



Intégration temporelle basée sur la resommation de séries divergentes: Où en est-on ?

Dina Razafindralandy, Aziz Hamdouni, Ahmad Deeb

Laboratoire des Sciences de l'Ingénieur pour l'Environnement
LaSIE - UMR CNRS 7356
Université de La Rochelle - France

Journée Méthodes Asymptotiques Numériques 2017
Metz

- ▶ Asymptotic Num. Meth. for unsteady problems: $u(t) = \sum_{n \geq 0} u_n t^n$
- ▶ But the convergence radius δt may be very small, or even vanish

Heat equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$

$$u(t = 0, x) = \frac{1}{1-x}$$

$$u(t, x) = \sum_{n \geq 0} \frac{(2k)!}{k!} \frac{t^k}{(1-x)^{2k+1}}$$

$$\delta t = 0 \text{ for all } x$$

- ▶ Goal:
 - ★ Take advantage of the divergence, using a suitable summation theory
 - ★ Build numerical schemes based on this approach
- ▶ Overview:
 - ★ Use of divergent series in history
 - ★ Borel-Laplace summation
 - ★ Summation algorithm
 - ★ Applications

► Astronomers: n -body problems

- ★ 3 bodies = 2 bodies + perturbation:

$$u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

- ★ Good agreement with measures
- ★ Yet, the series is **divergent** in the sense of geometers



► Astronomers: n -body problems

- ★ 3 bodies = 2 bodies + perturbation:

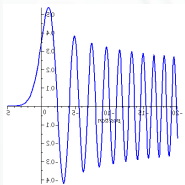
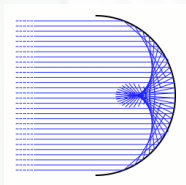
$$u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

- ★ Good agreement with measures
- ★ Yet, the series is **divergent** in the sense of geometers



► Airy: Study of rainbow (1838)

- ★ Ray reflection in the neighbourhood of a caustics
- ★ Intensity of light proportional to $Ai(t) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{x^3}{3} + tx\right) dx$



- ★ Fringe when $Ai(t) = 0$

- ▶ Airy: Asymptotic development at zero: **convergent series**

$$Ai(t) \sim 3^{-2/3} \sum_{n=0}^{\infty} \frac{t^{3n}}{9^n n! \Gamma\left(n + \frac{2}{3}\right)} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{t^{3n+1}}{9^n n! \Gamma\left(n + \frac{4}{3}\right)}$$

The terms increase before becoming small \longrightarrow Need many term

\longrightarrow Very tedious calculations

Airy: 1 fringe with 4 digits

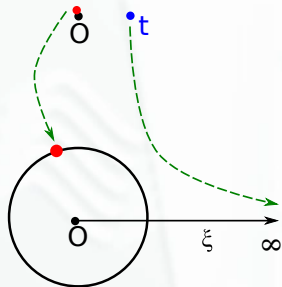
- ▶ Stokes: Change of variable (Asymptotic dev. at ∞): **divergent series**

$$Ai(t) \sim \frac{t^{-1/4} e^{-\frac{2}{3}t^{3/2}} 3^{-\frac{2}{3}}}{\pi^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{5}{6}\right) \Gamma\left(n + \frac{1}{6}\right)}{n!} \frac{3^n}{4^n} (-t)^{-\frac{3n}{2}}$$

The terms decrease before “exploding” \longrightarrow Good approximation with few terms

Summation at the smallest term

Stokes: 30 fringes with 4 digits



▶ Initial: $\hat{u}(t) = \sum_n u_n t^n \in \mathbb{C}[[t]]$

▶ “Zoom”: Borel transformation

$$\mathcal{B}\hat{u}(\xi) = \sum_n \frac{u_n}{n!} \xi^n \in \mathbb{C}[[\xi]]$$

$\mathcal{B}\hat{u}(\xi) \in \mathbb{C}\{\xi\}$ (convergent) if

Gevrey series: $|u_n| < CA^n n!$

▶ Analytical prolongation on \mathbb{R}^+

▶ Unzoom with Laplace transformation

$$\begin{array}{ccc}
 \hat{u}(t) = u_0 + \sum_{n \geq 1} u_n t^n & \stackrel{\mathcal{G}}{\sim} & \mathcal{S}\hat{u}(t) = u_0 + \int_d P(\xi) e^{-\xi/t} d\xi \\
 \downarrow \text{Borel} & & \uparrow \text{Laplace} \\
 \mathcal{B}\hat{u}(\xi) = \sum_{n \geq 0} \frac{u_{n+1}}{n!} \xi^n & \xrightarrow{\text{Prolongation}} & P(\xi)
 \end{array}$$

