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Heuristic Modes of Decision Making and Survival in Financial Markets*

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Abstract

We consider the impact of partial awareness in the form of a restriction of the state space. When a decision maker is aware of his unawareness, his perception of the state space can be represented by a partition. Such a decision maker behaves as if he restricts his portfolio choice to a subset of the available assets with payoffs measurable w.r.t. his awareness partition and thus avoids "surprises". Such a heuristic allows the partially aware agent to survive, but only if his beliefs on his awareness partition are at least as close to the truth as the beliefs of a more aware agent with a finer partition. Introducing a second heuristic, aversion to unforeseen unfavorable surprises, as in Grant and Quiggin (2015), allows us to establish survival of partially aware agents, regardless of whether their beliefs are correct. This heuristic implies that the agent holds a minimal number of bonds in his portfolio. Asset prices in such an economy reflect the potentially incorrect beliefs of the fully aware agents and replicate the well-known equity premium puzzle.

1 Introduction

Financial markets play a crucial role in modern economies. They drive major investment decisions and account for a large share of corporate profits and top labor incomes. Moreover, financial crises are commonly associated with deep and long-lasting recessions. It is therefore important to understand the modes of decision-making that determine the behavior of financial markets.

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Two contrasting perspectives have emerged. The first is centered on the ‘(Strong) Efficient Markets Hypothesis’, which postulates that financial markets make the best possible use of available information. This hypothesis is closely associated with rational choice models of decision-making in which agents act to maximize a well-defined objective function such as expected value or expected utility.

The second approach, commonly associated with the term ‘behavioral finance’ begins with the observation that observed decisions are often inconsistent with the requirements of rational choice models such as expected utility. This observation may be developed in various ways, but the approach most commonly associated with ‘behavioral finance’ is centered on the notion that decisions are made using ‘heuristics’ or ‘rules of thumb’. Goldstein and Gigerenzer (1996) and Gigerenzer (2007) argue that heuristics may be ‘ecologically rational’ in the environments to which they are adapted and provide examples related to portfolio choice, see Gigerenzer (2007, Chapter 2). The literature also gives examples of the potentially suboptimal results of reliance on heuristics in environments to which they are ill-adapted (the survey by Benartzi and Thaler, 2007 is useful, see also Shefrin, 2007).

A crucial argument in favor of strong-form EMH, discussed by Blume and Easley (2006) and Sandroni (2000), is the idea that markets favor the best-informed and most rational traders.¹ Trades in a financial market may be seen as ‘betting one’s beliefs’ about the relative probabilities of different states of nature, and the resulting returns on assets. Over time, traders who correctly judge these probabilities and make rational investment choices based on their beliefs will accumulate wealth at the expense of others. In the limit, only these rational well-informed traders will survive, and market prices will reflect their beliefs.

In the standard version of the Blume and Easley (2006), agents attempt to optimize rationally but fail to do so because their beliefs, represented by a probability distribution over possible states of the world, are incorrect. This analysis leaves open the possibility that agents who do not attempt rational optimization, but instead make decisions based on heuristics, might survive, even in the presence of more rational investors.

To examine this question it is necessary to formulate a model of bounded rationality. We will do this using recent developments in the theory of unawareness, in particular those of Grant and Quiggin (2013a, b, 2015),². Grant and Quiggin (2015) distinguish between two forms of differential awareness: ‘coarsening’, in which some distinctions between states are disregarded, and ‘restriction’, in which some states of nature are not considered.

The concept of restricted awareness is closely related to the idea of Black

¹This result is conditional on the assumption that markets are complete, see Coury and Sciuabba (2012), as well as that endowments are bounded, see Kogan et al. (2006, 2011), Yan (2008). Its robustness to preference specifications is still being explored, see Condie (2008), Da Silva (2011), Eichberger and Guerdjikova (2018), Guerdjikova and Sciuabba (2015), Easley and Yang (2014), Borovicka (2014).

²See Schipper (2018) for an extensive bibliography on unawareness.

Swans', which has received considerable attention in popular discussions of financial markets and financial crisis, Taleb (2007). This concept has received more limited attention in the economic and finance-theoretic literature, see Orlik and Veldkamp (2015), Chichilnisky (2010) and Modica et al. (1998). In particular, there has been little discussion of the implications of unforeseen contingencies for financial market efficiency and for the dynamics of survival in financial markets.

The concept of 'black swans' is obviously relevant to the questions of survival and asset pricing in financial markets. Traders who failed to foresee, or to protect themselves against, the breakdown of derivatives markets in 2008 suffered severe losses, in some cases leading to bankruptcy as a result. The question of how to represent unforeseen contingencies in financial market models is, therefore, of considerable interest in relation to the Blume-Easley analysis.

The failure of highly sophisticated traders to foresee relevant events also raises questions about rational investment strategies. From the ex ante perspective of a potential trader who has no particular reason to believe themselves to be uniquely gifted, participation in financial markets looks like a long-odds bet, with an infinitesimal probability of great wealth being offered against a near-certainty of disaster. This raises the question of why people participate in financial markets at all, and if they do, whether 'betting your beliefs' is a sensible strategy.

As is argued by Grant and Quiggin (2013a,b), sophisticated but boundedly rational agents will infer from experience that they are likely to be surprised by unforeseen contingencies. Such agents will choose to constrain their choices by heuristics designed to reduce vulnerability to unfavorable surprises (and, if possible, to remain open to favorable surprises). The Precautionary Principle, commonly used in environmental decision making to deal with poorly understood risks, may be seen as an example of such a heuristic.

We should therefore consider whether boundedly rational agents with incorrect beliefs can survive if they follow plausible heuristics. e.g., 'don't invest in assets you don't understand'. The present paper addresses this question. In Guerdjikova and Quiggin (2016), drawing on the analysis of Heifetz, Meier and Schipper (2006), we analyze the implications for survival in financial markets of differential awareness as coarsening. In the present paper, we consider the implications of coarse and restricted awareness, and show that the combination of the two provides a natural source of incorrect beliefs.

In particular, we look at agents who are unaware of certain states, but cognisant of the fact that their representation of uncertainty might be incomplete. We show that this generates a coarse partition of the state space, such that agents are unaware of some of the states associated with particular elements of the partition. For example, they may be aware of the event 'stock market crash' but fail to consider, even implicitly, some possible states that would be associated with a crash.

Using the decision-theoretic model of Grant and Quiggin (2015) adapted to a multi-period setting, we then study the implications of such partial awareness on portfolio choice, equilibrium outcomes and survival. We show that agents

who are partially aware will behave as if they are following certain heuristics to avoid "surprises". This, in turn, allows them to survive in the presence of "more aware" agents as long as their beliefs on the coarse partition are correct.

The condition that a partially aware agent have correct beliefs over his awareness partition turns out, however to be unduly strong (see Section 3.3). In a second step, we modify the agents' preferences by assuming aversion to unforeseen unfavorable surprises, as in Grant and Quiggin (2015). The portfolios of such agents are characterized by a strictly positive lower bound on bond holdings, which ensures that agents averse to unforeseen unfavorable surprises survive regardless of their beliefs.

The paper is organized as follows: in Section 2, we discuss the role of heuristics in financial markets with partially aware agents. In Section 3, we describe the model of the economy, drawing a distinction between the true model of uncertainty and the model of limited awareness used to describe the agents' perception of the economy. In Section 4, we derive our main results on survival in markets with differential awareness and show that having beliefs (over a coarse partition of the state space) closer to the truth than those of an agent with a finer partition is a necessary condition for the survival of a partially aware agent. Section 5 introduces aversion to unforeseen surprises and shows that partially aware agents who are averse to unforeseen unfavorable surprises survive regardless of whether their beliefs are correct. Section 6 concludes. In Appendix A, we provide an axiomatization of the intertemporal version of the Grant and Quiggin (2015) preferences used in this paper. Appendix B contains all proofs.

2 The Role of Heuristics in Financial Markets with Partially Aware Agents

2.1 Optimizing versus Heuristic Approaches to Choice

It is usual, in discussion of decisions under uncertainty, to distinguish between rational optimization and heuristics. However, this distinction is problematic. Any complete and transitive ordering over risky prospects may be represented as the maximization of an appropriately chosen function. (If the domain is specified to include the set of alternatives as well as the prospect under consideration, transitivity may be dispensed with, as in some versions of regret theory, and similarly for completeness.)

This point obviously applies to heuristics such as 'satisficing' or the precautionary principle. These may be interpreted as optimization taking account of costs of search and cognition.

The crucial difference between heuristic and optimizing approaches lies in the way in which they are justified. For optimizing approaches, such as expected utility maximization, justification takes the form of a representation theorem. Assuming preferences satisfy some set of axioms, typically presented as evidently rational, a representation theorem shows that choices consistent with these pref-

erences will maximize the value of an appropriately chosen function. It follows that an agent who assents to the axioms and who can specify the parameters of the relevant function can achieve security; that is, the assurance that they have made the best possible choice given their preferences, the choice set and the available information.

Nevertheless, as noted by Forster (1999), such rules may not "fit to reality": "according to such theories, decision makers act rationally even when their beliefs are entirely false or inappropriate"³. As noted earlier, in financial markets, agents with such false beliefs eventually disappear, suggesting that such rules might not be adapted to the relevant environment and thus fail the criterion of "ecologic rationality".

By contrast, heuristics may be justified (or not) as ecologically rational. An ecologically rational heuristic is one that achieves an optimal outcome in some given environment. However, it is not assumed that the agents who follow heuristics can assure themselves that the environment is one for which their heuristic is ecologically rational. At most, agents can work on the basis of inductive reasoning as in case-based decision theory.

In this paper, we combine the two approaches by considering individuals which are partially aware and thus, limited in their ability to formulate correct beliefs. We model the decisions of such agents as resulting from rational optimization of axiomatized preferences, which however, takes into account the agent's limited understanding of the economic environment. We show that such decisions can be related to the use of heuristics. Moreover, we demonstrate that decisions which mimic the use of heuristics can indeed be ecologically rational, in that they allow partially aware agents to survive in environments, in which fully aware agents with wrong beliefs would vanish.

2.2 Examples of Heuristics in Financial Markets

What kind of heuristics might be applied to trade in financial markets? Consider a partially aware agent, who understands that they are not fully aware of all relevant contingencies, and who can follow the reasoning that suggests that a strategy of betting on incorrect beliefs will almost surely lead to the loss of all wealth.

Such an agent might adopt two types of heuristics. The first is directly related to his level of understanding of the relevant state space. It amounts to effectively restricting the set of available assets to those the payoff structure of which the agent understands and which do not expose him to "surprises", i.e., differences of consumption levels, to which he cannot assign numerical probabilities. This heuristic will have two effects on the agent's portfolio. First, the agent will be more willing to buy insurance across events he is only partially aware of (i.e., he is less willing to expose himself to risks he does not understand). With respect to such events, his portfolio will be safer than that of a fully aware agent. Second, the agent's portfolio will appear underdiversified

³See also the discussion in Gilboa, Postlewaite and Schmeidler (2004).

across events he understands and he might be exposed to some idiosyncratic risk.

Both of these heuristics are commonly observed in financial markets: e.g., Goetzman and Kumar⁴ (2008) find that US investors hold on average 4-7 assets in their portfolios on average. They provide evidence of insufficient diversification and furthermore show that it correlates with individuals' experience in financial markets, education and sophistication, rather than with the cost associated with holding more diversified portfolios.

While Goetzman and Kumar document that "better diversified investors are likely to have better stock selection abilities, perhaps due to their higher levels of financial sophistication", they also write: "Most surprisingly, we find that high-turnover, under-diversified portfolios perform better than high-turnover, better-diversified portfolios", thus implying that there might be gains to under-diversification at least in some cases. Further support to this type of heuristics is provided by Borges et al. (1999), who show that the recognition heuristic, which consists in holding a portfolio consisting of the most-commonly recognized stocks can significantly outperform both the market and at times, even major managed funds.

The second type of heuristics is related to the desire to avoid certain undesirable outcomes. An agent might, e.g., adopt a heuristic of allocating a proportion of initial wealth to bonds, yielding sufficient income to sustain a minimum acceptable level of consumption in all periods. We will demonstrate that partially aware agents, who are averse to unforeseen undesirable surprises will behave as if they follow this heuristic.

A number of observations may be made about this heuristic. First, although a riskless unit bond is equivalent to a collection of unit Arrow securities, one for each state of nature, the bond investor need not be aware of all possible states, or, indeed of the existence of more than one state of nature.

Second, the position where the agent has exactly the wealth required to satisfy the heuristic constraint is, in the language of Markov processes, an absorbing state. Having reached this position, the agent undertakes no trade except to roll over the bond and consume the interest. In particular, therefore, she cannot vanish.

Third, consider the equilibrium in which all investors follow a heuristic of this kind (we can allow the trivial case where the required level of bond holding is set at zero). The arguments of Blume and Easley (2006) suggest that all but the most informed trader(s) will be driven to the absorbing state where they hold only bonds. Thus, the relative prices of asset claims will be determined by

- (i) the beliefs of the most informed trader(s);
- (ii) the demand for bonds from less informed traders.

In this context, the price of bonds will be higher and the price of the market portfolio lower than can be explained by risk aversion in a model with fully rational traders. Hence, we will observe the risk-free rate puzzle and equity premium puzzle, as documented by Mehra and Prescott (1985) and Weil (1989).

⁴See also the references therein.

Other heuristics may also be considered, such as investment in an index fund. Provided such heuristics generate an absorbing state or set of states at a wealth level below initial wealth, they ensure that agents who adopt them will not vanish. We are thus led to a set of market equilibria in which a small number of active investors trade their beliefs, while the remainder of the population follows a heuristic investment strategy independent of beliefs.

As explained above, heuristics have to be evaluated in regard to their ecological rationality, i.e., their performance relative to other rules. In this paper, we are interested in the issue of survival in financial markets. We thus define:

Definition 1 *A heuristic is ecologically rational for an agent, if the agent survives when using the heuristic and vanishes, otherwise.*

In Sections 4 and 5 we will examine the behavior of partially aware agents in financial markets and the conditions under which they can survive. We will demonstrate that some of the heuristics described above are indeed ecologically rational in this sense.

3 Financial Markets with Unforeseen Contingencies

In the standard Blume-Easley model, all traders are fully aware, and financial markets are implicitly assumed to span the state space. Traders may therefore be regarded as choosing consumption paths, without specifying the associated portfolio of assets. There is no loss of generality in assuming that the set of assets consists of Arrow securities, one for each node σ_t . Any redundant assets may be priced in terms of the replicating portfolio of Arrow securities.

These features of the model remain even if some traders incorrectly impute zero probability to some events that are in fact possible. Such traders will seek, to the extent permitted by market constraints, to short-sell securities that pay off only in such states.

The situation with reduced awareness is different. We will show below that traders will voluntarily choose to not trade in assets specific to contingencies of which they are unaware. Hence, their portfolio cannot be represented entirely in terms of Arrow securities. This creates a non-trivial role for 'traditional' securities such as bonds and equity.

To understand this fundamental difference, we will first specify the "true" model of the economy, which incorporates all possible contingencies and imputes to them objectively correct probabilities. This model corresponds to a state of full awareness and mimics the framework of Blume and Easley (2006).

In a second step, we will introduce differential awareness and explain how it affects the agents' beliefs and utility functions.

Finally, we will study the implications of differential awareness for survival.

3.1 Time and Uncertainty

Let $\mathbb{N} = \{0; 1; 2; \dots\}$ denote the set of time periods. Uncertainty is modelled through a sequence of random variables $\{\mathcal{S}_t\}_{t \in \mathbb{N}}$ each of which takes value from a finite set S . We set $S_0 = \{s_0\}$, i.e., no information is revealed in period 0. Denote by $s_t \in S$ the realization of random variable \mathcal{S}_t . Denote by $\Omega = \prod_{t \in \mathbb{N}} S$ the set of all possible observation paths, with representative element

$\sigma = (s_0; s_1; s_2 \dots s_t \dots)$. Finally denote by $\Omega_t = \prod_{\tau=0}^t S$ the collection of all finite

paths of length t , with representative element $\sigma_t = (s_0; s_1; s_2 \dots s_t)$. Each finite observation path σ_t identifies a decision/observation node and the set of all possible observation paths Ω can also be seen as the set of all nodes.

We can represent the information revelation process in this economy through a sequence of finite partitions of the state space Ω . In particular, define the cylinder with base on $\sigma_t \in \Omega_t$, $t \in \mathbb{N}$ as $Z(\sigma_t) = \{\sigma \in \Omega | \sigma = (\sigma_t \dots)\}$. Let $\mathbb{F}_t = \{Z(\sigma_t) : \sigma_t \in \Omega_t\}$ be a partition of the set Ω . Clearly, $\mathbb{F} = (\mathbb{F}_0 \dots \mathbb{F}_t \dots)$ denotes a sequence of finite partitions of Ω such that $\mathbb{F}_0 = \Omega$ and \mathbb{F}_t is finer than \mathbb{F}_{t-1} . We assume that all fully aware agents have identical information and that the information revelation process for them is represented by the sequence \mathbb{F} .

