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► **To cite this version:**

Fabien Soulier, Nicolas Dollinger, Patrick Lagonotte. From Cauer synthesis to Legendre polynomials - The case of the lossless transmission line. 4th IFAC Workshop on Time Delay Systems (TDS 2003), International Federation of Automatic Control (IFAC), Sep 2003, Rocquencourt, France. pp.391-396, 10.1016/S1474-6670(17)33358-X . hal-02087528

HAL Id: hal-02087528

<https://hal.science/hal-02087528>

Submitted on 16 Apr 2024

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FROM CAUER SYNTHESIS TO LEGENDRE POLYNOMIALS — THE CASE OF THE LOSSLESS TRANSMISSION LINE

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Abstract: In this article, we study a simple Time Delay System (TDS), the short-circuited lossless transmission line. By realizing the Cauer synthesis of the input impedance, we can put in evidence a special expression of the impulse response of this system. Furthermore, we illustrate on this example a particular link between the Cauer synthesis of infinite order systems and the Legendre polynomials.

Keywords: Transmission line, partial differential equations, circuit models, networks, impulse response, polynomial transform.

1. INTRODUCTION

Two years ago, we started to study the extension of the Cauer synthesis in order to model distributed parameter systems. We first use continued fractions as a complementary tool to reduce the order of thermal models (Trigeol *et al.*, 2002). Then, we have study propagative models like electrical transmission lines. These systems sometimes appear as pure TDS. And in this case, it has turned out that the Cauer representation we were using can be linked to the Legendre polynomials in the time domain.

Although the telegrapher's equation is well-known in both electrotechnical and telecommunication domains, we shall remind the physical model it comes from. Then, we give some classical results on the equivalent input impedance and the impulse response of a short-circuited line. Secondly, we detailed the Cauer synthesis of this infinite order system and its ladder network representation. Eventually, the explicit calculation of the voltages and the currents within the network puts in evidence the orthogonal polynomials.

2. A SIMPLE TIME-DELAY SYSTEM: THE LOSSLESS TRANSMISSION LINE

2.1 Input impedance

A transmission line is a distributed parameter system. In the lossless case, the variations of the voltage and the current can be described by a set of partial differential equations, or PDEs,

$$\frac{\partial v}{\partial x} = -l \frac{\partial i}{\partial t}, \quad \frac{\partial i}{\partial x} = -c \frac{\partial v}{\partial t}$$

where l and c are per unit length inductance and capacitance (figure 1). In this article the line is assumed to be homogeneous, thus l and c are constant parameters.

It is convenient to solve this system thanks to the Laplace transform. However, for the sake of simplicity, we keep the same notations (v and i) for the voltage and the current and their transforms. Let s be the Laplace variable. Introducing the widespread notations for the *longitudinal impedance* $z = ls$ and the *transversal admittance*

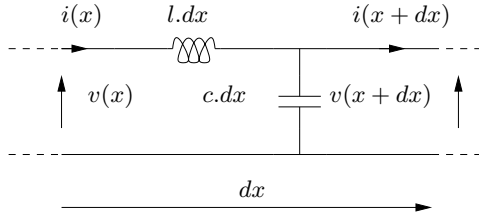


Fig. 1. Elementary segment of a lossless transmission line.

$y = cs^{-1}$, one can rewrite the equations as

$$\frac{dv}{dx} = -zi, \quad \frac{di}{dx} = -yv.$$

In the particular case of a short-circuited transmission line of length x , these equations lead to the input impedance

$$\mathcal{Z} = \frac{v}{i} = \sqrt{\frac{z}{y}} \tanh \sqrt{zy} x = \sqrt{\frac{Z}{Y}} \tanh \sqrt{ZY}, \quad (1)$$

where $Z = zx$ and $Y = yx$ (figure 2).

