

This document contains the proofs of the properties and theorem presented in the following conference article:

D. S. Maia, J. Cousty, L. Najman, and B. Perret. Watershedding hierarchies. In *ISMM*. 2019.

A Proof of Property 3

Property 16. *Let P be a map from $\mathcal{R}(\mathcal{B})$ to \mathbb{R}^+ . The map P is an extinction map for w if and only if the following statements hold true:*

1. $\text{range}(P) = \{0, \dots, n\}$;
2. for any two minima M_1 and M_2 if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and
3. for any region R of \mathcal{B} , we have $P(R) = \vee\{P(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$.

We prove the forward and backward implications of Property 3 in Property 17 and Property 18, respectively.

Property 17. *Let P be a map from $\mathcal{R}(\mathcal{B})$ to \mathbb{R}^+ . If the map P is an extinction map for w , then the following statements hold true:*

1. $\text{range}(P) = \{0, \dots, n\}$;
2. for any two minima M_1 and M_2 if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and
3. for any region R of \mathcal{B} , we have $P(R) = \vee\{P(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$.

Proof. Let P be an extinction map. Then there is a sequence $S = (M_1, \dots, M_n)$ of minima of w such that P is the extinction map for S . By Definition 2, for any minimum M_i , for i in $\{1, \dots, n\}$, we have $P(M_i) = i$ because M_i is the only minimum of w included in M_i . Therefore, for any two distinct minima M_i and M_j , for i in $\{1, \dots, n\}$, we have $P(M_i) \neq P(M_j)$, which proves the condition 2 of Property 17. Since w has n minima, the extinction value of any region which includes a minimum (i.e. any non leaf region) is in the set $\{1, \dots, n\}$. For any leaf region R which do not include any minimum of w , we have $P(R) = 0$ by Definition 2. Therefore, the range of P is $\{0, \dots, n\}$, which corresponds to the first condition of Property 17. The third condition of Property 17 is part of the Definition 2, so its proof is trivial. \square

Property 18. *Let P be a map from $\mathcal{R}(\mathcal{B})$ to \mathbb{R}^+ such that:*

1. $\text{range}(P) = \{0, \dots, n\}$;
2. for any two minima M_1 and M_2 if $P(M_1) = P(M_2)$, then $M_1 = M_2$; and
3. for any region R of \mathcal{B} , we have $P(R) = \vee\{P(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$.

The map P is an extinction map for w .

Proof. Let P be a map from $\mathcal{R}(\mathcal{B})$ to \mathbb{R}^+ for which the statements 1, 2 and 3 hold true. To prove that P is an extinction map, we have to show that there exists a sequence S of n pairwise distinct minima of w such that, for any region R of \mathcal{B} , the value $P(R)$ is the extinction value of R for S .

Let $S = (M_1, \dots, M_n)$ be a sequence of n pairwise distinct minima of w ordered in non-decreasing order for P , i.e., for any two distinct minima M_i and M_j , for i and j in $\{1, \dots, n\}$, if $i < j$ then $P(M_i) \leq P(M_j)$.

By the statement 2, the sequence S is unique. By the statement 3, for any region R of \mathcal{B} such that there is no minimum of w included in R , $P(R) = \vee\{\} = 0$, so $P(R)$ is the extinction value of R for S .

Since w has n minima, for any minimum M of w , the value $P(M)$ is in $\{1, \dots, n\}$. Otherwise, if there existed a minimum M' of w such that $P(M') = 0$, then there would be a value i in $\{1, \dots, n\}$ such that for any minimum M'' of w the value $P(M'')$ is different from i . Consequently, the range of P would be $\{0, \dots, n\} \setminus \{i\}$, which contradicts the statement 1. Therefore, for any minimum M_i , for i in $\{1, \dots, n\}$, we have that $P(M_i) = i$, so $P(M_i)$ is the extinction value of M_i for S .

It follows that, by the statement 3, for any region R such that there is a minimum of \mathcal{B} included in R , the value $P(R)$ is the maximum value i in $\{1, \dots, n\}$ such that M_i is included in R .

