)$$

where we used that  $\phi(\cdot)$  is non-increasing. As a consequence of the weak dissipation (14), one further has

$$X(s) = X(t) + \int_t^s \dot{X}(\sigma)d\sigma \leq X(t) + \tau V(0).$$

for all  $s \in [t, t + \tau]$ . By (17), and recalling again that  $\phi(\cdot)$  is non-increasing, it holds

$$\begin{aligned} & B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}(s, \mathbf{x}(s))ds\right) \mathbf{w}, \mathbf{w}\right) \\ & \geq \frac{\phi(2\sqrt{N}(X(t) + \tau V(0)))}{2N^2} \sum_{i,j=1}^N \left(\frac{1}{\tau}\int_t^{t+\tau} \xi_{ij}(s)ds\right) |v_i - v_j|^2 \\ & = \phi(2\sqrt{N}(X(t) + \tau V(0)))B\left(\left(\frac{1}{\tau}\int_t^{t+\tau} \mathbf{L}_\xi(s)ds\right) \mathbf{w}, \mathbf{w}\right) \\ & \geq \mu\phi(2\sqrt{N}(X(t) + \tau V(0)))B(\mathbf{w}, \mathbf{w}), \end{aligned}$$

where we used  $(\text{PE}_{\tau, \mu})$  in the last inequality. ■

We now define the candidate Lyapunov function

$$\mathcal{V}_\tau(t) := \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\mathbf{v}(t), \mathbf{v}(t))} \quad (18)$$

where  $\psi_\tau(\cdot)$  is defined in (4) and  $\lambda(\cdot)$  is a tuning curve, smooth with the respect to time. We have the following lemma.

*Lemma 2*: For any  $\epsilon_0 > 0$ , there exists  $T_{\epsilon_0}^* > 0$  such that for almost every  $t \in [0, 2T_{\epsilon_0}^*)$ , it holds

$$\dot{\mathcal{V}}_\tau(t) \leq -\frac{\mu\phi(2\sqrt{N}(X(t) + \tau V(0)))}{2\sqrt{(1+c^2)\tau}}V(t). \quad (19)$$

*Proof*: Following the proof of Theorem 1, we have

$$\begin{aligned} \dot{\mathcal{V}}_\tau(t) & \leq \frac{\left(\frac{1}{2\sqrt{\tau}} + \frac{c^3\tau}{2\epsilon(t)} - \sqrt{\tau}\lambda(t)\right)}{\sqrt{\tau}V(t)} B(\mathbf{L}(t, \mathbf{x}(t))\mathbf{v}(t), \mathbf{v}(t)) \\ & \quad - \left(\frac{\mu\phi(2\sqrt{N}(X(t) + \tau V(0)))}{2\sqrt{(1+c^2)\tau}} - \frac{c^3\sqrt{\tau}}{2}\epsilon(t) - \dot{\lambda}(t)\right) V(t). \end{aligned} \quad (20)$$

The two differences with respect to the proof of Theorem 1 are the choice of *time-dependent families of parameters*  $(\lambda(\cdot), \epsilon(\cdot))$  and the use of (16) instead of  $(\text{PE}_{\tau, \mu})$ .

Choose

$$\lambda(t) = \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon(t)}. \quad (21)$$

This implies in particular that  $\dot{\lambda}(t) = -\frac{c^3\sqrt{\tau}}{2\epsilon^2(t)}\dot{\epsilon}(t)$ . Choose now  $\epsilon(\cdot)$  as the solution of

$$\dot{\epsilon}(t) = \epsilon^3(t), \quad \epsilon(0) = \epsilon_0,$$

for a given constant  $\epsilon_0 > 0$ , i.e.  $\epsilon(\cdot)$  defined by

$$\epsilon(t) = \frac{\epsilon_0}{\sqrt{1-2\epsilon_0^2 t}}, \quad (22)$$

for  $t \in [0, 1/2\epsilon_0^2)$ . Then, (20) reads as

$$\dot{\mathcal{V}}_\tau(t) \leq -\frac{\mu\phi(2\sqrt{N}(X(t) + \tau V(0)))}{2\sqrt{(1+c^2)\tau}}V(t),$$

and (19) holds with  $2T_{\epsilon_0}^* = 1/2\epsilon_0^2$ . ■

Observe that (19) involves both  $\mathcal{V}_\tau(\cdot)$  and  $V(\cdot)$ . We now aim to find an estimate involving  $V(\cdot)$  only.