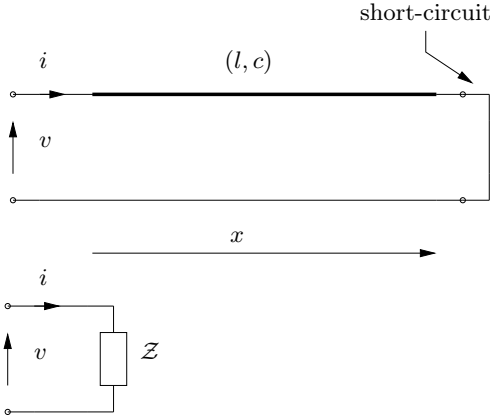


Fig. 2. Input impedance of a short-circuited transmission line.

2.2 Impulse response

Rewriting the equation (1) in form of the Ohm's law allows to regard the impedance \mathcal{Z} as an infinite order transfer function.

$$u = \mathcal{Z}i.$$

From this point of view, the current i is the input of the process, whereas the voltage v is the output.

To make the transcendental nature of the transfer function obvious, we can express the hyperbolic tangent by the mean of the exponential function.

$$\mathcal{Z} = \frac{v}{i} = \sqrt{\frac{Z}{Y}} \frac{1 - e^{-2\sqrt{ZY}}}{1 + e^{-2\sqrt{ZY}}}$$

¹ In the most general case, losses are taken into account by introducing resistive terms. So the distributed parameters become $z = r + ls$ and $y = g + cs$.

Thus, we have the relation

$$v = \sqrt{\frac{Z}{Y}} i - \sqrt{\frac{Z}{Y}} i e^{-2\sqrt{ZY}} - v e^{-2\sqrt{ZY}}.$$

Since the line is lossless, we have

$$\sqrt{ZY} = \sqrt{lc} xs = \tau s$$

and

$$\sqrt{\frac{Z}{Y}} = \sqrt{\frac{l}{c}} = Z_c$$

with $(\tau, Z_c) \in \mathbb{R}^2$. Therefore, back in the time domain, the exponentials become pure delays.

$$v(t) = Z_c i(t) - Z_c i(t - 2\tau) - v(t - 2\tau) \quad (2)$$

In terms of propagation, 2τ is the time spent by a progressive wave to travel back and forth after reflection at end of the line. On the other hand, Z_c is usually called the *characteristic impedance of the line*.

Assume that $i(t) = \delta(t)$ (the Dirac distribution). The impulse response $v(t)$ is then obtained thanks to the recurrence equation (2).

$$v(t) = Z_c \left(\delta(t) + 2 \sum_{n=1}^{\infty} (-1)^n \delta(t - 2n\tau) \right)$$

Thus, the impulse response is a serie of Dirac with alternate weights.

3. EQUIVALENT LADDER NETWORK

3.1 Cauer synthesis of the line impedance

We have previously discussed the generalization of the Cauer synthesis to infinite order systems, and shown its efficiency as a method to obtain reduced order systems (Soulie and Lagonotte, 2002). The synthesis basically consists in the continued fraction expansion of the impedance (Cauer, 1926).

$$\mathcal{Z} = \sqrt{\frac{Z}{Y}} \tanh \sqrt{ZY} = \frac{1}{\frac{1}{Z} + \frac{1}{\frac{3}{Y} + \frac{1}{\frac{5}{Z} + \dots}}}$$

This expression shows that \mathcal{Z} can be seen as the equivalent impedance of a infinite network of discrete elements (figure 3). In other words, we have obtained a lumped parameter system equivalent to the initial distributed parameter system *without* any approximation.

3.2 Recurrence relations in the ladder network

The ladder structure on the figure 3 allows to express some useful relations between the voltages and the currents inside the network. For

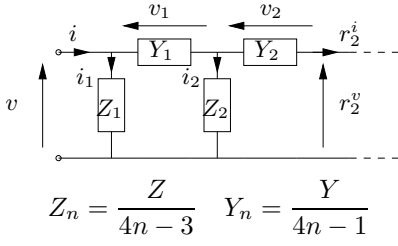


Fig. 3. Ladder network obtained by the Cauer synthesis.