For a convergent series, $\hat{u}(t) = \mathcal{S}\hat{u}(t)$ inside the convergent disc

A series $\hat{u} \in \mathbb{C}[[t]]$ is **Gevrey asymptotic** (of order 1) to a function S , analytic within a sector $S \subset \mathbb{C}^*$ of apex O , if for any relatively compact subsector $W \subset\subset S$, there exists $C > 0, A > 0$,

$$\left| S(t) - \sum_{n=0}^{N-1} u_n t^n \right| = CA^N N! t^N, \quad \forall t \in W, N \in \mathbb{N}$$

s -Borel transform, $s > 0$)

$$\mathcal{B}\hat{u}(\xi) = \sum_{n \geq 0} \frac{u_{n+1}}{(n!)^s} \xi^n \in \mathbb{C}[[\xi]]$$

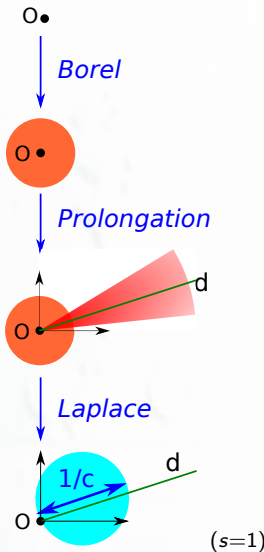
s -Gevrey series if

$$|u_n| < CA^n (n!)^s \quad \forall n \in \mathbb{N}$$

s -Laplace transform along a direction $d = [0, e^{i\theta} \infty[$

$$\mathcal{L}_d P(t) = \int_d P(\xi) e^{-\left(\frac{\xi}{t}\right)^s} d\xi^s.$$

Exponential growth: $P(\xi) = O(e^{c\xi^s})$ along d at ∞



Borel-(Padé-)Laplace integrator

$$\hat{u}^N(t) = \sum_{n=0}^N u_n t^n$$

Borel

$$\mathcal{B}\hat{u}^N(\xi) = \sum_{n=0}^{N-1} \frac{u_{n+1}}{n!} \xi^n$$

Padé

$$\mathcal{S}^N \hat{u}(t) = u_0 + \sum_{i=1}^{NG} \tilde{P}(\xi_i) \omega_i$$

Gauss-Laguerre

$$P(\xi) = \frac{P_0 + P_1 \xi + \dots + P_L \xi^L}{1 + Q_1 \xi + \dots + Q_M \xi^M}$$

- ▶ Padé approximants efficient on $\mathcal{B}\hat{u}(\xi)$ (convergent series)
- ▶ Exponential growth of $|P(\xi)|$ at infinity
- ▶ Choice of the Laplace direction d : poles of $P(\xi)$

$$t^2 \frac{du}{dt} + u = t, u(0) = 0$$

$$\hat{u}(t) = \sum_{n \geq 0} (-1)^n n! t^{n+1}$$

Borel ↓

$$\mathcal{B}\hat{u}(\xi) = \sum_{n \geq 0} (-1)^n \xi^n$$

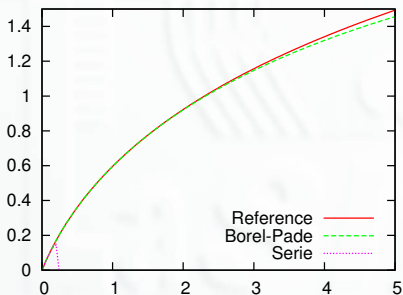
→ Prolongation

$$\int_0^{+\infty} \frac{1}{1+\xi} e^{-\xi/t} d\xi, \operatorname{Re}(t) \geq 0$$

↑ Laplace

$$P(\xi) = \frac{1}{1+\xi}$$

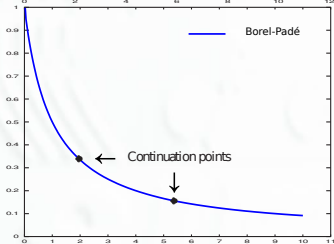
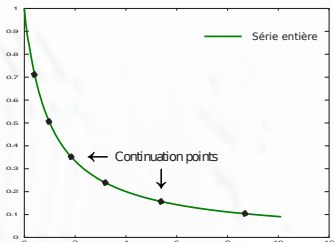
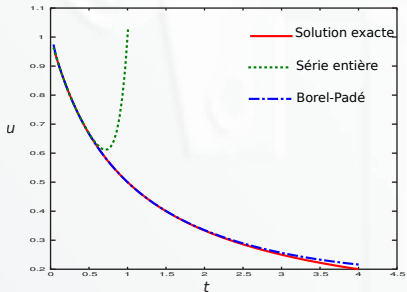
- N=10



