



**HAL**  
open science

# Intertemporal Portfolio Choice with Incorrect Beliefs and Aversion to Surprise

Ani Guerdjikova, John Quiggin

► **To cite this version:**

Ani Guerdjikova, John Quiggin. Intertemporal Portfolio Choice with Incorrect Beliefs and Aversion to Surprise. 2018 Risk, Uncertainty and Decision Conference, Jun 2018, Heidelberg, Germany. hal-02086151

**HAL Id: hal-02086151**

**<https://hal.science/hal-02086151>**

Submitted on 30 Jan 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Intertemporal Portfolio Choice with Incorrect Beliefs and Aversion to Surprise\*

Ani Guerdjikova<sup>†</sup>                      and                      John Quiggin<sup>‡</sup>  
University of Grenoble-Alpes, IUF                      University of Queensland

February 28, 2018

## Abstract

In this paper, we provide an extension of the Grant and Quiggin (2015) model of preferences subject to surprises to an intertemporal setting. We show that even if decision makers have correct information about the relative likelihoods of the states they are aware of, differential unawareness will in general lead to belief heterogeneity: agents with identical awareness partitions need not have identical beliefs and "less aware" agents might entertain beliefs closer to the truth than "more aware" agents. We use these findings to study the issue of survival with differential unawareness in the form of a restriction of the state space. We show that partially aware decision makers behave as if they restrict their portfolio choice to a subset of the available assets with payoffs measurable w.r.t. their awareness partition and thus avoid "surprises". Such a heuristic allows the partially aware agent to survive, but only if his beliefs on his awareness partition are at least as close to the truth as the beliefs of a more aware agent with a finer partition. Introducing aversion to unforeseen unfavorable surprises, as in Grant and Quiggin (2015), allows us to establish survival of partially aware agents, regardless of whether their beliefs are correct. Such agents hold a minimal number of bonds in their portfolios. Asset prices in such an economy reflect the potentially incorrect beliefs of the fully aware agents and replicate the well-known equity premium puzzle.

---

\*This research was supported by IUF and by an Australian Research Council Laureate Fellowship. An alternative version of this paper, entitled "Heuristic Modes of Decision Making and Survival in Financial Markets" has been submitted to DTEA 2018.

<sup>†</sup>University of Grenoble-Alpes, CNRS, INRA, Grenoble INP, GAEL, 1241 rue des Résidences, 38400 Saint Martin d'Hères, Tel.: +33 4 56 52 85 78, email: ani.guerdjikova@univ-grenoble-alpes.fr

<sup>‡</sup>University of Queensland, School of Economics, Faculty of Business, Economics and Law, j.quiggin@uq.edu.au, Tel.: +61 7 334 69646

# 1 Introduction

Boundedly rational agents (that is, possibly existent agents) can never be aware of all the possibilities relevant to a decision. The attempt to integrate this self-evident fact into decision theory has produced a large literature, centered on the concept of unawareness (Schipper 2018 gives a bibliography). An issue that has received relatively limited attention (Halpern and Rego, 2009, Karni and Viero, 2013, 2014) is whether, and in what sense, agents can be aware of their own unawareness.

Grant and Quiggin (2013a) argue that agents cannot be aware (in the modal-logical sense) of unawareness. Nonetheless they can reason inductively from past surprises (discovery of states/propositions of which they were previously unaware) to the conclusion that they will be surprised in future.

Hence, as is argued by Grant and Quiggin (2013b), sophisticated but boundedly rational agents will infer from experience that they are likely to be surprised by unforeseen contingencies. Such agents will choose to constrain their choices by heuristics designed to reduce vulnerability to unfavorable surprises (and, if possible, to remain open to favorable surprises).

Grant and Quiggin (2015) provide an axiomatization of choice subject to surprise. In this model, agents understand the possibility of an unfavorable surprise, but are subject to bounded awareness of the states of nature in which such a surprise may take place. They can however, judge whether available actions are subject to unfavorable surprises. Under the given axiomatization, agents always prefer actions that are not subject to unfavorable surprises.

The exclusion of actions subject to unfavorable surprises may be seen as an example of a heuristic constraint on choice. Goldstein and Gigerenzer (1996) and Gigerenzer (2007) argue that heuristics may be 'ecologically rational' in the environments to which they are adapted and provide examples related to portfolio choice, see Gigerenzer (2007, Chapter 2).

The model of Grant and Quiggin (2015) represents one-shot decisions subject to surprise. In this paper we propose a model of intertemporal decisions with aversion to surprise. We extend the axiomatization of Grant and Quiggin to this case.

A natural application of this analysis is to financial markets<sup>1</sup>. Trades in a financial market may be seen as 'betting one's beliefs' about the relative probabilities of different states of nature, and the resulting returns on assets. Blume and Easley (2006) and Sandroni (2000) argue that traders who correctly judge these probabilities and make rational investment choices based on their beliefs will accumulate wealth at the expense of others. In the limit, only these

---

<sup>1</sup>The concept of restricted awareness is closely related to the idea of Black Swans', which has received considerable attention in popular discussions of financial markets and financial crisis, Taleb (2007). This concept has received more limited attention in the economic and finance-theoretic literature, see Orlik and Veldkamp (2015), Chichilnisky (2010) and Modica et al. (1998). In particular, there has been little discussion of the implications of unforeseen contingencies for financial market efficiency and for the dynamics of survival in financial markets, except for the work of Guerdjikova and Quiggin (2016), who analyze the case of coarsening of the state space.

rational well-informed traders will survive, and market prices will reflect their beliefs.

However, agents with preferences of the kind modelled here do not simply bet their beliefs. Rather their decisions are constrained by heuristics that help them to avoid unfavorable surprises. The portfolios of such agents are characterized by a strictly positive lower bound on bond holdings, which ensures that agents averse to unforeseen unfavorable surprises survive regardless of their beliefs. That is, these heuristics are ecologically rational.

The paper is organized as follows: in Section 2, we provide an axiomatization of the inter-temporal version of the Grant and Quiggin (2015) preferences used in this paper. Section 3 defines financial assets and financial markets in an infinite-time setting, taking account of the possibility of surprises. In Section 4, we derive our main results on survival in markets with differential awareness and show that having correct beliefs (over a coarse partition of the state space) is a necessary condition for the survival of a partially aware agent. Section 5 introduces aversion to unforeseen surprises and shows that partially aware agents who are averse to unforeseen unfavorable surprises survive regardless of whether their beliefs are correct. Section 6 deals with the role of heuristics and the contrast between heuristics and rational optimization. Section 7 concludes.

## 2 Axiomatization of Intertemporal Preferences Subject to Surprise

### 2.1 The Model of Differential Unawareness as a Restriction of the State Space

In this paper, we look at decision makers who trade in financial markets, without being aware of all relevant contingencies. While the actual model of the economy is described by a set of states  $S$  such that in every period one of the states is realized, the decision maker is only aware of a subset of these states,  $S^i \subseteq S$ . Despite his limited knowledge, the partially aware agent understands that there are contingencies he might not be taking into account. Formally, for each state  $s \in S$ , there is a corresponding state  $a^i(s) \in S^i$  with the interpretation that the partially aware consumer is aware of  $a^i(s)$ , but is also aware that some unforeseen contingencies ( $s$ ) might occur instead of  $a^i(s)$ . We postulate that for  $s \in S^i$ ,  $a^i(s) = s$ , i.e., whenever the agent is aware of a state, he is also aware that no unforeseen contingencies can occur in this state. The awareness of each agent  $i$  is thus described by two elements,  $S^i \subseteq S$  and  $a^i : S \rightarrow S^i$ .

Let  $W^i$  denote a partition of  $S$  with elements

$$w = \{s' \in S \mid a(s') = s \text{ for some } s \in S^i\}.$$

I.e., each element of the partition  $W^i$  contains a state  $s \in S^i$  of which the agent is aware, as well as all those states that he is unaware of and which are mapped into  $s$  by  $a^i$ . It follows that the partition  $W^i$  has  $|S^i|$  elements. A special case is given by the fully aware agent, for whom  $W^i = \{\{s\}_{s \in S}\}$ .

**Remark 1** *Our framework generalizes the model of Grant and Quiggin (2015), in which the partition of  $S$  (as defined in our paper) consists of tuples, i.e., for each state in  $S^i$ , there is a single corresponding state in  $S \setminus S^i$ , of which the decision maker is unaware. Formally, Grant and Quiggin (2015) separate the states in  $S$  into two disjoint subsets,  $(s; 0) \in S_0$ , i.e., those states of which the decision maker is aware and  $(s; 1) \in S_1$ , the unforeseen states (surprises). This corresponds to  $S^i = S_0$ ,  $a(s; 1) = (s; 0)$  and the corresponding partition is given by  $W^i = \{(s; 0); (s; 1)\}_{s \in S^i}$ . In particular, the generalization we propose allows us to model decision makers who face the same set of objective contingencies,  $S$ , but differ w.r.t. their degree of awareness. Our next definition formalizes this statement.*

Consider two agents,  $i$  and  $j$  such that  $S^j \subset S^i$  and  $a^i(s) = a^j(s)$  for all  $s \in S \setminus S^i$ , i.e., all states that  $i$  and  $j$  are both unaware of are mapped into the same states that they are both aware of. The corresponding partition  $W^j$  will then be strictly coarser than  $W^i$ . This gives rise to the following definition which provides a partial ordering of agents w.r.t. their awareness level.

**Definition 2** *Agent  $i$  is "more aware" than agent  $j$  if  $S^j \subset S^i$  and  $W^i$  is a refinement of  $W^j$ .*

## 2.2 Introducing Time

Let  $\mathbb{N} = \{0; 1; 2; \dots\}$  denote the set of time periods. Denote by  $s_t \in S$  the realization of the state at time  $t \geq 1$ . We set  $S_0 = \{s_0\}$ . Denote by  $\Omega = \prod_{t \in \mathbb{N}} S$  the set of all possible observation paths, with representative element  $\sigma = (s_0; s_1; s_2 \dots s_t \dots)$ .

Finally denote by  $\Omega_t = \prod_{\tau=0}^t S$  the collection of all finite paths of length  $t$ , with representative element  $\sigma_t = (s_0; s_1; s_2 \dots s_t)$ . Each finite observation path  $\sigma_t$  identifies a decision/observation node and the set of all possible observation paths  $\Omega$  can also be seen as the set of all nodes.