Thus, for any region R of \mathcal{B} , the value $P(R)$ is the extinction value of R for S . Therefore, the map P is an extinction map of w . \square

B Proof of Property 8

Property 19. Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . The map f is the saliency map of a hierarchical watershed of (G, w) if and only if the map ξ_f is an extinction map.

We prove the forward and backward implications of Property 8 in Property 20 and Property 29, respectively.

Property 20. Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If f is the saliency map of a hierarchical watershed, then ξ_f is an extinction map.

By Theorem 3 of [10], if f is the saliency map of a hierarchical watershed, then f is a one-side increasing map, which implies that:

1. $\text{range}(f) = \{0, \dots, n - 1\}$;
2. for any u in E , $f(u) > 0$ if and only if $u \in WS(w)$, where $WS(w)$ is the set of watershed-cut edges of w ; and
3. for any u in E , there exists a child R of R_u such that $f(u) \geq \vee\{f(v)$ such that R_v is included in $R\}$.

In order to prove Property 20, we prove that three conditions for ξ_f to be an extinction map (Property 3) are satisfied in Properties 21, 22 and 26.

Property 21. *Let f be the saliency map of a hierarchical watershed and let ξ_f be the approximated extinction map of f . Then $\text{range}(\xi_f) = \{0, \dots, n\}$.*

Proof. We need to prove that:

1. for any i in $\{0, \dots, n\}$, there is a region R of \mathcal{B} such that $\xi_f(R) = i$; and
2. for any region R of \mathcal{B} , we have $\xi_f(R)$ in $\{0, \dots, n\}$.

Proof of 1:

Let $R = V$, then $\xi_f(R) = \vee^f(R) + 1$. Since the range of f is $\{0, \dots, n-1\}$, we have $\vee^f(V) = \vee\{f(u) \mid R_u \subseteq V\} = n-1$. Then, $\xi_f(R) = n-1+1 = n$.

Let R be a leaf region of \mathcal{B} . Then the building edge of $\text{parent}(R)$ is not in $WS(w)$. Let u be the building edge of $\text{parent}(R)$. Since f is a one-side increasing map and since u is not in $WS(w)$, we have $f(u) = 0$. Since R is a leaf region, R is not a dominant region for f and then, by Definition 7, we have $\xi_f(R) = f(u) = 0$.

Now, we have to prove that, for any i in $\{1, \dots, n-1\}$, there is a region R of \mathcal{B} such that $\xi_f(R) = i$. Since w has n minima, we can conclude that w has $n-1$ watershed-cut edges. Since the range of f is $\{0, \dots, n-1\}$ and the weight of the watershed-cut edges of w is strictly greater than zero, we can conclude that the watershed-cut edges of w have pairwise distinct weights for f from 1 to $n-1$. Given any watershed-cut edge u , we can affirm that exactly one of the children of R_u is a dominant region of f because, given the children X and Y of R_u , we know that both X and Y are not leaf regions and we have either $X \prec^{(f,w)} Y$ or $Y \prec^{(f,w)} X$. Let Y be the child of R_u which is not a dominant region for f . By Definition 7, we have $\xi_f(Y) = f(u)$. Therefore, for any i in $\{1, \dots, n-1\}$, there is a watershed-cut edge u in $WS(w)$ such that $f(u) = i$ and such that there is a child Y of R_u such that $\xi_f(Y) = f(u) = i$.

Proof of 2:

Let R be a region of \mathcal{B} and let u be the building edge of the parent of R if $R \neq V$. By Definition 7, the value $\xi_f(R)$ is: $\vee^f(V) + 1$, $f(u)$ or $\xi_f(\text{parent}(R))$. It is enough to prove that $\vee^f(V) + 1$ and $f(u)$ are in $\{0, \dots, n\}$. Since the range of f is $\{0, \dots, n-1\}$, it is clear that $f(u)$ is in $\{0, \dots, n\}$. We can see that $\vee^f(V) = n-1$ because any region of \mathcal{B} is included in V . Then $\vee^f(V) + 1 = n$ which is in $\{0, \dots, n\}$. \square

Property 22. *Let f be the saliency map of a hierarchical watershed and let ξ_f be the approximated extinction map for f . For any two minima M_1 and M_2 of w , if $\xi_f(M_1) = \xi_f(M_2)$, then $M_1 = M_2$.*

As established by Theorem 3 of [10], if f is the saliency map of a hierarchical watershed of (G, w) , then f is one-side increasing for \mathcal{B} . To prove Property 22, we first present the Properties 23, 24 and 25.