convenience, we introduce two more notations: the *residual voltages* $r_n^v = v - \sum_{k=1}^n v_k$ and the *residual currents* $r_n^i = i - \sum_{k=1}^n i_k$. Provided the series converge, the Kirchoff's laws imply

$$v = \sum_{n=1}^{\infty} v_n \quad i = \sum_{n=1}^{\infty} i_n, \quad (3)$$

then

$$\lim_{n \rightarrow \infty} r_n^i = \lim_{n \rightarrow \infty} r_n^v = 0.$$

Moreover, assuming that $r_0^v = v$ and $r_0^i = i$. We can write the following recurrence relations (in the frequential domain) for all $n \geq 1$.

$$\begin{cases} i_n = \frac{r_{n-1}^v}{Z_n} \\ r_n^i = r_{n-1}^i - i_n \\ v_n = \frac{r_n^i}{Y_n} \\ r_n^v = r_{n-1}^v - v_n \end{cases}$$

To translate these relations back into time-domain, we just have to notice that in the lossless case, $\frac{1}{Z_n}$ and $\frac{1}{Y_n}$ represent pure integrators.

$$\frac{1}{Z_n} = \frac{(4n-3)}{lx} \frac{1}{s}, \quad \frac{1}{Y_n} = \frac{(4n-1)}{cx} \frac{1}{s}.$$

Therefore, the recurrence relations result in

$$\begin{cases} i_n(t) = \frac{4n-3}{lx} \int_0^t r_{n-1}^v(u) du \\ r_n^i(t) = r_{n-1}^i(t) - i_n(t) \\ v_n(t) = \frac{4n-1}{cx} \int_0^t r_n^i(u) du \\ r_n^v(t) = r_{n-1}^v(t) - v_n(t) \end{cases}$$

4. IMPULSE RESPONSE DECOMPOSITION

4.1 Reduced equations

In order to make the demonstration more readable, we shall use reduced equations. The pertinent variable changes are

$$\begin{aligned} \tilde{v}_n &= \sqrt{cx} v_n & \tilde{r}_n^v &= \sqrt{cx} r_n^v & \tilde{t} &= \frac{t}{\tau} \\ \tilde{i}_n &= \sqrt{lx} i_n & \tilde{r}_n^i &= \sqrt{lx} r_n^i & \tilde{u} &= \frac{u}{\tau}. \end{aligned}$$

Let $\tilde{i} = \delta(\tilde{t})$, the impulse response now is

$$\tilde{v}(\tilde{t}) = \delta(\tilde{t}) + 2 \sum_{n=1}^{\infty} (-1)^n \delta(\tilde{t} - 2n). \quad (4)$$

While the recurrence relations lead to a reduced system where the electrical constants do not appear anymore

$$\begin{cases} \tilde{i}_n(\tilde{t}) = (4n-3) \int_0^{\tilde{t}} \tilde{r}_{n-1}^v(\tilde{u}) d\tilde{u} \\ \tilde{r}_n^i(\tilde{t}) = \tilde{r}_{n-1}^i(\tilde{t}) - \tilde{i}_n(\tilde{t}) \\ \tilde{v}_n(\tilde{t}) = (4n-1) \int_0^{\tilde{t}} \tilde{r}_n^i(\tilde{u}) d\tilde{u} \\ \tilde{r}_n^v(\tilde{t}) = \tilde{r}_{n-1}^v(\tilde{t}) - \tilde{v}_n(\tilde{t}) \end{cases}$$

One step further is achieved by another variable change hiding the voltages and the currents. So the two parts of the recurrence equations can be merged into one.

$$\begin{cases} a_{2n} = \tilde{v}_n & r_{2n}^a = \tilde{r}_n^v \\ a_{2n-1} = \tilde{i}_n & r_{2n-1}^a = \tilde{r}_n^i \\ \begin{cases} r_{-1}^a = \tilde{i}, & r_0^a = \tilde{v} \\ a_n(\tilde{t}) = (2n-1) \int_0^{\tilde{t}} r_{n-1}^a(\tilde{u}) d\tilde{u}, & \text{for } n \geq 1 \\ r_n^a(\tilde{t}) = r_{n-2}^a(\tilde{t}) - a_n(\tilde{t}), & \text{for } n \geq 1 \end{cases} \end{cases} \quad (5)$$

4.2 Legendre polynomial form

We shall express the newly defined a_n – thus the voltages and the currents inside the network – in form of Legendre polynomials. More precisely, we shall demonstrate the following property.