$$\frac{du}{dt} + u^2 = 0, \quad u(0) = 1$$

Solution: $u(t) = \frac{1}{1+t}, \quad \hat{u} = \sum_{n \geq 0} (-1)^n$

Truncation : $N = 8$



Borel-Padé: 2 itérations versus 6

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

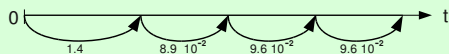
► Spatial discretization:

$$\frac{du^i}{dt} = \frac{u^{i-1} - 2u^i + u^{i+1}}{\Delta x^2}$$

► Solution: $e^{-t} \sin x$. $N = 10$

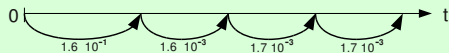
► Mesh: $N_x = 16$.

Explicit Euler $\Delta t_E = \Delta x^2/2 = 1.7 \cdot 10^{-3}$

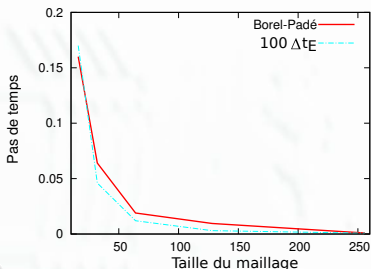


$830 \times \Delta t_E$

► Mesh : $N_x = 128$. $\Delta t_E = 3.0 \cdot 10^{-5}$



$5300 \times \Delta t_E$



	N=16	N=128
Euler	22	1913
BPL	2	343
Ratio	10%	18%

CPU (milliseconds) over 10 s

► Burger's, Navier-Stokes



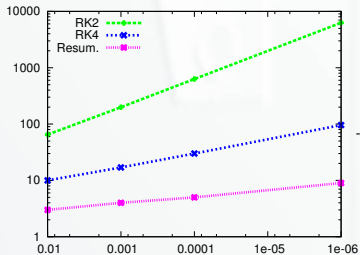
$$\begin{cases} \dot{q} = \nabla_p H(q, p, t) \\ \dot{p} = -\nabla_q H(q, p, t) \end{cases}$$



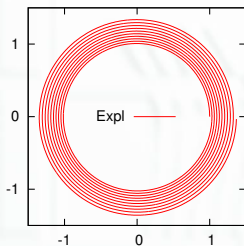
$dp(t) \wedge dq(t)$
conserved



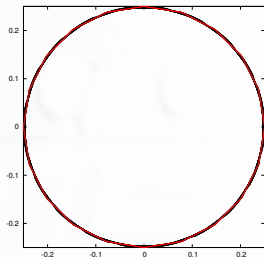
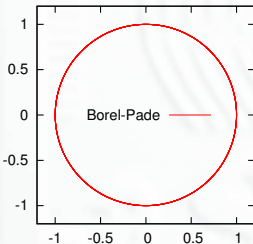
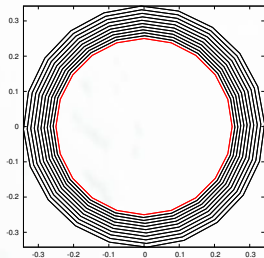
Iterations / 1 period



Trajectory 10 periods



Symplecticity



- ▶ Completely integrable Lax system

$$\begin{cases} \dot{q}_i = \nabla_{p_i} H \\ \dot{p}_i = -\nabla_{q_i} H \end{cases} \quad i = 1, \dots, n$$

- ▶ Lax pair $\dot{L} = [L, M]$

- ▶ Theorem: $\frac{d\lambda_i}{dt} = 0, \quad i = 1, \dots, n$

- ▶ Completely integrable Lax system

$$\begin{cases} \dot{q}_i = \nabla_{p_i} H \\ \dot{p}_i = -\nabla_{q_i} H \end{cases} \quad i = 1, \dots, n$$

- ▶ Lax pair $\dot{L} = [L, M]$

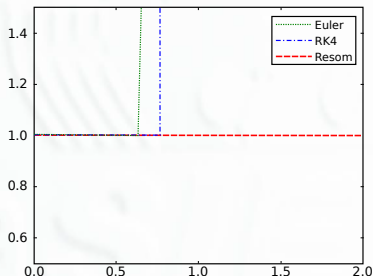
- ▶ Theorem: $\frac{d\lambda_i}{dt} = 0, \quad i = 1, \dots, n$