Let \mathcal{F}_t be the σ -algebra generated by partition \mathbb{F}_t . \mathcal{F}_0 is the trivial σ -algebra. Let \mathcal{F} be the σ -algebra generated by $\cup_{t \in \mathbb{N}} \mathcal{F}_t$. It can be shown that $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$ is a filtration.

We define on $(\Omega; \mathcal{F})$ a probability distribution π and throughout we assume $\pi(Z(\sigma_t)) > 0$ for all σ_t . Intuitively, π describes the evolution of the state process in the economy. In what follows, for brevity, we abuse notation slightly by denoting $\pi(Z(\sigma_t)) = \pi(\sigma_t) = \pi(s_0; s_1; s_2 \dots s_t)$. The one-step-ahead probability distribution $\pi(s_{t+1} | \sigma_t)$ at node σ_t is determined by:

$$\pi(s_{t+1} | \sigma_t) = \pi(s_0 \dots s_t; s_{t+1} | s_0 \dots s_t) = \frac{\pi(s_0 \dots s_t; s_{t+1})}{\pi(s_0 \dots s_t)} \text{ for any } s_{t+1} \in S$$

In words, $\pi(s_{t+1} | \sigma_t)$ is the probability under distribution π^n that the next observation will be s_{t+1} given that we have reached node σ_t . We will assume that the true process of the economy is i.i.d. and write $\pi(s_{t+1} = s | \sigma_t) =: \pi(s)$. Note that this does not restrict the endowment process to be i.i.d..

3.2 The Model of Differential Unawareness: Restriction of the State Space and Awareness of Unawareness

In most of the existing literature agents know the true model of the economy, except for the probability distribution π . In contrast, we consider agents who might be unaware of the existence of some of the states. In the following, we extend the model of Grant and Quiggin (2015) to model agents who are partially aware, but also aware of their limited perception of the world.

In particular, agent i is assumed to be aware of a set of states $S^i \subseteq S$. Despite his limited knowledge, the partially aware agent understands that there are contingencies he might not be taking into account. Formally, for each state $s \in S$, there is a corresponding state $a^i(s) \in S^i$ with the interpretation that the partially aware consumer is aware of $a^i(s)$, but is also aware that some unforeseen contingencies (s) might occur instead of $a^i(s)$. We postulate that for $s \in S^i$, $a^i(s) = s$, i.e., whenever the agent is aware of a state, he is also aware that no unforeseen contingencies can occur in this state. The awareness of each agent i is thus described by two elements, $S^i \subseteq S$ and $a^i : S \rightarrow S^i$.

Let W^i denote a partition of S with elements

$$w^i = \{s' \in S \mid a^i(s') = s \text{ for some } s \in S^i\}.$$

I.e., each element of the partition W^i contains a state $s \in S^i$ of which the agent is aware, as well as all those states that he is unaware of and which are mapped into s by a^i . It follows that the partition W^i has $|S^i|$ elements. A special case is given by the fully aware agent, for whom $W^i = \{\{s\}_{s \in S}\}$.

Hence, we can describe the perception of the uncertainty by agent i as a coarsening of the set of paths Ω . Rather than observing s_t at time t , agent i observes the corresponding element of the partition, w , which contains s_t . Therefore, for a partially aware consumer i , the paths he is aware of can be written as $\Omega^i = \prod_{t \in \mathbb{N}} W^i$ with a representative element $\omega^i = (w_0 = s_0; w_1^i \dots w_t^i \dots)$. Denote by Ω_t^i the set of paths of length t .

Consider two agents, i and j such that $S^j \subset S^i$ and $a^i(s) = a^j(s)$ for all $s \in S \setminus S^i$, i.e., all states that i and j are both unaware of are mapped into the same states that they are both aware of. The corresponding partition W^j will then be strictly coarser than W^i , and hence, Ω^j will be strictly coarser than Ω^i . This gives rise to the following definition which provides a partial ordering of agents w.r.t. their awareness level.

Definition 2 *Agent i is "more aware" than agent j if $S^j \subset S^i$ and W^i is a refinement of W^j .*

3.3 Beliefs

From the point of view of consumer i , the information revelation is described by finite partitions of the set Ω^i , $(\mathbb{F}_t^i)_{t \in \mathbb{N}}$ defined in analogy to $(\mathbb{F}_t)_{t \in \mathbb{N}}$. Note that for each t , \mathbb{F}_t^i is coarser than the corresponding \mathbb{F}_t . We will denote by \mathcal{F}_t^i the σ -algebra generated by partition \mathbb{F}_t^i . $\mathcal{F}_0^i = \mathcal{F}_0$ is the trivial σ -algebra. Let \mathcal{F}^i be the σ -algebra generated by $\cup_{t \in \mathbb{N}} \mathcal{F}_t^i$. Just as above, $\{\mathcal{F}_t^i\}_{t \in \mathbb{N}}$ is a filtration.

Obviously, \mathcal{F} is finer than \mathcal{F}^i and hence, any probability distribution π on $(\Omega; \mathcal{F})$ specifies a probability distribution on $(\Omega^i; \mathcal{F}^i)$ with

$$\pi(\omega_t^i) = \pi\{\sigma_t \mid s_\tau \in w_\tau^i \text{ for all } \tau \in \{1 \dots t\}\}.$$

The one-step ahead probability distribution $\pi(w_{t+1}^i | \omega_t^i)$ is defined analogously to $\pi(s_{t+1} | \sigma_t)$.

The subjective beliefs of consumer i , π^i are defined on $(\Omega^i, \mathcal{F}^i)$ and need not in general coincide with the truth. i 's beliefs are correct if they coincide with the restriction of π to Ω^i .

For the rest of the paper, we will restrict attention to beliefs which describe an i.i.d. process, $\pi^i(w_{t+1}^i = w^i | \omega_t^i) = \pi^i(w^i)$.

Just as in the case of a fully aware agent, our theory cannot and does not aim to explain how agents choose their priors⁵. Here, we will show that even agents who share "correct beliefs" on the events of which they are aware, might end up with heterogeneous beliefs as a result of differential unawareness. Formally,

Definition 3 *We will say that consumer i , has awareness-biased beliefs if the relative likelihoods assigned to states he is aware of are correct, $\frac{\pi^i(s)}{\pi^i(s')} = \frac{\pi(s)}{\pi(s')}$ for any $s, s' \in S^i$ and if the probability assigned to an element of the partition W^i , w^i equals the probability of the state $s \in w \cap S^i$ of which i is aware, $\pi^i(w^i) = \pi^i(w \cap S^i)$.*

Awareness-biased beliefs are correct w.r.t. to the states of which a consumer is aware in that the relative likelihoods assigned to such states coincide with the truth. However, lacking any further information about the likelihood of states he is unaware of, such a consumer identifies the probability of an event containing states he is unaware of with the probability of the states he understands. It is clear that in general, the awareness-biased beliefs π^i will not be correct.

Note that whenever $\pi(s) > 0$ for all $s \in S$ (as in Assumption 3 below), $\pi^i(w^i) > 0$ will hold for all $w^i \in W^i$. Hence, i 's one-step ahead beliefs will be absolutely continuous w.r.t. the truth. The extent to which i 's beliefs w.r.t. W^i are "wrong" can then be quantified by the Kullback-Leibler distance between / relative entropy of $\pi(s | S^i)$ and $\pi(w^i)$, i.e.,

$$\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^i | S^i)} \quad (1)$$

Our definition of awareness-biased beliefs shows that differential awareness can lead to belief heterogeneity even when agents share the same information about likelihoods. The following two remarks will help illustrate this point:

Remark 4 *The awareness-biased beliefs of two agents with identical awareness partitions need not coincide. Indeed, consider agents i and j , who form a common awareness partition on the state $S = \{1; 2; 3; 4\}$, $W = \{w = \{1; 2\}; w' = \{3; 4\}\}$. If i is aware of states $S^i = \{1; 3\}$, whereas j is aware of states $S^j = \{1; 4\}$, and*

⁵Dietrich (2017), as well as Ahn and Ergin (2010) provide axiomatizations, in which the beliefs of partially aware agents and depend on the partition used by the decision maker. Thus, beliefs under partial awareness are consistent with beliefs derived for the finest possible partition only in special cases.

unless $\pi(3) = \pi(4)$, agents' awareness-biased beliefs will satisfy

$$\frac{\pi^i(w)}{\pi^i(w')} = \frac{\pi(1)}{\pi(3)} \neq \frac{\pi^j(w)}{\pi^j(w')} = \frac{\pi(1)}{\pi(4)}.$$

Remark 5 *An agent with a coarser awareness partition can entertain awareness-biased beliefs closer to the truth than an agent with a finer awareness partition. Indeed, consider $S = \{1\dots 5\}$, suppose that $\pi(1) = \frac{3}{8}$, $\pi(2) = \pi(3) = \pi(4) = \frac{1}{8}$ and $\pi(5) = \frac{1}{4}$. For two agents, i and j , let $S^i = \{2; 3\}$, $S^j = \{2; 3; 5\}$ and*

$$\begin{aligned} W^i &= \{w^i = \{1; 2; \}; w^{i'} = \{3; 4; 5\}\} \\ W^j &= \{w^j = \{1; 2; \}; w^{j'} = \{3; 4\}; w^{j''} = \{5\}\} \end{aligned}$$

According to our definition, agent j is thus more aware than agent i . Yet, i 's awareness-biased beliefs are correct:

$$\pi^i(w^i) = \pi^i(w^{i'}) = \frac{1}{2} = \pi(w^i) = \pi(w^{i'}),$$

whereas those of j w.r.t. to i 's partition⁶ are not:

$$\pi^j(w^i) = \frac{1}{4}, \pi^j(w^{i'}) = \frac{3}{4}$$

Our results below, see Proposition 13, will demonstrate that markets select for agents who have beliefs closer to the truth in the sense defined above, i.e., agents for whom the relative entropy in (1) is minimal, unless the wrong beliefs are compensated by a finer awareness partition.

Hence, even though the results we derive in Section 4 show that market forces do not select against partially aware agents, the fact that it might be less likely that these agents have correct beliefs would imply that these agents would also be less likely to survive and impact prices in the limit. In contrast, our result in Section 5 does not depend on agents' beliefs.

Finally, we remark that the results derived in Sections 4 and 5 do not depend on the assumption of awareness-biased beliefs, but merely on the property of such beliefs to be absolutely continuous w.r.t. the truth. Nevertheless, we believe that this model of beliefs provides some useful intuition and allows us to better evaluate the conditions under which survival occurs.

3.4 The Model of the Economy

There is a single good and I infinitely lived consumers, each with consumption set \mathbb{R}_+ . A consumption plan $c : \Omega \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}_+$ is a sequence of \mathbb{R}_+ -valued functions $\{c(\sigma_t)\}_{t \in \mathbb{N}}$ in which each $c(\sigma_t)$ is \mathcal{F}_t -measurable. Each consumer is endowed with a particular consumption plan, called i 's endowment stream and denoted e^i . $e = \sum_{i \in I} e^i$ stands for the total endowment of the economy.

⁶This is the relevant comparison for the results derived in Section 4.

We follow Grant and Quiggin (2015) to model the preference of partially aware agents over infinite consumption streams. Note that in general, the consumption stream of a partially aware agent, c^i , is not measurable w.r.t. Ω^i . Since the consumer is not aware of some of the contingencies which lead to different amounts of consumption on a given path, his consumption stream c^i associates with each node consumer i is aware of, $\sigma_t \in \Omega^i$ a set of possible outcomes / levels of consumption:

$$C^i(\omega_t^i) = \{c^i(\sigma_t) \mid \sigma_t \in \omega_t^i\}.$$

This set is a singleton, only when the consumption at ω_t is "surprise-free" in the language of Grant and Quiggin (2015), i.e., $c^i(\sigma_t) = c^i(\sigma'_t)$ for all $\sigma_t, \sigma'_t \in \omega_t^i$.

We extend the representation of Gul and Pesendorfer (2014) and Grant and Quiggin (2015) to a multi-period setting:

$$\begin{aligned} V_0^i(c^i) &= u_i(c^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega^i} \beta_i^t \pi^i(\omega_t^i) u_i(C^i(\omega_t^i)) \\ &= u_i(c^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\}; \max\{C^i(\omega_t^i)\}) \end{aligned}$$

where β_i is agent i 's discount factor. An axiomatization for the intertemporal version of their model used here is provided in Appendix A. There are two major differences between this representation and the standard EUM model. First, subjective probabilities are well-defined only for coarse events, i.e., those in Ω^i , but not for those events, on which surprises can occur. This in turn, implies that in states associated with surprises, the consumer must take into account two different outcomes — the minimal and the maximal possible consumption — when evaluating his utility of consumption in this state. This second property leads to the utility function u_i over outcomes being defined over intervals rather than single values.

In principle, many different specifications of u_i are possible. Here, we will concentrate on the case of⁷

$$u_i(C^i(\omega_t^i)) = u_i(\min\{C^i(\omega_t^i)\})$$

Intuitively, if the consumer is partially aware, but perceives the possibility of a surprise, he concentrates only on the worst possible outcome.

We will impose the following assumptions on utility functions and endowments, which are standard in the survival literature:

Assumption 1 The functions $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ are twice continuously differentiable, strictly concave, and satisfy $\lim_{c \rightarrow 0} u'_i(c) = \infty$ and $\lim_{c \rightarrow \infty} u'_i(c) = 0$.

⁷Part (iv) of Axiom 3 stated in Appendix A provides the behavioral condition which implies that u_i is a function only of $\min C^i(\omega_t^i)$.

Assumption 2 Individual endowments are uniformly bounded away from zero and uniformly bounded from above. Formally, there is an $m > 0$ such that $e^i(\sigma_t) > m$ for all i, σ_t ; moreover, there is an $m' > m > 0$ such that $\sum_{i \in I} e^i(\sigma_t) < m'$ for all σ_t .

Assumption 3 $\pi(s) > 0$ for all $s \in S$ and all agents $i \in I$ have awareness-biased beliefs. In particular, $\pi^i(w^i) > 0$ for all $w^i \in W^i$.

Assumption 1 implies that the agent would never choose zero minimal consumption on an event ω_t^i he believes to have a positive probability. Assumption 2 ensures that endowments are uniformly bounded away from 0 and above. Given the i.i.d. structure imposed on the true process and on beliefs, Assumption 3 states that one-step-ahead probabilities of all states of the world are positive and that all subjective beliefs assign a positive one-step-ahead probability to every element in their respective partitions. As we showed above, the assumption that agents form awareness-biased beliefs implies the absolute continuity of subjective beliefs with respect to the true probability distribution (as in Blume and Easley, 2006). Taken together, Assumptions 1 and 3 ensure that no agent vanishes in finite time.

In economies with bounded endowments and complete markets, and populated by expected utility maximizers, only beliefs and discount factors matter for survival. In particular, if all agents are equally patient, agents with incorrect beliefs vanish a.s. in the presence of agents with correct beliefs. By contrast, in unbounded economies, risk attitudes also matter for survival, and agents with incorrect beliefs can survive. In order to disentangle the effects of partial awareness on survival from those of risk attitude, we restrict our attention to the case of bounded economies.

3.5 Assets

We have so far described the economy only in terms of state-contingent consumption. We now introduce assets explicitly.

In particular, we define the set of period t Arrow-securities in the economy to be

$$\tilde{A}_t^0 = \{a_{\sigma_t} \text{ with } a_{\sigma_t}(\sigma_t) = \mathbf{1}_{\sigma_t}\} \quad (2)$$

We will also define for each agent i and the corresponding partition Ω^i , the set of unit-securities that pay conditional on events in Ω^i :

$$\tilde{A}_t^i = \left\{ a_{\omega_t^i} \text{ with } a_{\omega_t^i}(\sigma_t) = 1 \text{ if } \sigma_t \in \omega_t^i, a_{\omega_t^i}(\sigma_t) = 0, \text{ else for all } \omega_t^i \in \Omega_t^i \right\}. \quad (3)$$

\tilde{A}_t^i is thus the set of assets with payoffs measurable w.r.t. to agent i 's partition Ω_t^i . Intuitively, such assets do not involve "surprises" from the point of view of agent i .

Note that while in general $\tilde{A}_t^i \not\subset \tilde{A}_t$, we have that for $i = 1 \dots n$, the set of payoffs generated by the set of assets in \tilde{A}_t^i ,

$$\tilde{M}_t^i = \left\{ \sum_{\omega_t^i \in \Omega_t^i} \mathbf{1}_{\omega_t^i} \theta(\omega_t^i) \mid (\theta(\omega_t^i))_{\omega_t^i \in \Omega_t^i} \in \mathbb{R}^{|\Omega_t^i|} \right\}$$

satisfies

$$\tilde{M}_t^i \subseteq \tilde{M}_t^0 = \mathbb{R}^{|\Omega_t|}.$$

We can now rewrite the endowment of the economy in terms of assets in $\tilde{A}^0 = \cup_{t=0}^{\infty} \tilde{A}_t^0$ by assigning to each agent i , an initial portfolio $\bar{\theta}^i$ with $\bar{\theta}^i(\sigma_t) = e^i(\sigma_t)$. Let $(\theta^i(\sigma_t))_{\sigma_t \in \Omega}$ denote the portfolio holdings of agent i . Clearly, his consumption is then given by $c^i(\sigma_t) = \theta^i(\sigma_t)$.

