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1 Introduction

The physical properties of a phenomenon which has been modelized into a set of equations are contained in the symmetry group of these equations. This fact is well illustrated by Nœther's theorem [5, 7] which states that for a Lagrangian system, each symmetry of the equations corresponds to a conservation law. For example, a system having a time-translation symmetry is energy conserving.

In fluid mechanics, the symmetry group theory has been used to study, among others, the dynamics of vortices in a fluid flow [3, 4] and to investigate spectral properties of the flow [9]. Oberlack [6] also used the symmetry theory to deduce scaling laws, such as the algebraic law which has been confirmed in the center and in the near-wall region of a channel flow, the usual logarithmic wall law, a linear law which is observed in the center region of a couette flow. He also found a new exponential law which has been confirmed later in the mid-wake region of the boundary layer.

The first part of this paper extends the work of Oberlack to non-isothermal and parallel flow, such as a pipe flow. Scaling laws for both velocity and temperature will be given. The second part is devoted to the construction of a class of turbulence models of a non-isothermal flow, which respect the physics of the flow.

Indeed, in spite of many decades of studies, turbulence modelling still constitutes an important challenge in engineering science. No turbulence model, either with RANS method or with large-eddy simulation (LES) approach, is really satisfying regarding their mathematical and physical properties. Moreover, the subgrid heat flux model is, most of the time, simply deduced from the subgrid stress tensor model by a Reynolds analogy. This analogy limits the scope of the models for strongly coupled problems (natural and mixed convection). In this paper, we propose a general class of subgrid stress and heat flux models based on the preservation of the symmetry group (and then the underlying physical properties) of the non-isothermal Navier–Stokes equations (1).

2 Scaling laws in a parallel flow

Consider the Navier–Stokes equations for an incompressible, non-isothermal Newtonian fluid, with kinematic viscosity ν and thermal diffusivity κ:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \dfrac{1}{\rho}\nabla p - \operatorname{div}\, \tau_r - \beta g \theta\, \boldsymbol{e}_2 = 0 \\ \dfrac{\partial \theta}{\partial t} + \operatorname{div}(\theta \boldsymbol{u}) - \operatorname{div}\, \mathbf{h}_r = 0 \\ \operatorname{div}\, \boldsymbol{u} = 0 \end{cases} \tag{1}$$

In these expressions, $\tau_r = 2\nu\mathsf{S}$ and $\mathbf{h}_r = \kappa\nabla\theta$ are respectively the viscous stress tensor and the heat flux, and $\mathsf{S} = \frac{\nabla\boldsymbol{u}+^T\nabla\boldsymbol{u}}{2}$ is the strain rate tensor. β is the thermal expansion and g the acceleration due to gravity.

Using the RANS approach, we decompose the velocity, the pressure and the temperature into a mean and a fluctuating quantities:

$$f = F + f'. \tag{2}$$

As we are seeking for stationnary bidimensional scaling laws in a parallel flow, we assume that the mean pressure is of the form $P(x_2) - x_1 K$, that $U_2 = U_3 = 0$ and that U_1 and Θ depend only on the wall normal coordinate x_2. The fluctuating quantities verify the following set of equations:

$$\begin{cases} \dfrac{\partial u_i'}{\partial t} + U_1\dfrac{\partial u_i'}{\partial x_1} + u_2'\dfrac{\mathrm{d}U_1}{\mathrm{d}x_2}\delta_{i1} - \left(K + \nu\dfrac{\partial^2 U_1}{\partial x_2^2}\right)\delta_{i1} + \left(\dfrac{\partial P}{\partial x_2} - \beta g\Theta - \beta g\theta'\right)\delta_{i2} \\ \qquad + \dfrac{\partial u_i'u_k'}{\partial x_k} + \dfrac{\partial p'}{\partial x_i} - \nu\dfrac{\partial^2 u_i'}{\partial x_k^2} = 0 \\ \dfrac{\partial \theta'}{\partial t} + \dfrac{\partial \theta' u_j'}{\partial x_j} - \kappa\dfrac{\partial^2\theta'}{\partial x_j^2} - \kappa\dfrac{\partial^2\Theta}{\partial x_2^2} + u_2'\dfrac{\partial\Theta}{\partial x_2} + U_1\dfrac{\partial\theta'}{\partial x_1} = 0 \\ \dfrac{\partial u_k'}{\partial x_k} = 0 \end{cases} \tag{3}$$

For simplicity, we also designate system (3) by

$$E(t, \boldsymbol{x}, \boldsymbol{U}, \boldsymbol{u}', P, p', \Theta, \theta', \nu, \kappa) = 0 \tag{4}$$

Scaling laws are self-similar solutions of (3). In what follows, we use Lie's theory to find the symmetries of the equations and to deduce the self-similar solutions.

