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## Proofs of the properties presented in the article "Watershedding hierarchies"

This document contains the proofs of the properties and theorem presented in the following conference article:

D. S. Maia, J. Cousty, L. Najman, and B. Perret. Watershedding hierarchies. In *ISMM*. 2019.

### A Proof of Property 3

**Property 16.** *Let  $P$  be a map from  $\mathcal{R}(\mathcal{B})$  to  $\mathbb{R}^+$ . The map  $P$  is an extinction map for  $w$  if and only if the following statements hold true:*

1.  $\text{range}(P) = \{0, \dots, n\}$ ;
2. for any two minima  $M_1$  and  $M_2$  if  $P(M_1) = P(M_2)$ , then  $M_1 = M_2$ ; and
3. for any region  $R$  of  $\mathcal{B}$ , we have  $P(R) = \vee\{P(M) \mid M \text{ is a minimum of } w \text{ included in } R\}$ .

We prove the forward and backward implications of Property 3 in Property 17 and Property 18, respectively.

**Property 17.** *Let  $P$  be a map from  $\mathcal{R}(\mathcal{B})$  to  $\mathbb{R}^+$ . If the map  $P$  is an extinction map for  $w$ , then the following statements hold true:*

1.  $\text{range}(P) = \{0, \dots, n\}$ ;
2. for any two minima  $M_1$  and  $M_2$  if  $P(M_1) = P(M_2)$ , then  $M_1 = M_2$ ; and
3. for any region  $R$  of  $\mathcal{B}$ , we have  $P(R) = \vee\{P(M) \mid M \text{ is a minimum of } w \text{ included in } R\}$ .

*Proof.* Let  $P$  be an extinction map. Then there is a sequence  $\mathcal{S} = (M_1, \dots, M_n)$  of minima of  $w$  such that  $P$  is the extinction map for  $\mathcal{S}$ . By Definition 2, for any minimum  $M_i$ , for  $i$  in  $\{1, \dots, n\}$ , we have  $P(M_i) = i$  because  $M_i$  is the only minimum of  $w$  included in  $M_i$ . Therefore, for any two distinct minima  $M_i$  and  $M_j$ , for  $i$  in  $\{1, \dots, n\}$ , we have  $P(M_i) \neq P(M_j)$ , which proves the condition 2 of Property 17. Since  $w$  has  $n$  minima, the extinction value of any region which includes a minimum (i.e. any non leaf region) is in the set  $\{1, \dots, n\}$ . For any leaf region  $R$  which do not include any minimum of  $w$ , we have  $P(R) = 0$  by Definition 2. Therefore, the range of  $P$  is  $\{0, \dots, n\}$ , which corresponds to the first condition of Property 17. The third condition of Property 17 is part of the Definition 2, so its proof is trivial.  $\square$

**Property 18.** *Let  $P$  be a map from  $\mathcal{R}(\mathcal{B})$  to  $\mathbb{R}^+$  such that:*

1.  $\text{range}(P) = \{0, \dots, n\}$ ;
2. for any two minima  $M_1$  and  $M_2$  if  $P(M_1) = P(M_2)$ , then  $M_1 = M_2$ ; and

3. for any region  $R$  of  $\mathcal{B}$ , we have  $P(R) = \vee\{P(M) \mid M \text{ is a minimum of } w \text{ included in } R\}$ .

The map  $P$  is an extinction map for  $w$ .

*Proof.* Let  $P$  be a map from  $\mathcal{R}(\mathcal{B})$  to  $\mathbb{R}^+$  for which the statements 1, 2 and 3 hold true. To prove that  $P$  is an extinction map, we have to show that there exists a sequence  $\mathcal{S}$  of  $n$  pairwise distinct minima of  $w$  such that, for any region  $R$  of  $\mathcal{B}$ , the value  $P(R)$  is the extinction value of  $R$  for  $\mathcal{S}$ .

Let  $\mathcal{S} = (M_1, \dots, M_n)$  be a sequence of  $n$  pairwise distinct minima of  $w$  ordered in non-decreasing order for  $P$ , i.e., for any two distinct minima  $M_i$  and  $M_j$ , for  $i$  and  $j$  in  $\{1, \dots, n\}$ , if  $i < j$  then  $P(M_i) \leq P(M_j)$ .

By the statement 2, the sequence  $\mathcal{S}$  is unique. By the statement 3, for any region  $R$  of  $\mathcal{B}$  such that there is no minimum of  $w$  included in  $R$ ,  $P(R) = \vee\{\} = 0$ , so  $P(R)$  is the extinction value of  $R$  for  $\mathcal{S}$ .