While a priori, each agent has access to all assets in $\cup_t \tilde{A}_t^0$, we will see that agent i with preferences subject to surprises might find it optimal to restrict his investments to the set $\cup_t \tilde{A}_t^i$, i.e., to the set of assets he understands.

4 Survival with Differential Awareness

We will now consider an economy with agents who are partially aware, but aware of the possibility of surprises as described above.

Definition 6 *An equilibrium of an economy with partially aware agents who are aware of their unawareness is described by consumption streams $(c^i)_{i \in I}$ and an integrable⁸ price system $p(\sigma_t)$ such that each of the consumers maximizes their utility function given prices in the economy and markets clear:*

$$c^i = \arg \max_{c^i} V_0^i(c^i) = \arg \max_{c^i} \left[u_i(\bar{c}^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min \{C^i(\omega_t^i)\}) \right]$$

$$s.t. \quad \sum_{t \in \mathbb{N}} \sum_{\sigma_t \in \Sigma} p(\sigma_t) c^i(\sigma_t^i) \leq \sum_{t \in \mathbb{N}} \sum_{\sigma_t \in \Sigma} p(\sigma_t) e^i(\sigma_t^i)$$

$$\sum_{i \in I} c^i(\sigma_t) = \sum_{i \in I} e^i(\sigma_t) \quad \forall \sigma_t \in \Omega$$

It is easy to show that an equilibrium exists using Bewley's (1972) theorem.

An alternative definition of an equilibrium can be states in terms of asset trades:

Definition 7 *An equilibrium of an economy with assets \tilde{A}_t^0 with partially aware agents who are aware of their unawareness is described by portfolio holdings $(\theta^i)_{i \in I}$ and an integrable price system $p(\sigma_t)$ such that each of the consumers*

⁸ An integrable price system $p(\sigma_t)_{\sigma_t \in \Sigma}$ has the property that $\sum_{t \in \mathbb{N}} \sum_{\sigma_t \in \Sigma_t} p(\sigma_t) e^i(\sigma_t)$ is finite for any uniformly bounded sequence $e^i(\sigma_t)$.

maximizes their utility function given prices in the economy and asset markets clear:

$$\theta^i = \arg \max_{\theta^i} V_0^i(\theta^i) = \arg \max_{c^i} \left[u_i(\theta^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega^i} \beta_i^t \pi^i(\omega_t^i) u_i \left(\min_{\sigma_t \in \omega_t^i} \{\theta^i(\sigma_t)\} \right) \right]$$

$$s.t. \sum_{t \in \mathbb{N}} \sum_{\sigma_t \in \Sigma} p(\sigma_t) \theta^i(\sigma_t) \leq \sum_{t \in \mathbb{N}} \sum_{\sigma_t \in \Sigma} p(\sigma_t) \bar{\theta}^i(\sigma_t)$$

$$\sum_{i \in I} \theta^i(\sigma_t) = \sum_{i \in I} \bar{\theta}^i(\sigma_t) \quad \forall \sigma_t \in \Omega$$

Clearly, the equilibria satisfying Definition 7 specify the same equilibrium consumption and state prices as those satisfying Definition 6.

Our first result generalizes the main result of Blume and Easley (2006) to apply to agents with bounded, but identical levels of awareness.

Proposition 8 *Consider two agents i and j , with identical awareness partitions, $\Omega^i = \Omega^j$. Assume that the initial endowment of the economy is measurable w.r.t. Ω^i . If the two agents have identical beliefs, but different discount factors, then the agent with the lower discount factor vanishes a.s.. If the two agents have identical discount factors and different beliefs, the agent whose beliefs are further away from the truth vanishes a.s.. More generally, the agent with the lower survival index:*

$$\ln \beta_i - \sum_{w \in W} \pi(w) \ln \frac{\pi(w)}{\pi(w \cap S^i | S^i)}$$

vanishes a.s..

As we showed in Remark 4, agents with identical awareness partitions need not hold identical beliefs. The proposition shows that (for identical discount factors), it is the Kullback-Leibler distance between the awareness-biased beliefs and the truth on the relevant partition as defined in (1) that determines whether an agent survives or not. This distance, in turn is fully determined by the set of states of which the agent is aware, S^i .

Our next Proposition extends these results to a set of agents with nested awareness partitions.

Proposition 9 *Consider a population of agents with awareness partitions ordered with respect to inclusion, equal discount factors and correct beliefs. Assume that the initial endowment of the economy is measurable w.r.t. the finest awareness partition. All agents survive a.s.*

Our result shows that whenever agents have equal discount factors and correct beliefs relative to their awareness partition and the awareness partitions are nested, the agent's level of awareness is irrelevant for survival. In fact, all

agents survive. Note however, that awareness-biased beliefs will be correct only in very special cases. E.g., if all states are equiprobable, and for all agents all elements of the agent's partition contain the same number of states, all agents will have awareness-biased beliefs which are correct. In general, however, as shown in Section 3.3, differential awareness will imply belief heterogeneity and hence, the result above will not apply.

Nevertheless, the result is useful for understanding the properties of equilibria in economies with differential awareness. Note that unless $p(\sigma_t) = 0$ for some $\sigma_t \in \omega_t^i$, we have $c^i(\sigma_t) = c^i(\sigma'_t)$ for all $\sigma_t, \sigma'_t \in \omega_t^i$. I.e., when state prices are strictly positive, agent i 's consumption is measurable w.r.t. Ω^i . We conclude that in equilibrium with strictly positive state prices, a partially aware consumer i behaves as if his investment opportunities were restricted to belong to the set $\cup_t \tilde{A}_t^i$ with payoffs measurable w.r.t. Ω^i , even though in fact, he has access to a richer set of assets, $\cup_t \hat{A}_t^0$. Thus, his portfolio will appear underdiversified when compared to that of a fully aware agent. Furthermore, he will be overinvesting in insurance across states $\sigma_t \in \omega_t^i$, which might involve aggregate risk considered by agent i as a "surprise". A more aware agent j might provide such full insurance against aggregate risk. As a result, asset prices in such an economy will be distorted: the price for insurance across states belonging to the same elements of the partition ω_t^i will be higher than in an economy with fully aware agents.

We conclude that in equilibrium, a partially aware investor will behave as if he were following a heuristic to invest in a subset of assets he is familiar with and "understands", as described in Section 2. Furthermore, the result above implies that in our model, such heuristics are in line with ecologic rationality, in that they do not lead an investor to disappear. The phenomena generated by the use of such heuristics will persist in the long-run, as long as all agents have equal survival indices.

We next examine the impact of heterogeneity in discount factors and beliefs on survival when awareness partitions are nested. We first show that a less aware agent can only survive if his survival index is at least as large as that of a more aware agent. In particular, for given identical beliefs, the less aware agent can survive only if his discount factor is at least as high as that of the more aware agent and, for given identical discount factors, the less aware agent can only survive only if his beliefs are at least as close to the truth as those of the more aware agent. As we showed in Remark 5, it is easy to construct examples for which this is the case.

Proposition 10 *Suppose that an agent i has a strictly coarser awareness partition Ω^i than that of an agent j , Ω^j . Assume as well that the endowment of the economy is measurable w.r.t. the partition Ω^j . If*

$$\ln \frac{\beta_j}{\beta_i} + \left(\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^i | S^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^j | S^j)} \right) > 0$$

i vanishes a.s.

This result shows that in order for an agent i with a coarser partition than agent j to survive, it has to be that:

$$\ln \frac{\beta_j}{\beta_i} + \left(\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^i | S^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^j | S^j)} \right) \leq 0.$$

Hence, we will consider an economy, in which the agents with coarser awareness partitions have (weakly) larger survival indices. To formulate our result, we will have to understand when partial awareness matters in the long-term. We will use the following definitions:

Definition 11 *The unawareness of agent i , given by the partition Ω^i , is irrelevant in the limit if for any $\omega^i \in \Omega^i$ and any $\sigma, \sigma' \in \omega^i$*

$$\lim_{t \rightarrow \infty} e(\sigma_t) - e(\sigma'_t) = 0.$$

The unawareness of agent i , given by the partition Ω^i is relevant in the limit if for some $w^i \in W^i$, s and $s' \in w^i$, there is an $\epsilon > 0$ such that for any $\sigma, \sigma' \in \omega^i$,

$$\limsup_{t \rightarrow \infty} [e(\sigma_t; s) - e(\sigma'_t; s')] > \epsilon. \quad (4)$$

The unawareness of agent i is considered irrelevant if, in the limit, the total endowment of the economy is measurable with respect to agent i 's partition. Such an agent is not exposed to surprises w.r.t. the total endowment process of the economy in the limit. In contrast, agent i 's unawareness is relevant even in the limit, if there are at least two states that i cannot distinguish and in which the total endowment of the economy remains distinct.

Definition 12 *Let agent i 's awareness partition Ω^i be finer than that of j . The unawareness of agent j given by the partition Ω^j , is irrelevant in the limit w.r.t. that of agent i given by partition Ω^i if for any $\omega^i \in \Omega^i$ and $\omega^j \in \Omega^j$ s.t. $\omega^i \subseteq \omega^j$,*

$$\lim_{t \rightarrow \infty} \min_{\sigma_t \in \omega_t^i} e(\sigma_t) - \min_{\tilde{\omega}_t^i \subseteq \omega_t^j} e(\tilde{\omega}_t^i) = 0.$$

The unawareness of agent j is relevant in the limit w.r.t. that of agent i if there is an $\epsilon > 0$ such that for any $\omega^i \in \Omega^i$ and $\omega^j \in \Omega^j$ s.t. $\omega^i \subseteq \omega^j$,

$$\limsup_{t \rightarrow \infty} \left[\min_{\sigma_t \in \omega_t^i} e(\sigma_t) - \min_{\tilde{\omega}_t^i \subseteq \omega_t^j} e(\tilde{\omega}_t^i) \right] > \epsilon.$$

To understand the definition note that in general, the initial endowment of the economy is not measurable w.r.t. to Ω^i or Ω^j . Hence, if prices are strictly positive, the maximum consumption i can thus obtain given the initial endowment of the economy is given by $\min_{\sigma_t \in \omega_t^i} e(\sigma_t)$. Since i 's partition is finer than j 's, the maximum consumption j can obtain given the initial endowment of the economy is given by $\min_{\tilde{\omega}_t^i \subseteq \omega_t^j} e(\tilde{\omega}_t^i)$. j 's unawareness is called relevant in the limit w.r.t. to i 's whenever the maximal consumption streams these two agents can obtain in the limit differ and irrelevant if they are the same.

Proposition 13 Consider a population of agents with nested awareness partitions, Ω^1 strictly finer than $\Omega^2 \dots$ strictly finer than Ω^n , and ordered survival indices such that:

$$\ln \beta_k - \sum_{w^{k'}} \pi(w^{k'}) \ln \frac{\pi(w^{k'})}{\pi(w^{k'} \cap S^k | S^k)} < \ln \beta_{k'} - \sum_{w^{k'}} \pi(w^{k'}) \ln \frac{\pi(w^{k'})}{\pi(w^{k'} \cap S^{k'} | S^{k'})}$$

holds for all $k < k'$. Assume that the initial endowment of the economy is measurable w.r.t. Ω^1 . Assume that \tilde{k} is the agent with the finest awareness partition, whose unawareness is relevant in the limit ($\tilde{k} > 1$), whereas the unawareness of all agents $i < \tilde{k}$ is irrelevant in the limit. Then there exists an equilibrium of the economy, in which

- (i) agents $k < (\tilde{k} - 1)$ vanish a.s.;
- (ii) agent $(\tilde{k} - 1)$ survives a.s.;
- (iii) agent $n > k > \tilde{k} - 1$ survives a.s. if the unawareness of agent $k + 1$ is relevant in the limit w.r.t. that of k and vanishes a.s. otherwise;
- (iv) agent n a.s. survives.

The proposition demonstrates that partially aware agents can survive in the presence of a fully aware agent as long as their survival index exceeds that of the fully aware agent. In such an economy, an agent survives regardless of the value of his survival index as long as the awareness of the agent with the finest partition coarser than his is relevant in the limit and vanishes otherwise. This result is interesting, because it shows that more aware agents can survive even when their survival index is not maximal in the economy.

This requires however the presence of agents whose unawareness is relevant even in the limit. In such a scenario, the partially aware agents do not wish to consume the entire endowment of the economy: such a consumption stream would expose them to surprises. Hence, it is the agents with lower survival indexes, but higher awareness who consume in such states (hold the assets that pay in such states) and thus survive.

Note that the results for an economy with naively unaware agents, who act as if they assign 0-probability to possible events, and that with agents who are aware of their own limited perception of the world as in Grant and Quiggin (2015) are qualitatively different. While the former vanish a.s. in finite time, the latter, provided that they entertain correct beliefs over their awareness partition, effectively insure themselves against all surprises and survive a.s., even in the presence of a fully aware agent.

This result, however, hinges on the assumption that partially aware agents can have a survival index which exceeds that of fully aware agents, or, for equal discount factors, that they can adopt correct beliefs. While it is easy to generate examples, in which this would be the case, in general, this assumption

need not be satisfied. In particular, if the most aware agent in the economy is fully aware, then his (awareness-biased) beliefs would be correct and hence, the partially aware agents would only survive if their beliefs were also correct. The discussion above suggests that this is not likely. In the next section, we will thus extend the model by introducing aversion to unfavorable surprises. This extension will allow to establish survival results, which do not depend on the assumption of correct beliefs for partially aware agents.

5 Aversion to Unforeseen Unfavorable Surprises

We next extend the model to introduce the possibility of unforeseen surprises. An unforeseen unfavorable surprise is an outcome \ominus , which is considered worse than any of the foreseen outcomes. An unforeseen favorable surprise is an outcome \oplus , which is considered better than any of the foreseen outcomes.

As in Grant and Quiggin (2015), we can define for a given consumer i , the set of consumption streams not subject to unforeseen surprises by:

$$\bar{C}^i = \{c^i \mid C^i(\omega_t^i) \cap \{\ominus; \oplus\} = \emptyset \text{ for all } \omega_t^i \in \Omega^i\}.$$

For consumer i , the set of consumption streams subject to unforeseen unfavorable surprises is:

$$C^{uu, i} = \{c^i \mid \ominus \in C^i(\omega_t^i) \text{ for some } \omega_t^i \in \Omega^i\}$$

and that of consumption streams subject to unforeseen favorable surprises is:

$$C^{uf, i} = \{c^i \mid \oplus \in C^i(\omega_t^i) \text{ for some } \omega_t^i \in \Omega^i\}.$$

Grant and Quiggin (2015) then impose three additional axioms on preferences: strict aversion to unforeseen unfavorable surprises, preference for unforeseen favorable surprises and comparability of unforeseen surprises, see Appendix A, Axioms 6, 7, 8 for details. Strict aversion to unforeseen unfavorable surprises states that all consumption streams that contain unfavorable surprises are considered worse than the consumption stream that delivers the worst perceived consumption L at all σ_t . Preference for unforeseen favorable surprises means that consumption streams that contain favorable, but no unfavorable surprises are considered better than the consumption stream that pays the maximal perceived outcome M at all σ_t .

To define comparability of unforeseen surprises, they define for each consumption stream the following projection: for a given consumption stream c^i , which contains unforeseen surprises, an unfavorable unforeseen surprise is "replaced" by the minimal perceived outcome, let's call it L . Symmetrically, an unforeseen favorable surprise, is replaced by a maximal perceived outcome, M . Formally,

$$\bar{C}^i(\omega_t^i) = \begin{cases} C^i(\omega_t^i) & \text{if } C^i(\omega_t^i) \cap \{\ominus; \oplus\} = \emptyset \\ C^i(\omega_t^i) \setminus \{\ominus\} \cup \{L\} & \text{if } \ominus \in C^i(\omega_t^i), \oplus \notin C^i(\omega_t^i) \\ C^i(\omega_t^i) \setminus \{\oplus\} \cup \{M\} & \text{if } \ominus \notin C^i(\omega_t^i), \oplus \in C^i(\omega_t^i) \\ C^i(\omega_t^i) \setminus \{\ominus; \oplus\} \cup \{L; M\} & \text{if } \ominus, \oplus \in C^i(\omega_t^i) \end{cases}$$

Comparability of unforeseen surprises requires consumption streams which belong to the same category ($\mathcal{C}^{uu, i}$, or $\mathcal{C}^{uf, i}$) to be ranked identically to their projections, \bar{C}^i . These three axioms are easy to rewrite in our context and imply the following representation of preferences over consumption streams:

$$\tilde{V}_0^i(c^i) = \begin{cases} \left[\begin{array}{l} u_i(c^i(\sigma_0)) + \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\}) \end{array} \right] & \text{if } c^i \in \bar{C}^i \\ \left[\begin{array}{l} u_i(c^i(\sigma_0)) + u(L) - u(M) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\}) \end{array} \right] & \text{if } c^i \in \mathcal{C}^{uu, i} \\ \left[\begin{array}{l} u_i(c^i(\sigma_0)) - u(L) + u(M) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\}) \end{array} \right] & \text{if } c^i \in \mathcal{C}^{uf, i} \setminus \mathcal{C}^{uu, i} \end{cases}$$

To understand how this model of unforeseen surprises can be applied to our setting, assume that an unforeseen unfavorable surprise occurs whenever the consumption at some node is in the set $[0; L)$. For instance, we could set, $L \leq m$, since m is the minimal endowment of an agent with the interpretation that an agent is not able to imagine an outcome that is lower than those specified by his initial endowment stream. Similarly, we could say that an unforeseen favorable surprise occurs, whenever the consumption exceeds some value M . Since preferences for unforeseen favorable surprises lead to non-convex preferences, we will here set $M \geq m'$, the maximal endowment of the economy. In this way, while à priori possible, unforeseen favorable surprises are de facto excluded by the specification of the endowment process.