A (Lie point-)symmetry of (3) is a transformation

$$(t, \boldsymbol{x}, \boldsymbol{U}, \boldsymbol{u}', P, p', \Theta, \theta', \nu, \kappa) \mapsto (\widehat{t}, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{U}}, \widehat{\boldsymbol{u}}', \widehat{P}, \widehat{p}', \widehat{\Theta}, \widehat{\theta}', \widehat{\nu}, \widehat{\kappa}) \tag{5}$$

depending continuously on a parameter a, which maps a solution of (3) into another solution. Knowing transformation (5) is equivalent to knowing its generator

$$X = \xi_t\frac{\partial}{\partial t} + \xi_{x_i}\frac{\partial}{\partial x_i} + \xi_{U_1}\frac{\partial}{\partial U_1} + \xi_P\frac{\partial}{\partial P} + \xi_\Theta\frac{\partial}{\partial\Theta} + \xi_{u_i'}\frac{\partial}{\partial u_i'} + \xi_{p'}\frac{\partial}{\partial p'} + \xi_{\theta'}\frac{\partial}{\partial\theta'} \tag{6}$$

where

$$\xi_q = \left.\frac{\partial\widehat{q}}{\partial a}\right|_{a=0}. \tag{7}$$

According to Lie's theory, transformation (5) is a symmetry of (4) if and only if

$$E = 0 \;\Rightarrow\; X.E = 0. \tag{8}$$

This last condition permits to compute the components of the symmetry generators X of (3), which are

$$
\begin{aligned}
\xi_{\nu} &= n\nu, \\
\xi_{\kappa} &= m\kappa, \\
\xi_{t} &= [n-2m]\,t + c_1, \\
\xi_{x_1} &= [n-m]\,x_1 + \alpha_1(t) + c_2, \\
\xi_{x_2} &= [n-m]\,x_2 + c_3, \\
\xi_{x_3} &= [n-m]\,x_3 + \alpha_2(t) + g, \\
\xi_{U_1} &= mU_1 - \alpha_3(t) + \alpha_4(x_2) + \dot{\alpha}_1(t), \\
\xi_{P} &= 2mP + \beta g x_2 \alpha_5(t, x_1, x_3) + \alpha_6(t, x_1, x_3) + \alpha_9(t, x_2), \\
\xi_{\Theta} &= [3m-n]\,\Theta - \alpha_7(x_2) + \alpha_5(t, x_1, x_3), \\
\xi_{u_1'} &= mu_1' + \alpha_3(t) - \alpha_4(x_2), \\
\xi_{u_2'} &= mu_2', \\
\xi_{u_3'} &= mu_3' + \dot{\alpha}_2(t), \\
\xi_{p'} &= 2mp' + [K(n-3m) - \dot{\alpha}_3(t)]\,x_1 - \ddot{\alpha}_2(t)x_3 + \alpha_8(t) - \alpha_9(t, x_2), \\
\xi_{\theta'} &= [3m-n]\,\theta' + \alpha_7(x_2).
\end{aligned}
\tag{9}
$$

where n, m, the c_i's and the α_i's, which are arbitrary constants and functions, are the parameters of the symmetries.

(U_1, Θ) represents a self-similar solution of (3) if

$$X.U_1 = 0 \qquad \text{and} \qquad X.\Theta = 0,$$

that is, if

$$\frac{\mathrm{d}U_1}{\xi_{U_1}} = \frac{\mathrm{d}\Theta}{\xi_{\Theta}} = \frac{\mathrm{d}x_2}{\xi_{x_2}} \tag{10}$$

In the particular case where the α_i's are constant, the resolution of (10) leads to particular scaling laws.