Theorem 1. Let a_n be the reduced voltages and currents in the ladder network with an impulse input. Then,

$$\forall \tilde{t} \in [0, 2[, \quad a_n(\tilde{t}) = (-1)^{n+1} (2n-1) P_{n-1}(\tilde{t}-1), \quad (6)$$

with P_n being the orthogonal Legendre polynomials.

PROOF. We proceed by mathematical induction. Let's begin by proving the property for $n = 1$ and $n = 2$. This is simply achieved by using the relations (5), the impulse response (4) and the fact that \tilde{t} is limited to the interval $[0, 2[$.

$$\begin{aligned} a_1(\tilde{t}) &= \int_0^{\tilde{t}} r_0^a(\tilde{u}) d\tilde{u} = \int_0^{\tilde{t}} \tilde{v}(\tilde{u}) d\tilde{u} \\ &= \int_0^{\tilde{t}} \left(\delta(\tilde{u}) + 2 \sum_{n=1}^{\infty} (-1)^n \delta(\tilde{u} - 2n) \right) d\tilde{u} \\ &= U(\tilde{t}) + 2 \sum_{n=1}^{\infty} (-1)^n U(\tilde{t} - 2n). \end{aligned} \quad (7)$$

With U being the Heaviside function, $a_1 = 1$ for all $\tilde{t} \in [0, 2[$.

In the same manner, we determine a_2 by integration

$$\begin{aligned} a_2(\tilde{t}) &= 3 \int_0^{\tilde{t}} (r_{-1}^a(\tilde{u}) - a_1(\tilde{u})) d\tilde{u} \\ &= 3 \int_0^{\tilde{t}} (\delta(\tilde{u}) - a_1(\tilde{u})) d\tilde{u} \\ &= \begin{cases} 3(4k+1-\tilde{t}) & \text{pour } \tilde{t} \in [4k, 4k+2[, \\ 3(\tilde{t}-4k-3) & \text{pour } \tilde{t} \in [4k+2, 4k+4[\\ & k \in \mathbb{N} \end{cases} \end{aligned} \quad (8)$$

Thus, $a_2(\tilde{t}) = 3(1-\tilde{t})$ for \tilde{t} limited to $[0, 2[$.

Since $P_0(x) = 1$ and $P_1(x) = x$, the property is true up to $n = 2$. Let's go on with the induction step. Assume that the relation (6) is true for any integer lower or equal to n (induction hypothesis). Then,

$$\begin{aligned} a_{n+1}(\tilde{t}) &= (2n+1) \int_0^{\tilde{t}} r_n^a(\tilde{u}) d\tilde{u} \\ &= (2n+1) \int_0^{\tilde{t}} (r_{n-2}^a(\tilde{u}) - a_n(\tilde{u})) d\tilde{u} \\ &= \frac{2n+1}{2n-3} a_{n-1}(\tilde{t}) - \int_0^{\tilde{t}} a_n(\tilde{u}) d\tilde{u} \end{aligned}$$

Because of the induction hypothesis, the relation (6) holds for n and $n-1$. Thus we can rewrite a_{n+1} with Legendre polynomials.

$$\begin{aligned} a_{n+1}(\tilde{t}) &= (-1)^n (2n+1) P_{n-2}(\tilde{t}-1) \\ &\quad - (-1)^{n+1} (2n+1)(2n-1) \int_0^{\tilde{t}} P_{n-1}(\tilde{u}-1) d\tilde{u} \end{aligned}$$

Let's notice the following property for the Legendre polynomials, resulting from the Rodrigues' formula (see for example (Sansone, 1959), p.178).