- ▶ Example:

$$\star L(t) = \begin{pmatrix} \tan(t + \frac{\pi}{4}) & 1 \\ -\cos^{-2}(t + \frac{\pi}{4}) & -\tan(t + \frac{\pi}{4}) \end{pmatrix}$$

$$\star \lambda_1 = i, \quad \lambda_2 = -i$$

★ Figure: evolution of $|\lambda_1|$



- ▶ Only very few explicit schemes work
- ▶ Exponential integrators (ETD: exponential time differencing) semi-explicit

- ★ $\frac{du}{dt} = Lu + N(t, u)$

- ★ Exact integration of the linear part: $u(t) = e^{tL} u_0 + \int_0^t e^{(t-\tau)L} N(\tau, u(\tau)) d\tau$

- ★ Runge-Kutta RK4 for the resolution of the integral \longrightarrow ETRK4

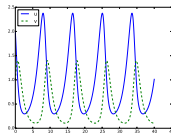
- ▶ Borel-Padé-Laplace integrator, $N = 10$

- ★ $u_{n+1} = \frac{1}{n+1} [Lu_n + N_n(u_0, \dots, u_n)]$

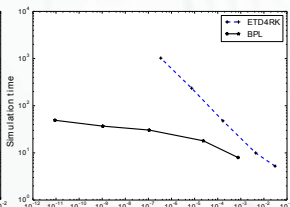
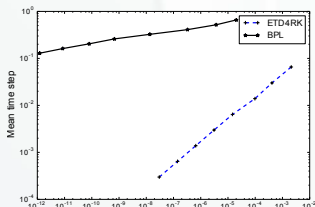
- ★ Summation

- ▶ Direct Adaptive RK4
- ▶ Direct Adaptive RK, with 5 Gauss-quadrature based implicit internal stages

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \beta uv \\ \gamma uv \end{pmatrix}$$



- ▶ First integral: $I(t) = \beta v + \gamma u - \alpha \ln v - \delta \ln u$
- ▶ Non-stiff case: $\alpha = \frac{2}{3}, \delta = 1, \beta = \frac{4}{3}, \gamma = 1$
- ▶ Calibration: $\epsilon = \frac{|\bar{I}(t) - I(0)|}{|I(0)|} \leq \epsilon_I \quad \text{for } t \in [0, 500]$



$$\epsilon_I = 10^{-6},$$

$$CPU_{ETDRK4} =$$

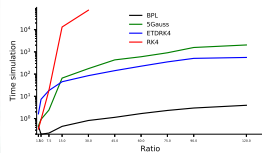
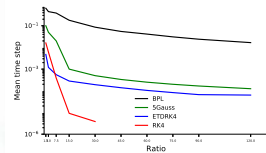
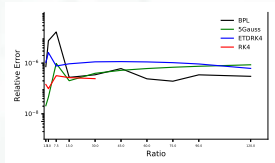
$$25 \times CPU_{BPL}$$

Evolution of the time step and CPU with $\epsilon_I, t \in [0, 500]$

★ Increasing stiffness ratio $r = \frac{\max(\alpha, \delta)}{\min(\alpha, \delta)}$

★ $N = 5$

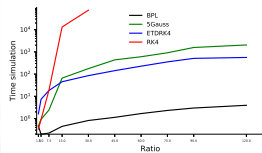
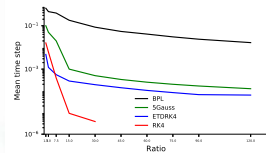
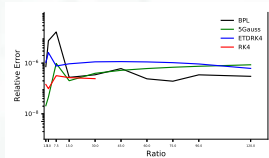
Parameter calibration: Approximately the same order of precision



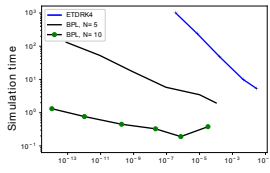
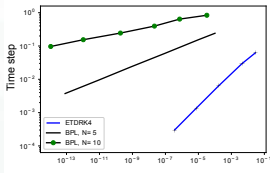
★ Increasing stiffness ratio $r = \frac{\max(\alpha, \delta)}{\min(\alpha, \delta)}$

★ $N = 5$

Parameter calibration: Approximately the same order of precision



★ $N = 5$ and $N = 10$



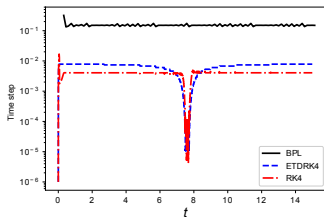
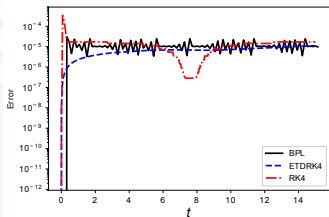
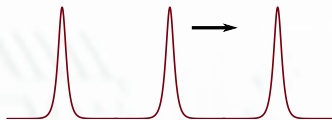
For $r = 120$ and $N = 10$