Since  $w$  has  $n$  minima, for any minimum  $M$  of  $w$ , the value  $P(M)$  is in  $\{1, \dots, n\}$ . Otherwise, if there existed a minimum  $M'$  of  $w$  such that  $P(M') = 0$ , then there would be a value  $i$  in  $\{1, \dots, n\}$  such that for any minimum  $M''$  of  $w$  the value  $P(M'')$  is different from  $i$ . Consequently, the range of  $P$  would be  $\{0, \dots, n\} \setminus \{i\}$ , which contradicts the statement 1. Therefore, for any minimum  $M_i$ , for  $i$  in  $\{1, \dots, n\}$ , we have that  $P(M_i) = i$ , so  $P(M_i)$  is the extinction value of  $M_i$  for  $\mathcal{S}$ .

It follows that, by the statement 3, for any region  $R$  such that there is a minimum of  $\mathcal{B}$  included in  $R$ , the value  $P(R)$  is the maximum value  $i$  in  $\{1, \dots, n\}$  such that  $M_i$  is included in  $R$ .

Thus, for any region  $R$  of  $\mathcal{B}$ , the value  $P(R)$  is the extinction value of  $R$  for  $\mathcal{S}$ . Therefore, the map  $P$  is an extinction map of  $w$ .  $\square$

## B Proof of Property 8

**Property 19.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . The map  $f$  is the saliency map of a hierarchical watershed of  $(G, w)$  if and only if the map  $\xi_f$  is an extinction map.

We prove the forward and backward implications of Property 8 in Property 20 and Property 29, respectively.

**Property 20.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $f$  is the saliency map of a hierarchical watershed, then  $\xi_f$  is an extinction map.

By Theorem 3 of [11], if  $f$  is the saliency map of a hierarchical watershed, then  $f$  is a one-side increasing map, which implies that:

1.  $\text{range}(f) = \{0, \dots, n - 1\}$ ;
2. for any  $u$  in  $E$ ,  $f(u) > 0$  if and only if  $u \in WS(w)$ , where  $WS(w)$  is the set of watershed-cut edges of  $w$ ; and

3. for any  $u$  in  $E$ , there exists a child  $R$  of  $R_u$  such that  $f(u) \geq \vee\{f(v)$  such that  $R_v$  is included in  $R\}$ .

In order to prove Property 20, we prove that three conditions for  $\xi_f$  to be an extinction map (Property 3) are satisfied in Properties 21, 22 and 26.

**Property 21.** *Let  $f$  be the saliency map of a hierarchical watershed and let  $\xi_f$  be the approximated extinction map of  $f$ . Then  $\text{range}(\xi_f) = \{0, \dots, n\}$ .*

*Proof.* We need to prove that:

1. for any  $i$  in  $\{0, \dots, n\}$ , there is a region  $R$  of  $\mathcal{B}$  such that  $\xi_f(R) = i$ ; and
2. for any region  $R$  of  $\mathcal{B}$ , we have  $\xi_f(R)$  in  $\{0, \dots, n\}$ .

*Proof of 1:*

Let  $R = V$ , then  $\xi_f(R) = \vee^f(R) + 1$ . Since the range of  $f$  is  $\{0, \dots, n - 1\}$ , we have  $\vee^f(V) = \vee\{f(u) \mid R_u \subseteq V\} = n - 1$ . Then,  $\xi_f(R) = n - 1 + 1 = n$ .

Let  $R$  be a leaf region of  $\mathcal{B}$ . Then the building edge of  $\text{parent}(R)$  is not in  $WS(w)$ . Let  $u$  be the building edge of  $\text{parent}(R)$ . Since  $f$  is a one-side increasing map and since  $u$  is not in  $WS(w)$ , we have  $f(u) = 0$ . Since  $R$  is a leaf region,  $R$  is not a dominant region for  $f$  and then, by Definition 7, we have  $\xi_f(R) = f(u) = 0$ .