- If $m \neq 0$, $n \neq m$, $n \neq 3m$, we get the following algebraic scaling law:

$$U_1 = C_1 \left[x_2 + \frac{b}{n-m}\right]^{\frac{m}{n-m}} + \frac{C_3}{m}, \qquad \Theta = C_2 \left[x_2 + \frac{b}{n-m}\right]^{\frac{3m-n}{n-m}} + \frac{C_4}{3m-n}. \tag{11}$$

- A logarithmic velocity profile arises if $m = 0$ and $n \neq 0$:

$$U_1 = \frac{C_1}{n} \ln\left(x_2 + \frac{b}{n}\right) + C_3, \qquad \Theta = C_2 \left[x_2 + \frac{b}{n}\right]^{-1} + \frac{C_4}{n}. \tag{12}$$

- A linear law appears when $m = n = 0$:

$$U_1 = C_1 x_2 + C_3, \qquad \Theta = C_2 x_2 + C_4. \tag{13}$$

- We obtain an exponential law when $m = n \neq 0$:

$$U_1 = C_1 \exp\left(\frac{m x_2}{b}\right) + \frac{C_3}{m}, \qquad \Theta = C_2 \exp\left(\frac{2m x_2}{b}\right) + \frac{C_4}{2m}. \tag{14}$$

As noticed in the introduction, the corresponding scaling laws in the isothermal case have all been confirmed either by experiments or by DNS data.

As we could see, the symmetry group are of paramount importance in modelling physical phenomena. It contains, among other, the scaling laws of fluid flows. In particular, turbulence models should also preserve the symmetry group of the equations. In the next section, we build a class of LES turbulence models, which are compatible with the symmetry group of the non-isothermal Navier–Stokes equations. These models should naturally reproduce the physical properties of the flow, such as scaling laws.

3 Symmetry-preserving turbulence models

Let us consider equations (1). Using the same approach as above, the symmetry generator of these equations can be computed. Its components are

$$\begin{array}{rcl}
\xi_t &=& c_1 + 2tc_3, \\
\xi_{x_1} &=& \quad c_2 x_2 + [c_3 + c_5]x_1 + \alpha_1(t), \\
\xi_{x_2} &=& -c_2 x_1 + [c_3 + c_5]x_2 + \alpha_2(t), \\
\xi_{x_3} &=& \qquad\quad [c_3 + c_5]x_3 + \alpha_3(t), \\
\xi_{u_1} &=& \quad c_2 u_2 + [c_5 - c_3]u_1 + \dot{\alpha}_1(t), \\
\xi_{u_2} &=& -c_2 u_1 + [c_5 - c_3]u_2 + \dot{\alpha}_2(t), \\
\xi_{u_3} &=& \qquad\quad [c_5 - c_3]u_1 + \dot{\alpha}_3(t), \\
\xi_p &=& \zeta(t) + 2[c_5 - c_3]p - \rho(\boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t)) + c_4 \beta g x_3, \\
\xi_\theta &=& [c_5 - 3c_3]\theta + c_4/\rho, \\
\xi_\nu &=& 2c_5\nu, \\
\xi_\kappa &=& 2c_5\kappa.
\end{array} \tag{15}$$

With the generator, one get the following list of symmetries of the non-isothermal Navier–Stokes equations (1):

- the group of *time translations*: $(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t + a, \boldsymbol{x}, \boldsymbol{u}, p, \theta)$
- the group of *pressure translations*: $(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}, \boldsymbol{u}, p + \zeta(t), \theta)$
- the group of *pressure-temperature translations*:
  $$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x}, \boldsymbol{u}, p + a\,\beta g\, x_3, \theta + a/\rho)$$
- the group of *horizontal rotations*: $(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \mathsf{R}\boldsymbol{x}, \mathsf{R}\boldsymbol{u}, p, \theta)$
  where R is a 2D (constant) rotation matrix,
- the group of *generalized Galilean transformations*:
  $$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (t, \boldsymbol{x} + \boldsymbol{\alpha}(t), \boldsymbol{u} + \dot{\boldsymbol{\alpha}}(t), p + \rho\, \boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t), \theta)$$
- the group of the *first scale transformations*:
  $$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \longmapsto (\mathrm{e}^{2a} t, \mathrm{e}^{a} \boldsymbol{x}, \mathrm{e}^{-a} \boldsymbol{u}, \mathrm{e}^{-2a} p, \mathrm{e}^{-3a} \theta)$$
- and the group of the *second scale transformations*:
  $$(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta, \nu, \kappa) \longmapsto (t, \mathrm{e}^{a} \boldsymbol{x}, \mathrm{e}^{a} \boldsymbol{u}, \mathrm{e}^{2a} p, \mathrm{e}^{a} \theta, \mathrm{e}^{2a} \nu, \mathrm{e}^{2a} \kappa).$$