We can represent the information revelation process in this economy through a sequence of finite partitions of the state space  $\Omega$ . In particular, define the cylinder with base on  $\sigma_t \in \Omega_t$ ,  $t \in \mathbb{N}$  as  $Z(\sigma_t) = \{\sigma \in \Omega | \sigma = (\sigma_t \dots)\}$ . Let  $\mathbb{F}_t = \{Z(\sigma_t) : \sigma_t \in \Omega_t\}$  be a partition of the set  $\Omega$ . Clearly,  $\mathbb{F} = (\mathbb{F}_0 \dots \mathbb{F}_t \dots)$  denotes a sequence of finite partitions of  $\Omega$  such that  $\mathbb{F}_0 = \Omega$  and  $\mathbb{F}_t$  is finer than  $\mathbb{F}_{t-1}$ . We assume that all fully aware agents have identical information and that the information revelation process for them is represented by the sequence  $\mathbb{F}$ .

Let  $\mathcal{F}_t$  be the  $\sigma$ -algebra generated by partition  $\mathbb{F}_t$ .  $\mathcal{F}_0$  is the trivial  $\sigma$ -algebra. Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by  $\cup_{t \in \mathbb{N}} \mathcal{F}_t$ . It can be shown that  $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$  is a filtration.

From the point of view of agent  $i$ , who perceives the state space as a partition  $W^i$ , the event tree of the economy can be seen as a coarsening of the set of paths  $\Omega$ . Rather than observing  $s_t$  at time  $t$ , agent  $i$  observes the corresponding element of his partition,  $w^i$ , which contains  $s_t$ . Therefore, for a partially aware

consumer  $i$ , the paths he is aware of can be written as  $\Omega^i = \prod_{t \in \mathbb{N}} W^i$  with a representative element  $\omega^i = (w_0 = s_0; w_1^i \dots w_t^i \dots)$ . Denote by  $\Omega_t^i$  the set of paths of length  $t$ .

In analogy to the construction for the objective information process, define the finite partitions of the set  $\Omega^i$ ,  $(\mathbb{F}_t^i)_{t \in \mathbb{N}}$  and the corresponding  $\sigma$ -algebras by  $\mathcal{F}_t^i$ . Let  $\mathcal{F}^i$  be the  $\sigma$ -algebra generated by  $\cup_{t \in \mathbb{N}} \mathcal{F}_t^i$ . Obviously,  $\mathcal{F}$  is finer than  $\mathcal{F}^i$ .

### 2.3 Representing Intertemporal Preferences Subject to Surprises

In our framework, agents express preferences over consumption streams  $c : \Omega \rightarrow \mathbb{R}_0^+$ . Since we will assume that the initial endowment of the economy is uniformly bounded above, we will further restrict the set of consumption streams to those with image in a convex and closed interval<sup>2</sup>,  $c : \Omega \rightarrow [\tilde{l}; \tilde{m}]$ , with  $\tilde{l} \geq 0$  and  $\tilde{m} > 0$ . For agent  $i$ , define a preference relation  $\succsim^i$  over consumption streams  $c : \Omega \rightarrow [\tilde{l}; \tilde{m}]$ .

We follow Grant and Quiggin (2015) to model the preference of partially aware agents over such infinite consumption streams. Note that in general, the consumption stream of a partially aware agent,  $c$ , is not measurable w.r.t.  $\Omega^i$ .

Yet, a consumer who is not aware of some of the contingencies, will in general not be able to form beliefs over the general state space  $\Omega$ . Rather, our representation below will have the property that the subjective beliefs of consumer  $i$ ,  $\pi^i$  are defined on his awareness partition,  $(\Omega^i; \mathcal{F}^i)$ .

The fact that consumption streams are in general not measurable w.r.t.  $\Omega^i$  means that consumer  $i$  will in general associate with each node  $\omega_t^i \in \Omega^i$  a set of possible outcomes / levels of consumption:

$$C(\omega_t^i) = \{c(\sigma_t) \mid \sigma_t \in \omega_t^i\}.$$

This set is a singleton, only when the consumption at  $\omega_t^i$  is "surprise-free" in the language of Grant and Quiggin (2015), i.e.,  $c(\sigma_t) = c(\sigma'_t)$  for all  $\sigma_t, \sigma'_t \in \omega_t^i$ .

We propose the following extension of the representation of Gul and Penderfer (2014) and Grant and Quiggin (2015) to a multi-period setting:  $c \succsim^i c'$  iff  $V_0^i(c) \geq V_0^i(c')$ , where  $V_0^i$  is defined as:

$$V_0^i(c) = u_i(c(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}; \max\{C(\omega_t^i)\})$$

where  $\beta_i$  is agent  $i$ 's discount factor.

In two special cases, the representation above boils down to discounted subjective expected utility maximization: the first is when the decision maker is

<sup>2</sup>One might wish to set  $\tilde{l} = 0$  and  $\tilde{m} = m'$  for the representation used in Section 4 and  $\tilde{l} = L$ ,  $\tilde{m} = M$  for that used in Section 5.

fully aware and thus,  $\Omega^i = \Omega$ ; the second is when the consumption streams are restricted to be measurable w.r.t. the awareness partition  $\Omega^i$ , and thus, the sets  $C(\omega_t^i)$  are all singletons. In general, however, there are two major differences between this representation and the standard EUM model. First, subjective probabilities  $\pi^i$  are well-defined only for coarse events, i.e., those in  $\Omega^i$ , but not for those events, on which surprises can occur. This in turn, implies that in states associated with surprises, the consumer must take into account two different outcomes — the minimal and the maximal possible consumption — when evaluating his utility of consumption in this state. This second property leads to the utility function  $u_i$  over outcomes being defined over intervals rather than single values.

In principle, many different specifications of  $u_i$  are possible. Here, we will concentrate on the case of<sup>3</sup>

$$u_i(C^i(\omega_t^i)) = u_i(\min\{C^i(\omega_t^i)\}) \quad (1)$$

Intuitively, if the consumer is partially aware, but perceives the possibility of a surprise, he concentrates only on the worst possible outcome. Furthermore, under this specification of the utility function preferences are convex and we can thus use standard results to characterize the optimal choices and show the existence of an equilibrium.

We thus start by axiomatizing preferences over consumption streams represented by:

$$V_0^i(c) = u_i(c(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \quad (2)$$

This representation will be used in Section 4 to study equilibria in markets with differential awareness.

## 2.4 The Axiomatization

To obtain the representation in (2), we combine the axiomatization for the static case of Gul and Pesendorfer (2014) with Koopmans' (1960) intertemporal framework and with Grant and Quiggin's (2015) model of preferences subject to surprises.

Denote by  $\mathcal{C}$  the set of all consumption streams  $c : \Omega \rightarrow [\tilde{l}; \tilde{m}]$ . Note that at each period,  $t$ , the consumption stream  $c$  uniquely identifies a Savage act  $\mathbf{c}_t : \mathbb{F}_t \rightarrow [\tilde{l}; \tilde{m}]$ , a mapping from the partition of the state space  $\Omega$  relevant at  $t$  into outcomes. Denote by  $\mathbf{C}_t = \{\mathbf{c}_t : \mathbb{F}_t \rightarrow [\tilde{l}; \tilde{m}]\}$  the set of all acts measurable w.r.t.  $\mathbb{F}_t$ . Furthermore,

$$\mathbf{C}_{t+1} = \{\mathbf{c}_{t+1} : \mathbb{F}_t \rightarrow [\tilde{l}; \tilde{m}]\} \supseteq \mathbf{C}_t = \{\mathbf{c}_{t+1} : \mathbb{F}_{t+1} \rightarrow [\tilde{l}; \tilde{m}]\}$$

<sup>3</sup>It is part (iv) of Axiom 3 below that restricts  $u_i$  to be only a function of  $\min C(\omega_t^i)$ . Dropping this condition gives the more general formulation above.

and hence, the set of all such Savage acts can be written as:

$$\mathbf{C} = \{\cup_{t=0}^{\infty} \mathbf{C}_t\}$$

In turn, any consumption stream  $c$  can thus be represented as a sequence of Savage acts.

Our first two axioms state that consumer  $i$  has well-defined preferences over consumption streams and that these preferences correspond to well-defined (time-independent) preferences over Savage acts,  $\mathbf{c}_{\infty} : \{\sigma \in \Omega\} \rightarrow [\tilde{l}; \tilde{m}]$ :

**Axiom 1** Preferences over  $\mathcal{C}$ ,  $\succsim^i$ , are complete and transitive.

To state the next axiom, for a consumption stream  $\tilde{c}$  and a Savage act  $\mathbf{c} \in \mathbf{C}_t$ , denote by  $\mathbf{c}_t \tilde{c}_{-t}$  the consumption stream which differs from  $\tilde{c}$  only in its  $t^{\text{th}}$  component, which is replaced by the Savage act  $\mathbf{c}$ :

$$\mathbf{c}_t \tilde{c}_{-t} = (\tilde{c}_1 \dots \tilde{c}_{t-1}; \mathbf{c}_t; \tilde{c}_{t+1} \dots \tilde{c}_T \dots)$$

**Axiom 2** For any two Savage acts measurable w.r.t.  $\mathbb{F}_t$ ,  $\mathbf{c}'_t$  and  $\mathbf{c}''_t \in \mathbf{C}_t$ , any  $t' \geq t$  and any consumption streams  $c, \tilde{c}$ ,

$$\begin{aligned} \mathbf{c}'_t c_{-t} &\succsim \mathbf{c}''_t c_{-t} \text{ iff} \\ \mathbf{c}'_{t'} \tilde{c}_{-t'} &\succsim \mathbf{c}''_{t'} \tilde{c}_{-t'} \end{aligned}$$

Note that a repeated recursive application of Axiom 2 implies that

$$\begin{aligned} \mathbf{c}'_t c_{-t} &\succsim \mathbf{c}'_{t'} \tilde{c}_{-t'} \text{ iff} \\ \mathbf{c}''_{t'} c_{-t'} &\succsim \mathbf{c}''_{t'} \tilde{c}_{-t'} \end{aligned} \tag{3}$$

This axiom provides a strong time-separability condition, which states that the comparison between two Savage acts in  $\mathbf{C}_t$  available in a given period does not depend on the period at which they are obtained, nor on the rest of the consumption stream.

This axiom allows us to define a static preference relation on Savage acts  $\succsim_s^i$  on  $\mathbf{C}$  defined by:

$$\mathbf{c}'_{t'} \succsim_s^i \mathbf{c}''_{t''}$$

iff there exists a  $t \geq \max\{t'; t''\}$  and a consumption stream  $c$  such that

$$\mathbf{c}'_t c_{-t} \succsim_t^i \mathbf{c}''_t c_{-t}.$$

Note that  $\succsim_s^i$  is complete and transitive.

For any two Savage acts  $\mathbf{c}_t$  and  $\mathbf{c}'_{t'}$  and event  $E \in \mathcal{F}$ , we write  $\mathbf{c}_{tE} \mathbf{c}'_{t'}$  for the Savage act, which coincides with  $\mathbf{c}_t$  on  $E$  and with  $\mathbf{c}'_{t'}$  on  $E^c$ .