Proposition 14 *Consider an economy, in which agents are partially aware and aware of their unawareness. Assume as well that an unforeseen unfavorable surprise is associated with a consumption less than L , with $L \leq m$ and that an unforeseen favorable surprise is associated with consumption larger than M , $M \geq m'$. Consider, furthermore, a population of agents $I = \{1 \dots n\}$ with nested partitions, $\Omega^1 = \Omega$ strictly finer than Ω^2 , ... strictly finer than Ω^n . Suppose that the partially aware agents exhibit strict aversion to unforeseen unfavorable surprises. Under assumptions 1–3 and if the unawareness of agent 2 is relevant in the limit, there is an equilibrium of the economy, in which all partially aware agents do not experience unforeseen surprises and thus survive on all paths σ , whereas the fully aware agent 1 (who does not exhibit strict aversion to unforeseen unfavorable surprises) survives a.s.*

Our last proposition establishes survival of partially aware agents, who are averse to unforeseen unfavorable surprises, independently of their beliefs. Since such agents wish at all cost to avoid consumption lower than L , their consumption in equilibrium is always bounded away from 0 on all paths (even those they are unaware of). In terms of asset holdings, such agents hold a minimal number L of bonds paying on Ω_t for each t . Hence, they follow the second heuristic described in Section 2. As in Grant and Quiggin (2013 a, b), this heuristic can be interpreted as an instance of the Precautionary Principle, which avoids the worst possible outcomes, in this case, consumption below L (or converging to 0) on paths with strictly positive probability.

According to our definition in Section 2, such a heuristic is ecologically rational:

Corollary 15 *The heuristic "aversion to unforeseen unfavorable surprises" is ecologically rational for any partially aware agent whose beliefs are further away from the truth than those of a fully aware agent.*

Indeed, under the conditions of Proposition 14, the partially aware agent would survive when applying the heuristic, whereas according to Proposition 10, he would vanish.

An interesting case to consider is that of partially aware agents with incorrect beliefs and a fully aware agent with correct beliefs. In this case, the economy a.s. reaches a node, from which on the constraint requiring the agent to avoid an unfavorable surprise is binding. From this point on, the partially aware agents with incorrect beliefs consume L in each state of the world, whereas the fully aware agent with correct beliefs will hold the rest of the endowment of the economy. The equilibrium prices on such paths will then be determined by the beliefs of the fully aware agent and thus be potentially biased. Furthermore, since the partially aware agents create an excess demand for bonds, prices will exhibit the equity premium puzzle.

6 Conclusion

In this paper, we considered the impact of partial awareness in the form of a restriction of the state space. We showed that when a decision maker is aware of his unawareness, his perception of the state space can be represented by a partition. Such a decision maker behaves as if he restricts his portfolio choice to a subset of the available assets with payoffs measurable w.r.t. his awareness partition and thus avoids "surprises". Such a heuristic allows the partially aware agent to survive, but only if his beliefs on his awareness partition are at least as close to the truth as the beliefs of a more aware agent with a finer partition. As we show, however, formulating correct beliefs might be a challenging task for a partially aware person. Introducing a second heuristic, aversion to unforeseen unfavorable surprises, as in Grant and Quiggin (2015), allows us to establish survival of partially aware agents, regardless of whether their beliefs are correct. This heuristic implies that the agent holds a minimal number of bonds in his portfolio. Asset prices in such an economy will reflect the potentially incorrect beliefs of the fully aware agents and replicate the well-known equity premium puzzle.

Our analysis illustrates how the use of heuristics naturally arises in situations, in which agents have limited ability to describe the uncertainty they face, but are also conscious of their limitations. While the preference representation we use has axiomatic foundations, the behavior modelled by such preferences mimics popular heuristics documented in the literature. Furthermore, as we show, these heuristics turn out to be "ecologically rational" in that they permit

partially aware agents to survive without formulating correct beliefs in environments, where having correct beliefs is a necessary condition for survival.

In this sense, our paper can be viewed as a bridge between two strands of the literature: the literature on behavioral finance, which explains empirical evidence from financial markets by assuming certain biases and heuristics, but without providing an explanation as to how such heuristics can survive over time; and the literature on survival in financial markets, which assumes that agents are rational and strives to uncover the characteristics that can affect prices in the long-run.

7 Appendix A: Decision-theoretic foundations: Axiomatization of "intertemporal" preferences subject to surprise

7.1 Representing Intertemporal Preferences Subject to Surprises

In our framework, agents express preferences over consumption streams $c : \Omega \rightarrow \mathbb{R}_0^+$. Since we will assume that the initial endowment of the economy is uniformly bounded above, we will further restrict the set of consumption streams to those with image in a convex and closed interval⁹, $c : \Omega \rightarrow [\tilde{l}; \tilde{m}]$, with $\tilde{l} \geq 0$ and $\tilde{m} > 0$. For agent i , define a preference relation \succsim^i over consumption streams $c : \Omega \rightarrow [\tilde{l}; \tilde{m}]$.

We follow Grant and Quiggin (2015) to model the preference of partially aware agents over such infinite consumption streams. Note that in general, the consumption stream of a partially aware agent, c , is not measurable w.r.t. Ω^i .

Yet, a consumer who is not aware of some of the contingencies, will in general not be able to form beliefs over the general state space Ω . Rather, our representation below will have the property that the subjective beliefs of consumer i , π^i are defined on his awareness partition, $(\Omega^i; \mathcal{F}^i)$.

The fact that consumption streams are in general not measurable w.r.t. Ω^i means that consumer i will in general associate with each node $\omega_t^i \in \Omega^i$ a set of possible outcomes / levels of consumption:

$$C(\omega_t^i) = \{c(\sigma_t) \mid \sigma_t \in \omega_t^i\}.$$

This set is a singleton, only when the consumption at ω_t^i is "surprise-free" in the language of Grant and Quiggin (2015), i.e., $c(\sigma_t) = c(\sigma'_t)$ for all $\sigma_t, \sigma'_t \in \omega_t^i$.

We propose the following extension of the representation of Gul and Pesendorfer (2014) and Grant and Quiggin (2015) to a multi-period setting: $c \succsim^i c'$

⁹One might wish to set $\tilde{l} = 0$ and $\tilde{m} = m'$ for the representation used in Section 3 and $\tilde{l} = L$, $\tilde{m} = M$ for that used in Section 4.

iff $V_0^i(c) \geq V_0^i(c')$, where V_0^i is defined as:

$$V_0^i(c) = u_i(c(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}; \max\{C(\omega_t^i)\})$$

where β_i is agent i 's discount factor.

In two special cases, the representation above boils down to discounted subjective expected utility maximization: the first is when the decision maker is fully aware and thus, $\Omega^i = \Omega$; the second is when the consumption streams are restricted to be measurable w.r.t. the awareness partition Ω^i , and thus, the sets $C(\omega_t^i)$ are all singletons. In general, however, there are two major differences between this representation and the standard EUM model. First, subjective probabilities π^i are well-defined only for coarse events, i.e., those in Ω^i , but not for those events, on which surprises can occur. This in turn, implies that in states associated with surprises, the consumer must take into account two different outcomes — the minimal and the maximal possible consumption — when evaluating his utility of consumption in this state. This second property leads to the utility function u_i over outcomes being defined over intervals rather than single values.

In principle, many different specifications of u_i are possible. Here, we will concentrate on the case of¹⁰

$$u_i(C^i(\omega_t^i)) = u_i(\min\{C^i(\omega_t^i)\}) \quad (5)$$

Intuitively, if the consumer is partially aware, but perceives the possibility of a surprise, he concentrates only on the worst possible outcome. Furthermore, under this specification of the utility function preferences are convex and we can thus use standard results to characterize the optimal choices and show the existence of an equilibrium.

We thus start by axiomatizing preferences over consumption streams represented by:

$$V_0^i(c) = u_i(c(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \quad (6)$$

7.2 The Axiomatization

To obtain the representation in (6), we combine the axiomatization for the static case of Gul and Pesendorfer (2014) with Koopmans' (1960) intertemporal framework and with Grant and Quiggin's (2015) model of preferences subject to surprises.

Denote by \mathcal{C} the set of all consumption streams $c : \Omega \rightarrow [\tilde{l}; \tilde{m}]$. Note that at each period, t , the consumption stream c uniquely identifies a Savage act

¹⁰It is part (iv) of Axiom 3 below that restricts u_i to be only a function of $\min C(\omega_t^i)$. Dropping this condition gives the more general formulation above.

$\mathbf{c}_t : \mathbb{F}_t \rightarrow [\tilde{l}; \tilde{m}]$, a mapping from the partition of the state space Ω relevant at t into outcomes. Denote by $\mathbf{C}_t = \{\mathbf{c}_t : \mathbb{F}_t \rightarrow [\tilde{l}; \tilde{m}]\}$ the set of all acts measurable w.r.t. \mathbb{F}_t . Furthermore,

$$\mathbf{C}_{t+1} = \{\mathbf{c}_{t+1} : \mathbb{F}_t \rightarrow [\tilde{l}; \tilde{m}]\} \supseteq \mathbf{C}_t = \{\mathbf{c}_{t+1} : \mathbb{F}_{t+1} \rightarrow [\tilde{l}; \tilde{m}]\}$$

and hence, the set of all such Savage acts can be written as:

$$\mathbf{C} = \{\cup_{t=0}^{\infty} \mathbf{C}_t\}$$

In turn, any consumption stream c can thus be represented as a sequence of Savage acts $(\mathbf{c}_t)_{t=1}^{\infty}$.

Our first two axioms state that consumer i has well-defined preferences over consumption streams and that these preferences correspond to well-defined (time-independent) preferences over Savage acts:

Axiom 1 Preferences over \mathcal{C} , \succsim^i , are complete and transitive.

To state the next axiom, for a consumption stream \tilde{c} and a Savage act $\mathbf{c} \in \mathbf{C}_t$, denote by $\mathbf{c}_t \tilde{c}_{-t}$ the consumption stream which differs from \tilde{c} only in its t^{th} component, which is replaced by the Savage act \mathbf{c} :

$$\mathbf{c}_t \tilde{c}_{-t} = (\tilde{c}_1 \dots \tilde{c}_{t-1}; \mathbf{c}; \tilde{c}_{t+1} \dots \tilde{c}_T \dots)$$

Axiom 2 For any two Savage acts measurable w.r.t. \mathbb{F}_t , \mathbf{c}'_t and $\mathbf{c}''_t \in \mathbf{C}_t$, any $t' \geq t$ and any consumption streams c, \tilde{c} ,

$$\begin{aligned} \mathbf{c}'_t c_{-t} &\succsim \mathbf{c}''_t c_{-t} \text{ iff} \\ \mathbf{c}'_{t'} \tilde{c}_{-t'} &\succsim \mathbf{c}''_{t'} \tilde{c}_{-t'} \end{aligned}$$

Note that a repeated recursive application of Axiom 2 implies that

$$\begin{aligned} \mathbf{c}'_t c_{-t} &\succsim \mathbf{c}'_{t'} \tilde{c}_{-t'} \text{ iff} \\ \mathbf{c}''_{t'} c_{-t'} &\succsim \mathbf{c}''_{t'} \tilde{c}_{-t'} \end{aligned} \tag{7}$$

This axiom provides a strong time-separability condition, which states that the comparison between two Savage acts in \mathbf{C}_t available in a given period does not depend on the period at which they are obtained, nor on the rest of the consumption stream.

This axiom allows us to define a static preference relation on Savage acts \succsim_s^i on \mathbf{C} defined by:

$$\mathbf{c}'_{t'} \succsim_s^i \mathbf{c}''_{t''}$$

iff there exists a $t \geq \max\{t'; t''\}$ and a consumption stream c such that

$$\mathbf{c}'_t c_{-t} \succsim_t^i \mathbf{c}''_t c_{-t}.$$

Note that \succsim_s^i is complete and transitive.

For any two Savage acts \mathbf{c}_t and $\mathbf{c}'_{t'}$ and event $E \in \mathcal{F}$, we write $\mathbf{c}_{tE}\mathbf{c}'_{t'}$ for the Savage act, which coincides with \mathbf{c}_t on E and with $\mathbf{c}'_{t'}$ on E^c .

We will now show how preferences \succsim_s^i combined with our definition of partial awareness can be interpreted in the spirit of Gul and Pesendorfer's (2014) Expected Uncertain Utility. We first recall the definition of an ideal event:

Definition 16 *A subset $E \in \mathcal{F}$ is an ideal event if for every pair of Savage acts \mathbf{c}_t and $\mathbf{c}'_{t'} \in \mathbf{C}$*

$$\mathbf{c}_t \succsim \mathbf{c}_{tE}\mathbf{c}'_{t'}$$

implies

$$\mathbf{c}_{tE}\mathbf{c}'_{t'} \succsim \mathbf{c}'_{t'}.$$

Let \mathcal{I} denote the set of ideal events in \mathcal{F} .

Gul and Pesendorfer (2014) explain that ideal events correspond to events, on which subjective probabilities are well-defined. In our context, ideal events are events on which the decision maker does not experience any surprises, provided that the Savage act is measurable w.r.t. such an event. In particular, for agent i , the elements of the partition \mathcal{F}^i correspond to ideal events. E.g., a Savage act which is constant across nodes σ_t belonging to the same ω_t^i , $\mathbf{c}_t = f(\omega_t^i)$ for $\omega_t^i \in \mathcal{F}_t^i$ is measurable w.r.t. to ideal events and thus, not subject to surprises from the point of view of agent i . As explained above, agent i will be able to assign additive probabilities to such ideal events.

In contrast, if both σ_t and $\sigma'_t \in \omega_t^i$, neither of the events σ_t and σ'_t are ideal. Indeed, an act \mathbf{c}_t with $\mathbf{c}_t(\sigma_t) \neq \mathbf{c}_t(\sigma'_t)$ is subject to a surprise from the point of view of agent i . Furthermore, agent i will not be able to assign probabilities to such events. Nevertheless, the probabilities of certain non-ideal events can be bounded by probabilities of ideal events. For instance, the probabilities of σ_t and σ'_t can be bounded above by the probability assigned to the ideal event ω_t^i (and below by 0).

In contrast, there are certain non-ideal events, the probability of which can only be bounded (trivially) by the universal event Ω and the empty set. Suppose, e.g., that each ω_t^i contains at least two elements and consider all $\omega_t^i \in \Omega_t^i$. For each ω_t^i select exactly one $\sigma_t(\omega_t^i) \in \omega_t^i$. The probability of the event $D = \left\{ (\sigma_t(\omega_t^i))_{\omega_t^i \in \Omega_t^i} \right\}$ can now be bounded above only by that of Ω , 1 and below by the empty set. Such an event is called diffuse. Hence, an event is diffuse if no non-null ideal event is included in the event, or its complement.

Definition 17 *A subset $E \in \mathcal{F}$ is a null event if for every pair of Savage acts \mathbf{c}_t and $\mathbf{c}'_{t'} \in \mathbf{C}$*

$$\mathbf{c}_t \sim \mathbf{c}'_{t'E}\mathbf{c}_t.$$

Let \mathcal{N} denote the set of null events.

Definition 18 *An event $D \in \mathcal{F}$ is diffuse if for any $E \in \mathcal{I} \setminus \mathcal{N}$, $E \cap D \neq \emptyset$ and $E \cap (\Omega \setminus D) \neq \emptyset$.*

As in Grant and Quiggin (2015), the fact that the decision maker is subject to unquantifiable uncertainty in the face of non-ideal events and hence, faces the possibility of a surprise, is captured by a utility function defined on sets of payoffs, denoted above by $C^i(\omega_t^i)$. As shown in Gul and Pesendorfer (2014, Lemma 1) for the purposes of the desired representation, each such set can be uniquely identified by its lower and upper bound, $[\min C^i(\omega_t^i); \max C^i(\omega_t^i)]$. It is then on such intervals that the utility function u is defined. Since ideal events (in particular, ω_t^i) are assigned unique probabilities, π , the uncertainty expected utility representation combines in a multiplicative way the utility function u defined on sets of outcomes obtained on ideal events with their probability.