Equations (1) own other known symmetries. These symmetries are not local continuous transformations like the previous ones and cannot be deduced from Lie's theory. They are:

- the *reflections* which are discrete symmetries:: $(t, \boldsymbol{x}, \boldsymbol{u}, p, \theta) \mapsto (t, \Lambda\boldsymbol{x}, \Lambda\boldsymbol{u}, p, \iota_3\theta)$
  where Λ is a reflection matrix, $\Lambda = diag(\iota_1, \iota_2, \iota_3)$ with $\iota_i = \pm 1$,
- and the *material indifference* in the limit of a 2D horizontal flow [2] which is a time-dependent rotation: $(t, \boldsymbol{x}, \boldsymbol{u}, p) \mapsto (t, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}}, \widehat{p})$
  with
  $$\widehat{\boldsymbol{x}} = \mathsf{R}(t)\ \boldsymbol{x}, \qquad \widehat{\boldsymbol{u}} = \mathsf{R}(t)\ \boldsymbol{u} + \dot{\mathsf{R}}(t)\ \boldsymbol{x}, \qquad \widehat{p} = p - 3\omega\boldsymbol{\phi} + \omega^2 \|\boldsymbol{x}\|^2/2$$

where $\mathsf{R}(t)$ is an horizontal 2D rotation matrix with angle ωt, ω a real parameter, $\boldsymbol{\phi}$ the usual 2D stream function such that ($\boldsymbol{u} = \mathrm{curl}(\boldsymbol{\phi}\boldsymbol{e}_3)$) and $\|\bullet\|$ indicates the Euclidian norm. The material indifference constitutes a non-local symmetry of the equations.

The combination of all these symmetries constitutes the symmetry group of (1).

In LES approach, one does not solve directly equations (1). Instead, one computes an approximate solution $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ which contains only the large scales of the actual velocity $\boldsymbol{u}$, pressure p and temperature θ of the fluid. The separation of large and small scales is defined by a filtering. The equations of $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ are:

$$\begin{cases} \dfrac{\partial \overline{\boldsymbol{u}}}{\partial t} + \mathrm{div}(\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + \dfrac{1}{\rho}\nabla \overline{p} - \mathrm{div}(\overline{\tau}_r - \tau) - \beta g \overline{\theta}\, \boldsymbol{e}_2 = 0 \\ \dfrac{\partial \overline{\theta}}{\partial t} + \mathrm{div}(\overline{\theta}\overline{\boldsymbol{u}}) - \mathrm{div}(\overline{\mathbf{h}}_r - \mathbf{h}) = 0 \\ \mathrm{div}\, \overline{\boldsymbol{u}} = 0. \end{cases} \tag{16}$$

In these equations, $\tau = \overline{\boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}$ is the subgrid stress tensor and $\mathbf{h} = \overline{\theta \boldsymbol{u}} - \overline{\theta}\overline{\boldsymbol{u}}$ the subgrid heat flux. In order to close the equations, these two terms must be modelled, *i.e.*, replaced by functions of $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$, called turbulence model. A "good" turbulence model is one with which $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ has the same properties as $(\boldsymbol{u}, p, \theta)$ from certain point of view. In our approach, we require that $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$ has the same symmetry properties as $(\boldsymbol{u}, p, \theta)$. This requirement is important because, as underlined earlier, the symmetry group contains important physical information on the flow.

More precisely, the model should be such that each symmetry of (1) applied to $(\boldsymbol{u}, p, \theta)$ is also a symmetry of (16) applied to $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\theta})$. When it is the case, the model will be said *invariant*. Unfortunately, as analyzed in ([8]), the major part of existing LES turbulence models do not comply with this requirement. In what follows, we propose new models based on the preservation of the symmetry group of the equations.

Time, pressure, and pressure-temperature translations, and the galilean transformations remain symmetries of (16) if τ and $\mathbf{h}$ depend only on $\overline{\mathsf{S}}$ and $\overline{\mathbb{T}} = \nabla\overline{\theta}$:

$$-\tau = -\tau(\overline{S}, \overline{\mathbb{T}}), \qquad -\mathbf{h} = -\mathbf{h}(\overline{S}, \overline{\mathbb{T}}) \tag{17}$$