Now, we have to prove that, for any  $i$  in  $\{1, \dots, n - 1\}$ , there is a region  $R$  of  $\mathcal{B}$  such that  $\xi_f(R) = i$ . Since  $w$  has  $n$  minima, we can conclude that  $w$  has  $n - 1$  watershed-cut edges. Since the range of  $f$  is  $\{0, \dots, n - 1\}$  and the weight of the watershed-cut edges of  $w$  is strictly greater than zero, we can conclude that the watershed-cut edges of  $w$  have pairwise distinct weights for  $f$  from 1 to  $n - 1$ . Given any watershed-cut edge  $u$ , we can affirm that exactly one of the children of  $R_u$  is a dominant region of  $f$  because, given the children  $X$  and  $Y$  of  $R_u$ , we know that both  $X$  and  $Y$  are not leaf regions and we have either  $X \prec^{(f,w)} Y$  or  $Y \prec^{(f,w)} X$ . Let  $Y$  be the child of  $R_u$  which is not a dominant region for  $f$ . By Definition 7, we have  $\xi_f(Y) = f(u)$ . Therefore, for any  $i$  in  $\{1, \dots, n - 1\}$ , there is a watershed-cut edge  $u$  in  $WS(w)$  such that  $f(u) = i$  and such that there is a child  $Y$  of  $R_u$  such that  $\xi_f(Y) = f(u) = i$ .

*Proof of 2:*

Let  $R$  be a region of  $\mathcal{B}$  and let  $u$  be the building edge of the parent of  $R$  if  $R \neq V$ . By Definition 7, the value  $\xi_f(R)$  is:  $\vee^f(V) + 1$ ,  $f(u)$  or  $\xi_f(\text{parent}(R))$ . It is enough to prove that  $\vee^f(V) + 1$  and  $f(u)$  are in  $\{0, \dots, n\}$ . Since the range of  $f$  is  $\{0, \dots, n - 1\}$ , it is clear that  $f(u)$  is in  $\{0, \dots, n\}$ . We can see that  $\vee^f(V) = n - 1$  because any region of  $\mathcal{B}$  is included in  $V$ . Then  $\vee^f(V) + 1 = n$  which is in  $\{0, \dots, n\}$ .  $\square$

**Property 22.** *Let  $f$  be the saliency map of a hierarchical watershed and let  $\xi_f$  be the approximated extinction map for  $f$ . For any two minima  $M_1$  and  $M_2$  of  $w$ , if  $\xi_f(M_1) = \xi_f(M_2)$ , then  $M_1 = M_2$ .*

As established by Theorem 3 of [11], if  $f$  is the saliency map of a hierarchical watershed of  $(G, w)$ , then  $f$  is one-side increasing for  $\mathcal{B}$ . To prove Property 22, we first present the Properties 23, 24 and 25.

**Property 23.** *Let  $f$  be a one-side increasing map and let  $\xi_f$  be the approximated extinction map of  $f$ . For any region  $X$  of  $\mathcal{B}$ , we denote by  $u_X$  the building edge of  $X$ . For any region  $X$  such that there is a minimum of  $w$  strictly included in  $X$ , there is a child  $Y$  of  $X$  such that:*

- $\xi_f(Y) = \xi_f(X)$ ;
- $\xi_f(\text{sibling}(Y)) = f(u_X)$ ; and
- there is a minimum of  $w$  included in  $Y$ .

*Proof.* Let  $X$  be a region such that there is at least one minimum of  $w$  strictly included in  $X$ . By the definition of dominant regions (Definition 6), at most one of the children of  $X$  is a dominant region of  $\mathcal{B}$ . Since there is at least one minimum of  $w$  strictly included in  $X$ , then there is a minimum included in at least one child of  $X$ . Therefore, there is a child  $Y$  of  $X$  such that  $Y$  is a dominant region of  $\mathcal{B}$  and, therefore  $\xi_f(Y) = \xi_f(X)$  (condition 2 of Definition 7). Thus,  $\text{sibling}(Y)$  is not a dominant region of  $\mathcal{B}$  and  $\xi(\text{sibling}(Y)) = f(u)$ , where  $u$  is the building edge of  $X$  (condition 3 of Definition 7).  $\square$

**Property 24.** *Let  $u$  be any watershed edge of  $w$  and let  $f$  be a one-side increasing map. There is a minimum  $M$  of  $w$  such that  $\xi_f(M) = f(u)$ .*