We now impose the axioms of Gul and Pesendorfer (2014) and Grant and Quiggin (2015) on \succsim_s^i to obtain the static EUU representation:

Axiom 3 (EUU-representation)

- (i) Every event in \mathcal{F}^i is ideal.
- (ii) Any event of the type $\left\{(\sigma(\omega^i))_{\omega^i \in \Omega^i}\right\} \notin \mathcal{F}^i$, where $\sigma(\omega^i) \in \omega^i$ is diffuse.
- (iii) Preferences over Savage acts \succsim_s^i satisfy Axioms 1-6 of Gul and Pesendorfer (2014).
- (iv) Let D be a diffuse event and consider a Savage act $\mathbf{c}_D \mathbf{c}'$ with $\mathbf{c}(\sigma_t) = a$ for all σ_t and $\mathbf{c}'(\sigma_t) = b$ for all σ_t , $\tilde{m} \geq a > b \geq \tilde{l}$. Then, $\mathbf{c}_D \mathbf{c}' \sim \mathbf{c}'$.

Part (iii) imposes the axioms of Gul and Pesendorfer (2014) on preferences over Savage acts, \succsim_s^i . Indeed, the representation of Gul and Pesendorfer (2014) implies the existence of a representation of preferences on the set of ideal events \mathcal{I} with a non-atomic prior on \mathcal{I} . Parts (i) and (ii) are the counterparts of Axiom 1 (i) and (ii) in Grant and Quiggin (2015) and serve to specify the set \mathcal{I} , as well as the set of diffuse events. These two conditions imply that the probability distribution π^i will be defined on $(\Omega^i; \mathcal{F}^i)$. Note that if some $\left\{(\sigma(\omega^i))_{\omega^i \in \Omega^i}\right\} \in \mathcal{F}^i$, then all such sets are in \mathcal{F}^i — this corresponds to the scenario, where $\Omega^i = \Omega$ and the decision maker is fully aware. Finally, Part (iv) implies that the function u_i depends only on $\min C^i(\omega_t^i)$. We obtain:

Proposition 19 *Preferences over Savage acts \succsim_s^i satisfy Axiom 3 iff there exists a countably-additive and non-atomic probability measure π^i on $(\Omega^i; \mathcal{F}^i)$ and an interval utility¹¹ u_i such that \succsim_s^i are represented by:*

$$U_i(\mathbf{c}_t) = \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i(\min C^i(\omega_t^i)).$$

¹¹Interval utility u_i is defined in Gul and Pesendorfer (2014) as a utility function, which is continuous and strictly monotone in its arguments.

Proof of Proposition 19:

The proof of Theorem 1 in Grant and Quiggin (2015) implies the existence of a representation given by:

$$U_i(\mathbf{c}_t) = \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i(\min C^i(\omega_t^i); \max C^i(\omega_t^i)).$$

When $C^i(\omega_t^i)$ is a singleton, u_i reduces to the standard von-Neumann-Morgenstern utility function. In particular, u_i on the set of singletons is identified by the Savage axioms on acts measurable w.r.t. \mathcal{F}^i . Given u_i restricted to singletons, u_i is extended to intervals as follows. Given a diffuse event D , consider the certainty equivalent $\mathbf{c}''(\sigma_t) = x \in [\tilde{l}; \tilde{m}]$ for all $\sigma_t \in \Omega$ of a bet on D , $\mathbf{c}_D \mathbf{c}'$ with $\mathbf{c}(\sigma_t) = a$ for all σ_t and $\mathbf{c}'(\sigma_t) = b$ for all σ_t , $\tilde{m} \geq a > b \geq \tilde{l}$.

$$\mathbf{c}'' \sim \mathbf{c}_D \mathbf{c}'.$$

Then,

$$u_i(b; a) = u(x).$$

By part (iv) of Axiom 3, we have that

$$u_i(b; a) = u(b)$$

for any $b < a$ and hence, $u_i(\min C^i(\omega_t^i); \max C^i(\omega_t^i)) = u_i(\min C^i(\omega_t^i))$. ■

We now provide the intertemporal part of the representation. We do so by combining the framework developed so far with the Koopmans' (1960) setting. We first note that for each Savage act, \mathbf{c}_t , there is a constant Savage act denoted $x(\mathbf{c}_t) \in [\tilde{l}; \tilde{m}]$ such that:

$$\mathbf{c}_t \sim_c^i x(\mathbf{c}_t)$$

Next note that applying inductively Axiom 2 implies that each consumption stream $c = (\mathbf{c}_t)_{t=0}^\infty$ is indifferent to the consumption stream:

$$x(c) = (x(\mathbf{c}_t))_{t=0}^\infty.$$

Let X denote the set of all certain consumption streams. We can apply to the preference relation \succsim^i restricted to the set X the axioms of Koopmans (1960) to derive the discounted utility representation.

Since part of Postulate 1 and Postulate 3 are implied by Axiom 1 and Axiom 2, respectively and since Koopman's postulates are stated in terms of utility rather than preferences, we here restate the necessary conditions:

Axiom 4 Preferences \succsim^i on X satisfy:

- (i) **Uniform Continuity:** For $x, x', x'' \in X$ if $x' \succ x \succ x''$, there is a δ such that for any $y \in X$, $\lim_{t \rightarrow \infty} \sup |x_t - y_t| < \delta$ implies $x' \succ y \succ x''$.
- (ii) **Monotonicity:** If $x > x'$, $x \succ x'$.

(iii) **Stationarity:** For any $(x_t)_{t=0}^\infty, (x'_t)_{t=0}^\infty$ such that $x_0 = x'_0, (x_t)_{t=0}^\infty \succsim (x'_t)_{t=0}^\infty$ iff $(x_{t+1})_{t=0}^\infty \succsim (x'_{t+1})_{t=0}^\infty$.

Note that we have replaced Koopmans' weaker Postulate 2, Sensitivity and Postulate 5, Extreme Programs, by Monotonicity. The reason for this is that monotonicity on X is implied by a combination of Axiom 2 in Gul and Pesendorfer (2014) and our Axiom 2. Hence, we will need neither Postulates 2 and 5, nor Monotonicity in our final representation.

Proposition 20 (Koopmans, 1960) *Axioms 1, 2 and 4 imply the existence of a function $v_i : [\tilde{l}; \tilde{m}] \rightarrow \mathbb{R}$ and a discount factor $\beta_i \in (0; 1)$ such that for any consumption streams c, c' ,*

$$x(c) \succsim x(c') \text{ iff } \sum_{t=0}^{\infty} \beta_i^t v_i(x(c_t)) \geq \sum_{t=0}^{\infty} \beta_i^t v_i(x(c'_t))$$

Proof of Proposition 20:

As noted above, Axiom 2 implies Koopmans' Postulates 3 and 3'. Monotonicity implies Postulates 2 and 5. Postulate 1 is implied by a combination of Axiom 7.2 and Uniform Continuity. The proof in Koopmans (1960) shows that these conditions imply the desired result. ■

It remains thus to show that the function v_i coincides with the function u_i (i.e., that the utility index captures at the same time risk- and time-preferences). Note, however, that while u_i is unique up to an affine-linear transformation, the same is not true for v_i . Indeed, we can, w.l.o.g., normalize $v_i(\tilde{l}) = 0$ and $v_i(\tilde{m}) = 1$. For a given discount factor β_i and some outcome $z \in [\tilde{l}; \tilde{m}]$, the utility of z , $v_i(z)$ is uniquely identified by the representation only if there exists a (possibly infinite) set of periods, $(\tau_k)_k$ such that the act that pays \tilde{m} on all periods in $(\tau_k)_k$ and \tilde{l} , else (and thus generates utility equal to $\sum_k \beta_i^{\tau_k}$) is indifferent to the constant act that pays z at each period t :

$$\left((\tilde{m}_{\tau_k})_k \tilde{l}_{-(\tau_k)_k} \right) \sim_i ((z_t)_{t=0}^\infty)$$

In this case, clearly

$$v_i(z) = \sum_k \beta_i^{\tau_k}$$

is uniquely determined by the value of β_i . However, since time is discrete, such a construction is in general not possible for each outcome¹² z .

To ensure that u_i and v_i can be chosen to be identical, we impose:

¹²As an example, consider an outcome $y \in (\tilde{l}; \tilde{m})$ with a utility value $v_i(y)$ provided by the representation. Since $v_i(y) < 1$, we have that $\frac{1}{1-\beta} > v_i(y)$. Assume now that $v_i(y) > \frac{\beta}{1-\beta}$.

Axiom 5 Suppose that for some ideal event $E \in \mathcal{F}_t$ and some sequence of periods $(\tau_k)_k$,

$$\left(\underbrace{\tilde{l} \dots \tilde{l}}_{t-1}; \underbrace{\tilde{m}_E \tilde{l}}_t; \tilde{l} \dots \tilde{l} \dots \right) \sim \left(\tilde{m}_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right)$$

then

$$\left(\underbrace{\tilde{l} \dots \tilde{l}}_{t-1}; \underbrace{z_E \tilde{l}}_t; \tilde{l} \dots \tilde{l} \dots \right) \sim \left(z_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right)$$

for all $z \in [\tilde{l}; \tilde{m}]$.

Proposition 21 *If Axiom 5 holds, v_i can be chosen to coincide with u_i on $[\tilde{l}; \tilde{m}]$. Hence, \succsim^i and the corresponding \succsim_s^i satisfies Axioms 1, 2, 3, 4 (i) and (iii) and 5, iff there exist a countably-additive and non-atomic probability measure π^i on $(\Omega^i; \mathcal{F}^i)$, an interval utility¹³ u_i and a discount factor $\beta_i \in (0; 1)$ such that \succsim^i is represented by:*

$$V_i(c^i) = u_i(c^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\}).$$

Proof of Proposition 21:

Combining Propositions 19 and 20, and noting as above that Koopmans' Monotonicity is implied by Axiom 2 in Gul and Pesendorfer (2014) and our Axiom 2, we obtain the representation:

$$V_i(c^i) = v_i(c^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t v_i(u_i^{-1}(\pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\})))$$

It remains thus to show that the function v_i can be chosen to coincide with u_i . To do so, normalize both u_i and v_i so that $u_i(\tilde{l}) = v_i(\tilde{l}) = 0$ and $u_i(\tilde{m}) = v_i(\tilde{m}) = 1$.

Note that

$$\begin{aligned} \max_{(\tau_k)_k} \sum_k \beta_i^{\tau_k} &= \frac{1}{1-\beta} \\ \max_{(\tau_k)_k | \tau_k \neq 0 \text{ for all } k} \sum_k \beta_i^{\tau_k} &= \frac{\beta}{1-\beta} \end{aligned}$$

i.e., the maximum is obtained when we sum over all possible periods, but this maximum is too high to match $v_i(y)$. In fact, any sum that includes $\tau_k = 0$ will exceed 1 and thus $v_i(y)$. In contrast, any sum that contains only periods $\tau_k > 0$ cannot exceed $\frac{\beta}{1-\beta} < v_i(y)$. We conclude that on the interval of payoffs which correspond to utility values $\left(\frac{\beta}{1-\beta}; 1\right)$, any monotonic uniformly continuous transformation of v_i will represent the same preferences.

¹³Interval utility u_i is defined in Gul and Pesendorfer (2014) as a utility function, which is continuous and strictly monotone in its arguments.

Next note that

$$\left(\underbrace{\tilde{l} \dots \tilde{l}}_{t-1}; \underbrace{\tilde{m}_E \tilde{l}}_t; \tilde{l} \dots \tilde{l} \dots \right) \sim \left(\tilde{m}_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right)$$

implies

$$\pi^i(E) = \sum_k \beta_i^{\tau_k}$$

and hence, Axiom 5 requires that for any $\delta = \sum_k \beta_i^{\tau_k}$ for some sequence $(\tau_k)_k$, and any $z \in [\tilde{l}; \tilde{m}]$,

$$u_i^{-1}[\delta u_i(z)] = v_i^{-1}[\delta v_i(z)].$$

Let then $y = u_i^{-1}[\delta u_i(z)] = v_i^{-1}[\delta v_i(z)]$. We then have

$$\frac{u_i(y)}{u_i(z)} = \frac{v_i(y)}{v_i(z)} = \delta$$

for all z and all y as above. This, in turn implies that for $z = 1$, $y_0 = u_i^{-1}(\delta) = v_i^{-1}(\delta)$, $u_i(y_0) = v_i(y_0)$. Setting $y_1 = u_i^{-1}(\delta y_0) = v_i^{-1}(\delta y_0)$ gives $u_i(y_1) = v_i(y_1)$ and proceeding by induction, we obtain the equality of the functions u_i and v_i for all z in the set Z defined recursively as follows

$$\begin{aligned} Z_0 &= \left\{ z \in [\tilde{l}; \tilde{m}] \mid \text{there is a sequence } (\tau_k)_k \text{ such that } \left(z; \tilde{l} \dots \tilde{l} \dots \right) \sim \left(\tilde{m}_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right) \right\} \\ Z_1 &= \left\{ z \in [\tilde{l}; \tilde{m}] \mid \text{there is a sequence } (\tau_k)_k \text{ and } z' \in Z_0 \text{ such that } \left(z; \tilde{l} \dots \tilde{l} \dots \right) \sim \left(z'_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right) \right\} \\ &\dots \\ Z_K &= \left\{ z \in [\tilde{l}; \tilde{m}] \mid \text{there is a sequence } (\tau_k)_k \text{ and } z' \in Z_{K-1} \text{ such that } \left(z; \tilde{l} \dots \tilde{l} \dots \right) \sim \left(z'_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right) \right\} \\ &\dots \\ Z &= \bigcup_{\kappa=0}^{\infty} Z_{\kappa} \end{aligned}$$

Note, furthermore, that differently from u_i , v_i is only unique up to an affine-linear transformation on Z and we have shown $u_i(z) = v_i(z)$ for all $z \in Z$.

The continuity of the functions u_i and v_i ensures that Z is closed. Consider an outcome $y \notin Z$. It follows that $y \in (\underline{y}; \bar{y}) \subset [\tilde{l}; \tilde{m}]$ such that $u_i(z) = v_i(z)$ for $z \in \{\underline{y}; \bar{y}\}$. Furthermore, any monotonic uniformly continuous transformation of v_i on $(\underline{y}; \bar{y})$, including u_i , represents the same preferences. Hence, we can set $v_i(y) = u_i(y)$ for all $y \in (\underline{y}; \bar{y})$.

This gives us the desired representation. ■

7.3 Unforeseen Surprises

We next extend the model to introduce the possibility of unforeseen surprises. An unforeseen unfavorable surprise is an outcome \ominus , which is considered worse

than any of the foreseen outcomes. An unforeseen favorable surprise is an outcome \oplus , which is considered better than any of the foreseen outcomes.

As in Grant and Quiggin (2015), we can define for a given consumer i , the set of consumption streams not subject to unforeseen surprises:

$$\bar{\mathcal{C}} = \{c \mid \{\sigma_t \in \Omega_t \mid \mathbf{c}_t(\sigma_t) \cap \{\ominus; \oplus\}\} \in \mathcal{N} \text{ for all } t\}.$$

The set of consumption streams subject to unforeseen unfavorable surprises is:

$$\mathcal{C}^{uu} = \{c \mid \{\sigma_t \in \Omega_t \mid \mathbf{c}_t(\sigma_t) = \ominus\} \notin \mathcal{N} \text{ for some } t\}$$

and that of consumption streams subject to unforeseen favorable surprises is:

$$\mathcal{C}^{uf} = \{c \mid \{\sigma_t \in \Omega_t \mid \mathbf{c}_t(\sigma_t) = \oplus\} \notin \mathcal{N} \text{ for some } t\}.$$

Here we axiomatize preferences over consumption streams c^i representable by:

$$\tilde{V}_0^i(c) = \begin{cases} \left[\begin{array}{l} u_i(c(\sigma_0)) + \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \bar{\mathcal{C}} \\ \left[\begin{array}{l} u_i(c(\sigma_0)) + u(\tilde{l}) - u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uu} \\ \left[\begin{array}{l} u_i(c(\sigma_0)) - u(\tilde{l}) + u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu} \end{cases} \quad (8)$$

To obtain such an axiomatization in a one-period setting, Grant and Quiggin (2015) impose three additional axioms on preferences: strict aversion to unforeseen unfavorable surprises, preference for unforeseen favorable surprises and comparability of unforeseen surprises, which we restate below to apply to infinite consumption streams:

Axiom 6 The preferences \succsim^i exhibit strict aversion to unforeseen unfavorable surprises, i.e., $(\tilde{l} \dots \tilde{l} \dots) \succ^i c$ for any $c \in \mathcal{C}^{uu}$.