Next, rotations, reflections and the material indifference are symmetries of (16) if the model has the following form

$$\begin{cases} -\tau^d = E_1 \overline{S} \ + E_2(\,\mathrm{Adj}\,\overline{S})^d \ + E_3(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d \ + E_4 \left[\overline{S}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})\right]^d \ + E_5 \left[\overline{S}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})\overline{S}\right]^d \\ -\mathbf{h} = E_6 \overline{\mathbb{T}} \ + E_7 \overline{S}\ \overline{\mathbb{T}} \ + E_8 \overline{S}^2 \overline{\mathbb{T}} \end{cases} \tag{18}$$

where the coefficients E_i are scalar functions of:

$$\mathcal{X} = \mathrm{tr}\,\overline{S}^2, \qquad \xi = \det \overline{S}, \qquad \vartheta = \overline{\mathbb{T}}^2, \qquad \omega_1 = \overline{\mathbb{T}} \cdot \overline{S}\ \overline{\mathbb{T}}, \qquad \omega_2 = \overline{S}\ \overline{\mathbb{T}} \cdot \overline{S}\ \overline{\mathbb{T}} \tag{19}$$

and Adj is the adjoint operator. Next, equations (16) are invariant under the first scale transformations if

$$\widehat{\tau} = \mathrm{e}^{-2a}\,\tau \qquad \text{and} \qquad \widehat{\overline{\mathbb{T}}} = \mathrm{e}^{-4a}\,\overline{\mathbb{T}}. \tag{20}$$

This condition is fullfilled if

$$E_i(\mathcal{X},\ \xi,\ \vartheta,\ \omega_1,\ \omega_2) = \mathcal{X}^{s_i} E'_i(v_1, v_2, v_3, v_4) \tag{21}$$

where the v_i's are the invariants:

$$v_1 = \frac{\xi}{\mathcal{X}^{3/2}}, \quad v_2 = \frac{\vartheta}{\mathcal{X}^2}, \quad v_3 = \frac{\omega_1}{\mathcal{X}^{5/2}}, \quad v_4 = \frac{\omega_2}{\mathcal{X}^{3.}} \tag{22}$$

and

$$s_1 = 0, \quad s_2 = -\tfrac{1}{2}, \quad s_3 = -\tfrac{3}{2}, \quad s_4 = -2, \quad s_5 = -\tfrac{5}{2}, \quad s_6 = 0, \quad s_7 = -\tfrac{1}{2}, \quad s_8 = -1$$

Finally, the second scale transformation are symmetries of equations (16) if

$$E'_i = \nu F_i, \qquad i = 1, ..., 5 \qquad \text{and} \qquad E'_i = \kappa F_i, \qquad i = 6, ..., 8.$$

To sum up, we get the following class of subgrid models which are consistent with the symmetry group of (1):

$$\begin{cases} -\tau^d = \nu F_1 \overline{S} + \nu \mathcal{X}^{-1/2} F_2 \,\mathrm{Adj}^d\, \overline{S} + \nu \mathcal{X}^{-3/2} F_3 (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d \\ \qquad + \nu \mathcal{X}^{-2} F_4 [\overline{S}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})]^d] + \nu \mathcal{X}^{-5/2} F_5 \overline{S} [(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}}) \overline{S}]^d] \\ -\mathbf{h} = \kappa \left(F_6 + \mathcal{X}^{-1/2} F_7 \overline{S} + \mathcal{X}^{-1} F_8 \overline{S}^2 \right) \overline{\mathbb{T}} \end{cases} \tag{23}$$

In order to reduce the degree of freedom of the model, we propose to restrain the class (23) to models which derive from a potential. This restriction is legitimated by the fact that, like τ_r and $\mathbf{h}_r$, τ and $\mathbf{h}$ represent respectively a (subgrid) stress and a (subgrid) heat flux. Moreover, τ_r and $\mathbf{h}_r$ are derived from the potentials $\nu\ \mathrm{tr}\,\overline{\mathsf{S}}^2$ and $\kappa||\overline{\mathbb{T}}||^2/2$, in the sense that

$$\tau_r = \frac{\partial \nu\ \mathrm{tr}\,\overline{\mathsf{S}}^2}{\partial \overline{\mathsf{S}}} \qquad \text{and} \qquad \mathbf{h}_r = \frac{\partial}{\partial \mathbb{T}} \left(\frac{1}{2} \kappa ||\overline{\mathbb{T}}||^2 \right) \tag{24}$$