Property 23. *Let f be a one-side increasing map and let ξ_f be the approximated extinction map of f . For any region X of \mathcal{B} , we denote by u_X the building edge of X . For any region X such that there is a minimum of w strictly included in X , there is a child Y of X such that:*

- $\xi_f(Y) = \xi_f(X)$;

- $\xi_f(\text{sibling}(Y)) = f(u_X)$; and
- there is a minimum of w included in Y .

Proof. Let X be a region such that there is at least one minimum of w strictly included in X . By the definition of dominant regions (Definition 6), at most one of the children of X is a dominant region of \mathcal{B} . Since there is at least one minimum of w strictly included in X , then there is a minimum included in at least one child of X . Therefore, there is a child Y of X such that Y is a dominant region of \mathcal{B} and, therefore $\xi_f(Y) = \xi_f(X)$ (condition 2 of Definition 7). Thus, $\text{sibling}(Y)$ is not a dominant region of \mathcal{B} and $\xi(\text{sibling}(Y)) = f(u)$, where u is the building edge of X (condition 3 of Definition 7). \square

Property 24. Let u be any watershed edge of w and let f be a one-side increasing map. There is a minimum M of w such that $\xi_f(M) = f(u)$.

Proof. Let u be a watershed-cut edge of w and let f be a one-side increasing map. By Property 23, there is a child X_1 of R_u such that $\xi_f(X_1) = f(u)$. Since u is a watershed edge, X_1 cannot be a leaf node. If X_1 is a minimum of w , then the property holds true. Otherwise, by Property 23, there is a child X_2 of X_1 such that $\xi_f(X_2) = \xi_f(X_1) = f(u)$ and such that there is a minimum of w included in X_2 . We can see that we define a sequence (X_1, \dots, X_p) where X_p is a minimum of w and such that $\xi_f(X_p) = \dots = \xi_f(X_1) = f(u)$ and $X_i \subset X_{i-1}$ for any i in $\{2, \dots, p\}$. Therefore, there is a minimum X_p included in R_u such that $\xi_f(X_p) = f(u)$. \square

Property 25. Let X be a non-leaf region of \mathcal{B} . There exists a minimum M of w such that $\xi_f(M) = \xi_f(X)$.

Proof. If X is a minimum of w , then the proof is trivial. Otherwise, there is a minimum of w strictly contained in X . By Property 23, there is a child X_1 of X such that $\xi_f(X_1) = \xi_f(X)$ and such that there is a minimum of w included in X_1 . If X_1 is a minimum of w , then the property holds true. Otherwise, by Property 23, there is a child X_2 of X_1 such that $\xi_f(X_2) = \xi_f(X_1) = \xi_f(X)$ and such that there is a minimum of w included in X_2 . We can see that we define a sequence (X_1, \dots, X_p) where X_p is a minimum of w and such that $\xi_f(X_p) = \dots = \xi_f(X_1) = \xi_f(X)$ and $X_i \subset X_{i-1}$ for any i in $\{2, \dots, p\}$. Therefore, there is a minimum X_p included in X such that $\xi_f(X_p) = \xi_f(X)$. \square

Proof (Property 22).