Axiom 7 The preferences \succsim^i exhibit preference to unforeseen favorable surprises, i.e., $c \succ^i (\tilde{m} \dots \tilde{m} \dots)$ for any $c \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu}$.

Axiom 6 (which corresponds to Axiom 2 Strict aversion to unforeseen unfavorable surprises in Grant and Quiggin, 2015) states that all consumption streams that contain unfavorable surprises are considered worse than the consumption stream that delivers the worst perceived consumption \tilde{l} at all σ_t . Axiom 7 (which corresponds to Axiom 3 Preference for unforeseen favorable surprises in Grant and Quiggin, 2015) means that consumption streams than contain favorable, but no unfavorable surprises are considered better than the consumption stream that pays the maximal perceived outcome \tilde{m} at all σ_t . The implication of these two axioms is that the decision maker will at all cost avoid

being exposed to unforeseen unfavorable surprises and seek to be exposed to exposed to unforeseen favorable surprises.

As in Grant and Quiggin (2015), we define for each consumption stream c , its projection, a consumption stream \bar{c} such that \bar{c} coincides with c whenever c contains no unforeseen surprises. If c contains unforeseen surprises, an unfavorable unforeseen surprise is "replaced" in \bar{c} by the minimal perceived outcome, \tilde{l} . Symmetrically, an unforeseen favorable surprise, is replaced in \bar{c} by the maximal perceived outcome, \tilde{m} . Formally,

$$\bar{c}(\sigma_t) = \begin{cases} c(\sigma_t) & \text{if } c(\sigma_t) \notin \{\ominus; \oplus\} \\ \tilde{l} & \text{if } c(\sigma_t) = \ominus \\ \tilde{m} & \text{if } c(\sigma_t) = \oplus \end{cases}$$

Let $\bar{\mathcal{C}}$ be the set of all projections.

The last axiom of Grant and Quiggin (2015) ensures that acts within one of the three sets defined above can be compared using their projections:

Axiom 8 The preferences \succsim^i permit comparability of unforeseen surprises if for any pair of acts c and c' if either $c, c' \in \bar{\mathcal{C}}$, or if $c, c' \in \mathcal{C}^{uu}$, or if $c, c' \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu}$, then

$$c \succsim^i c' \text{ iff } \bar{c} \succsim^i \bar{c}'.$$

Comparability of unforeseen surprises requires consumption streams which belong to the same category (\mathcal{C}^{uu} , or $\mathcal{C}^{uf} \setminus \mathcal{C}^{uu}$) to be ranked identically to their projections, $\bar{\mathcal{C}}$.

Using Theorem 3 in Grant and Quiggin (2015) and combining it with the result derived in Proposition 21, we obtain:

Proposition 22 \succsim^i and the corresponding \succsim_s^i satisfies Axioms 1, 2, 3, 4 (i) and (iii), 5, 6, 7 and 8, iff it can be represented as:

$$\tilde{V}_0^i(c) = \begin{cases} \left[\begin{array}{l} u_i(c(\sigma_0)) + \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \bar{\mathcal{C}} \\ \left[\begin{array}{l} u_i(c(\sigma_0)) + u(\tilde{l}) - u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uu} \\ \left[\begin{array}{l} u_i(c(\sigma_0)) - u(\tilde{l}) + u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu} \end{cases}$$

Remark 23 Just as Grant and Quiggin (2015), for the purposes of the axiomatization, we take the partition W^i and the implied partition Ω^i on the event tree of the economy as given and observable. In fact, the axiomatization of Gul and Pesendorfer (2014) used to establish the results does not require such observability. Without additional assumptions, in our context, a direct application of their theorem would imply a partition of Ω , Ω^i . However, in general, this partition need not satisfy the multiplicative structure $\Omega^i = \prod_{t=0}^{\infty} W^i$ imposed above. We conjecture that adding the following axiom characterizing the set of ideal events would provide a sufficient condition for the multiplicative structure assumed:

Axiom 9 For any t subsets of S , $w^1 \dots w^t \subseteq S$, $\omega_t = \{(s_0; s_1 \dots s_t) \mid s_1 \in w^1 \dots s_t \in w^t\}$ is ideal iff all of the events $\omega_1^t = \{(s_0; s)_{s \in w^t}\}$ are ideal.

8 Appendix B: Proofs

To simplify the proofs of the following results, we state and prove the following:

Lemma 24 Consider two agents i and j such that j is fully aware and i is partially aware. In equilibrium, for any path ω^i , if i 's consumption is measurable w.r.t. Ω^i ,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\left(\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} \mid \omega_{T+1}^i) \right)} \\ &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \left(\sum_{w^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right). \end{aligned}$$

Proof of Lemma 24:

We will use the analogue of the Blume and Easley (2006) decomposition. The f.o.c.'s of two agents, i , who is partially aware and j , who is fully aware give:

$$\frac{u'_i(c^i(\sigma_0))}{\beta_i u'_i(c^i(\omega_{T+1}^i)) \pi^i(\omega_{T+1}^i)} = \frac{u'_j(c^j(\sigma_0))}{\beta_j \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1})}.$$

Hence,

$$\frac{u'_i(c^i(\sigma_0))}{\beta_i u'_i(c^i(\omega_{T+1}^i)) \pi^i(\omega_{T+1}^i)} = \frac{u'_j(c^j(\sigma_0))}{\beta_j \pi^j(\omega_{T+1}^i) \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} \mid \omega_{T+1}^i)}, \quad (9)$$

which reduces to:

$$\begin{aligned} \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} \mid \omega_{T+1}^i)} &= \frac{\beta_j \pi^j(\omega_{T+1}^i) u'_i(c^i(\sigma_0))}{\beta_i \pi^i(\omega_{T+1}^i) u'_j(c^j(\sigma_0))} \\ &= \frac{u'_i(c^i(\sigma_0))}{u'_j(c^j(\sigma_0))} \prod_{t=1}^{T+1} \frac{\beta_j \pi^j(w_t^i)}{\beta_i \pi^i(w_t^i)} \end{aligned}$$

and we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} \mid \omega_{T+1}^i)} &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} \ln \frac{\pi^j(w_t^i)}{\pi^i(w_t^i)} \\ &+ \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\sigma_0))}{u'_j(c^j(\sigma_0))}. \end{aligned}$$

Since $u'_i(c^i(\sigma_0))$ and $u'_j(c^j(\sigma_0))$ are finite, the third term on the r.h.s. converges to 0, furthermore,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)} \\ &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} (\ln \pi^j(w_t^i) - \ln \pi(w_t^i)) + \\ & \quad + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} (\ln \pi(w_t^i) - \ln \pi^i(w_t^i)). \end{aligned}$$

Since $\ln \frac{\pi^j(w_t^i)}{\pi(w_t^i)}$ and $\ln \frac{\pi(w_t^i)}{\pi^i(w_t^i)}$ are i.i.d and are equal in expectations to the relative entropy of i 's and j 's beliefs with respect to the truth, we obtain that a.s.,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)} \\ &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \left(\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right). \end{aligned}$$

Proof of Proposition 8:

Note that when the initial endowment of the economy is measurable w.r.t. $\Omega^i = \Omega^j$, i and j 's consumption in equilibrium will be measurable w.r.t. Ω^i . Indeed, suppose that there is a node ω_t^i such that $c^i(\sigma_t) < c^i(\sigma'_t)$ for some $\sigma_t, \sigma'_t \in \omega_t^i$. It then follows that $p(\sigma'_t) = 0$. Indeed, otherwise i can increase his overall utility by selling $c^i(\sigma'_t) - c^i(\sigma_t)$ units at σ'_t and using the strictly positive revenue to increase strictly his consumption (and hence, his utility) at a set of nodes ω_t^i .

If in the economy, there is a consumer k for whom σ_t and σ'_t

Since the endowment of the economy is measurable w.r.t. Ω^i , it further follows that there is at least one consumer k for whom $c^k(\sigma_t) > c^k(\sigma'_t)$. Note that this would be suboptimal for k , unless, $c^k(\sigma''_t) \leq c^k(\sigma'_t) < c^k(\sigma_t)$ for some $\sigma''_t \in \omega_t^k$, $\sigma''_t \neq \sigma'_t$. Then $p(\sigma_t) = 0$, as above. And hence, if i 's consumption is optimal, then there is a σ'''_t such that

$$c^i(\sigma'''_t) < c^i(\sigma_t) < c^i(\sigma'_t)$$

and a k' such that

$$c^{k'}(\sigma'''_t) > c^{k'}(\sigma_t)$$

which, in turn, can only be optimal $p(\sigma'''_t) = 0$. Proceeding by induction, we conclude that if the aggregate endowment is measurable w.r.t. Ω^i , then $c^i(\sigma_t) < c^i(\sigma'_t)$ for some $\sigma_t, \sigma'_t \in \omega_t^i$ can only obtain iff $p(\sigma_t) = 0$ for all $\sigma_t \in \omega_t^i$. Recall, however, that i places a strictly positive probability on ω_t^i

and his utility is strictly increasing in $\min C^i(\omega_t^i)$. Hence, if $p(\sigma_t) = 0$ for all $\sigma_t \in \omega_t^i$, i will demand an infinite amount of consumption at each $\sigma_t \in \omega_t^i$, which is inconsistent with an equilibrium.

Since the same argument applies to j , we can use Lemma 24 and replacing the original state space Ω by the common partition, Ω^i , we obtain that for two such agents, i and j , with $\beta^i > \beta^j$:

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}))}{u'_j(c^j(\omega_{T+1}))} = \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} < 0$$

and hence $u'_j(c^j(\omega_{T+1})) \rightarrow \infty$, or $c^j(\omega_{T+1}) \rightarrow 0$.

When beliefs differ,

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}))}{u'_j(c^j(\omega_{T+1}))} = \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} \ln \frac{\pi^j(w_t)}{\pi^i(w_t)}$$

and both beliefs and the actual distribution over states of the world are i.i.d.,

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}))}{u'_j(c^j(\omega_{T+1}))} = \ln \frac{\beta_j}{\beta_i} + \left(\sum_{w \in W^i} \pi(w) \ln \frac{\pi(w)}{\pi^i(w)} - \sum_{w \in W^i} \pi(w) \ln \frac{\pi(w)}{\pi^j(w)} \right).$$

Hence, for equal discount factors, the agent whose beliefs with respect to the common partition are closer to the truth survives, while the other vanishes. When both discount factors and beliefs differ, we conclude that lower discount factors can be offset by having beliefs closer to the truth and vice versa. In particular, if

$$\ln \frac{\beta_j}{\beta_i} + \left(\sum_w \pi(w) \ln \frac{\pi(w)}{\pi^i(w)} - \sum_w \pi(w) \ln \frac{\pi(w)}{\pi^j(w)} \right) < 0,$$

$u'_j(c^j(\omega_{T+1})) \rightarrow \infty$, or $c^j(\omega_{T+1}) \rightarrow 0$ and j vanishes.

Finally noting that for agents who form awareness-biased beliefs,

$$\sum_{w \in W} \pi(w) \ln \frac{\pi(w)}{\pi^i(w)} = \sum_{w \in W} \pi(w) \ln \frac{\pi(w)}{\pi(w \cap S^i | S^i)}$$

completes the proof.

Proof of Proposition 9:

First note that w.l.o.g., we can identify the finest partition of an agent in the economy with the state space Ω . Order the agents from 1 to n w.r.t. the coarseness of their partitions with 1 being the agent with the finest partition. Then, we can set $\Omega^1 = \Omega$. Since the total endowment of the economy is measurable w.r.t. this partition, so will be the consumption of agent 1. Furthermore, since agent 1 assigns strictly positive probability to each node $\sigma_t \in \Omega$, $p(\sigma_t) > 0$ will hold for all σ_t . Finally, since all partitions $\Omega^2 \dots \Omega^n$ are coarser than that of 1, we conclude that each agent's equilibrium consumption will be measurable w.r.t. his own partition, see the proof of Proposition 8.

Lemma 25 *The agent with the finest partition, agent 1, cannot be the only one to survive on a path $\sigma \in \Omega$.*

Proof of Lemma 25:

Take a path σ_t and assume that on this path, only agent 1 survives, whereas all agents with coarser partitions, $i > 1$ vanish. Measurability of consumption implies that any agent $i > 1$ also vanishes on the set of paths ω^i such that $\sigma \in \omega^i$, whereas agent 1 survives on the set of paths ω^2 such that $\sigma \in \omega^2$ for the agent with the second finest partition. Then, for agents 1 and 2, (9) implies:

$$\frac{u'_2(c^2(\omega_{T+1}^2))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^2} u'_1(c^1(\tilde{\sigma}_{T+1})) \pi^1(\tilde{\sigma}_{T+1} | \omega_{T+1}^2)} = \frac{u'_2(c^2(\sigma_0)) \beta_1 \pi^1(\omega_{T+1}^2)}{u'_1(c^1(\sigma_0)) \beta_2 \pi^2(\omega_{T+1}^2)}$$

Note that the first ratio on the r.h.s. is finite (marginal utilities in period 0). Since $\beta_1 = \beta_2$, the second ratio is 1. And since both 1 and 2 have identical beliefs on ω^2 , the third ratio is also 1. Hence, the expression on the r.h.s. remains bounded, whereas the fact that 2 vanishes on ω^2 , whereas 1 survives on all $\sigma \in \omega^2$, imply that the denominator on the l.h.s. remains bounded, whereas the numerator converges to infinity, a contradiction to the equality.

It follows that at least one agent, $i > 1$ has to survive on σ , and thus on ω^i such that $\sigma \in \omega^i$.

Lemma 26 *For any $\omega_t^n \in \Omega^n$*

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n) \leq \max_{i \in \{1 \dots n\}} u'_i(m) \frac{u'_1(c^1(\sigma_0))}{u'_i(c^i(\sigma_0))} \quad (10)$$

Proof of Lemma 26:

Take a path $\sigma \in \Omega$ and note that for every σ_t , there is an $i(\sigma_t) \in I$ such that $c_i(\sigma_t) \geq m$. It follows that for any $\sigma_t \in \Omega$ and the corresponding $i(\sigma_t)$ and $\omega^{i(\sigma_t)}$ such that $\sigma \in \omega^{i(\sigma_t)}$, we have:

$$\frac{\sum_{\tilde{\sigma}_t \subseteq \omega_t^{i(\sigma_t)}} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^{i(\sigma_t)})}{u'_{i(\sigma_t)}(c^{i(\sigma_t)}(\omega_t^{i(\sigma_t)}))} = \frac{u'_1(c^1(\sigma_0))}{u'_{i(\sigma_t)}(c^{i(\sigma_t)}(\sigma_0))} \frac{\beta_{i(\sigma_t)} \pi^{i(\sigma_t)}(\omega_t^{i(\sigma_t)})}{\beta_1 \pi^1(\omega_t^{i(\sigma_t)})}$$

and since all agents have identical discount factors and beliefs,

$$\sum_{\tilde{\sigma}_t \subseteq \omega_t^{i(\sigma_t)}} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^{i(\sigma_t)}) \leq u'_{i(\sigma_t)}(m) \frac{u'_1(c^1(\sigma_0))}{u'_{i(\sigma_t)}(c^{i(\sigma_t)}(\sigma_0))}$$

It follows that for any σ_t and $\omega_t^{i(\sigma_t)} \in \Omega^{i(\sigma_t)}$,

$$\sum_{\tilde{\sigma}_t \subseteq \omega_t^{i(\sigma_t)}} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^{i(\sigma_t)}) \leq \max_{i \in \{1 \dots n\}} u'_i(m) \frac{u'_1(c^1(\sigma_0))}{u'_i(c^i(\sigma_0))} := P$$

Consider $\omega_t^n \in \Omega^n$ and note that we can partition ω_t^n into at most $n - 1$ subsets $x_k(\omega_t^n) \subseteq \Omega_t$ such that on $\sigma_t \in x_k(\omega_t^n)$ agent k is the agent with the coarsest partition and consumption $c^k(\sigma_t) \geq m$. Note furthermore that $x_k(\omega_t^n) \subseteq \Omega_t^k$. It follows that for every $\omega_t^k \in x_k(\omega_t^n)$,

$$\sum_{\tilde{\sigma}_t \subseteq \omega_t^k} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^k) \leq P$$

and thus,

$$\sum_{\omega_t^k \in x_k(\omega_t^n)} \pi^1(\omega_t^k | x_k(\omega_t^n)) \sum_{\tilde{\sigma}_t \subseteq \omega_t^k} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^k) \leq P$$

Finally, since the number of the elements of the partition $(x_k(\omega_t^n))_{k \in \{1 \dots n\}}$ is finite, we have:

$$\begin{aligned} & \sum_{k=1}^n \pi^1(x_k(\omega_t^n) | \omega_t^n) \sum_{\tilde{\sigma}_t \subseteq x_k(\omega_t^n)} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | x_k(\omega_t^n)) \\ &= \sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n) \leq P \end{aligned}$$

for any $\omega_t^n \in \Omega^n$, and thus,

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n) \leq P$$

proving the statement of the Lemma.