Hence, τ and $\mathbf{h}$ should also be derived from potentials. This condition leads to the following class of models (see [8]):

$$\begin{cases} -\tau^d = \nu \left[2g_m - 3v_1 \dfrac{\partial g_m}{\partial v_1} - 4v_2 \dfrac{\partial g_m}{\partial v_2} - 5v_3 \dfrac{\partial g_m}{\partial v_3} - 6v_4 \dfrac{\partial g_m}{\partial v_4} \right] \overline{S} \\ \qquad + \nu \left[\mathcal{X}^{-1/2} \dfrac{\partial g_m}{\partial v_1} \,\mathrm{Adj}^d\, \overline{S} + \mathcal{X}^{-3/2} \dfrac{\partial g_m}{\partial v_3} (\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})^d + 2\mathcal{X}^{-2} \dfrac{\partial g_m}{\partial v_4} [\overline{S}(\overline{\mathbb{T}} \otimes \overline{\mathbb{T}})]^d \right] \\ -\mathbf{h} = \kappa \left(\dfrac{\partial g_t}{\partial v_2} I_3 + \mathcal{X}^{-1/2} \dfrac{\partial g_t}{\partial v_3} \overline{S} + \mathcal{X}^{-1} \dfrac{\partial g_t}{\partial v_4} \overline{S}^2 \right) \overline{\mathbb{T}} \end{cases} \tag{25}$$

where g_m and g_t are arbitrary functions of the v_i's. Note that the assumption that the model derives from a convex potential ensures the stability of the model [8].

Further simplifications can be done on class (25) according to the type of model that we wish to obtain. For example,

- if g_m and g_t are only functions of v_1 and v_2, we get a strongly coupled model:

$$-\tau^d = \nu\left(2g_m - 3v_1\frac{\partial g_m}{\partial v_1} - 4v_2\frac{\partial g_m}{\partial v_2}\right)\overline{S} + \nu\frac{1}{||\overline{S}||}\frac{\partial g_m}{\partial v_1}\ \mathrm{Adj}^d\,\overline{S}, \qquad -\mathbf{h} = \kappa h_t\overline{\mathbb{T}} \tag{26}$$

  where $h_t = \dfrac{\partial g_t}{\partial v_2}$.

- A decoupled model can be obtained if g_m is function only of v_1 and g_t function of v_2:

$$-\tau^d = \nu(2g_m - 3v\dot{g}_m)\overline{S} + \nu\frac{1}{||\overline{S}||}\,\dot{g}_m\ \mathrm{Adj}^d\,\overline{S}, \qquad -\mathbf{h} = \kappa h_t\ \overline{\mathbb{T}}. \tag{27}$$

- Finally, we get a linear model when g_m and h_t are linear functions of v:

$$-\tau^d = \nu C_m\left(-\det\overline{S}\frac{1}{\left\|\overline{S}\right\|^3}\overline{S} + \mathrm{Adj}^d\,\overline{S}\frac{1}{\|\overline{S}\|}\right), \qquad -\mathbf{h} = \kappa C_t\frac{\det\overline{S}}{\left\|\overline{S}\right\|^3}\,\overline{\mathbb{T}} \tag{28}$$

  where C_m are C_t are the constants of the model, which may depend on the grid size. These constants may also be evaluated dynamically, using the Germano-Lilly procedure.

The simpliest model (28) (called *invariant* model) and its dynamical version (called *dynamic invariant* model) have been implemented for the simulation of an air flow in a ventilated 3D room (see Figure 1). The floor is heated at $35°C$ and the other walls are maintained at $15°C$. The code used for the simulation is based on a finite volume scheme, with Crank-Nicholson scheme in time [1]. The grid size is $(86 \times 86 \times 12)$.

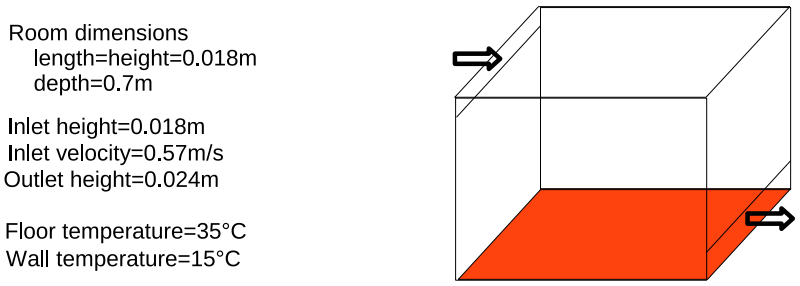

Figure 1: Ventilated and heated room.