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

And because of the fact that $P_{n+1}(-1) = P_{n-1}(-1) (= \pm 1, \text{ according to } n)$ the above formula is integrated into

$$(2n+1) \int_0^{\tilde{t}} P_n(\tilde{u}-1) d\tilde{u} = P_{n+1}(\tilde{t}-1) - P_{n-1}(\tilde{t}-1)$$

Eventually,

$$\begin{aligned} a_{n+1}(\tilde{t}) &= (-1)^{n+2} (2n+1) \\ &\quad \left(P_{n-2}(\tilde{t}-1) + (2n-1) \int_0^{\tilde{t}} P_{n-1}(\tilde{u}-1) d\tilde{u} \right) \\ &= (-1)^{n+2} (2n+1) \\ &\quad (P_{n-2}(\tilde{t}-1) + P_n(\tilde{t}-1) - P_{n-2}(\tilde{t}-1)) \\ &= (-1)^{n+2} (2n+1) P_n(\tilde{t}-1). \end{aligned}$$

So the relation holds for all $n \geq 1$.

4.3 Periodicity

We need to extend the results found on $a_n(\tilde{t})$ within the interval $\tilde{t} \in [0, 2[$ to the whole semi axis $\tilde{t} \geq 0$. To achieve this goal, we have to show the following properties about the symmetry and the periodicity of a_n .

Theorem 2. Let a_n be the reduced voltages and currents in the ladder network with an impulse input. Then,

- (i) $\forall \tilde{t} \geq 0, a_n(\tilde{t}+2) = -a_n(\tilde{t})$ (symmetry),
- (ii) $\forall \tilde{t} \geq 0, a_n(\tilde{t}+4) = a_n(\tilde{t})$ (periodicity).

PROOF. Once again, we carry out a mathematical induction. The previously found expressions (7) and (8) of a_1 and a_2 immediately show the property (i) for $n = 1$ and $n = 2$.

Let's assume that the relation (i) is true up to $n \geq 2$. We have to deal with two cases according to the parity of n . If n is even, let $n = 2p, p \in \mathbb{N}$. Then,

$$\begin{aligned} a_{n+1}(\tilde{t}+2) &= a_{2p+1}(\tilde{t}+2) \\ &= (4p+1) \int_0^{\tilde{t}+2} r_{2p}^a(\tilde{u}) d\tilde{u} \\ &= (4p+1) \int_0^{\tilde{t}+2} \left(r_0^a(\tilde{u}) - \sum_{k=1}^p a_{2k}(\tilde{u}) \right) d\tilde{u} \\ &= (4p+1) \\ &\quad \left(\int_0^{\tilde{t}+2} r_0^a(\tilde{u}) d\tilde{u} - \sum_{k=1}^p \int_0^{\tilde{t}+2} a_{2k}(\tilde{u}) d\tilde{u} \right) \end{aligned}$$

On the one hand, let's notice that

$$\begin{aligned} \int_0^{\tilde{t}+2} r_0^a(\tilde{u}) d\tilde{u} &= a_1(\tilde{t}+2) = -a_1(\tilde{t}) \\ &= - \int_0^{\tilde{t}} r_0^a(\tilde{u}) d\tilde{u} \end{aligned}$$

On the other hand, since the a_n are Legendre polynomials times a constant coefficient, one can write (due to the orthogonality):

$$\int_0^2 a_1(\tilde{u}) d\tilde{u} = 2, \text{ and } \int_0^2 a_n(\tilde{u}) d\tilde{u} = 0, \text{ if } n \neq 1.$$

Then, we deduce from the induction hypothesis that

$$\begin{aligned} &\int_0^{\tilde{t}+2} a_{2k}(\tilde{u}) d\tilde{u} \\ &= \int_0^2 a_{2k}(\tilde{u}) d\tilde{u} + \int_0^{\tilde{t}} a_{2k}(\tilde{u} + 2) d\tilde{u} \\ &= \int_0^{\tilde{t}} -a_{2k}(\tilde{u}) d\tilde{u} \end{aligned}$$