*Proof.* Let  $u$  be a watershed-cut edge of  $w$  and let  $f$  be a one-side increasing map. By Property 23, there is a child  $X_1$  of  $R_u$  such that  $\xi_f(X_1) = f(u)$ . Since  $u$  is a watershed edge,  $X_1$  cannot be a leaf node. If  $X_1$  is a minimum of  $w$ , then the property holds true. Otherwise, by Property 23, there is a child  $X_2$  of  $X_1$  such that  $\xi_f(X_2) = \xi_f(X_1) = f(u)$  and such that there is a minimum of  $w$  included in  $X_2$ . We can see that we define a sequence  $(X_1, \dots, X_p)$  where  $X_p$  is a minimum of  $w$  and such that  $\xi_f(X_p) = \dots = \xi_f(X_1) = f(u)$  and  $X_i \subset X_{i-1}$  for any  $i$  in  $\{2, \dots, p\}$ . Therefore, there is a minimum  $X_p$  included in  $R_u$  such that  $\xi_f(X_p) = f(u)$ .  $\square$

**Property 25.** *Let  $X$  be a non-leaf region of  $\mathcal{B}$ . There exists a minimum  $M$  of  $w$  such that  $\xi_f(M) = \xi_f(X)$ .*

*Proof.* If  $X$  is a minimum of  $w$ , then the proof is trivial. Otherwise, there is a minimum of  $w$  strictly contained in  $X$ . By Property 23, there is a child  $X_1$  of  $X$  such that  $\xi_f(X_1) = \xi_f(X)$  and such that there is a minimum of  $w$  included in  $X_1$ . If  $X_1$  is a minimum of  $w$ , then the property holds true. Otherwise, by Property 23, there is a child  $X_2$  of  $X_1$  such that  $\xi_f(X_2) = \xi_f(X_1) = \xi_f(X)$  and such that there is a minimum of  $w$  included in  $X_2$ . We can see that we define a sequence  $(X_1, \dots, X_p)$  where  $X_p$  is a minimum of  $w$  and such that  $\xi_f(X_p) = \dots = \xi_f(X_1) = \xi_f(X)$  and  $X_i \subset X_{i-1}$  for any  $i$  in  $\{2, \dots, p\}$ . Therefore, there is a minimum  $X_p$  included in  $X$  such that  $\xi_f(X_p) = \xi_f(X)$ .  $\square$

*Proof (Property 22).*

Let  $f$  be a one-side increasing map for  $\mathcal{B}$  and let  $\xi_f$  be the estimated extinction map for  $f$ . We need to prove that, for any two minima  $M_1$  and  $M_2$  of  $w$ , if  $\xi_f(M_1) = \xi_f(M_2)$ , then  $M_1 = M_2$ . By Property 24, we know that for

any watershed edge  $u$  of  $w$ , there is a minimum  $M$  such that  $\xi_f(M) = f(u)$ . By Property 25, we can say that there is a minimum  $M$  of  $w$  such that  $\xi_f(M) = \xi_f(V) = n$ . Since the range of  $f$  for the set of watershed edges is  $\{1, \dots, n-1\}$ , we can conclude, by Properties 24 and 25, that the range of  $\xi_f$  for the set of minima of  $w$  is  $\{1, \dots, n\}$ . Since  $w$  has  $n$  minima, it implies that the values  $\xi_f(M_1)$  and  $\xi_f(M_2)$  should be distinct for any pair  $(M_1, M_2)$  of distinct minima of  $w$ .  $\square$

**Property 26.** *Let  $f$  be the saliency map of a hierarchical watershed and let  $\xi_f$  be the estimated extinction map for  $f$ . For any region  $R$  in  $\mathcal{R}(\mathcal{B})$ , we have  $\xi_f(R) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$ .*

As established by Theorem 3 of [11], if  $f$  is the saliency map of a hierarchical watershed of  $(G, w)$ , then  $f$  is one-side increasing for  $\mathcal{B}$ . To prove this lemma, we introduce properties 27 and 28.

**Property 27.** *Let  $f$  be a one-side increasing map and let  $X$  be a region of  $\mathcal{B}$ . Then  $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$ .*

*Proof.* Let  $X$  be a region of  $\mathcal{B}$ . We will prove that this property holds true in all the cases of the definition of approximated extinction maps (Definition 7).