The mean velocity and temperature profiles along the vertical line $(x = 0.502, z = 0.35)$ are shown on Figure 2. A good agreement with experimental data [10], and better results compared to Smagorinsky and its dynamical version can be observed. These results are particularly good near the walls (no wall model were used). This may be explained by the preservation of scaling laws in the construction of the invariant models.

4 Conclusions

We showed that the symmetry theory provides a powerfull modelling tool in fluid mechanics, as well for theoretical as for numerical developpements. In particular, we saw that it could be used to establish scaling laws, which are essential for the understanding of turbulence. The symmetry approach has also been used for the construction of turbulence models which naturally have good physical properties, since they preserve the symmetries of the equations. The numerical test confirms this good behaviour.

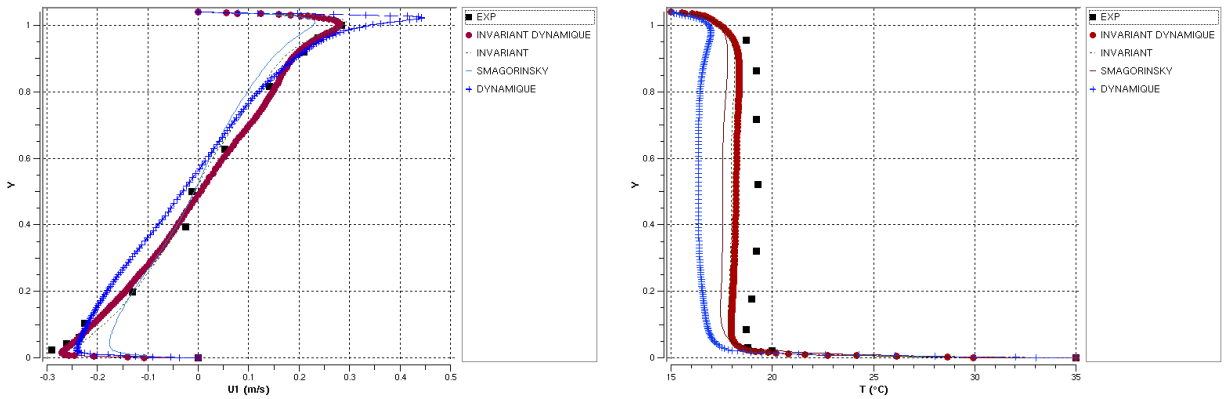

Figure 2: Mean streamwise velocity (left) and mean temperature (right) at $x = 0.502$ m.

References

[1] F. Archambeau, N. Mehitoua, and M. Sakiz. Code saturne: a finite volume code for the computation of turbulent incompressible flows - industrial applications. *International Journal On Finite Volumes*, (1), 2004.

[2] B.J. Cantwell. Similarity transformations for the two-dimensional, unsteady, stream-function equation. *Journal of Fluid Mechanics*, 85:257–271, 1978.

[3] V. Grassi, R.A. Leo, G. Soliani, and P. Tempesta. Vorticies and invariant surfaces generated by symmetries for the 3D Navier-Stokes equation. *Physica A*, 286:79–108, 2000.

[4] V. Grassi, R.A. Leo, G. Soliani, and P. Tempesta. Temperature behaviour of vortices of a 3D thermoconducting viscous fluid. *Physica A: Statistical Mechanics and its Applications*, 305(3-4):371 – 380, 2002.

[5] E. Nœther. Invariante Variationsprobleme. In *Königliche Gesellschaft der Wissenschaften*, pages 235–257, 1918.

[6] M. Oberlack. A unified approach for symmetries in plane parallel turbulent shear flows. *Journal of Fluid Mechanics*, 427:299–328, 2001.

[7] P. Olver. *Applications of Lie groups to differential equations*. Graduate texts in mathematics. Springer-Verlag, 1986.

[8] D. Razafindralandy, A. Hamdouni, and M. Oberlack. Analysis and development of subgrid turbulence models preserving the symmetry properties of the Navier–Stokes equations. *European Journal of Mechanics/B*, 26, 2007.

[9] G. Ünal. Application of equivalence transformations to inertial subrange of turbulence. *Lie Group and Their Applications*, 1(1):232–240, 1994.

[10] W. Zhang and Q. Chen. Large eddy simulation of indoor air flow with a filtered dynamic subgrid scale model. *International Journal of Heat and Mass Transfer*, 43:3219–3231, 2000.