Let f be a one-side increasing map for \mathcal{B} and let ξ_f be the estimated extinction map for f . We need to prove that, for any two minima M_1 and M_2 of w , if $\xi_f(M_1) = \xi_f(M_2)$, then $M_1 = M_2$. By Property 24, we know that for any watershed edge u of w , there is a minimum M such that $\xi_f(M) = f(u)$. By Property 25, we can say that there is a minimum M of w such that $\xi_f(M) = \xi_f(V) = n$. Since the range of f for the set of watershed edges is $\{1, \dots, n-1\}$, we can conclude, by Properties 24 and 25, that the range of ξ_f for the set of minima of w is $\{1, \dots, n\}$. Since w has n minima, it implies that the values $\xi_f(M_1)$ and $\xi_f(M_2)$ should be distinct for any pair (M_1, M_2) of distinct minima of w . \square

Property 26. *Let f be the saliency map of a hierarchical watershed and let ξ_f be the estimated extinction map for f . For any region R in $\mathcal{R}(\mathcal{B})$, we have $\xi_f(R) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$.*

As established by Theorem 3 of [10], if f is the saliency map of a hierarchical watershed of (G, w) , then f is one-side increasing for \mathcal{B} . To prove this lemma, we introduce properties 27 and 28.

Property 27. *Let f be a one-side increasing map and let X be a region of \mathcal{B} . Then $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$.*

Proof. Let X be a region of \mathcal{B} . We will prove that this property holds true in all the cases of the definition of approximated extinction maps (Definition 7).

1. If $X = V$, then $\xi_f(X) = n$ (first case of Definition 7). Since the range of f is $\{0, \dots, n-1\}$, we have $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$.
2. If X is non-dominant region of \mathcal{B} and if X is a leaf region of \mathcal{B} . Then $\{f(v) \mid R_v \subseteq X\} = \emptyset$. Since $\vee\emptyset = 0$, we have $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$.
3. If X is non-dominant region and a non-leaf region of \mathcal{B} , then $\text{sibling}(X)$ is a dominant region of \mathcal{B} , which implies that $\vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\} \leq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } \text{sibling}(X)\}$ by the definition of dominant regions. Since f is a one-side increasing map, then $f(u_{\text{parent}(X)}) \geq \vee\{f(v) \text{ such that } R_v \text{ is included in } Z\}$ for a child Z of $\text{parent}(X)$. Consequently, $f(u_{\text{parent}(X)}) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$ and, therefore, $\xi_f(X) = f(u_{\text{parent}(X)}) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$.
4. If X is a dominant region of \mathcal{B} , then $\xi_f(X) = \xi_f(\text{parent}(X))$. We will prove that $\xi_f(X) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$ by induction.
 - Base step: if $\text{parent}(X)$ is V , then $\xi_f(X) = \xi_f(V) = n$ and our property holds true.
 - Inductive step: if the property holds for $\text{parent}(X)$, then we have to show that it holds for X as well. If $\xi_f(\text{parent}(X)) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } \text{parent}(X)\}$ then $\xi_f(X) = \xi_f(\text{parent}(X)) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$ because every descendant of X is a descendant of $\text{parent}(X)$ as well. \square

Property 28. *Let X be a non-leaf region of \mathcal{B} . Then, for any region Y such that $Y \subseteq X$, the value $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$*

Proof. By induction:

- Base step: if X is a minimum of w . Let u be the building edge of X . For any child Y of X , we can affirm that Y is a non-dominant region of \mathcal{B} and then $\xi_f(Y) = f(u)$, which is equal to zero because f is a one-side increasing map. Thus, $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$.
- Inductive step: if X is not a minimum and the property holds for both children of X . By Property 23, we know that there is a child Y of X such that $\xi_f(Y) = f(u_X)$ and $\xi_f(\text{sibling}(Y)) = \xi_f(X)$. Therefore, for any region Y such that $Y \subseteq X$, the value $\xi_f(Y)$ is in $\{\xi_f(Y), 0\} \cup \{f(u) \mid R_u \subseteq X\}$.