1. If  $X = V$ , then  $\xi_f(X) = n$  (first case of Definition 7). Since the range of  $f$  is  $\{0, \dots, n-1\}$ , we have  $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$ .
2. If  $X$  is non-dominant region of  $\mathcal{B}$  and if  $X$  is a leaf region of  $\mathcal{B}$ . Then  $\{f(v) \mid R_v \subseteq X\} = \emptyset$ . Since  $\vee\emptyset = 0$ , we have  $\xi_f(X) \geq \vee\{f(v) \mid R_v \subseteq X\}$ .
3. If  $X$  is non-dominant region and a non-leaf region of  $\mathcal{B}$ , then  $\text{sibling}(X)$  is a dominant region of  $\mathcal{B}$ , which implies that  $\vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\} \leq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } \text{sibling}(X)\}$  by the definition of dominant regions. Since  $f$  is a one-side increasing map, then  $f(u_{\text{parent}(X)}) \geq \vee\{f(v) \text{ such that } R_v \text{ is included in } Z\}$  for a child  $Z$  of  $\text{parent}(X)$ . Consequently,  $f(u_{\text{parent}(X)}) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$  and, therefore,  $\xi_f(X) = f(u_{\text{parent}(X)}) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$ .
4. If  $X$  is a dominant region of  $\mathcal{B}$ , then  $\xi_f(X) = \xi_f(\text{parent}(X))$ . We will prove that  $\xi_f(X) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$  by induction.
  - Base step: if  $\text{parent}(X)$  is  $V$ , then  $\xi_f(X) = \xi_f(V) = n$  and our property holds true.
  - Inductive step: if the property holds for  $\text{parent}(X)$ , then we have to show that it holds for  $X$  as well. If  $\xi_f(\text{parent}(X)) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } \text{parent}(X)\}$  then  $\xi_f(X) = \xi_f(\text{parent}(X)) \geq \vee\{f(v) \text{ such that } R_v \text{ is a descendant of } X\}$  because every descendant of  $X$  is a descendant of  $\text{parent}(X)$  as well.  $\square$

**Property 28.** *Let  $X$  be a non-leaf region of  $\mathcal{B}$ . Then, for any region  $Y$  such that  $Y \subseteq X$ , the value  $\xi_f(Y)$  is in  $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$*

*Proof.* By induction:

- *Base step:* if  $X$  is a minimum of  $w$ . Let  $u$  be the building edge of  $X$ . For any child  $Y$  of  $X$ , we can affirm that  $Y$  is a non-dominant region of  $\mathcal{B}$  and then  $\xi_f(Y) = f(u)$ , which is equal to zero because  $f$  is a one-side increasing map. Thus,  $\xi_f(Y)$  is in  $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$ .
- *Inductive step:* if  $X$  is not a minimum and the property holds for both children of  $X$ . By Property 23, we know that there is a child  $Y$  of  $X$  such that  $\xi_f(Y) = f(u_X)$  and  $\xi_f(\text{sibling}(Y)) = \xi_f(X)$ . Therefore, for any region  $Y$  such that  $Y \subseteq X$ , the value  $\xi_f(Y)$  is in  $\{\xi_f(Y), 0\} \cup \{f(u) \mid R_u \subseteq Y\} \cup \{\xi_f(\text{sibling}(Y)), 0\} \cup \{f(u) \mid R_u \subseteq \text{sibling}(Y)\} \cup \{\xi_f(X)\}$  which is equivalent to  $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$ .  $\square$

*Proof (Property 26).* We can now prove that, for any region  $R$  of  $\mathcal{B}$ , we have  $\xi_f(R) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$ . Given a region  $X$  of  $\mathcal{B}$ :

- If there is no minimum of  $w$  included in  $X$ , then  $X$  is a leaf region and  $X$  is a non-dominant region of  $\mathcal{B}$ . Then  $\xi_f(Y) = f(u)$  (third condition of Definition 7), which is equal to zero because  $f$  is a one-side increasing map. Therefore,  $\xi_f(X) = \vee\{\xi_f(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\} = \vee\emptyset = 0$
- Otherwise, for any region  $Y \subseteq X$ ,  $\xi_f(Y)$  is in  $\{\xi_f(X), 0\} \cup \{f(u) \mid R_u \subseteq X\}$  by Property 28. By Property 27,  $\xi_f(X) \geq \{f(v) \mid R_v \subseteq X\}$ . Therefore,  $\xi_f(X) \geq \xi_f(Y)$ . Then,  $\xi_f$  is increasing on the hierarchy  $\mathcal{B}$ , i.e., for any region  $X$ , we have  $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X\}$ . By Property 23, there is a minimum  $M$  of  $w$  such that  $\xi_f(X) = \xi_f(M)$ . Hence,  $\xi_f(X) = \vee\{\xi_f(Y) \mid Y \subseteq X \text{ and } Y \text{ is a minimum of } w\}$ .  $\square$

**Property 29.** *Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $\xi_f$  is an extinction map, then  $f$  is the saliency map of a hierarchical watershed.*

If  $\xi_f$  is an extinction map, then, by Property 3, we have:

- $\text{range}(P) = \{0, \dots, n\}$ ;
- for any two distinct minima  $M_1$  and  $M_2$ ,  $P(M_1) \neq P(M_2)$ ; and
- for any region  $R$  of  $\mathcal{B}$ , we have  $P(R) = \vee\{P(M) \text{ such that } M \text{ is a minimum of } w \text{ included in } R\}$ .