Lemma 27 *On any $\omega^n \in \Omega^n$, agent n survives. Furthermore, the consumption of agent n on each path is uniformly bounded below.*

Proof of Lemma 27:

Consider agents 1 and n . We have that:

$$\frac{\sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n)}{u'_n(c^n(\omega_t^n))} = \frac{u'_1(c^1(\sigma_0))}{u'_n(c^n(\sigma_0))}$$

and since by (10) the numerator on the l.h.s. remains bounded by P , we have that

$$\limsup_{t \rightarrow \infty} u'_n(c^n(\omega_t^n)) \leq P \frac{u'_n(c^n(\sigma_0))}{u'_1(c^1(\sigma_0))} \quad (11)$$

and hence, n survives on any ω_t^n . Furthermore, (11) implies

$$\liminf_{t \rightarrow \infty} c^n(\omega_t^n) \geq u_n'^{-1} \left(P \frac{u'_n(c^n(\sigma_0))}{u'_1(c^1(\sigma_0))} \right) > 0.$$

Lemma 28 For any $j \in \{2 \dots n - 1\}$ and any $\omega_t^n \in \Omega^n$

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\omega}_t^j \subseteq \omega_t^n} u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) \pi^j \left(\tilde{\omega}_t^j \mid \omega_t^n \right) \leq P \frac{u'_j \left(c^j \left(\sigma_0 \right) \right)}{u'_1 \left(c^1 \left(\sigma_0 \right) \right)} \quad (12)$$

Proof of Lemma 28:

The result follows by combining the fact that since all agents have identical beliefs and discount factors, for any j and any ω_t^n

$$\frac{\sum_{\tilde{\omega}_t^j \subseteq \omega_t^n} u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) \pi^j \left(\tilde{\omega}_t^j \mid \omega_t^n \right)}{u'_n \left(c^n \left(\omega_t^n \right) \right)} = \frac{u'_j \left(c^j \left(\sigma_0 \right) \right)}{u'_n \left(c^i \left(\sigma_0 \right) \right)}$$

and u'_n is bounded as in (11).

Lemma 29 Agent $j \in \{1 \dots n - 1\}$ survives π -a.s. on Ω .

Proof of Lemma 29:

Suppose in a manner of contradiction that there exists a set $\tilde{\Omega}^j \subseteq \Omega$ with positive measure w.r.t. π , on which 1 vanishes a.s. Then, on this set we would have:

$$\limsup_{t \rightarrow \infty} u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) = \infty \text{ a.s. on } \tilde{\Omega}^j$$

and hence,

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\omega}_t^j \in \tilde{\Omega}^j} \pi \left(\tilde{\omega}_t^j \right) u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) = \infty$$

Now let $\tilde{\Omega}^n$ denote the smallest measurable event on Ω^n such that for every $\sigma \in \tilde{\Omega}$, there is an $\omega^n \in \tilde{\Omega}^n$ with $\sigma \in \omega^n$. We then have that if 1 were to vanish a.s. on $\tilde{\Omega}$, then

$$\limsup_{t \rightarrow \infty} \sum_{\omega_t^n \in \tilde{\Omega}^n} \pi \left(\omega_t^n \right) \sum_{\substack{\tilde{\omega}_t^j \in \omega_t^n \\ \tilde{\omega}_t^j \in \tilde{\Omega}^j}} \pi \left(\tilde{\omega}_t^j \mid \omega_t^n \right) u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) = \infty$$

Since,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \sum_{\omega_t^n \in \tilde{\Omega}^n} \pi \left(\omega_t^n \right) \sum_{\substack{\tilde{\omega}_t^j \in \omega_t^n \\ \tilde{\omega}_t^j \in \tilde{\Omega}^j}} \pi \left(\tilde{\omega}_t^j \mid \omega_t^n \right) u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) = \\ & \leq \limsup_{T \rightarrow \infty} \sum_{\omega_t^n \in \tilde{\Omega}^n} \pi \left(\omega_t^n \right) \sum_{\substack{\tilde{\omega}_t^j \in \omega_t^n \\ \tilde{\omega}_t^j \in \tilde{\Omega}^j}} \pi^j \left(\tilde{\omega}_t^j \mid \omega_t^n \right) u'_j \left(c^j \left(\tilde{\omega}_t^j \right) \right) \end{aligned}$$

where the inequality follows from the definition of $\tilde{\Omega}^j$ and the fact that j 's beliefs satisfy

$$\pi^j \left(\tilde{\omega}_t^j \mid \omega_t^n \right) = \pi \left(\tilde{\omega}_t^j \mid \omega_t^n \right)$$

and since (10) and (12) hold, we obtain a contradiction for any $j \in \{1 \dots n - 1\}$. It follows that j π -a.s. survives on Ω^j and thus, by measurability on Ω .

Proof of Propositions 10:

Suppose that agent i has the coarser partition. By the proof of Proposition 9, i 's consumption will be measurable w.r.t. Ω^i . We can then use the proof of Lemma 24 by replacing the original state space by the finer of the two partitions, Ω^j . We conclude that

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\left(\sum_{\tilde{\omega}_{T+1}^j \in \omega_{T+1}^i} u'_j(c^j(\tilde{\omega}_{T+1}^j)) \pi^j(\tilde{\omega}_{T+1}^j | \omega_{T+1}^i) \right)} \quad (13) \\ &= \ln \frac{\beta_j}{\beta_i} + \left(\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right). \end{aligned}$$

Since consumption is bounded, and hence, $u'_j(c^j(\tilde{\omega}_{T+1}^j))$ cannot become 0, and since the sum of the probabilities in the denominator is 1, if

$$\ln \frac{\beta_j}{\beta_i} + \left(\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right) > 0,$$

i vanishes a.s. Finally applying (1) gives the desired result.

Proof of Proposition 13:

Let $(c; p)$ be an equilibrium of the economy. Suppose that for some $i \in I$, $\sigma_t, \sigma'_t \in \omega_t^i$, $c^i(\sigma_t) > c^i(\sigma'_t)$. Obviously, this implies that $p(\sigma_t) = 0$ and $\sigma_t, \sigma'_t \in \omega_t^j$ for all agents $j \in I$. Then consider the allocation \tilde{c} , which differs from c only in that consumption at σ_t is reallocated from agent i to agent \tilde{k} , the agent with the finest partition, whose unawareness is relevant in the limit. Formally,

$$\begin{aligned} \tilde{c}^i(\sigma_t) &= \min \{ C^i(\omega_t^i) \} \\ \tilde{c}^{\tilde{k}}(\sigma_t) &= c^{\tilde{k}}(\sigma_t) + c^i(\sigma_t) - \tilde{c}^i(\sigma_t) \\ \tilde{c}^j(\tilde{\sigma}_t) &= c^j(\tilde{\sigma}_t), \end{aligned}$$

otherwise. Obviously, $(\tilde{c}; p)$ is also an equilibrium of the economy. Hence, for each equilibrium of the economy, we can construct an equilibrium with the same price system, in which the consumption streams of each of the consumers but agent k are measurable w.r.t. his awareness partition. Furthermore, it is easy to see that the f.o.c. for this equilibrium are exactly the same as those in an economy, in which both the initial endowments and the equilibrium allocations of the consumers are given by: $e^i = \tilde{c}^i$, $e^{\tilde{k}}(\sigma_t) = \min \tilde{c}^{\tilde{k}}(\omega_t^{\tilde{k}})$ for $\sigma_t \in \omega_t^{\tilde{k}}$. We will now show that this type of equilibria satisfy the statement of the Proposition.

Lemma 30 *Suppose that the unawareness given by the partition $\Omega^{\tilde{k}}$ is relevant. A.s., at least one of the agents $1 \dots (\tilde{k} - 1)$ survives.*

Proof of Lemma 30:

Assume that a.s., only agents with indices $k \geq \tilde{k}$ survive. For a given path σ , let $i \geq \tilde{k}$ be the minimal index of an agent who survives on σ . Let $\sigma \in \omega^i \in \Omega^i$. Clearly, if i survives on σ , he also survives on any $\sigma' \in \omega^i$. According to Definition 11, there are s and $s' \in w^{\tilde{k}} \subseteq w^i$ and $\epsilon > 0$ such that for any $\sigma, \sigma' \in \omega^{\tilde{k}} \subseteq \omega^i$,

$$\limsup_{t \rightarrow \infty} [e(\sigma_t; s) - e(\sigma'_t; s')] > \epsilon.$$

Since for every $k > i$, k 's consumption is measurable with respect to ω^i and since the partitions are nested, we have that for every t , every $\sigma_t \in \omega_t^i$ and $s' \in w_{t+1}^i$,

$$\sum_{k \geq i} c^k(\omega_{t+1}^i) \leq e(\sigma_t; s'), \quad (14)$$

and hence, for every $\sigma \in \omega^i$, on which states s and s' occur infinitely often,

$$\begin{aligned} \limsup_{T \rightarrow \infty} \sum_{j < i} c^j(\sigma_T) &= \limsup_{T \rightarrow \infty} \left[e(\sigma_T) - \sum_{k \geq i} c^k(\sigma_T) \right] \\ &\geq \limsup_{T \rightarrow \infty} [e(\sigma_T) - \min e(\sigma_{T-1}; s')] \\ &\geq \limsup_{T \rightarrow \infty} [e(\sigma_{T-1}; s) - e(\sigma_{T-1}; s')] > \epsilon. \end{aligned}$$

Since s occurs infinitely often on almost every path σ , and since there is a finite number of agents in the economy, and since the unawareness of all agents with indices larger than $\tilde{k} - 1$ is relevant in the limit, a.s., at least one of the agents $j \leq \tilde{k} - 1$ survives on σ .

Lemma 31 *Suppose that there are no two agents i and j such that $\pi^i = \pi^j$ and $\beta_i = \beta_j$. Then, all agents $i < \tilde{k} - 1$ a.s. vanish, whereas all agents $i \geq \tilde{k} - 1$ a.s. survive.*

Proof of Lemma 31:

First consider agents 1 and 2. Note that since in the limit, the total consumption of the economy is measurable w.r.t. the partition defined by $\Omega^{\tilde{k}-1}$, in the limit, agent 1's consumption has to be measurable w.r.t. the partition of agent 2, Ω^2 , i.e.

$$\lim_{t \rightarrow \infty} |c^1(\sigma_t) - c^1(\sigma'_t)| = 0 \text{ if } \sigma, \sigma' \in \omega^2 \text{ for some } \omega^2 \in \Omega^2.$$

Suppose first that agent 2 survives on a path $\sigma \in \Omega$ and thus, by measurability of consumption on $\omega^2 \in \Omega^2$ such that $\sigma \in \omega^2$. It follows that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_2(c^2(\omega_{T+1}^2))}{\left(\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^2} u'_1(c^1(\tilde{\sigma}_{T+1})) \pi^k(\tilde{\sigma}_{T+1} | \omega_{T+1}^2) \right)} = \\
& = \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_2(c^2(\omega_{T+1}^2))}{u'_1(c^1(\omega_{T+1}^2))} = \\
& = \ln \frac{\beta_1}{\beta_2} + \left(\sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^2(w^2)} - \sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^1(w^2)} \right) < 0.
\end{aligned}$$

Since, by assumption, agent 2 survives on ω^2 , it follows that the numerator in the \ln term is bounded. If the denominator were similarly bounded, the expression on the second line would converge to 0, in contradiction to the fact that the third line is strictly negative. We thus have to conclude that $\lim_{T \rightarrow \infty} \sup u'_1(c^1(\omega_{T+1}^2)) = \infty$ and thus, $\lim_{T \rightarrow \infty} \inf c^1(\omega_{T+1}^2) = 0$, i.e., 1 vanishes on σ and on ω^2 .

Next consider the case, in which 2 vanishes on σ and thus on ω^2 . Then,

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left(\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^2} u'_1(c^1(\tilde{\sigma}_{T+1})) \pi^1(\tilde{\sigma}_{T+1} | \omega_{T+1}^2) \right)}{u'_2(c^2(\omega_{T+1}^2))} = \\
& = \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_1(c^1(\omega_{T+1}^2))}{u'_2(c^2(\omega_{T+1}^2))} = \\
& = \ln \frac{\beta_2}{\beta_1} + \left(\sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^1(w^2)} - \sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^2(w^2)} \right) > 0.
\end{aligned}$$

By assumption, $\lim_{T \rightarrow \infty} \inf c^2(\omega_{T+1}^2) = 0$ and thus, $\lim_{T \rightarrow \infty} \sup u'_2(c^2(\omega_{T+1}^2)) = \infty$. Hence, if u'_1 were bounded, the term

$$\lim_{T \rightarrow \infty} \ln \frac{u'_1(c^1(\omega_{T+1}^2))}{u'_2(c^2(\omega_{T+1}^2))} = -\infty$$

and thus the second line would be less or equal to 0 in contradiction to the third line being strictly positive. $\lim_{T \rightarrow \infty} \sup u'_1(c^1(\omega_{T+1}^2)) = \infty$ and thus, $\lim_{T \rightarrow \infty} \inf c^1(\omega_{T+1}^2) = 0$, i.e., 1 vanishes on σ and on ω^2 .

Hence, regardless of whether agent 2 survives or not, agent 1 vanishes a.s.

We now proceed by induction. Suppose that we have shown that all agents $1 \dots k-1 < \tilde{k}-2$ a.s. vanish. In the discussion that follows, restrict attention to the set of paths with measure 1, on which all agents $1 \dots k-1$ vanish. Consider agents k and $k+1$. Note that since all agents $1 \dots k-1$ vanish, and since the unawareness defined by partition Ω^k ($k < \tilde{k}-2$) is not relevant in the limit, the consumption of agent k has to be measurable w.r.t. to the partition of agent $k+1$, Ω^{k+1} .

Suppose first that agent $k+1$ survives on a set of paths $\omega^k \in \Omega^k$ and thus, by measurability of consumption on $\omega^{k+1} \in \Omega^{k+1}$ such that $\omega^k \subseteq \omega^{k+1}$. It follows that

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))}{\left(\sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \right)} = \\ & \left(= \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))}{u'_k(c^k(\omega_{T+1}^{k+1}))} = \right) \\ & = \ln \frac{\beta_k}{\beta_{k+1}} + \left(\sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^{k+1}(w^{k+1})} - \sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^k(w^{k+1})} \right) < 0. \end{aligned}$$

Since, by assumption, agent $k+1$ survives on ω^2 , it follows that the numerator in the \ln term is bounded. If the denominator were similarly bounded, the expression on the second line would converge to 0, in contradiction to the fact that the second (third) line is strictly negative. We thus have to conclude that

$$\limsup_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) = \infty$$

Next consider the case, in which $k+1$ vanishes on ω^k and thus on ω^{k+1} . Then,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left(\sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \right)}{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))} = \\ & \left(= \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_k(c^k(\omega_{T+1}^{k+1}))}{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))} = \right) \\ & = \ln \frac{\beta_{k+1}}{\beta_k} + \left(\sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^k(w^{k+1})} - \sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^{k+1}(w^{k+1})} \right) > 0. \end{aligned}$$

By assumption, $\lim_{T \rightarrow \infty} \inf c^{k+1}(\omega_{T+1}^{k+1}) = 0$ and thus, $\lim_{T \rightarrow \infty} \sup u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1})) = \infty$. Hence, if the term

$$\left(\sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \right)$$

were bounded, the term

$$\lim_{T \rightarrow \infty} \ln \frac{\sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1})}{u'_2(c^2(\omega_{T+1}^2))} = -\infty$$

and thus the first line would be less or equal to 0 in contradiction to the third line being strictly positive. We thus conclude that

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) = \infty$$

Hence, whether $k + 1$ survives or vanishes, π -a.s. on ω^{k+1} ,

$$\lim_{T \rightarrow \infty} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) = \infty$$

Furthermore, π -a.s., $\lim_{T \rightarrow \infty} |c^k(\tilde{\omega}_T^k) - c^k(\tilde{\omega}'_T^k)| = 0$ if $\tilde{\omega}^k$ and $\tilde{\omega}'^k \in \omega^{k+1}$ for some $\omega^{k+1} \in \tilde{\Omega}^{k+1}$.

Consider a set $\tilde{\Omega}^{k+1}$ with a strictly positive measure w.r.t. π and note that:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} \pi(\omega_{T+1}^{k+1}) \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \\ &= \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi(\tilde{\omega}_{T+1}^k) \frac{\pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1})}{\pi(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1})} \\ &= \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^{k+1})) \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} \pi(\tilde{\omega}_{T+1}^{k+1}) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \\ &= \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^{k+1})) \pi(\tilde{\omega}_{T+1}^{k+1}) \\ &= \infty \end{aligned}$$

Since $\tilde{\Omega}^{k+1}$ is an arbitrary set with strictly positive measure, we conclude that π -a.s.

$$\lim_{T \rightarrow \infty} u'_k(c^k(\tilde{\omega}_{T+1}^{k+1})) \rightarrow \infty$$

and hence, k vanishes.