*Proposition 2*: There exists a constant  $X_M > 0$  such that  $\sup_{t \geq 0} X(t) \leq X_M$ . Moreover, for any  $\epsilon_0 \in (0, 1]$  it holds

$$V(T_{\epsilon_0}^*) \leq \left(\frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0}\right) V(0) \exp\left(-\frac{\mu\phi(2\sqrt{N}(X_M + \tau V(0)))}{4(\alpha_3 + \beta_3 \epsilon_0)\epsilon_0}\right), \quad (23)$$

where  $\{\alpha_k, \beta_k\}_{k=1}^3$  are positive constants depending on  $(c, \tau)$  only.

*Proof*: Choose  $\epsilon_0 \in (0, 1]$  and denote by  $(\lambda(\cdot), \epsilon(\cdot))$  the corresponding functions given by (21)-(22) respectively.

Similarly to (6), it holds that

$$\sqrt{\tau}V(t) \leq \sqrt{B(\psi_\tau(t)\mathbf{v}(t), \mathbf{v}(t))} \leq \sqrt{(1+c^2)\tau}V(t).$$

By definition of  $\mathcal{V}_\tau(\cdot)$  in (18), it then holds that

$$\begin{cases} \mathcal{V}_\tau(0) \leq \left(\sqrt{(1+c^2)\tau} + \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{\tau}}{2\epsilon_0}\right) V(0), \\ \mathcal{V}_\tau(t) \geq \left(\sqrt{\tau} + \frac{1}{2\sqrt{\tau}} + \frac{c^3\sqrt{2\tau}}{4\epsilon_0}\right) V(t), \end{cases}$$

for any  $t \in [0, T_{\epsilon_0}^*]$ , where we used the fact that  $\epsilon(t) \leq \sqrt{2}\epsilon_0$  on this time interval. By simple identification of the coefficients, these estimates can be rewritten as

$$\mathcal{V}_\tau(0) \leq \left(\frac{\alpha_1}{\epsilon_0} + \beta_1\right) V(0) \text{ and } \mathcal{V}_\tau(t) \geq \left(\frac{\alpha_2}{\epsilon_0} + \beta_2\right) V(t)$$

for positive constants  $\{\alpha_k, \beta_k\}_{k=1}^2$  depending on  $(c, \tau)$ . Integrate (19) on  $[0, t]$ , to obtain

$$\begin{aligned} \mathcal{V}_\tau(t) &\leq \mathcal{V}_\tau(0) \\ &\quad - \frac{\mu}{2\sqrt{(1+c^2)\tau}} \int_0^t \phi(2\sqrt{N}(X(s) + \tau V(0))) V(s) ds. \end{aligned}$$

that in turn implies

$$\begin{aligned} V(t) &\leq \left(\frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0}\right) V(0) \\ &\quad - \frac{\mu \epsilon_0}{\alpha_3 + \beta_3 \epsilon_0} \int_0^t \phi(2\sqrt{N}(X(s) + \tau V(0))) V(s) ds. \end{aligned} \quad (24)$$

where  $(\alpha_3, \beta_3) = 2\sqrt{(1+c^2)\tau}(\alpha_2, \beta_2)$ . Recall now that  $\dot{X}(s) \leq V(s)$  by (14) and apply the change of variable  $r = X(s)$  in (24). It then holds that

$$\begin{aligned} V(t) &\leq \left(\frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0}\right) V(0) \\ &\quad - \frac{\mu \epsilon_0}{\alpha_3 + \beta_3 \epsilon_0} \int_{X(0)}^{X(t)} \phi(2\sqrt{N}(r + \tau V(0))) dr. \end{aligned} \quad (25)$$

Define  $\bar{V} := V(0)/\epsilon_0$  and observe that the strong interaction condition (13) implies

$$\bar{V} < \frac{\mu \alpha_2}{(\alpha_1 + \beta_1)(\alpha_3 + \beta_3)} \int_{X(0)}^{+\infty} \phi(2\sqrt{N}(r + \tau \bar{V})) dr.$$

By strict positivity of  $\phi(\cdot)$ , one can choose  $X_M > 0$  independent from  $\epsilon_0$  such that

$$\bar{V} < \frac{\mu \alpha_2}{(\alpha_1 + \beta_1)(\alpha_3 + \beta_3)} \int_{X(0)}^{X_M} \phi(2\sqrt{N}(r + \tau \bar{V})) dr.$$

Going back to  $V(0)$  and recalling that  $\epsilon_0 \in (0, 1]$  and  $\phi(\cdot)$  is non-increasing, it holds

$$V(0) < \frac{\epsilon_0 \mu \alpha_2}{(\alpha_1 + \beta_1 \epsilon_0)(\alpha_3 + \beta_3 \epsilon_0)} \int_{X(0)}^{X_M} \phi(2\sqrt{N}(r + \tau V(0))) dr.$$

Plugging this estimate into (25) yields

$$V(t) < \frac{\mu \epsilon_0}{\alpha_3 + \beta_3 \epsilon_0} \int_{X(t)}^{X_M} \phi(2\sqrt{N}(r + \tau V(0))) dr.$$

Since  $V(\cdot)$  is a positive quantity by definition and  $\phi(\cdot)$  is strictly positive, one necessarily has that  $X(t) < X_M$  on  $[0, T_{\epsilon_0}^*]$ . Since the uniform bound  $X_M$  is independent from  $\epsilon_0$  and  $\lim_{\epsilon_0 \rightarrow 0^+} T_{\epsilon_0}^* = +\infty$ , it holds  $\sup_{t \geq 0} X(t) \leq X_M$ .