Now we have to prove that  $f$  is a one-side increasing map. We need to prove that the three conditions for  $f$  to be a one-side increasing map are satisfied.

**Property 30.** *Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $\xi_f$  is an extinction map, then  $\text{range}(f) = \{0, \dots, n-1\}$ .*

*Proof.* We need to prove that:

1. for any  $i$  in  $\{0, \dots, n-1\}$ , there is an edge  $u$  in  $E$  such that  $f(u) = i$ ; and
2. for any edge  $u$  in  $E$ , we have  $f(u)$  in  $\{0, \dots, n-1\}$ .

Proof of 1:

For  $i = 0$ : Since  $\xi_f$  is an extinction map, for any leaf region  $R$  of  $\mathcal{B}$ , we have  $\xi_f(R) = \vee\{\xi_f(M) \mid M \text{ is a minimum of } w \text{ included in } R\} = 0$ . Let  $R$  be a leaf region. Since  $R$  is not a dominant region for  $f$ , this means that  $\xi_f(R) = f(u)$ , where  $u$  is the building edge of  $\text{parent}(R)$ , and, since  $\xi_f(R) = 0$ , this implies that there exists an edge  $u$  in  $E$  such that  $f(u) = 0$ .

For  $i$  in  $\{1, \dots, n-1\}$ : Since  $\xi_f$  is an extinction map, then  $\text{range}(\xi_f) = \{0, \dots, n\}$ . Then, for any  $i$  in  $\{1, \dots, n-1\}$  there is a region  $R$  of  $\mathcal{B}$  such that  $\xi_f(R) = i$ . Let  $i$  be any value in  $\{1, \dots, n-1\}$  and let  $R$  be a region of  $\mathcal{B}$  such that  $\xi_f(R) = i$ . If  $R$  is not a dominant region for  $R$ , then  $\xi_f(R) = f(u)$ , where  $u$  is the building edge of the parent of  $R$  and, then, we can affirm that there exists an edge in  $E$  whose weight for  $f$  is  $i$ . Otherwise, if  $R$  is a dominant region for  $f$ , then  $\xi_f(R) = \xi_f(\text{parent}(R))$ . If  $\text{parent}(R)$  is not a dominant region for  $f$ , then  $\xi_f(\text{parent}(R)) = \xi_f(v)$ , where  $v$  is the building edge of the parent of  $\text{parent}(R)$  and we have our property. Otherwise, if  $\text{parent}(R)$  is a dominant region of  $\mathcal{B}$ , then  $\xi_f(\text{parent}(R)) = \xi_f(\text{parent}(\text{parent}(R)))$ . We can see that, at some point, we will have  $\xi_f(R) = \xi_f(\text{parent} \dots (\text{parent}(R))) = f(y)$  for an edge  $y$  in  $E$ .

Proof of 2: By contradiction, let us assume that there is an edge  $u$  in  $E$  such that  $f(u)$  is not in  $\{0, \dots, n-1\}$ . We can affirm that any non leaf region of  $\mathcal{B}$  has a child which is not a dominant region for  $f$ . So, we can affirm that there is a child  $X$  of  $R_u$  such that  $\xi_f(X) = f(u)$ . Since  $\xi_f$  is an extinction map, the range of  $\xi_f$  is  $\{0, \dots, n\}$ . Then,  $\xi_f(X) = f(u)$  should be in  $\{0, \dots, n\}$  as well. Therefore, the only value that  $f(u)$  could have and that is not in  $\{0, \dots, n-1\}$  is  $n$ . So, let us assume that  $f(u) = n$ . In this case, we would have  $\vee^f(V) = n+1$ , which contradicts the fact that  $\text{range}(\xi_f) = \{0, \dots, n\}$ . Therefore, we may conclude that, for any edge  $u$  in  $E$ , we have  $f(u)$  in  $\{0, \dots, n-1\}$ .  $\square$

**Property 31.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $\xi_f$  is an extinction map, then for any  $u$  in  $E$ ,  $f(u) > 0$  if and only if  $u \in WS(w)$ .