And eventually,

$$\begin{aligned}
a_{n+1}(\tilde{t} + 2) &= (4p + 1) \\
&\left(- \int_0^{\tilde{t}} r_0^a(\tilde{u}) d\tilde{u} - \sum_{k=1}^p \int_0^{\tilde{t}} -a_{2k}(\tilde{u}) d\tilde{u} \right) \\
&= -(4p + 1) \\
&\int_0^{\tilde{t}+2} \left(r_0^a(\tilde{u}) - \sum_{k=1}^p a_{2k}(\tilde{u}) \right) d\tilde{u} \\
&= -a_{2p+1}(\tilde{t}) = -a_{n+1}(\tilde{t})
\end{aligned}$$

In the odd case, we can carry out a similar process. Let $n = 2p + 1$,

$$\begin{aligned}
a_{n+1}(\tilde{t} + 2) &= a_{2p+2}(\tilde{t} + 2) = (4p + 3) \\
&\left(\int_0^{\tilde{t}+2} r_{-1}^a(\tilde{u}) d\tilde{u} - \sum_{k=0}^p \int_0^{\tilde{t}+2} a_{2k+1}(\tilde{u}) d\tilde{u} \right)
\end{aligned}$$

On the one hand, since $r_{-1}^a(\tilde{u}) = \delta(\tilde{u})$, we have

$$\int_0^{\tilde{t}+2} r_{-1}^a(\tilde{u}) d\tilde{u} = 1.$$

And on the other hand, if $k > 0$,

$$\begin{aligned}
&\int_0^{\tilde{t}+2} a_{2k+1}(\tilde{u}) d\tilde{u} \\
&= \int_0^2 a_{2k+1}(\tilde{u}) d\tilde{u} + \int_0^{\tilde{t}} a_{2k+1}(u + 2) d\tilde{u} \\
&= - \int_0^{\tilde{t}} a_{2k+1}(\tilde{u}) d\tilde{u}
\end{aligned}$$

And if $k = 0$,

$$\begin{aligned}
&\int_0^{\tilde{t}+2} a_1(\tilde{u}) d\tilde{u} \\
&= \int_0^2 a_1(\tilde{u}) d\tilde{u} + \int_0^{\tilde{t}} a_1(u + 2) d\tilde{u} \\
&= 2 - \int_0^{\tilde{t}} a_1(\tilde{u}) d\tilde{u}
\end{aligned}$$

Eventually,

$$\begin{aligned}
&\sum_{k=0}^p \int_0^{\tilde{t}+2} a_{2k+1}(\tilde{u}) d\tilde{u} \\
&= 2 - \sum_{k=0}^p \int_0^{\tilde{t}} a_{2k+1}(\tilde{u}) d\tilde{u},
\end{aligned}$$

thus,

$$\begin{aligned}
a_{n+1}(\tilde{t} + 2) &= (4p + 3) \left(-1 + \sum_{k=0}^p \int_0^{\tilde{t}} a_{2k+1}(\tilde{u}) d\tilde{u} \right) \\
&= -(4p + 3) \\
&\left(\int_0^{\tilde{t}} r_{-1}^a(\tilde{u}) d\tilde{u} - \sum_{k=0}^p \int_0^{\tilde{t}} a_{2k+1}(\tilde{u}) d\tilde{u} \right) \\
&= -a_{2p+2}(\tilde{t}) = -a_{n+1}(\tilde{t})
\end{aligned}$$

The property (i) is proven, and (ii) directly results from (i).

Therefore, it appears that the series (3) represent a decomposition on the orthogonal basis of the Legendre polynomials. The first terms of these series are plotted on the figure 4 along with their partial sums on the figures 5 and 6. This makes it possible to observe the convergence *in the sense of distributions*.