Since  $\phi(\cdot)$  is non-increasing, it further holds

$$\phi(2\sqrt{N}(X(t) + \tau V(0))) \geq \phi(2\sqrt{N}(X_M + \tau V(0))).$$

for all times  $t \geq 0$ . Then, (24) implies

$$\begin{aligned} V(t) &\leq \left(\frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0}\right) V(0) \\ &\quad - \frac{\mu \epsilon_0}{\alpha_3 + \beta_3 \epsilon_0} \phi(2\sqrt{N}(X_M + \tau V(0))) \int_0^t V(s) ds. \end{aligned}$$

By Grönwall's Lemma, it holds

$$V(T_{\epsilon_0}^*) \leq \left(\frac{\alpha_1 + \beta_1 \epsilon_0}{\alpha_2 + \beta_2 \epsilon_0}\right) V(0) \exp\left(-\frac{\mu \phi(2\sqrt{N}(X_M + \tau V(0)))}{4(\alpha_3 + \beta_3 \epsilon_0) \epsilon_0}\right)$$

where we used the fact that  $T_{\epsilon_0}^* = 1/4\epsilon_0^2$ . ■

We now prove Theorem 2, that is our main result.

*Proof:* By Proposition 2, it holds  $\sup_{t \geq 0} X(t) \leq X_M$ , hence for each  $i \in \{1, \dots, N\}$  it holds

$$\sup_{t \geq 0} |x_i(t) - \bar{x}(t)| < +\infty.$$

We now prove convergence of the velocity variables. First fix  $\delta > 0$ . In (23), observe that

$$\lim_{\epsilon_0 \rightarrow 0^+} \exp\left(-\frac{\mu \phi(2\sqrt{N}(X_M + \tau V(0)))}{4(\alpha_3 + \beta_3 \epsilon_0) \epsilon_0}\right) = 0, \text{ hence there exists } \epsilon_0 \text{ such that } V(T_{\epsilon_0}^*) \leq \delta \text{ where } T_{\epsilon_0}^* = 1/4\epsilon_0^2.$$

Since (CSM<sub>2</sub>) is dissipative with respect to the velocity variable, then  $t \geq T_{\epsilon_0}^*$  implies  $V(t) \leq \delta$ . Since this estimate holds for any  $\delta > 0$ , it holds  $\lim_{t \rightarrow +\infty} V(t) = 0$ .

By definition of  $V(\cdot)$ , this implies that

$$\lim_{t \rightarrow +\infty} |v_i(t) - \bar{v}(t)| = 0,$$

for all  $i \in \{1, \dots, N\}$ . ■

#### IV. NUMERICAL EXAMPLES

In this section, we provide numerical simulations of convergence to flocking for a system with a particular class of random-in-time failures. We consider the 2D Cucker-Smale system with interaction kernel  $\phi(r) = 1/(1+r)$ , that is strongly interacting. Given  $\Delta t > 0$ , we take the  $\xi_{ij}(\cdot)$  to be realizations of piecewise constant Bernoulli processes with parameter  $p \in [0, 1]$ , i.e.

$$\mathbb{P}(\xi_{ij}(\cdot) \equiv 1) = p, \quad \mathbb{P}(\xi_{ij}(\cdot) \equiv 0) = 1 - p, \quad (26)$$

over each time interval  $[n\Delta t, (n+1)\Delta t]$  with  $n \in \mathbb{N}$ . We sample randomly the initial state of the  $N$  agents with uniform distributions in  $[0, X_0]^2 \times [0, V_0]^2$ .

We first study the case of  $N = 15$  agents with parameters  $X_0 = 4, V_0 = 2, \Delta t = 0.2, p = 3/4$ . In Figure 1-top, we plot the trajectories of agents in the plane, while in Figure 2-top we plot the evolution of the norms of the velocities in time. We compare these results with the same dynamics with full communication rates, i.e. when  $\xi_{ij}(\cdot) \equiv 1$ , for which we plot both the trajectories (Figure 1-bottom) and the evolution of the norms of the velocities (Figure 2-bottom).