$Y\} \cup \{\xi_f(\text{ sibling}(Y)), 0\} \cup \{f(u) \mid R_u \subseteq \text{ sibling}(Y)\} \cup \{\xi_f(X)\}$ which is equivalent to $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$. \square

Proof (Property 26). We can now prove that, for any region R of \mathcal{B} , we have $\xi_f(R) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$. Given a region X of \mathcal{B} :

- If there is no minimum of w included in X , then X is a leaf region and X is a non-dominant region of \mathcal{B} . Then $\xi_f(Y) = f(u)$ (third condition of Definition 7), which is equal to zero because f is a one-side increasing map. Therefore, $\xi_f(X) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\} = \vee\emptyset = 0$
- Otherwise, for any region $Y \subseteq X$, $\xi_f(Y)$ is in $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$ by Property 28. By Property 27, $\xi_f(X) \geq \{f(v) \mid R_v \subseteq X\}$. Therefore, $\xi_f(X) \geq \xi_f(Y)$. Then, ξ_f is increasing on the hierarchy \mathcal{B} , i.e., for any region X , we have $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X\}$. By Property 23, there is a minimum M of w such that $\xi_f(X) = \xi_f(M)$. Hence, $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X \text{ and } Y \text{ is a minimum of } w\}$. \square

Property 29. *Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If ξ_f is an extinction map, then f is the saliency map of a hierarchical watershed.*

If ξ_f is an extinction map, then, by Property 3, we have:

- $\text{range}(P) = \{0, \dots, n\}$;
- for any two distinct minima M_1 and M_2 , $P(M_1) \neq P(M_2)$; and
- for any region R of \mathcal{B} , we have $P(R) = \vee\{P(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$.

Now we have to prove that f is a one-side increasing map. We need to prove that the three conditions for f to be a one-side increasing map are satisfied.

Property 30. *Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If ξ_f is an extinction map, then $\text{range}(f) = \{0, \dots, n-1\}$.*

Proof. We need to prove that:

1. for any i in $\{0, \dots, n-1\}$, there is an edge u in E such that $f(u) = i$; and
2. for any edge u in E , we have $f(u)$ in $\{0, \dots, n-1\}$.

Proof of 1:

For $i = 0$: Since ξ_f is an extinction map, for any leaf region R of \mathcal{B} , we have $\xi_f(R) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\} = 0$. Let R be a leaf region. Since R is not a dominant region for f , this means that $\xi_f(R) = f(u)$, where u is the building edge of $\text{parent}(R)$, and, since $\xi_f(R) = 0$, this implies that there exists an edge u in E such that $f(u) = 0$.

For i in $\{1, \dots, n-1\}$: Since ξ_f is an extinction map, then $\text{range}(\xi_f) = \{0, \dots, n\}$. Then, for any i in $\{1, \dots, n-1\}$ there is a region R of \mathcal{B} such that

$\xi_f(R) = i$. Let i be any value in $\{1, \dots, n-1\}$ and let R be a region of \mathcal{B} such that $\xi_f(R) = i$. If R is not a dominant region for R , then $\xi_f(R) = f(u)$, where u is the building edge of the parent of R and, then, we can affirm that there exists an edge in E whose weight for f is i . Otherwise, if R is a dominant region for f , then $\xi_f(R) = \xi_f(\text{parent}(R))$. If $\text{parent}(R)$ is not a dominant region for f , then $\xi_f(\text{parent}(R)) = \xi_f(v)$, where v is the building edge of the parent of $\text{parent}(R)$ and we have our property. Otherwise, if $\text{parent}(R)$ is a dominant region of \mathcal{B} , then $\xi_f(\text{parent}(R)) = \xi_f(\text{parent}(\text{parent}(R)))$. We can see that, at some point, we will have $\xi_f(R) = \xi_f(\text{parent} \dots (\text{parent}(R))) = f(y)$ for an edge y in E .

Proof of 2: By contradiction, let us assume that there is an edge u in E such that $f(u)$ is not in $\{0, \dots, n-1\}$. We can affirm that any non leaf region of \mathcal{B} has a child which is not a dominant region for f . So, we can affirm that there is a child X of R_u such that $\xi_f(X) = f(u)$. Since ξ_f is an extinction map, the range of ξ_f is $\{0, \dots, n\}$. Then, $\xi_f(X) = f(u)$ should be in $\{0, \dots, n\}$ as well. Therefore, the only value that $f(u)$ could have and that is not in $\{0, \dots, n-1\}$ is n . So, let us assume that $f(u) = n$. In this case, we would have $\vee^f(V) = n+1$, which contradicts the fact that $\text{range}(\xi_f) = \{0, \dots, n\}$. Therefore, we may conclude that, for any edge u in E , we have $f(u)$ in $\{0, \dots, n-1\}$. \square

Property 31. Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If ξ_f is an extinction map, then for any u in E , $f(u) > 0$ if and only if $u \in WS(w)$.