We will now show how preferences  $\succsim_s^i$  combined with our definition of partial awareness can be interpreted in the spirit of Gul and Pesendorfer's (2014) Expected Uncertain Utility. We first recall the definition of an ideal event:



**Definition 3** A subset  $E \in \mathcal{F}$  is an ideal event if for every pair of Savage acts  $\mathbf{c}_t$  and  $\mathbf{c}'_{t'} \in \mathbf{C}$

$$\mathbf{c}_t \succsim \mathbf{c}_{tE} \mathbf{c}'_{t'}$$

implies

$$\mathbf{c}_{tE} \mathbf{c}'_{t'} \succsim \mathbf{c}'_{t'}.$$

Let  $\mathcal{I}$  denote the set of ideal events in  $\mathcal{F}$ .

Gul and Pesendorfer (2014) explain that ideal events correspond to events, on which subjective probabilities are well-defined. In our context, ideal events are events on which the decision maker does not experience any surprises, provided that the Savage act is measurable w.r.t. such an event. In particular, for agent  $i$ , the elements of the partition  $\mathcal{F}^i$  correspond to ideal events. E.g., a Savage act which is constant across nodes  $\sigma_t$  belonging to the same  $\omega_t^i$ ,  $\mathbf{c}_t = f(\omega_t^i)$  for  $\omega_t^i \in \mathcal{F}^i$  is measurable w.r.t. to ideal events and thus, not subject to surprises from the point of view of agent  $i$ . As explained above, agent  $i$  will be able to assign additive probabilities to such ideal events.

In contrast, if both  $\sigma_t$  and  $\sigma'_t \in \omega_t^i$ , neither of the events  $\sigma_t$  and  $\sigma'_t$  are ideal. Indeed, an act  $\mathbf{c}_t$  with  $\mathbf{c}_t(\sigma_t) \neq \mathbf{c}_t(\sigma'_t)$  is subject to a surprise from the point of view of agent  $i$ . Furthermore, agent  $i$  will not be able to assign probabilities to such events. Nevertheless, the probabilities of certain non-ideal events can be bounded by probabilities of ideal events. For instance, the probabilities of  $\sigma_t$  and  $\sigma'_t$  can be bounded above by the probability assigned to the ideal event  $\omega_t^i$  (and below by 0).

In contrast, there are certain non-ideal events, the probability of which can only be bounded (trivially) by the universal event  $\Omega$  and the empty set. Suppose, e.g., that each  $\omega_t^i$  contains at least two elements and consider all  $\omega_t^i \in \Omega_t^i$ . For each  $\omega_t^i$  select exactly one  $\sigma_t(\omega_t^i) \in \omega_t^i$ . The probability of the event  $D = \left\{ (\sigma_t(\omega_t^i))_{\omega_t^i \in \Omega_t^i} \right\}$  can now be bounded above only by that of  $\Omega$ , 1 and below by the empty set. Such an event is called diffuse. Hence, an event is diffuse if no non-null ideal event is included in the event, or its complement.

**Definition 4** A subset  $E \in \mathcal{F}$  is a null event if for every pair of Savage acts  $\mathbf{c}_t$  and  $\mathbf{c}'_{t'} \in \mathbf{C}$

$$\mathbf{c}_t \sim \mathbf{c}'_{t'E} \mathbf{c}_t.$$

Let  $\mathcal{N}$  denote the set of null events.

**Definition 5** An event  $D \in \mathcal{F}$  is diffuse if for any  $E \in \mathcal{I} \setminus \mathcal{N}$ ,  $E \cap D \neq \emptyset$  and  $E \cap (\Omega \setminus D) \neq \emptyset$ .

As in Grant and Quiggin (2015), the fact that the decision maker is subject to unquantifiable uncertainty in the face of non-ideal events and hence, faces the possibility of a surprise, is captured by a utility function defined on sets of payoffs, denoted above by  $C(\omega_t^i)$ . As shown in Gul and Pesendorfer (2014, Lemma 1) for the purposes of the desired representation, each such set can be uniquely identified by its lower and upper bound,  $[\min C(\omega_t^i); \max C(\omega_t^i)]$ . It

is then on such intervals that the utility function  $u$  is defined. Since ideal events (in particular,  $\omega_t^i$ ) are assigned unique probabilities,  $\pi$ , the uncertainty expected utility representation combines in a multiplicative way the utility function  $u$  defined on sets of outcomes obtained on ideal events with their probability.

We now impose the axioms of Gul and Pesendorfer (2014) and Grant and Quiggin (2015) on  $\succsim_s^i$  to obtain the static EUU representation:

**Axiom 3** (EUU-representation)

- (i) Every event in  $\mathcal{F}^i$  is ideal.
- (ii) Any set of the type  $\left\{(\sigma(\omega^i))_{\omega^i \in \Omega^i}\right\} \notin \mathcal{F}^i$ , where  $\sigma(\omega^i) \in \omega^i$  is diffuse.
- (iii) Preferences over Savage acts  $\succsim_s^i$  satisfy Axioms 1-6 of Gul and Pesendorfer (2014).
- (iv) Let  $D$  be a diffuse event and consider a Savage act  $\mathbf{c}_D \mathbf{c}'$  with  $\mathbf{c}(\sigma_t) = a$  for all  $\sigma_t$  and  $\mathbf{c}'(\sigma_t) = b$  for all  $\sigma_t$ ,  $\tilde{m} \geq a > b \geq \tilde{l}$ . Then,  $\mathbf{c}_D \mathbf{c}' \sim \mathbf{c}'$ .

Part (iii) imposes the axioms of Gul and Pesendorfer (2014) on preferences over Savage acts,  $\succsim_s^i$ . Indeed, the representation of Gul and Pesendorfer (2014) implies the existence of a representation of preferences on the set of ideal events  $\mathcal{I}$  with a non-atomic prior on  $\mathcal{I}$ . Parts (i) and (ii) are the counterparts of Axiom 1 (i) and (ii) in Grant and Quiggin (2015) and serve to specify the set  $\mathcal{I}$ , as well as the set of diffuse events. These two conditions imply that the probability distribution  $\pi^i$  will be defined on  $(\Omega^i, \mathcal{F}^i)$ . Note that if some  $\left\{(\sigma(\omega^i))_{\omega^i \in \Omega^i}\right\} \in \mathcal{F}^i$ , then all such sets are in  $\mathcal{F}^i$  — this corresponds to the scenario, where  $\Omega^i = \Omega$  and the decision maker is fully aware. Finally, Part (iv) implies that the function  $u_i$  depends only on  $\min C(\omega_t^i)$ . We obtain:

**Proposition 6** *Preferences over Savage acts  $\succsim_s^i$  satisfy Axiom 3 iff there exists a countably-additive and non-atomic probability measure  $\pi^i$  on  $(\Omega^i, \mathcal{F}^i)$  and an interval utility<sup>4</sup>  $u_i$  such that  $\succsim_s^i$  are represented by:*

$$U_i(\mathbf{c}_t) = \sum_{\omega_t^i \in \Omega^i} \pi^i(\omega_t^i) u_i(\min C(\omega_t^i)).$$

We now provide the intertemporal part of the representation. We do so by combining the framework developed so far with the Koopmans' (1960) setting. We first note that for each Savage act,  $\mathbf{c}_t$ , there is a constant Savage act denoted  $x(\mathbf{c}_t) \in [\tilde{l}, \tilde{m}]$  such that:

$$\mathbf{c}_t \sim_c^i x(\mathbf{c}_t)$$

---

<sup>4</sup>Interval utility  $u_i$  is defined in Gul and Pesendorfer (2014) as a utility function, which is continuous and strictly monotone in its arguments.

Next note that applying inductively Axiom 2 implies that each consumption stream  $c = (\mathbf{c}_t)_{t=0}^\infty$  is indifferent to the consumption stream:

$$x(c) = (x(\mathbf{c}_t))_{t=0}^\infty.$$

Let  $X$  denote the set of all certain consumption streams. We can apply to the preference relation  $\succsim^i$  restricted to the set  $X$  the axioms of Koopmans (1960) to derive the discounted utility representation.

Since part of Postulate 1 and Postulate 3 are implied by Axiom 1 and Axiom 2, respectively and since Koopman's postulates are stated in terms of utility rather than preferences, we here restate the necessary conditions:

**Axiom 4** Preferences  $\succsim^i$  on  $X$  satisfy:

- (i) **Uniform Continuity:** For  $x, x', x'' \in X$  if  $x' \succ x \succ x''$ , there is a  $\delta$  such that for any  $y \in X$ ,  $\lim_{t \rightarrow \infty} \sup |x_t - y_t| < \delta$  implies  $x' \succ y \succ x''$ .
- (ii) **Monotonicity:** If  $x \succ x', x \succ x'$ .
- (iii) **Stationarity:** For any  $(x_t)_{t=0}^\infty, (x'_t)_{t=0}^\infty$  such that  $x_0 = x'_0, (x_t)_{t=0}^\infty \succsim (x'_t)_{t=0}^\infty$  iff  $(x_{t+1})_{t=0}^\infty \succsim (x'_{t+1})_{t=0}^\infty$ .

Note that we have replaced Koopmans' weaker Postulate 2, Sensitivity and Postulate 5, Extreme Programs, by Monotonicity. The reason for this is that monotonicity on  $X$  is implied by a combination of Axiom 2 in Gul and Pendorfer (2014) and our Axiom 2. Hence, we will need neither Postulates 2 and 5, nor Monotonicity in our final representation.

**Proposition 7** (Koopmans, 1960) *Axioms 1, 2 and 4 imply the existence of a function  $v_i : [\tilde{l}; \tilde{m}] \rightarrow \mathbb{R}$  and a discount factor  $\beta_i \in (0; 1)$  such that for any consumption streams  $c, c'$ ,*

$$x(c) \succsim x(c') \text{ iff } \sum_{t=0}^{\infty} \beta_i^t v_i(x(\mathbf{c}_t)) \geq \sum_{t=0}^{\infty} \beta_i^t v_i(x(\mathbf{c}'_t))$$

It remains thus to show that the function  $v_i$  coincides with the function  $u_i$  (i.e., that the utility index captures at the same time risk- and time-preferences). Note, however, that while  $u_i$  is unique up to an affine-linear transformation, the same is not true for  $v_i$ . Indeed, we can, w.l.o.g., normalize  $v_i(\tilde{l}) = 0$  and  $v_i(\tilde{m}) = 1$ . For a given discount factor  $\beta_i$  and some outcome  $z \in [\tilde{l}; \tilde{m}]$ , the utility of  $z$ ,  $v_i(z)$  is uniquely identified by the representation only if there exists a (possibly infinite) set of periods,  $(\tau_k)_k$  such that the act that pays  $\tilde{m}$  on all periods in  $(\tau_k)_k$  and  $\tilde{l}$ , else (and thus generates utility equal to  $\sum_k \beta_i^{\tau_k}$ ) is indifferent to the constant act that pays  $z$  at each period  $t$ :

$$\left( (\tilde{m}_{\tau_k})_k \tilde{l}_{-(\tau_k)_k} \right) \sim_i ((z_t)_{t=0}^\infty)$$

In this case, clearly

$$v_i(z) = \sum_k \beta_i^{\tau_k}$$

is uniquely determined by the value of  $\beta_i$ . However, since time is discrete, such a construction is in general not possible for each outcome<sup>5</sup>  $z$ .