• Time step
BPL: $1.7 \cdot 10^{-2}$
ETDRK4: $6.4 \cdot 10^{-5}$

• CPU :
BPL: 3.95.
ETDRK4: 516.14 (131 times)

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \frac{\alpha}{2} \frac{\partial u^2}{\partial x} = 0$$

- ▶ Spectral discretisation in space
- ▶ $u(t, x) = h / \cosh^2(\kappa(x - ct))$

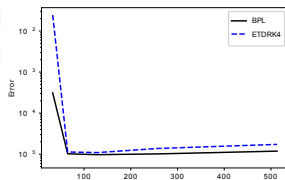


Over one period, with $d = 128$

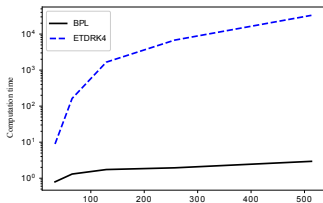
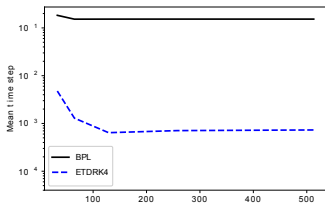
	BPL	ETDRK4
L^2 error at $t = T$	$9.76 \cdot 10^{-06}$	$1.10 \cdot 10^{-05}$
Mean time step	$1.53 \cdot 10^{-01}$	$6.41 \cdot 10^{-04}$ (/238)
Simulation time	1.74	$1.66 \cdot 10^{+03}$ ($\times 954$)

- ▶ Increasing the size d of the problem (refining spectral grid)
- ▶ Δt decreases slowly with BPL and ETD RK4
- ▶ CPU growth rate between $d = 256$ and $d = 512$

BPL: 0.51 ETD RK4: 3.81



Evolution of Δt and CPU with d



Integration by parts of $\mathcal{S}u(t)$

Inverse factorial series algorithm

$$\hat{u}(t) = u_0 + \sum_{n \geq 1} u_n t^n \quad \longrightarrow \quad \mathcal{S}\hat{u}(t) = u_0 + \sum_{n \geq 0} \frac{b_n n!}{\frac{1}{t} \cdots (\frac{1}{t} + n)}$$

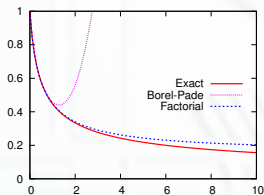
Borel
Laplace

$$B\hat{u}(\xi) = \sum_{n \geq 0} \frac{u_{n+1}}{n!} \xi^n \quad \xrightarrow{\text{Prolongation}} \quad P(\xi)$$

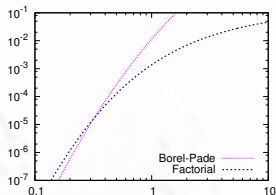
- ▶ $b_n = \frac{1}{n!} \sum_{k=1}^{n+1} |S_{k-1}^n| u_k$ S_k^n Stirling numbers of 1st kind
- ▶ Unique, absolutely convergent for $\text{Re}(1/t)$ big enough
- ▶ Avoids quadrature error, artificial poles in Padé, non unique Padé

$$\frac{du}{dt} + u^3 = 0,$$

$$u(0) = 1$$



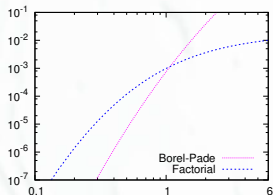
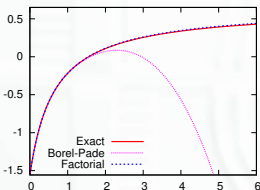
Approximate solution



Error

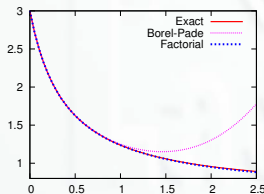
$$\frac{du}{dt} - \frac{1 + u^2}{1 + t^2} = 0,$$

$$u(0) = -1.5$$



Population dynamics

$$\frac{du}{dt} = u(1 - u) - \frac{1}{2}$$



- ▶ Interesting qualities
 - ★ Large time steps, low CPU
 - ★ High-order Symplecticity
 - ★ Isospectrality
 - ★ Ability to cross some types of singularities
 - ★ Efficiency for stiff problems, despite explicit character

- ▶ Theoretical questions
 - ★ Gevrey order
 - ★ Laplace direction
 - ★ \longrightarrow Optimization

- ▶ Prospects: large size ODE