Proof. If ξ_f is an extinction map, then only the leaf nodes do not include any minimum of w , which implies that only the leaf nodes has a value equal to zero for ξ_f . We can say that any leaf region R is not a dominant region of \mathcal{B} , and then $\xi_f(R) = f(u)$ where u is the building edge of the parent of R . We can say that an edge is a watershed-cut edge if and only if it has no leaf regions as children. This implies that $\xi_f(R) = 0$ if and only if the parent of R is not a watershed-cut edge. \square

Property 32. Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If ξ_f is an extinction map, then for any u in E , there exists a child R of R_u such that $f(u) \geq \vee\{f(v) \mid R_v \text{ is included in } R\}$.

In order to prove Property 32, we first present properties 33 and 34.

Property 33. Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If ξ_f is an extinction map then, for any region R of \mathcal{B} , $\xi_f(R) \geq \vee\{\xi_f(X) \mid X \subseteq R\}$.

Proof. The proof is straightforward if we consider the third condition of Property 3. \square

Property 34. Let f be a map from E into \mathbb{R}^+ and let ξ_f be the approximated extinction map of f . If ξ_f is an extinction map then, for any region R of \mathcal{B} , $\xi_f(R) \geq f(u)$, where u is the building edge of R .

Proof. Let R be a region of \mathcal{B} :

- If $R = V$, then $\xi_f(R) = \vee^f(V) + 1$, which means that $\xi_f(R) > f(u)$.
- If R is not a dominant region, then $\xi_f(R) = f(v)$ where v is the building edge of the parent of R . By Property 33, we know that $\xi_f(\text{parent}(R)) \geq \vee\{\xi_f(X) \mid X \subseteq R\}$, which means that $\xi_f(\text{parent}(R)) \geq \xi_f(R) = f(v)$, so the property holds for the parent of R . \square

Proof (Property 32).

Let R be any region of \mathcal{B} . We have $\xi_f(R) \geq \{\xi_f(X) \mid X \subseteq R\}$. Then, we have $\xi_f(R) \geq \{f(u) \mid u \text{ is the building edge of } X \subseteq R\}$. If R is a non-dominant region, then $\xi_f(R) = f(v)$ where v is the building edge of the parent of R . So, we will have $f(v) \geq \{f(u) \mid u \text{ is the building edge of } X \subseteq R\}$. Since R is a child of R_v , we can say that there is a child of R_v such that the property holds. \square

Proof (Property 8).

Properties 20 and 29 correspond to the forward and backward implications of Property 8. \square

C Proof of Property 10

Let f be a map from E into \mathbb{R}^+ and let \mathcal{S}_f be the estimated sequence of minima of f . If f is the saliency map of a hierarchical watershed, then ξ_f is an extinction map by Property 8. First, we have to prove that ξ_f is the extinction map for the sequence \mathcal{S}_f . Then, we have to prove that, for any edge u , we have $f(u) = \min\{\xi_f(X) \mid X \subseteq R_u\}$.