To ensure that  $u_i$  and  $v_i$  can be chosen to be identical, we impose:

**Axiom 5** Suppose that for some ideal event  $E \in \mathcal{F}_t$  and some sequence of periods  $(\tau_k)_k$ ,

$$\left( \underbrace{\tilde{l} \dots \tilde{l}}_{t-1}; \underbrace{\tilde{m}_E \tilde{l}}_t; \tilde{l} \dots \tilde{l} \dots \right) \sim \left( \tilde{m}_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right)$$

then

$$\left( \underbrace{\tilde{l} \dots \tilde{l}}_{t-1}; \underbrace{z_E \tilde{l}}_t; \tilde{l} \dots \tilde{l} \dots \right) \sim \left( z_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right)$$

for all  $z \in [\tilde{l}; \tilde{m}]$ .

**Proposition 8** *If Axiom 5 holds,  $v_i$  can be chosen to coincide with  $u_i$  on  $[\tilde{l}; \tilde{m}]$ . Hence,  $\succsim^i$  and the corresponding  $\succsim_s^i$  satisfies Axioms 1, 2, 3, 4 (i) and (iii) and 5, iff there exist a countably-additive and non-atomic probability measure  $\pi^i$  on  $(\Omega^i; \mathcal{F}^i)$ , a strictly monotone and continuous utility function  $u_i$  and a discount factor  $\beta_i \in (0; 1)$  such that  $\succsim^i$  is represented by:*

$$V_i(c) = u_i(c(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}).$$

---

<sup>5</sup>As an example, consider an outcome  $y \in (\tilde{l}; \tilde{m})$  with a utility value  $v_i(y)$  provided by the representation. Since  $v_i(y) < 1$ , we have that  $\frac{1}{1-\beta} > v_i(y)$ . Assume now that  $v_i(y) > \frac{\beta}{1-\beta}$ . Note that

$$\begin{aligned} \max_{(\tau_k)_k} \sum_k \beta_i^{\tau_k} &= \frac{1}{1-\beta} \\ \max_{(\tau_k)_k | \tau_k \neq 0 \text{ for all } k} \sum_k \beta_i^{\tau_k} &= \frac{\beta}{1-\beta} \end{aligned}$$

i.e., the maximum is obtained when we sum over all possible periods, but this maximum is too high to match  $v_i(y)$ . In fact, any sum that includes  $\tau_k = 0$  will exceed 1 and thus  $v_i(y)$ . In contrast, any sum that contains only periods  $\tau_k > 0$  cannot exceed  $\frac{\beta}{1-\beta} < v_i(y)$ . We conclude that on the interval of payoffs which correspond to utility values  $\left(\frac{\beta}{1-\beta}; 1\right)$ , any monotonic continuous transformation of  $v_i$  will represent the same preferences.

## 2.5 Unforeseen Surprises

We next extend the model to introduce the possibility of unforeseen surprises. An unforeseen unfavorable surprise is an outcome  $\ominus$ , which is considered worse than any of the foreseen outcomes. An unforeseen favorable surprise is an outcome  $\oplus$ , which is considered better than any of the foreseen outcomes.

As in Grant and Quiggin (2015), we can define for a given consumer  $i$ , the set of consumption streams not subject to unforeseen surprises:

$$\bar{\mathcal{C}} = \{c \mid \{\sigma_t \in \Omega_t \mid \mathbf{c}_t(\sigma_t) \cap \{\ominus; \oplus\}\} \in \mathcal{N} \text{ for all } t\}.$$

The set of consumption streams subject to unforeseen unfavorable surprises is:

$$\mathcal{C}^{uu} = \{c \mid \{\sigma_t \in \Omega_t \mid \mathbf{c}_t(\sigma_t) = \ominus\} \notin \mathcal{N} \text{ for some } t\}$$

and that of consumption streams subject to unforeseen favorable surprises is:

$$\mathcal{C}^{uf} = \{c \mid \{\sigma_t \in \Omega_t \mid \mathbf{c}_t(\sigma_t) = \oplus\} \notin \mathcal{N} \text{ for some } t\}.$$

Here we axiomatize preferences over consumption streams  $c$  representable by:

$$\tilde{V}_0^i(c) = \begin{cases} \left[ \begin{array}{l} u_i(c(\sigma_0)) + \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \bar{\mathcal{C}} \\ \left[ \begin{array}{l} u_i(c(\sigma_0)) + u(\tilde{l}) - u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uu} \\ \left[ \begin{array}{l} u_i(c(\sigma_0)) - u(\tilde{l}) + u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu} \end{cases} \quad (4)$$

To obtain such an axiomatization in a one-period setting, Grant and Quiggin (2015) impose three additional axioms on preferences: strict aversion to unforeseen unfavorable surprises, preference for unforeseen favorable surprises and comparability of unforeseen surprises, which we restate below to apply to infinite consumption streams:

**Axiom 6** The preferences  $\succsim^i$  exhibit strict aversion to unforeseen unfavorable surprises, i.e.,  $(\tilde{l} \dots \tilde{l} \dots) \succ^i c$  for any  $c \in \mathcal{C}^{uu}$ .

**Axiom 7** The preferences  $\succsim^i$  exhibit preference to unforeseen favorable surprises, i.e.,  $c \succ^i (\tilde{m} \dots \tilde{m} \dots)$  for any  $c \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu}$ .

Axiom 6 (which corresponds to Axiom 2 Strict aversion to unforeseen unfavorable surprises in Grant and Quiggin, 2015) states that all consumption streams that contain unfavorable surprises are considered worse than the consumption stream that delivers the worst perceived consumption  $\tilde{l}$  at all  $\sigma_t$ . Axiom 7 (which corresponds to Axiom 3 Preference for unforeseen favorable

surprises in Grant and Quiggin, 2015) means that consumption streams than contain favorable, but no unfavorable surprises are considered better than the consumption stream that pays the maximal perceived outcome  $\tilde{m}$  at all  $\sigma_t$ . The implication of these two axioms is that the decision maker will at all cost avoid being exposed to unforeseen unfavorable surprises and seek to be exposed to exposed to unforeseen favorable surprises.

As in Grant and Quiggin (2015), we define for each consumption stream  $c$ , its projection, a consumption stream  $\bar{c}$  such that  $\bar{c}$  coincides with  $c$  whenever  $c$  contains no unforeseen surprises. If  $c$  contains unforeseen surprises, an unfavorable unforeseen surprise is "replaced" in  $\bar{c}$  by the minimal perceived outcome,  $\tilde{l}$ . Symmetrically, an unforeseen favorable surprise, is replaced in  $\bar{c}$  by the maximal perceived outcome,  $\tilde{m}$ . Formally,

$$\bar{c}(\sigma_t) = \begin{cases} c(\sigma_t) & \text{if } c(\sigma_t) \notin \{\ominus; \oplus\} \\ \tilde{l} & \text{if } c(\sigma_t) = \ominus \\ \tilde{m} & \text{if } c(\sigma_t) = \oplus \end{cases}$$

Let  $\bar{\mathcal{C}}$  be the set of all projections.

The last axiom of Grant and Quiggin (2015) ensures that acts within one of the three sets defined above can be compared using their projections:

**Axiom 8** The preferences  $\succsim^i$  permit comparability of unforeseen surprises if for any pair of acts  $c$  and  $c'$  if either  $c, c' \in \bar{\mathcal{C}}$ , or if  $c, c' \in \mathcal{C}^{uu}$ , or if  $c, c' \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu}$ , then

$$c \succsim^i c' \text{ iff } \bar{c} \succsim^i \bar{c}'.$$

Comparability of unforeseen surprises requires consumption streams which belong to the same category ( $\mathcal{C}^{uu}$ , or  $\mathcal{C}^{uf} \setminus \mathcal{C}^{uu}$ ) to be ranked identically to their projections,  $\bar{\mathcal{C}}$ .

Using Theorem 3 in Grant and Quiggin (2015) and combining it with the result derived in Proposition 8, we obtain:

**Proposition 9**  $\succsim^i$  and the corresponding  $\succsim_s^i$  satisfies Axioms 1, 2, 3, 4 (i) and (iii), 5, 6, 7 and 8, iff it can be represented as:

$$\tilde{V}_0^i(c) = \begin{cases} \left[ \begin{array}{l} u_i(c(\sigma_0)) + \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \bar{\mathcal{C}} \\ \left[ \begin{array}{l} u_i(c(\sigma_0)) + u(\tilde{l}) - u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uu} \\ \left[ \begin{array}{l} u_i(c(\sigma_0)) - u(\tilde{l}) + u(\tilde{m}) \\ \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_t^i \pi^i(\omega_t^i) u_i(\min\{C(\omega_t^i)\}) \end{array} \right] & \text{if } c \in \mathcal{C}^{uf} \setminus \mathcal{C}^{uu} \end{cases}$$

**Remark 10** Just as Grant and Quiggin (2015), for the purposes of the axiomatization, we take the partition  $W^i$  and the implied partition  $\Omega^i$  on the event tree of the economy as given and observable. In fact, the axiomatization of Gul and

*Pesendorfer (2014) used to establish the results does not require such observability. Without additional assumptions, in our context, a direct application of their theorem would imply a partition of  $\Omega$ ,  $\Omega^i$ . However, in general, this partition need not satisfy the multiplicative structure  $\Omega^i = \prod_{t=0}^{\infty} W^i$  imposed above. We conjecture that adding the following axiom characterizing the set of ideal events would provide a sufficient condition for the multiplicative structure assumed:*

**Axiom 9** For any  $t$  subsets of  $S$ ,  $w^1 \dots w^t \subseteq S$ ,  $\omega_t = \{(s_0; s_1 \dots s_t) \mid s_1 \in w^1 \dots s_t \in w^t\}$  is ideal iff all of the events  $\omega_1^t = \{(s_0; s)_{s \in w^t}\}$  are ideal.