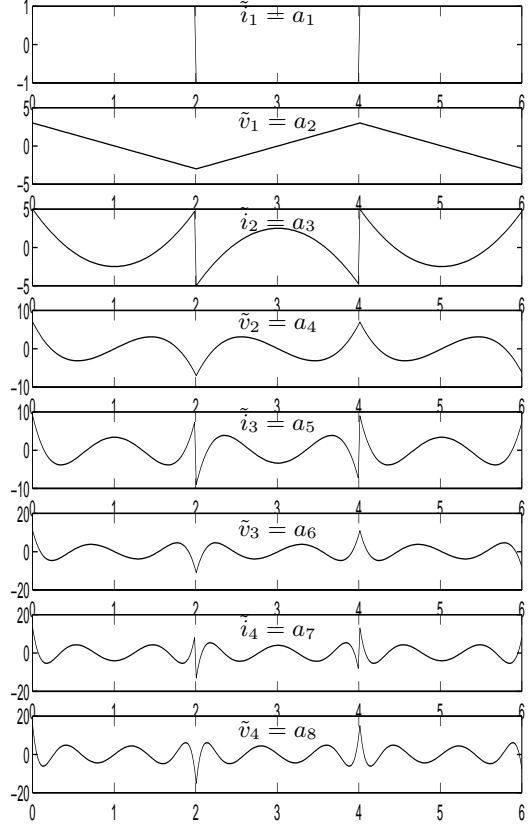


Fig. 4. The first components of the expansion of the input \tilde{i} and the corresponding impulse response \tilde{v} .

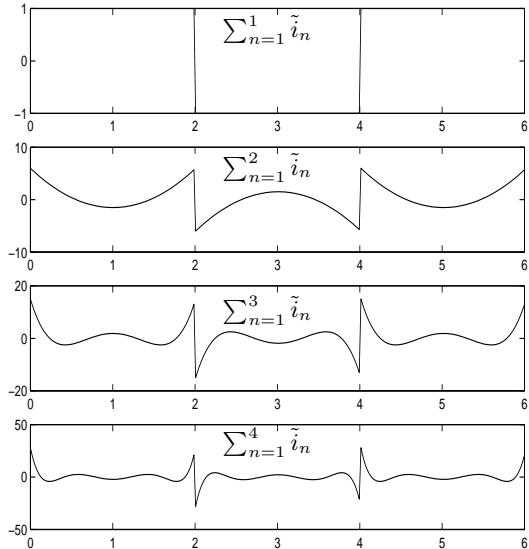


Fig. 5. The partial sums of the input \tilde{i} .

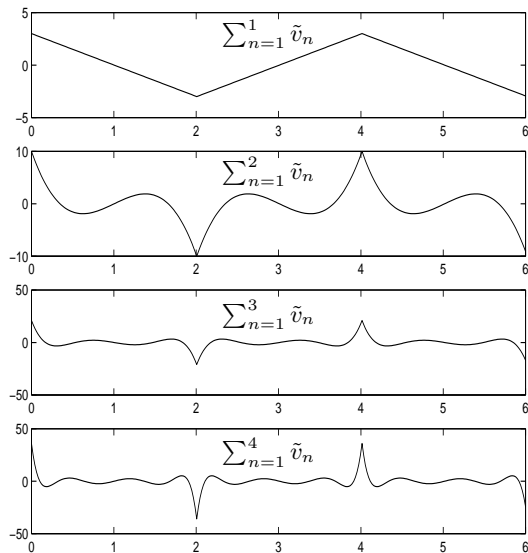


Fig. 6. The partial sums of the impulse response \tilde{v} .

5. CONCLUSION

The Cauer synthesis has proven to be particularly efficient in system modeling and in order reduction. This is due to its link with the Padé approximants and the rapid convergence of certain continued fractions.

Now, we just found the remarkable fact that for a simple propagative time-delay system, the Cauer synthesis is equivalent to representing signals with orthogonal polynomials. Thus, the coefficients of the continued fraction can be seen as the Fourier coefficients of the Legendre transform of the signals. In spite of some known connexions between the Padé approximants and the Legendre polynomials (Brezinski and Iseghem, 1983), such a physical representation of the Legendre transform of a delay transfer function seems to never have been use.

We hope this can lead to new approximate models of propagative systems and new simulation or identification methods.

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