Property 35. *Let f be a map from E into \mathbb{R}^+ and let \mathcal{S}_f be the estimated sequence of minima of f . If ξ_f is an extinction map, then ξ_f is the extinction map for the estimated sequence of minima \mathcal{S}_f of f .*

Proof. If ξ_f is an extinction map, then for any two distinct minima M_1 and M_2 , we have $\xi_f(M_1) \neq \xi_f(M_2)$. If we prove that $\xi_f(M) > 0$ for any minimum M , then we prove that the range of ξ_f for the set of minima is $\{1, \dots, n\}$. Let M be a minimum of w and let us assume that $\xi_f(M) = 0$. If M is not a dominant region of \mathcal{B} , then $\xi_f(M) = f(v) = 0$ where v is the building edge of $\text{parent}(M)$. However, if M is not a dominant region of \mathcal{B} , this implies that none of the children of R_v is a leaf region and then, v is a watershed-cut edge. Since f is one-side increasing, this implies that $f(v)$ is strictly greater than 0. If M is a dominant region of \mathcal{B} , this implies that $\xi_f(M) = \xi_f(\text{parent}(M))$. Since ξ_f is an extinction map, only the leaf regions of \mathcal{B} has a zero value for ξ_f . Since the parent of M is not a leaf region, we can affirm that $\xi_f(\text{parent}(M)) > 0$ and, therefore, $\xi_f(M) > 0$. So, we just proved that the range of ξ_f for the set of minima is $\{1, \dots, n\}$. Since ξ_f is an extinction map, we can say for any region R , we have $\xi_f(R) = \vee\{\xi_f(M) \mid M \text{ is a minimum}\}$. So, ξ_f is the extinction map for \mathcal{S}_f . \square

Property 36. *Let f be a map from E into \mathbb{R}^+ and let \mathcal{S}_f be the estimated sequence of minima of f . If ξ_f is an extinction map, then for any edge u , we have $f(u) = \min\{\xi_f(R_u), f(u)\}$.*

Proof. Any region has a child which is dominant and a child which is not a dominant region of \mathcal{B} . This implies that, for any edge u , there is a child X of u such that $\xi_f(X) = \xi_f(R_u)$ and another child Y such that $\xi_f(Y) = f(u)$. By Property 34, we have $\xi_f(R_u) \geq f(u)$, which implies that $f(u) = \min\{\xi_f(X) \mid X \subseteq R_u\} = \min\{\xi_f(R_u), f(u)\}$. \square

D Proof of Theorem 12

Theorem 37. *Let f be a map from E into \mathbb{R}^+ . The watershedding $\omega(f)$ of f is the saliency map of the hierarchical watershed of (G, w) for the estimated sequence of minima for f .*

Proof. By Property 8, the map ξ_f is an extinction map and, by Property 35, the map ξ_f is an extinction map for the estimated sequence of minima \mathcal{S}_f for f . Therefore, by the Definition 11 of watershedding, the watershedding $\omega(f)$ of f is the saliency map of the hierarchical watershed of (G, w) for the estimated sequence of minima for f . \square

E Proof of Property 13

Property 38. *Let f be a map from E into \mathbb{R}^+ . The watershedding $\omega(\omega(f))$ of $\omega(f)$ is equal to $\omega(f)$.*

Proof. By Theorem 12, we know that $\omega(f)$ is the saliency map of a hierarchical watershed. By Property 10, we know that $\xi_{\omega(f)}$ is the extinction map for the estimated sequence of minima $\mathcal{S}_{\omega(f)}$ for $\omega(f)$. By Property 36, we know that $\omega(f)(u) = \min\{\xi_{\omega(f)}(X) \mid X \subseteq R_u\}$. By Definition 11, we have $\omega(\omega(f)) = \min\{\xi_{\omega(f)}(X) \mid X \subseteq R_u\} = \omega(f)(u)$. \square

F Proof of Property 15

Property 39. *Let \mathcal{H} be a hierarchy and let f be the saliency map of \mathcal{H} . The hierarchy \mathcal{H} is a hierarchical watershed of (G, w) if and only if $\omega(f) = f$.*

Proof. Let \mathcal{H} be a hierarchical watershed and let f be the saliency map of \mathcal{H} . By Property 14, we may say that $\omega(f) = f$.

On the other hand, let $\omega(f) = f$. Then, for any edge u , we have $f(u) = \omega(f)(u) = \min\{P(R) \mid R \text{ is a child of } R_u\}$, where P is the extinction map for \mathcal{S}_f . Therefore, by the backward implication of Property 8, we can say that f is the saliency map of the hierarchical watershed for \mathcal{S}_f . \square