The representation derived in Proposition 9 will be used in Section 5 to model financial markets, in which agents are averse to unforeseen unfavorable surprises. To avoid non-convexities, we will assume that the unforeseen favorable surprises occur only for consumption quantities which exceed the maximal possible endowment of the economy. We will see that aversion to unforeseen unfavorable surprises ensures the survival of partially aware agents even when their beliefs  $\pi^i$  on  $\Omega^i$  are not objectively correct.

### 3 Financial Markets with Unforeseen Contingencies

In the standard Blume-Easley model, all traders are fully aware, and financial markets are implicitly assumed to span the state space. Traders may therefore be regarded as choosing consumption paths, without specifying the associated portfolio of assets. There is no loss of generality in assuming that the set of assets consists of Arrow securities, one for each node  $\sigma_t$ . Any redundant assets may be priced in terms of the replicating portfolio of Arrow securities.

These features of the model remain even if some traders incorrectly impute zero probability to some events that are in fact possible. Such traders will seek, to the extent permitted by market constraints, to short-sell securities that pay off only in such states.

The situation with reduced awareness is different. We will show below that traders will voluntarily choose to not trade in assets specific to contingencies of which they are unaware. Hence, their portfolio cannot be represented entirely in terms of Arrow securities. This creates a non-trivial role for 'traditional' securities such as bonds and equity.

To understand this fundamental difference, we will first specify the "true" process of the economy and compare it to the beliefs formed by partially aware agents. In particular, we will show how differential awareness can cause belief heterogeneity even when all agents have access to the same correct information about relative likelihoods of states they are aware of.

#### 3.1 The True Process of the Economy and Agents' Beliefs

We define on  $(\Omega; \mathcal{F})$  a probability distribution  $\pi$  and throughout we assume  $\pi(Z(\sigma_t)) > 0$  for all  $\sigma_t$ . Intuitively,  $\pi$  describes the evolution of the state

process in the economy. In what follows, for brevity, we abuse notation slightly by denoting  $\pi(Z(\sigma_t)) = \pi(\sigma_t) = \pi(s_0; s_1; s_2 \dots s_t)$ . The one-step-ahead probability distribution  $\pi(s_{t+1} | \sigma_t)$  at node  $\sigma_t$  is determined by:

$$\pi(s_{t+1} | \sigma_t) = \pi(s_0 \dots s_t; s_{t+1} | s_0 \dots s_t) = \frac{\pi(s_0 \dots s_t; s_{t+1})}{\pi(s_0 \dots s_t)} \text{ for any } s_{t+1} \in S$$

In words,  $\pi(s_{t+1} | \sigma_t)$  is the probability under distribution  $\pi^n$  that the next observation will be  $s_{t+1}$  given that we have reached node  $\sigma_t$ . We will assume that the true process of the economy is i.i.d. and write  $\pi(s_{t+1} = s | \sigma_t) =: \pi(s)$ . Note that this does not restrict the endowment process to be i.i.d..

From the point of view of consumer  $i$ , the information revelation is described by finite partitions of the set  $\Omega^i$  and the corresponding  $\sigma$ -algebra  $\mathcal{F}^i$ . Obviously,  $\mathcal{F}$  is finer than  $\mathcal{F}^i$  and hence, any probability distribution  $\pi$  on  $(\Omega; \mathcal{F})$  specifies a probability distribution on  $(\Omega^i; \mathcal{F}^i)$  with

$$\pi(\omega_t^i) = \pi\{\sigma_t | s_\tau \in w_\tau^i \text{ for all } \tau \in \{1 \dots t\}\}.$$

The one-step ahead probability distribution  $\pi(w_{t+1}^i | \omega_t^i)$  is defined analogously to  $\pi(s_{t+1} | \sigma_t)$ .

The subjective beliefs of consumer  $i$ ,  $\pi^i$  are defined on  $(\Omega^i; \mathcal{F}^i)$  and need not in general coincide with the truth.  $i$ 's beliefs are correct if they coincide with the restriction of  $\pi$  to  $\Omega^i$ .

For the rest of the paper, we will restrict attention to beliefs which describe an i.i.d. process,  $\pi^i(w_{t+1}^i = w^i | \omega_t^i) = \pi^i(w^i)$ .

Just as in the case of a fully aware agent, our theory cannot and does not aim to explain how agents choose their priors<sup>6</sup>. Here, we will show that even agents who have correct information about the likelihoods of the events of which they are aware, might end up with heterogeneous beliefs as a result of differential unawareness. Formally,

**Definition 11** *We will say that consumer  $i$ , has awareness-biased beliefs if the relative likelihoods assigned to states he is aware of are correct,  $\frac{\pi^i(s)}{\pi^i(s')} = \frac{\pi(s)}{\pi(s')}$  for any  $s, s' \in S^i$  and if the probability assigned to an element of the partition  $W^i$ ,  $w^i$  equals the probability of the state  $s \in w \cap S^i$  of which  $i$  is aware,  $\pi^i(w^i) = \pi^i(w \cap S^i)$ .*

Awareness-biased beliefs are correct w.r.t. to the states of which a consumer is aware in that the relative likelihoods assigned to such states coincide with the truth. However, lacking any further information about the likelihood of states he is unaware of, such a consumer identifies the probability of an event containing states he is unaware of with the probability of the states he understands. It is clear that in general, the awareness-biased beliefs  $\pi^i$  will not be correct.

<sup>6</sup>Dietrich (2017), as well as Ahn and Ergin (2010) provide axiomatizations, in which the beliefs of partially aware agents and depend on the partition used by the decision maker. Thus, beliefs under partial awareness are consistent with beliefs derived for the finest possible partition only in special cases.



Note that whenever  $\pi(s) > 0$  for all  $s \in S$  (as in Assumption 3 below),  $\pi^i(w^i) > 0$  will hold for all  $w^i \in W^i$ . Hence,  $i$ 's one-step ahead beliefs will be absolutely continuous w.r.t. the truth. The extent to which  $i$ 's beliefs w.r.t.  $W^i$  are "wrong" can then be quantified by the Kullback-Leibler distance between / relative entropy of  $\pi(s | S^i)$  and  $\pi(w^i)$ , i.e.,

$$\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^i | S^i)} \quad (5)$$

Our definition of awareness-biased beliefs shows that differential awareness can lead to belief heterogeneity even when agents share the same information about likelihoods. The following two remarks will help illustrate this point:

**Remark 12** *The awareness-biased beliefs of two agents with identical awareness partitions need not coincide. Indeed, consider agents  $i$  and  $j$ , who form a common awareness partition on the state  $S = \{1; 2; 3; 4\}$ ,  $W = \{w = \{1; 2\}; w' = \{3; 4\}\}$ . If  $i$  is aware of states  $S^i = \{1; 3\}$ , whereas  $j$  is aware of states  $S^j = \{1; 4\}$ , and unless  $\pi(3) = \pi(4)$ , agents' awareness-biased beliefs will satisfy*

$$\frac{\pi^i(w)}{\pi^i(w')} = \frac{\pi(1)}{\pi(3)} \neq \frac{\pi^j(w)}{\pi^j(w')} = \frac{\pi(1)}{\pi(4)}.$$

**Remark 13** *An agent with a coarser awareness partition can entertain awareness-biased beliefs closer to the truth than an agent with a finer awareness partition. Indeed, consider  $S = \{1..5\}$ , suppose that  $\pi(1) = \frac{3}{8}$ ,  $\pi(2) = \pi(3) = \pi(4) = \frac{1}{8}$  and  $\pi(5) = \frac{1}{4}$ . For two agents,  $i$  and  $j$ , let  $S^i = \{2; 3\}$ ,  $S^j = \{2; 3; 5\}$  and*

$$\begin{aligned} W^i &= \{w^i = \{1; 2; \}; w^{i'} = \{3; 4; 5\}\} \\ W^j &= \{w^j = \{1; 2; \}; w^{j'} = \{3; 4\}; w^{j''} = \{5\}\} \end{aligned}$$

*According to our definition, agent  $j$  is thus more aware than agent  $i$ . Yet,  $i$ 's awareness-biased beliefs are correct:*

$$\pi^i(w^i) = \pi^i(w^{i'}) = \frac{1}{2} = \pi(w^i) = \pi(w^{i'}),$$

*whereas those of  $j$  w.r.t. to  $i$ 's partition<sup>7</sup> are not:*

$$\pi^j(w^i) = \frac{1}{4}, \pi^j(w^{i'}) = \frac{3}{4}$$

Our results below, see Proposition 21, will demonstrate that markets select for agents who have correct beliefs in the sense defined above, i.e., agents for whom the relative entropy in (5) is 0, unless the wrong beliefs are compensated by a finer awareness partition.

Hence, even though the results we derive in Section 4 show that market forces do not select against partially aware agents, the fact that it might be

<sup>7</sup>This is the relevant comparison for the results derived in Section 4.

less likely that these agents have correct beliefs would imply that these agents would also be less likely to survive and impact prices in the limit. In contrast, our result in Section 5 does not depend on agents' beliefs.

Finally, we remark that the results derived in Sections 4 and 5 do not depend on the assumption of awareness-biased beliefs, but merely on the property of such beliefs to be absolutely continuous w.r.t. the truth. Nevertheless, we believe that this model of beliefs provides some useful intuition and allows us to better evaluate the conditions under which survival occurs.

### 3.2 The Model of the Economy

There is a single good and  $I$  infinitely lived consumers, each with consumption set  $\mathbb{R}_+$ . Consumption plans are defined as in Section 2. Each consumer is endowed with a particular consumption plan, called  $i$ 's endowment stream and denoted  $e^i$ .  $e = \sum_{i \in I} e^i$  stands for the total endowment of the economy.

We will consider consumers who maximize a utility function as in the representations (2) and (4) axiomatized in Section 2.

We will impose the following assumptions on utility functions and endowments, which are standard in the survival literature:

**Assumption 1** The functions  $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$  are twice continuously differentiable, strictly concave, and satisfy  $\lim_{c \rightarrow 0} u'_i(c) = \infty$  and  $\lim_{c \rightarrow \infty} u'_i(c) = 0$ .

**Assumption 2** Individual endowments are uniformly bounded away from zero and uniformly bounded from above. Formally, there is an  $m > 0$  such that  $e^i(\sigma_t) > m$  for all  $i, \sigma_t$ ; moreover, there is an  $m' > m > 0$  such that  $\sum_{i \in I} e^i(\sigma_t) < m'$  for all  $\sigma_t$ .

**Assumption 3**  $\pi(s) > 0$  for all  $s \in S$  and all agents  $i \in I$  have awareness-biased beliefs. In particular,  $\pi^i(w^i) > 0$  for all  $w^i \in W^i$ .