Clearly, the convergence to flocking of the system with random failures is slower than in the full communication case. To better quantify this phenomenon, we display in Figure 3 the first time  $T_\delta \equiv T_\delta(p)$  at which  $B(\mathbf{v}(T_\delta), \mathbf{v}(T_\delta)) \leq \delta$  as a function of  $p$ . In this case, we fixed  $\Delta t = 0.5, X_0 = V_0 = 0.5$  and  $\delta = 10^{-6}$ . We aim to prove this empirical result in much more generality in a future work.

#### V. CONCLUSION AND PERSPECTIVES

In this article, we proved two main results of convergence of multi-agent systems under random-in-time communication failures. If communication rates satisfy a persistence of excitation condition, then one has both convergence to consensus for first-order systems (Theorem 1) and convergence to flocking for Cucker-Smale systems under an additional strong interaction condition (Theorem 2).

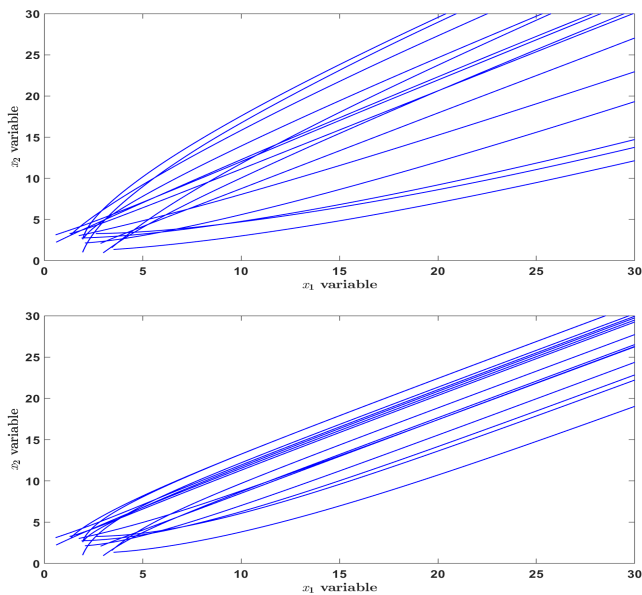


Fig. 1. Trajectories of agents with random-in-time failures (top) and with full communication rates (bottom)

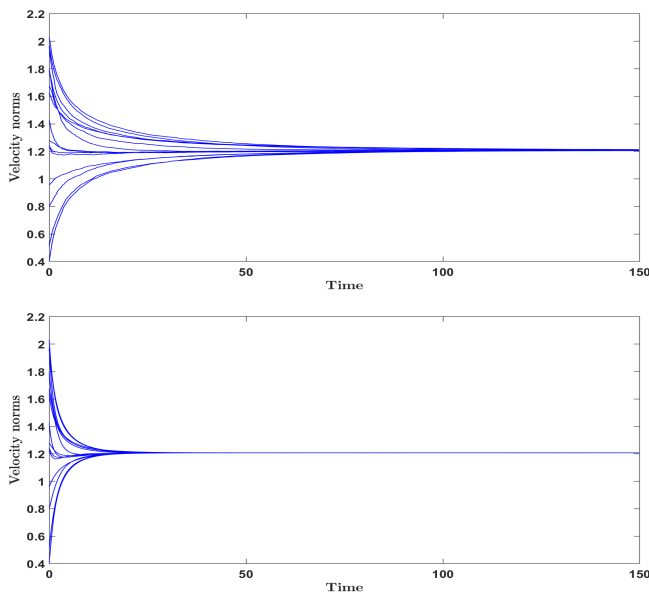


Fig. 2. Norm of velocities of agents with random-in-time failures (top) and with full communication rates (bottom)

In the future, we aim to improve such results in two directions. First, we will consider conditional flocking, i.e. we aim to find conditions on the initial configurations that ensure convergence even with non-strong interaction kernels. Second, we will consider communication failures as a result of a stochastic process and aim to estimate the probability of convergence to consensus and flocking.

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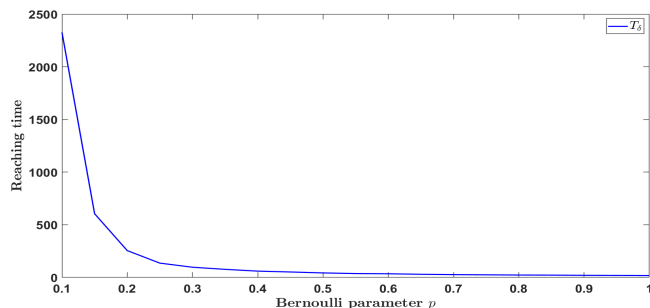


Fig. 3. Reaching time  $T_d$  of the  $\delta$ -neighbourhood of the origin as a function of the Bernoulli parameter  $p$

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