*Proof.* If  $\xi_f$  is an extinction map, then only the leaf nodes do not include any minimum of  $w$ , which implies that only the leaf nodes has a value equal to zero for  $\xi_f$ . We can say that any leaf region  $R$  is not a dominant region of  $\mathcal{B}$ , and then  $\xi_f(R) = f(u)$  where  $u$  is the building edge of the parent of  $R$ . We can say that an edge is a watershed-cut edge if and only if it has no leaf regions as children. This implies that  $\xi_f(R) = 0$  if and only if the parent of  $R$  is not a watershed-cut edge.  $\square$

**Property 32.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $\xi_f$  is an extinction map, then for any  $u$  in  $E$ , there exists a child  $R$  of  $R_u$  such that  $f(u) \geq \vee\{f(v) \mid v \text{ is included in } R\}$ .

In order to prove Property 32, we first present properties 33 and 34.

**Property 33.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $\xi_f$  is an extinction map then, for any region  $R$  of  $\mathcal{B}$ ,  $\xi_f(R) \geq \vee\{\xi_f(X) \mid X \subseteq R\}$ .



*Proof.* The proof is straightforward if we consider the third condition of Property 3.  $\square$

**Property 34.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\xi_f$  be the approximated extinction map of  $f$ . If  $\xi_f$  is an extinction map then, for any region  $R$  of  $\mathcal{B}$ ,  $\xi_f(R) \geq f(u)$ , where  $u$  is the building edge of  $R$ .

*Proof.* Let  $R$  be a region of  $\mathcal{B}$ :

- If  $R = V$ , then  $\xi_f(R) = \vee^f(V) + 1$ , which means that  $\xi_f(R) > f(u)$ .
- If  $R$  is not a dominant region, then  $\xi_f(R) = f(v)$  where  $v$  is the building edge of the parent of  $R$ . By Property 33, we know that  $\xi_f(\text{parent}(R)) \geq \vee\{\xi_f(X) \mid X \subseteq R\}$ , which means that  $\xi_f(\text{parent}(R)) \geq \xi_f(R) = f(v)$ , so the property holds for the parent of  $R$ .  $\square$

*Proof (Property 32).*

Let  $R$  be any region of  $\mathcal{B}$ . We have  $\xi_f(R) \geq \{\xi_f(X) \mid X \subseteq R\}$ . Then, we have  $\xi_f(R) \geq \{f(u) \mid u \text{ is the building edge of } X \subseteq R\}$ . If  $R$  is a non-dominant region, then  $\xi_f(R) = f(v)$  where  $v$  is the building edge of the parent of  $R$ . So, we will have  $f(v) \geq \{f(u) \mid u \text{ is the building edge of } X \subseteq R\}$ . Since  $R$  is a child of  $R_v$ , we can say that there is a child of  $R_v$  such that the property holds.  $\square$

*Proof (Property 8).*

Properties 20 and 29 correspond to the forward and backward implications of Property 8.  $\square$

## C Proof of Property 10

Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\mathcal{S}_f$  be the estimated sequence of minima of  $f$ . If  $f$  is the saliency map of a hierarchical watershed, then  $\xi_f$  is an extinction map by Property 8. First, we have to prove that  $\xi_f$  is the extinction map for the sequence  $\mathcal{S}_f$ . Then, we have to prove that, for any edge  $u$ , we have  $f(u) = \min\{\xi_f(X) \mid X \subseteq R_u\}$ .

**Property 35.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\mathcal{S}_f$  be the estimated sequence of minima of  $f$ . If  $\xi_f$  is an extinction map, then  $\xi_f$  is the extinction map for the estimated sequence of minima  $\mathcal{S}_f$  of  $f$ .