Assumption 1 implies that the agent would never choose zero minimal consumption on an event  $\omega_t^i$  he believes to have a positive probability. Assumption 2 ensures that endowments are uniformly bounded away from 0 and above. Given the i.i.d. structure imposed on the true process and on beliefs, Assumption 3 states that one-step-ahead probabilities of all states of the world are positive and that all subjective beliefs assign a positive one-step-ahead probability to every element in their respective partitions. As we showed above, the assumption that agents form awareness-biased beliefs implies the absolute continuity of subjective beliefs with respect to the true probability distribution (as in Blume and Easley, 2006). Taken together, Assumptions 1 and 3 ensure that no agent vanishes in finite time.

In economies with bounded endowments and complete markets, and populated by expected utility maximizers, only beliefs and discount factors matter for

survival<sup>8</sup>. In particular, if all agents are equally patient, agents with incorrect beliefs vanish a.s. in the presence of agents with correct beliefs. By contrast, in unbounded economies, risk attitudes also matter for survival, and agents with incorrect beliefs can survive. In order to disentangle the effects of partial awareness on survival from those of risk attitude, we restrict our attention to the case of bounded economies.

### 3.3 Assets

We have so far described the economy only in terms of state-contingent consumption. We now introduce assets explicitly.

In particular, we define the set of period  $t$  Arrow-securities in the economy to be

$$\tilde{A}_t^0 = \{a_{\sigma_t} \text{ with } a_{\sigma_t}(\sigma'_t) = \mathbf{1}_{\sigma_t}\} \quad (6)$$

We will also define for each agent  $i$  and the corresponding partition  $\Omega^i$ , the set of unit-securities that pay conditional on events in  $\Omega^i$ :

$$\tilde{A}_t^i = \left\{ a_{\omega_t^i} \text{ with } a_{\omega_t^i}(\sigma_t) = 1 \text{ if } \sigma_t \in \omega_t^i, a_{\omega_t^i}(\sigma_t) = 0, \text{ else for all } \omega_t^i \in \Omega_t^i \right\}. \quad (7)$$

$\tilde{A}_t^i$  is thus the set of assets with payoffs measurable w.r.t. to agent  $i$ 's partition  $\Omega_t^i$ . Intuitively, such assets do not involve "surprises" from the point of view of agent  $i$ .

Note that while in general  $\tilde{A}_t^i \not\subseteq \tilde{A}_t$ , we have that for  $i = 1 \dots n$ , the set of payoffs generated by the set of assets in  $\tilde{A}_t^i$ ,

$$\tilde{M}_t^i = \left\{ \sum_{\omega_t^i \in \Omega_t^i} \mathbf{1}_{\omega_t^i} \theta(\omega_t^i) \mid (\theta(\omega_t^i))_{\omega_t^i \in \Omega_t^i} \in \mathbb{R}^{|\Omega_t^i|} \right\}$$

satisfies

$$\tilde{M}_t^i \subseteq \tilde{M}_t^0 = \mathbb{R}^{|\Omega_t|}.$$

We can now rewrite the endowment of the economy in terms of assets in  $\tilde{A}^0 = \cup_{t=0}^{\infty} \tilde{A}_t^0$  by assigning to each agent  $i$ , an initial portfolio  $\bar{\theta}^i$  with  $\bar{\theta}^i(\sigma_t) = e^i(\sigma_t)$ . Let  $(\theta^i(\sigma_t))_{\sigma_t \in \Omega^0}$  denote the portfolio holdings of agent  $i$ . Clearly, his consumption is then given by  $c^i(\sigma_t) = \theta^i(\sigma_t)$ .

While a priori, each agent has access to all assets in  $\cup_t \tilde{A}_t^0$ , we will see that agent  $i$  with preferences subject to surprises might find it optimal to restrict his investments to the set  $\cup_t \tilde{A}_t^i$ , i.e., to the set of assets he understands.

<sup>8</sup>This result is conditional on the assumption that markets are complete, see Coury and Sciubba (2012), as well as that endowments are bounded, see Kogan et al. (2006, 2011), Yan (2008). Its robustness to preference specifications is still being explored, see Condie (2008), Da Silva (2011), Eichberger and Guerdjikova (2018), Guerdjikova and Sciubba (2015), Easley and Yang (2014), Borovicka (2014).

## 4 Survival with Differential Awareness

We will now consider an economy with agents who are partially aware, but aware of the possibility of surprises as described above. In this Section, we will assume that agents maximize a utility function given by (2).

**Definition 14** *An equilibrium of an economy with partially aware agents who are aware of their unawareness is described by consumption streams  $(c^i)_{i \in I}$  and an integrable<sup>9</sup> price system  $p(\sigma_t)$  such that each of the consumers maximizes their utility function given prices in the economy and markets clear:*

$$c^i = \arg \max_{c^i} V_0^i(c^i) = \arg \max_{c^i} \left[ u_i(\bar{c}^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega^i} \beta_i^t \pi^i(\omega_t^i) u_i(\min \{C^i(\omega_t^i)\}) \right]$$

$$\sum_{i \in I} c^i(\sigma_t) = \sum_{i \in I} e^i(\sigma_t) \quad \forall \sigma_t \in \Omega$$

It is easy to show that an equilibrium exists using Bewley's (1972) theorem.

An alternative definition of an equilibrium can be states in terms of asset trades:

**Definition 15** *An equilibrium of an economy with assets  $\tilde{A}_t^0$  with partially aware agents who are aware of their unawareness is described by portfolio holdings  $(\theta^i)_{i \in I}$  and an integrable price system  $p(\sigma_t)$  such that each of the consumers maximizes their utility function given prices in the economy and asset markets clear:*

$$\theta^i = \arg \max_{\theta^i} V_0^i(\theta^i) = \arg \max_{\theta^i} \left[ u_i(\theta^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega^i} \beta_i^t \pi^i(\omega_t^i) u_i\left(\min_{\sigma_t \in \omega_t^i} \{\theta^i(\sigma_t)\}\right) \right]$$

$$\sum_{i \in I} \theta^i(\sigma_t) = \sum_{i \in I} \bar{\theta}^i(\sigma_t) \quad \forall \sigma_t \in \Omega$$

Clearly, the equilibria satisfying Definition 15 specify the same equilibrium consumption and state prices as those satisfying Definition 14.

Our first result generalizes the main result of Blume and Easley (2006) to apply to agents with bounded, but identical levels of awareness.

**Proposition 16** *Consider two agents  $i$  and  $j$ , with identical awareness partitions,  $\Omega^i = \Omega^j$ . Assume that the initial endowment of the economy is measurable w.r.t.  $\Omega^i$ . If the two agents have identical beliefs, but different discount factors, then the agent with the lower discount factor vanishes a.s.. If the two agents have identical discount factors and different beliefs, the agent whose beliefs are*

<sup>9</sup> An integrable price system  $p(\sigma_t)_{\sigma_t \in \Sigma}$  has the property that  $\sum_{t \in \mathbb{N}} \sum_{\sigma_t \in \Sigma_t} p(\sigma_t) e^i(\sigma_t)$  is finite for any uniformly bounded sequence  $e^i(\sigma_t)$ .

further away from the truth vanishes a.s.. More generally, the agent with the lower survival index:

$$\ln \beta_i - \sum_{w \in W} \pi(w) \ln \frac{\pi(w)}{\pi(w \cap S^i | S^i)}$$

vanishes a.s..

As we showed in Remark 12, agents with identical awareness partitions need not hold identical beliefs. The proposition shows that (for identical discount factors), it is the Kullback-Leibler distance between the awareness-biased beliefs and the truth on the relevant partition as defined in (5) that determines whether an agent survives or not. This distance, in turn is fully determined by the set of states of which the agent is aware,  $S^i$ .

Our next Proposition extends these results to a set of agents with nested awareness partitions.

**Proposition 17** *Consider a population of agents with awareness partitions ordered with respect to inclusion, equal discount factors and correct beliefs. Assume that the initial endowment of the economy is measurable w.r.t. the finest awareness partition. All agents survive a.s.*

Our result shows that whenever agents have equal discount factors and correct beliefs relative to their awareness partition and the awareness partitions are nested, the agent's level of awareness is irrelevant for survival. In fact, all agents survive. Note however, that awareness-biased beliefs will be correct only in very special cases. E.g., if all states are equiprobable, and all agents' partitions have the same number of elements, all agents will have awareness-biased beliefs which are correct. In general, however, as shown above differential awareness will imply belief heterogeneity and hence, the result above will not apply.

Nevertheless, the analysis of an economy populated by agents with correct beliefs allows us to analyze the equilibrium allocation. Note that unless  $p(\sigma_t) = 0$  for some  $\sigma_t \in \omega_t^i$ , we have  $c^i(\sigma_t) = c^i(\sigma_t')$  for all  $\sigma_t, \sigma_t' \in \omega_t^i$ . I.e., when state prices are strictly positive, agent  $i$ 's consumption is measurable w.r.t.  $\Omega_t^i$ . We conclude that in equilibrium with strictly positive state prices, a partially aware consumer  $i$  behaves as if his investment opportunities were restricted to belong to the set  $\cup_t \tilde{A}_t^i$  with payoffs measurable w.r.t.  $\Omega^i$ , even though in fact, he has access to a richer set of assets,  $\cup_t \tilde{A}_t^0$ . Thus, his portfolio will appear underdiversified when compared to that of a fully aware agent. Furthermore, he will be overinvesting in insurance across states  $\sigma_t \in \omega_t^i$ , which might involve aggregate risk considered by agent  $i$  as a "surprise". A more aware agent  $j$  might provide such full insurance against aggregate risk. As a result, asset prices in such an economy will be distorted: the price for insurance across states belonging to the same elements of the partition  $\omega_t^i$  will be higher than in an economy with fully aware agents.

We conclude that in equilibrium, a partially aware investor will behave as if he were following a heuristic to invest in a subset of assets he is familiar with

and "understands", see Section 6. Furthermore, the result above implies that in our model, such heuristics are in line with ecologic rationality, in that they do not lead an investor to disappear. The phenomena generated by the use of such heuristics will persist in the long-run, as long as all agents have equal survival indices.

We next examine the impact of heterogeneity in discount factors and beliefs on survival when awareness partitions are nested. We first show that a less aware agent can only survive if his survival index is at least as large as that of a more aware agent. In particular, for given identical beliefs, the less aware agent can survive only if his discount factor is at least as high as that of the more aware agent and, for given identical discount factors, the less aware agent can only survive only if his beliefs are at least as close to the truth as those of the more aware agent. As we showed in Remark 13, it is easy to construct examples for which this is the case.