It follows that if all agents $1 \dots \tilde{k} - 1$ have strictly ordered survival indices then all agents $1 \dots \tilde{k} - 2$ a.s. vanish and agent $\tilde{k} - 1$ a.s. survives.

Note that on any path σ , on which state s occurs infinitely often, and hence, on any $\omega^{\tilde{k}-1}$ such that $\sigma \in \omega^{\tilde{k}-1}$, the consumption of agent $\tilde{k} - 1$ satisfies $\lim_{t \rightarrow \infty} \sup c^{\tilde{k}-1}(\omega_t^{\tilde{k}-1}) > \epsilon$ and hence, his marginal utility on such a path is bounded above:

$$\liminf_{t \rightarrow \infty} u'_{\tilde{k}-1}(c^{\tilde{k}-1}(\omega_t^{\tilde{k}-1})) < u'_{\tilde{k}-1}(\epsilon)$$

π -a.s., but also, by Assumption 3, $\pi^{\tilde{k}-1}$ -a.s.

Lemma 32 For any $n > j \geq \tilde{k} - 1$, $\lim_{t \rightarrow \infty} |c^j(\sigma_t) - \bar{c}^j(\sigma_t)| = 0$ π -a.s. on Ω , where for $\sigma_t \in \omega_t^j \subseteq \omega_t^{j+1}$,

$$\bar{c}^j(\sigma_t) = \min_{\sigma_t \in \omega_t^j} e(\sigma_t) - \min_{\tilde{\omega}_t^j \subseteq \omega_t^{j+1}} e(\tilde{\omega}_t^j)$$

Hence, j survives a.s. if his awareness is relevant in the limit w.r.t. those of $j+1$ and vanishes a.s. if it is irrelevant in the limit w.r.t. those of $j+1$.

Proof of Lemma 32:

Consider first agent $\bar{k}-1$. π -a.s.

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left(\sum_{\tilde{\omega}_{T+1}^{\bar{k}-1} \in \omega_{T+1}^j} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \right)}{u'_{\bar{k}} \left(c^{\bar{k}} \left(\omega_{T+1}^{\bar{k}} \right) \right)} = \\ & = \ln \frac{\beta_{\bar{k}}}{\beta_{\bar{k}-1}} + \left(\sum_{w^{\bar{k}} \in W^{\bar{k}}} \pi \left(w^{\bar{k}} \right) \ln \frac{\pi \left(w^{\bar{k}} \right)}{\pi^{\bar{k}-1} \left(w^{\bar{k}} \right)} - \sum_{w^{\bar{k}} \in W^{\bar{k}}} \pi \left(w^{\bar{k}} \right) \ln \frac{\pi \left(w^{\bar{k}} \right)}{\pi^{\bar{k}} \left(w^{\bar{k}} \right)} \right) > 0. \end{aligned}$$

and hence,

$$\lim_{t \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^j} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^j \right) = \infty$$

In an economy, in which all agents $1 \dots \bar{k}-2$ vanish, only agent $\bar{k}-1$ can consume the part of the initial endowment, which is not measurable relative to \bar{k} 's (and all subsequent agents') partitions. Hence, in the limit, at each $\omega^{\bar{k}-1} \subseteq \omega^{\bar{k}}$, we have:

$$\lim_{t \rightarrow \infty} \inf c_t^{\bar{k}-1} \left(\omega_t^{\bar{k}-1} \right) - \bar{c}_t^{\bar{k}-1} \left(\omega_t^{\bar{k}-1} \right) \geq 0,$$

where

$$\bar{c}_t^{\bar{k}-1} \left(\omega_t^{\bar{k}-1} \right) = e \left(\omega_t^{\bar{k}-1} \right) - \min_{\tilde{\omega}_t^{\bar{k}-1} \subseteq \omega_t^{\bar{k}}} e \left(\tilde{\omega}_t^{\bar{k}-1} \right)$$

For a given $\omega_t^{\bar{k}}$ denote by

$$\Omega_t^{\bar{k}-1} \left(\omega_t^{\bar{k}} \right) = \arg \min_{\tilde{\omega}_t^{\bar{k}-1} \subseteq \omega_t^{\bar{k}}} e \left(\tilde{\omega}_t^{\bar{k}-1} \right)$$

Similarly, for agent $j > \bar{k}-1$, define

$$\bar{c}_t^j \left(\omega_t^j \right) = \min_{\sigma_t \in \omega_t^j} e \left(\sigma_t \right) - \min_{\tilde{\omega}_t^j \subseteq \omega_t^{j+1}} e \left(\tilde{\omega}_t^j \right)$$

and

$$\Omega_t^j \left(\omega_t^{j+1} \right) = \arg \min_{\tilde{\omega}_t^j \subseteq \omega_t^{j+1}} e \left(\tilde{\omega}_t^j \right)$$

Intuitively, \bar{c}^j is the minimal consumption that has to be attributed to j provided that the consumption plans of all consumers $i < j$ satisfy $\lim_{t \rightarrow \infty} c^i \left(\sigma_t \right) = \lim_{t \rightarrow \infty} \bar{c}^i \left(\sigma_t \right)$. Note as well that on any of the sets $\Omega_t^j \left(\omega_t^{j+1} \right)$, the consumption

of all consumers $k > j$ is constant across $\sigma_t \in \Omega_t^j(\omega_t^{j+1})$ in the limit and since total endowment is constant in those states, we have that the consumption of j also has to be constant across $\sigma_t \in \Omega_t^j(\omega_t^{j+1})$ in the limit.

Note that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^j} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \\
&= \lim_{T \rightarrow \infty} \left[\sum_{\tilde{\omega}_{T+1}^{\bar{k}-1} \in \Omega_{T+1}^{\bar{k}-1}(\omega_{T+1}^{\bar{k}})} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \right. \\
&\quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^{\bar{k}} \\ \tilde{\omega}_{T+1}^{\bar{k}-1} \notin \Omega_{T+1}^{\bar{k}-1}(\omega_{T+1}^{\bar{k}})}} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \right] \\
&= \lim_{T \rightarrow \infty} \left[u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}} \right) \right) \pi^{\bar{k}-1} \left(\Omega_{T+1}^{\bar{k}-1} \left(\omega_{T+1}^{\bar{k}} \right) \right) \right. \\
&\quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^{\bar{k}} \\ \tilde{\omega}_{T+1}^{\bar{k}-1} \notin \Omega_{T+1}^{\bar{k}-1}(\omega_{T+1}^{\bar{k}})}} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \right]
\end{aligned}$$

where we write $c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}} \right)$ to express the fact that in the limit, on $\Omega_{T+1}^{\bar{k}-1} \left(\omega_{T+1}^{\bar{k}} \right)$, $\tilde{k} - 1$'s consumption is constant. Note further that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^{\bar{k}} \\ \tilde{\omega}_{T+1}^{\bar{k}-1} \notin \Omega_{T+1}^{\bar{k}-1}(\omega_{T+1}^{\bar{k}})}} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \\
&\leq \lim_{T \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^{\bar{k}} \\ \tilde{\omega}_{T+1}^{\bar{k}-1} \notin \Omega_{T+1}^{\bar{k}-1}(\omega_{T+1}^{\bar{k}})}} u'_{\bar{k}-1} \left(c^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \\
&= \lim_{T \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^{\bar{k}} \\ \tilde{\omega}_{T+1}^{\bar{k}-1} \notin \Omega_{T+1}^{\bar{k}-1}(\omega_{T+1}^{\bar{k}})}} u'_{\bar{k}-1} \left(e \left(\omega_{T+1}^{\bar{k}-1} \right) - \min_{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^{\bar{k}}} e \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left(\tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \\
&\leq u'_{\bar{k}-1} (\varepsilon)
\end{aligned}$$

where the last inequality follows from the fact that \tilde{k} 's unawareness is relevant in the limit.

Since

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^j} u'_{\tilde{k}-1} \left(c^{\tilde{k}-1} \left(\tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left(\tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^j \right) = \infty,$$

we thus have that:

$$\lim_{T \rightarrow \infty} u'_{\tilde{k}-1} \left(c^{\tilde{k}-1} \left(\tilde{\omega}_{T+1}^{\tilde{k}} \right) \right) \pi^{\tilde{k}-1} \left(\Omega_{T+1}^{\tilde{k}-1} \left(\omega_{T+1}^{\tilde{k}} \right) \right) = \infty$$

and hence

$$c^{\tilde{k}-1} \left(\left(\tilde{\omega}_{T+1}^{\tilde{k}} \right) \right) \rightarrow 0.$$

We thus conclude that

$$\lim_{T \rightarrow \infty} \left| c^{\tilde{k}-1} \left(\tilde{\omega}_{T+1}^{\tilde{k}} \right) - \bar{c}^{\tilde{k}-1} \left(\tilde{\omega}_{T+1}^{\tilde{k}} \right) \right| = 0$$

We can thus repeat the argument by induction for any agent with an index greater than $\tilde{k} - 1$. Indeed, suppose that for all agents $j \in \{\tilde{k} - 1 \dots k\}$ we have shown that $\lim_{T \rightarrow \infty} |c_j(\sigma_t) - \bar{c}_j(\sigma_t)| = 0$. Hence, in the limit, only agent $k + 1$ can consume the part of the initial endowment, $e - \sum_{j=\tilde{k}-1}^k \bar{c}_j$, which is not measurable relative to $k + 2$'s (and all subsequent agents') partitions. Hence,

$$\liminf_{t \rightarrow \infty} c_t^{k+1}(\omega_t^{k+1}) - \bar{c}_t^{k+1}(\omega_t^{k+1}) \geq 0$$

where

$$\begin{aligned} \bar{c}_t^{k+1}(\omega_t^{k+1}) &= \min_{\sigma_t \in \omega_t^{k+1}} e(\sigma_t) - \min_{\tilde{\omega}_t^{k+1} \subseteq \omega_t^{k+2}} e(\tilde{\omega}_t^{k+1}) \\ &= e(\omega_t^{k+1}) - \sum_{j=k-1}^k \bar{c}_j(\omega_t^{k+1}) - \min_{\tilde{\omega}_t^{k+1} \subseteq \omega_t^{k+2}} e(\tilde{\omega}_t^{k+1}) \end{aligned}$$

Since $k + 1$'s survival index is strictly smaller than that of $k + 2$, we have that a.s.

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left(\sum_{\tilde{\omega}_{T+1}^{k+1} \in \omega_{T+1}^j} u'_{k+1} \left(c^{k+1} \left(\tilde{\omega}_{T+1}^{k+1} \right) \right) \pi^{k+1} \left(\tilde{\omega}_{T+1}^{k+1} \mid \omega_{T+1}^{k+2} \right) \right)}{u'_{k+2} \left(c^{k+2} \left(\omega_{T+1}^{k+2} \right) \right)} = \\ &= \ln \frac{\beta_{k+2}}{\beta_{k+1}} + \left(\sum_{w^{k+2} \in W^{k+2}} \pi(w^{k+2}) \ln \frac{\pi(w^{k+2})}{\pi^{k+1}(w^{k+2})} - \sum_{w^{k+2} \in W^{k+2}} \pi(w^{k+2}) \ln \frac{\pi(w^{k+2})}{\pi^{k+2}(w^{k+2})} \right) > 0. \end{aligned}$$

and hence,

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{k+1} \in \omega_{T+1}^{k+2}} u'_{k+1} \left(c^{k+1} \left(\tilde{\omega}_{T+1}^{k+1} \right) \right) \pi^{k+1} \left(\tilde{\omega}_{T+1}^{k+1} \mid \omega_{T+1}^{k+2} \right) = \infty.$$

Suppose first that on all ω_{T+1}^{k+2} , $\lim_{t \rightarrow \infty} \bar{c}_{k+1} = 0$, i.e., $k+2$'s financial constraint is irrelevant in the limit w.r.t. that of $k+1$. Hence, in the limit, $k+1$'s consumption will be measurable w.r.t. Ω^{k+2} and thus,

$$\lim_{t \rightarrow \infty} u'_{k+1} (c^{k+1} (\omega_{T+1}^{k+2})) = \infty$$

implies $\lim_{T \rightarrow \infty} c^{k+1} (\omega_{T+1}^{k+2}) = \bar{c}^k (\omega_{T+1}^{k+2}) = 0$ and $k+1$ a.s. vanishes.

Suppose now that on all ω_{T+1}^{k+2} , $\lim_{t \rightarrow \infty} \bar{c}_{k+1} > \epsilon$, i.e., $k+2$'s financial constraint is relevant in the limit w.r.t. that of $k+1$. Then,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\ &= \lim_{T \rightarrow \infty} \inf \left[\sum_{\tilde{\omega}_{T+1}^{k+1} \in \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})} u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \right. \\ & \quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \right] \\ &= \lim_{T \rightarrow \infty} \inf \left[u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+2})) \pi^{k+1} (\Omega_{T+1}^{k+1} (\omega_{T+1}^{k+2})) \right. \\ & \quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \right] \end{aligned}$$

where we write $c^{k+1} (\tilde{\omega}_{T+1}^{k+2})$ to express the fact that in the limit, on $\Omega_{T+1}^{k+1} (\omega_{T+1}^{k+2})$, $k+1$'s consumption is constant. Note further that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\ & \leq \liminf_{t \rightarrow \infty} \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} (\bar{c}^{k+1} (\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\ & = \liminf_{t \rightarrow \infty} \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} \left(e(\omega_{T+1}^{k+1}) - \sum_{j=\bar{k}-1}^k \bar{c}_j (\omega_{T+1}^{k+1}) - \min_{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2}} e(\tilde{\omega}_{T+1}^{k+1}) \right) \pi^{k+1} (\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\ & \leq u'_{\bar{k}-1} (\epsilon) \end{aligned}$$

where the last inequality follows from the assumption that $k + 2$'s unawareness is relevant in the limit w.r.t. that of $k + 1$.

Since

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2}} u'_{k+1}(\bar{c}^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) = \infty,$$

we thus have that:

$$\lim_{T \rightarrow \infty} u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+2})) \pi^{k+1}(\Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})) = \infty$$

and hence on $\Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})$,

$$c^{k+1}(\tilde{\omega}_{T+1}^{k+2}) \rightarrow 0$$

since $c^{k+1}(\tilde{\omega}_{T+1}^{k+1}) - \bar{c}^{k+1}(\tilde{\omega}_{T+1}^{k+1}) \rightarrow_{T \rightarrow \infty} 0$, we conclude that:

$$\lim_{T \rightarrow \infty} |c^{k+1}(\sigma_T) - \bar{c}^{k+1}(\sigma_T)| = 0.$$

It follows that $k + 1$ a.s. survives if his unawareness is relevant in the limit w.r.t. those of $k + 2$.

Lemma 33 *Agent n a.s. survives.*

Proof of Lemma 33:

We have shown above that for agents $1 \dots \tilde{k} - 2$, consumption converges to 0 a.s., whereas for agents $j \in \{\tilde{k} - 1 \dots n - 1\}$, consumption converges a.s. to \bar{c}^j . It follows that for $\sigma_t \in \omega_t^n$, π -a.s.,

$$\lim_{t \rightarrow \infty} c^n(\sigma_t) = e(\sigma_t) - \sum_{j=\tilde{k}-1}^{n-1} \bar{c}^j(\sigma_t) = \min_{\tilde{\sigma}_t \in \omega_t^n} e(\tilde{\sigma}_t) \geq m$$

and hence, n a.s. survives.

Proof of Proposition 14:

Since the initial endowment of every agent specifies a consumption of at least m at every node σ_t , and since $L \leq m$, each agent i can afford to buy a consumption stream which avoids unforeseen unfavorable surprises, regardless of the price sequence. Since agents exhibit strict aversion to unfavorable surprises, in equilibrium, each partially aware agent will purchase a consumption stream not subject to unfavorable surprises. Hence, partially aware agents will not disappear as long as they are strictly averse to unforeseen unfavorable surprises. Hence, the equilibrium will be identical to that in an economy, in which we restrict partially aware agents to consumption bundles that satisfy $c^i(\sigma_t) \geq L$ for all $\sigma_t \in \Omega$. Since the initial endowment of the economy satisfies this restriction, and since preferences on the so-restricted consumption sets are convex and continuous, all the conditions of Bewley's (1972) theorem are

satisfied. Hence, an equilibrium of the economy exists. Furthermore, just as in the proof of Proposition 13, for each equilibrium, we can redistribute the equilibrium allocation and obtain a new equilibrium, in which the partially aware agents hold consumption streams which are measurable relative to their respective partitions \otimes^i . It is obvious that none of the partially aware agents vanish (since their consumption is bounded below by L at each σ_t). It is also obvious that with aggregate uncertainty in the limit, as long as prices are positive, the total demand of the partially aware agents will not sum up to the total demand of the economy and hence, a fully aware consumer will consume the remaining quantity of the consumption good.

9 References

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