*Proof.* If  $\xi_f$  is an extinction map, then for any two distinct minima  $M_1$  and  $M_2$ , we have  $\xi_f(M_1) \neq \xi_f(M_2)$ . If we prove that  $\xi_f(M) > 0$  for any minimum  $M$ , then we prove that the range of  $\xi_f$  for the set of minima is  $\{1, \dots, n\}$ . Let  $M$  be a minimum of  $w$  and let us assume that  $\xi_f(M) = 0$ . If  $M$  is not a dominant region of  $\mathcal{B}$ , then  $\xi_f(M) = f(v) = 0$  where  $v$  is the building edge of  $\text{parent}(M)$ . However, if  $M$  is not a dominant region of  $\mathcal{B}$ , this implies that none of the children of  $R_v$  is a leaf region and then,  $v$  is a watershed-cut edge. Since  $f$  is one-side increasing, this implies that  $f(v)$  is strictly greater than 0. If  $M$  is a dominant region of  $\mathcal{B}$ , this implies that  $\xi_f(M) = \xi_f(\text{parent}(M))$ . Since  $\xi_f$

is an extinction map, only the leaf regions of  $\mathcal{B}$  has a zero value for  $\xi_f$ . Since the parent of  $M$  is not a leaf region, we can affirm that  $\xi_f(\text{parent}(M)) > 0$  and, therefore,  $\xi_f(M) > 0$ . So, we just proved that the range of  $\xi_f$  for the set of minima is  $\{1, \dots, n\}$ . Since  $\xi_f$  is an extinction map, we can say for any region  $R$ , we have  $\xi_f(R) = \vee\{\xi_f(M) \mid M \text{ is a minimum}\}$ . So,  $\xi_f$  is the extinction map for  $\mathcal{S}_f$ .  $\square$

**Property 36.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$  and let  $\mathcal{S}_f$  be the estimated sequence of minima of  $f$ . If  $\xi_f$  is an extinction map, then for any edge  $u$ , we have  $f(u) = \min\{\xi_f(R_u), f(u)\}$ .

*Proof.* Any region has a child which is dominant and a child which is not a dominant region of  $\mathcal{B}$ . This implies that, for any edge  $u$ , there is a child  $X$  of  $u$  such that  $\xi_f(X) = \xi_f(R_u)$  and another child  $Y$  such that  $\xi_f(Y) = f(u)$ . By Property 34, we have  $\xi_f(R_u) \geq f(u)$ , which implies that  $f(u) = \min\{\xi_f(X) \mid X \subseteq R_u\} = \min\{\xi_f(R_u), f(u)\}$ .  $\square$

## D Proof of Theorem 12

**Theorem 37.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$ . The watershedding  $\omega(f)$  of  $f$  is the saliency map of the hierarchical watershed of  $(G, w)$  for the estimated sequence of minima for  $f$ .

*Proof.* By Property 8, the map  $\xi_f$  is an extinction map and, by Property 35, the map  $\xi_f$  is an extinction map for the estimated sequence of minima  $\mathcal{S}_f$  for  $f$ . Therefore, by the Definition 11 of watershedding, the watershedding  $\omega(f)$  of  $f$  is the saliency map of the hierarchical watershed of  $(G, w)$  for the estimated sequence of minima for  $f$ .  $\square$

## E Proof of Property 13

**Property 38.** Let  $f$  be a map from  $E$  into  $\mathbb{R}^+$ . The watershedding  $\omega(\omega(f))$  of  $\omega(f)$  is equal to  $\omega(f)$ .

*Proof.* By Theorem 12, we know that  $\omega(f)$  is the saliency map of a hierarchical watershed. By Property 10, we know that  $\xi_{\omega(f)}$  is the extinction map for the estimated sequence of minima  $\mathcal{S}_{\omega(f)}$  for  $\omega(f)$ . By Property 36, we know that  $\omega(f)(u) = \min\{\xi_{\omega(f)}(X) \mid X \subseteq R_u\}$ . By Definition 11, we have  $\omega(\omega(f)) = \min\{\xi_{\omega(f)}(X) \mid X \subseteq R_u\} = \omega(f)(u)$ .  $\square$

## F Proof of Property 15

**Property 39.** Let  $\mathcal{H}$  be a hierarchy and let  $f$  be the saliency map of  $\mathcal{H}$ . The hierarchy  $\mathcal{H}$  is a hierarchical watershed of  $(G, w)$  if and only if  $\omega(f) = f$ .

*Proof.* Let  $\mathcal{H}$  be a hierarchical watershed and let  $f$  be the saliency map of  $\mathcal{H}$ . By Property 14, we may say that  $\omega(f) = f$ .

On the other hand, let  $\omega(f) = f$ . Then, for any edge  $u$ , we have  $f(u) = \omega(f)(u) = \min\{P(R) \mid R \text{ is a child of } R_u\}$ , where  $P$  is the extinction map for  $S_f$ . Therefore, by the backward implication of Property 8, we can say that  $f$  is the saliency map of the hierarchical watershed for  $S_f$ .  $\square$