**Proposition 18** *Suppose that an agent  $i$  has a strictly coarser awareness partition  $\Omega^i$  than that of an agent  $j$ ,  $\Omega^j$ . Assume as well that the endowment of the economy is measurable w.r.t. the partition  $\Omega^j$ . If*

$$\ln \frac{\beta_j}{\beta_i} + \left( \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^i | S^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^j \cap S^j | S^j)} \right) > 0$$

*agent  $i$  vanishes a.s.*

This result shows that in order for an agent  $i$  with a coarser partition than agent  $j$  to survive, it has to be that:

$$\ln \frac{\beta_j}{\beta_i} + \left( \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^i \cap S^i | S^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi(w^j \cap S^j | S^j)} \right) \leq 0.$$

Hence, we will consider an economy, in which the agents with coarser awareness partitions have (weakly) larger survival indices. To formulate our result, we will have to understand when partial awareness matters in the long-term. We will use the following definitions:

**Definition 19** *The unawareness of agent  $i$ , given by the partition  $\Omega^i$ , is irrelevant in the limit if for any  $\omega^i \in \Omega^i$  and any  $\sigma, \sigma' \in \omega^i$*

$$\lim_{t \rightarrow \infty} e(\sigma_t) - e(\sigma'_t) = 0.$$

*The unawareness of agent  $i$ , given by the partition  $\Omega^i$  is relevant in the limit if for some  $w^i \in W^i$ ,  $s$  and  $s' \in w^i$ , there is an  $\epsilon > 0$  such that for any  $\sigma, \sigma' \in w^i$ ,*

$$\lim_{t \rightarrow \infty} \sup [e(\sigma_t; s) - e(\sigma'_t; s')] > \epsilon. \quad (8)$$

The unawareness of agent  $i$  is considered irrelevant if, in the limit, the total endowment of the economy is measurable with respect to agent  $i$ 's partition. Such an agent is not exposed to surprises w.r.t. the total endowment process of the economy in the limit. In contrast, agent  $i$ 's unawareness is relevant even in the limit, if there are at least two states that  $i$  cannot distinguish and in which the total endowment of the economy remains distinct.

**Definition 20** *Let agent  $i$ 's awareness partition  $\Omega^i$  be finer than that of  $j$ . The unawareness of agent  $j$  given by the partition  $\Omega^j$ , is irrelevant in the limit w.r.t. that of agent  $i$  given by partition  $\Omega^i$  if for any  $\omega^i \in \Omega^i$  and  $\omega^j \in \Omega^j$  s.t.  $\omega^i \subseteq \omega^j$ ,*

$$\lim_{t \rightarrow \infty} \min_{\sigma_t \in \omega_t^i} e(\sigma_t) - \min_{\tilde{\omega}_t^i \subseteq \omega_t^j} e(\tilde{\omega}_t^i) = 0.$$

*The unawareness of agent  $j$  is relevant in the limit w.r.t. that of agent  $i$  if there is an  $\epsilon > 0$  such that for any  $\omega^i \in \Omega^i$  and  $\omega^j \in \Omega^j$  s.t.  $\omega^i \subseteq \omega^j$ ,*

$$\limsup_{t \rightarrow \infty} \left[ \min_{\sigma_t \in \omega_t^i} e(\sigma_t) - \min_{\tilde{\omega}_t^i \subseteq \omega_t^j} e(\tilde{\omega}_t^i) \right] > \epsilon.$$

To understand the definition note that in general, the initial endowment of the economy is not measurable w.r.t. to  $\Omega^i$  or  $\Omega^j$ . Hence, if prices are strictly positive, the maximum consumption  $i$  can thus obtain given the initial endowment of the economy is given by  $\min_{\sigma_t \in \omega_t^i} e(\sigma_t)$ . Since  $i$ 's partition is finer than  $j$ 's, the maximum consumption  $j$  can obtain given the initial endowment of the economy is given by  $\min_{\tilde{\omega}_t^i \subseteq \omega_t^j} e(\tilde{\omega}_t^i)$ .  $j$ 's unawareness is called relevant in the limit w.r.t. to  $i$ 's whenever the maximal consumption streams these two agents can obtain in the limit differ and irrelevant if they are the same.

**Proposition 21** *Consider a population of agents with nested awareness partitions,  $\Omega^1$  strictly finer than  $\Omega^2$ ... strictly finer than  $\Omega^n$ , and ordered survival indices such that:*

$$\ln \beta_k - \sum_{w^{k'}} \pi(w^{k'}) \ln \frac{\pi(w^{k'})}{\pi(w^{k'} \cap S^k | S^k)} < \ln \beta_{k'} - \sum_{w^{k'}} \pi(w^{k'}) \ln \frac{\pi(w^{k'})}{\pi(w^{k'} \cap S^{k'} | S^{k'})}$$

*holds for all  $k < k'$ . Assume that the initial endowment of the economy is measurable w.r.t.  $\Omega^1$ . Assume that  $k$  is the agent with the finest awareness partition, whose unawareness is relevant in the limit ( $\tilde{k} > 1$ ), whereas the unawareness of all agents  $i < k$  is irrelevant in the limit. Then there exists an equilibrium of the economy, in which*

- (i) *agents  $k < (\tilde{k} - 1)$  vanish a.s.;*
- (ii) *agent  $(\tilde{k} - 1)$  survives a.s.;*

- (iii) *agent  $n > k > \tilde{k} - 1$  survives a.s. if the unawareness of agent  $k + 1$  is relevant in the limit w.r.t. that of  $k$  and vanishes a.s. otherwise;*
- (iv) *agent  $n$  a.s. survives.*

The proposition demonstrates that partially aware agents can survive in the presence of a fully aware agent as long as their survival index exceeds that of the fully aware agent. In such an economy, an agent survives regardless of the value of his survival index as long as the awareness of the agent with the finest partition coarser than his is relevant in the limit and vanishes otherwise. This result is interesting, because it shows that more aware agents can survive even when their survival index is not maximal in the economy.

This requires however the presence of agents whose unawareness is relevant even in the limit. In such a scenario, the partially aware agents do not wish to consume the entire endowment of the economy: such a consumption stream would expose them to surprises. Hence, it is the agents with lower survival indexes, but higher awareness who consume in such states (hold the assets that pay in such states) and thus survive.

Note that the results for an economy with naively unaware agents, who act as if they assign 0-probability to possible events, and that with agents who are aware of their own limited perception of the world as in Grant and Quiggin (2015) are qualitatively different. While the former vanish a.s. in finite time, the latter effectively insure themselves against all unforeseen contingencies and survive a.s., even in the presence of a fully aware agent with correct beliefs.

This result, however, hinges on the assumption that partially aware agents can have a survival index which exceeds that of fully aware agents, or, for equal discount factors, that they can adopt correct beliefs. While it is easy to generate examples, in which this would be the case, in general, this assumption need not be satisfied. In particular, if the most aware agent in the economy is fully aware, then his (awareness-biased) beliefs would be correct and hence, the partially aware agents would only survive if their beliefs were also correct. The discussion above suggests that this is not likely. In the next section, we will thus extend the model by introducing aversion to unfavorable surprises. This extension will allow to establish survival results, which do not depend on the assumption of correct beliefs for partially aware agents.

## 5 Aversion to Unforeseen Unfavorable Surprises

In this Section, we consider agents, whose preferences are averse to unforeseen unfavorable surprises and can thus be represented as in (4). To understand how this model of unforeseen surprises can be applied to our setting, assume that an unforeseen unfavorable surprise occurs whenever the consumption at some node is in the set  $[0; L)$ . For instance, we could set,  $L \leq m$ , since  $m$  is the minimal endowment of an agent with the interpretation that an agent is not able to imagine an outcome that is lower than those specified by his initial endowment stream. Similarly, we could say that an unforeseen favorable surprise



occurs, whenever the consumption exceeds some value  $M$ . Since preferences for unforeseen favorable surprises lead to non-convex preferences, we will here set  $M \geq m'$ , the maximal endowment of the economy. In this way, while à priori possible, unforeseen favorable surprises are de facto excluded by the specification of the endowment process.

**Proposition 22** *Consider an economy, in which agents are partially aware and aware of their unawareness. Assume as well that an unforeseen unfavorable surprise is associated with a consumption less than  $L$ , with  $L \leq m$  and that an unforeseen favorable surprise is associated with consumption larger than  $M$ ,  $M \geq m'$ . Consider, furthermore, a population of agents  $I = \{1 \dots n\}$  with nested partitions,  $\Omega^1 = \Omega$  strictly finer than  $\Omega^2$ , ... strictly finer than  $\Omega^n$ . Suppose that the partially aware agents exhibit strict aversion to unforeseen unfavorable surprises. Under assumptions 1—3 and if the unawareness of agent 2 is relevant in the limit, there is an equilibrium of the economy, in which all partially aware agents do not experience unforeseen surprises and thus survive on all paths  $\sigma$ , whereas the fully aware agent 1 (who does not exhibit strict aversion to unforeseen unfavorable surprises) survives a.s.*

Our last proposition establishes survival of partially aware agents, who are averse to unforeseen unfavorable surprises, independently of their beliefs. Since such agents wish at all cost to avoid consumption lower than  $L$ , their consumption in equilibrium is always bounded away from 0 on all paths (even those they are unaware of). In terms of asset holdings, such agents hold a minimal number  $L$  of bonds paying on  $\Omega_t$  for each  $t$ . Hence, they follow a heuristic consistent with the precautionary principle described in Section 5. As in Grant and Quiggin (2013a, b), this heuristic can be interpreted as an instance of the Precautionary Principle, which avoids the worst possible outcomes, in this case, consumption below  $L$  (or converging to 0) on paths with strictly positive probability.

An interesting case to consider is that of partially aware agents with incorrect beliefs and a fully aware agent with correct beliefs. In this case, the economy a.s. reaches a node, from which on the constraint requiring the agent to avoid an unfavorable surprise is binding. From this point on, the partially aware agents with incorrect beliefs consume  $L$  in each state of the world, whereas the fully aware agent with correct beliefs will hold the rest of the endowment of the economy. The equilibrium prices on such paths will then be determined by the beliefs of the fully aware agent and thus be potentially biased. Furthermore, since the partially aware agents create an excess demand for bonds, prices will exhibit the equity premium puzzle.

## 6 The Role of Heuristics in Financial Markets with Partially Aware Agents

### 6.1 Optimizing versus Heuristic Approaches to Choice

It is usual, in discussion of decisions under uncertainty, to distinguish between rational optimization and heuristics. However, this distinction is problematic. Any complete and transitive ordering over risky prospects may be represented as the maximization of an appropriately chosen function. (If the domain is specified to include the set of alternatives as well as the prospect under consideration, transitivity may be dispensed with, as in some versions of regret theory, and similarly for completeness.)

This point obviously applies to heuristics such as ‘satisficing’ or the precautionary principle. These may be interpreted as optimization taking account of costs of search and cognition.

The crucial difference between heuristic and optimizing approaches lies in the way in which they are justified. For optimizing approaches, such as expected utility maximization, justification takes the form of a representation theorem. Assuming preferences satisfy some set of axioms, typically presented as evidently rational, a representation theorem shows that choices consistent with these preferences will maximize the value of an appropriately chosen function. It follows that an agent who assents to the axioms and who can specify the parameters of the relevant function can achieve security; that is, the assurance that they have made the best possible choice given their preferences, the choice set and the available information.

Nevertheless, as noted by Forster (1999), such rules may not “fit to reality”: “according to such theories, decision makers act rationally even when their beliefs are entirely false or inappropriate”<sup>10</sup>. As noted earlier, in financial markets, agents with such false beliefs eventually disappear, suggesting that such rules might not be adapted to the relevant environment and thus fail the criterion of “ecologic rationality”.

By contrast, heuristics may be justified (or not) as ecologically rational. An ecologically rational heuristic is one that achieves an optimal outcome in some given environment. However, it is not assumed that the agents who follow heuristics can assure themselves that the environment is one for which their heuristic is ecologically rational. At most, agents can work on the basis of inductive reasoning as in case-based decision theory.

In this paper, we combine the two approaches by considering individuals which are partially aware and thus, limited in their ability to formulate correct beliefs. We model the decisions of such agents as resulting from rational optimization of axiomatized preferences, which however, takes into account the agent’s limited understanding of the economic environment. We show that such decisions can be related to the use of heuristics. Moreover, we demonstrate that decisions which mimic the use of heuristics can indeed be ecologically rational,

---

<sup>10</sup>See also the discussion in Gilboa, Postlewaite and Schmeidler (2004).

in that they allow partially aware agents to survive in environments, in which fully aware agents with wrong beliefs would vanish.

## 6.2 Examples of Heuristics in Financial Markets

What kind of heuristics might be applied to trade in financial markets? Consider a partially aware agent, who understands that they are not fully aware of all relevant contingencies, and who can follow the reasoning that suggests that a strategy of betting on incorrect beliefs will almost surely lead to the loss of all wealth.

Such an agent might adopt two types of heuristics. The first is directly related to his level of understanding of the relevant state space. It amounts to effectively restricting the set of available assets to those the payoff structure of which the agent understands and which do not expose him to "surprises", i.e., differences of consumption levels, to which he cannot assign numerical probabilities. This heuristic will have two effects on the agent's portfolio. First, the agent will be more willing to buy insurance across events he is only partially aware of (i.e., he is less willing to expose himself to risks he does not understand). With respect to such events, his portfolio will be safer than that of a fully aware agent. Second, the agent's portfolio will appear underdiversified across events he understands and he might be exposed to some idiosyncratic risk.

Both of these heuristics are commonly observed in financial markets: e.g., Goetzman and Kumar<sup>11</sup> (2008) find that US investors hold on average 4-7 assets in their portfolios on average. They provide evidence of insufficient diversification and furthermore show that it correlates with individuals' experience in financial markets, education and sophistication, rather than with the cost associated with holding more diversified portfolios.

While Goetzman and Kumar document that "better diversified investors are likely to have better stock selection abilities, perhaps due to their higher levels of financial sophistication", they also write: "Most surprisingly, we find that high-turnover, under-diversified portfolios perform better than high-turnover, better-diversified portfolios", thus implying that there might be gains to underdiversification at least in some cases. Further support to this type of heuristics is provided by Borges et al. (1999), who show that the recognition heuristic, which consists in holding a portfolio consisting of the most-commonly recognized stocks can significantly outperform both the market and at times, even major managed funds.

As we showed in Section 4, the behavior implied by this heuristic is considered optimal by partially aware agents who are aware of their unawareness and whose preferences are given by (2). As long as this agent's beliefs on his awareness partition are correct, he survives.

The second type of heuristics is related to the desire to avoid certain undesirable outcomes. An agent might, e.g., adopt a heuristic of allocating a

---

<sup>11</sup>See also the references therein.

proportion of initial wealth to bonds, yielding sufficient income to sustain a minimum acceptable level of consumption in all periods.

A number of observations may be made about this heuristic. First, although a riskless unit bond is equivalent to a collection of unit Arrow securities, one for each state of nature, the bond investor need not be aware of all possible states, or, indeed of the existence of more than one state of nature.

Second, the position where the agent has exactly the wealth required to satisfy the heuristic constraint is, in the language of Markov processes, an absorbing state. Having reached this position, the agent undertakes no trade except to roll over the bond and consume the interest. In particular, therefore, she cannot vanish.

Third, consider the equilibrium in which all investors follow a heuristic of this kind (we can allow the trivial case where the required level of bond holding is set at zero). The arguments of Blume and Easley (2006) suggest that all but the most informed trader(s) will be driven to the absorbing state where they hold only bonds. Thus, the relative prices of asset claims will be determined by

- (i) the beliefs of the most informed trader(s);
- (ii) the demand for bonds from less informed traders.

In this context, the price of bonds will be higher and the price of the market portfolio lower than can be explained by risk aversion in a model with fully rational traders. Hence, we will observe the risk-free rate puzzle and equity premium puzzle, as documented by Mehra and Prescott (1985) and Weil (1989).

As we showed in Section 5, agents who are averse to unforeseen unfavorable surprises behave as if they follow this heuristic. This heuristic can be understood as a precautionary principle: given his limited understanding of the uncertainty in the economy, the agent attempts to insure himself against unforeseen bad events. Our results demonstrate that in the context of financial markets, the precautionary principle implies survival even when agents have wrong beliefs.

Other heuristics may also be considered, such as investment in an index fund. Provided such heuristics generate an absorbing state or set of states at a wealth level below initial wealth, they ensure that agents who adopt them will not vanish. We are thus led to a set of market equilibria in which a small number of active investors trade their beliefs, while the remainder of the population follows a heuristic investment strategy independent of beliefs.

As explained above, such heuristics have to be evaluated in regard to their ecological rationality, i.e., their performance relative to other rules. In this paper, we are interested in the issue of survival in financial markets. We thus define:

**Definition 23** *A heuristic is ecologically rational for an agent, if the agent survives when using the heuristic and vanishes, otherwise.*

Combining this definition with the results of Propositions 18 and 22, we obtain the following:

**Corollary 24** *The heuristic "aversion to unforeseen unfavorable surprises" is*

*ecologically rational for any partially aware agent whose beliefs are further away from the truth than those of a fully aware agent.*

Indeed, under the conditions of Proposition 22, the partially aware agent would survive when applying the heuristic, whereas according to Proposition 18, he would vanish.

## 7 Conclusion

In this paper, we considered the impact of partial awareness in the form of a restriction of the state space. Using the models of Gul and Pesendorfer (2014), Grant and Quiggin (2015) and Koopmans (1960), we axiomatized two representations of intertemporal preferences subject to surprises. While the first representation expresses the decision maker's desire to smooth his consumption over contingencies he does not understand, the second representation includes aversion to unforeseen unfavorable surprises, i.e., aversion to outcomes which fall below a certain prespecified level of consumption, regardless of the event on which they occur.

We then used these representations to model agents' equilibrium consumption and to study survival in financial markets. We showed that when a decision maker is aware of his unawareness, his perception of the state space can be represented by a partition. Such a decision maker behaves as if he restricts his portfolio choice to a subset of the available assets with payoffs measurable w.r.t. his awareness partition and thus avoids "surprises". Such a heuristic allows the partially aware agent to survive, but only if his beliefs on his awareness partition are at least as close to the truth as the beliefs of a more aware agent with a finer partition.

The standard literature on survival in financial markets tends to assume heterogeneity in beliefs without discussing the foundations for such heterogeneity. As we show, formulating correct beliefs is a non-trivial task for a partially aware person. Hence, our analysis illustrates how incorrect beliefs naturally arise in situations, in which agents have limited, but distinct ability to describe the uncertainty they face, and are also conscious of their limitations. Furthermore, since correct beliefs turn out to be a necessary condition for the survival of a partially aware agent, the message of Section 4 is that such agents will tend to vanish relative to fully aware agents with correct beliefs.

Introducing aversion to unforeseen unfavorable surprises, allows us to establish survival of partially aware agents, regardless of whether their beliefs are correct. Agents who conform to this decision rule hold a minimal number of bonds in his portfolio. Asset prices in such an economy will reflect the potentially incorrect beliefs of the fully aware agents and replicate the well-known equity premium puzzle. Such results are in contrast with those obtained in markets with fully aware agents. In such markets, survival is fully determined by agents' beliefs (and discount factors) and agents with wrong beliefs vanish a.s.

While the preference representation we use has axiomatic foundations, the behavior modelled by such preferences mimics the precautionary principle discussed in the literature. Furthermore, as we show, this heuristic turns out to be "ecologically rational" in that it permits partially aware agents to survive without formulating correct beliefs in environments, where having correct beliefs is a necessary condition for survival.

In this sense, our paper can be viewed as a bridge between two strands of the literature: the literature on behavioral finance, which explains empirical evidence from financial markets by assuming certain biases and heuristics, but without providing an explanation as to how such heuristics emerge or can survive over time; and the literature on survival in financial markets, which assumes that agents are rational, but differ in terms of beliefs, and strives to uncover which beliefs can affect prices in the long-run.

## 8 Appendix

### Proof of Proposition 6:

The proof of Theorem 1 in Grant and Quiggin (2015) implies the existence of a representation given by:

$$U_i(\mathbf{c}_t) = \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i(\min C^i(\omega_t^i); \max C^i(\omega_t^i)).$$

When  $C^i(\omega_t^i)$  is a singleton,  $u_i$  reduces to the standard von-Neumann-Morgenstern utility function. In particular,  $u_i$  on the set of singletons is identified by the Savage axioms on acts measurable w.r.t.  $\mathcal{F}^i$ . Given  $u_i$  restricted to singletons,  $u_i$  is extended to intervals as follows. Given a diffuse event  $D$ , consider the certainty equivalent  $\mathbf{c}''(\sigma_t) = x \in [\tilde{l}; \tilde{m}]$  for all  $\sigma_t \in \Omega$  of a bet on  $D$ ,  $\mathbf{c}_D \mathbf{c}'$  with  $\mathbf{c}(\sigma_t) = a$  for all  $\sigma_t$  and  $\mathbf{c}'(\sigma_t) = b$  for all  $\sigma_t$ ,  $\tilde{m} \geq a > b \geq \tilde{l}$ .

$$\mathbf{c}'' \sim \mathbf{c}_D \mathbf{c}'.$$

Then,

$$u_i(b; a) = u(x).$$

By part (iv) of Axiom 3, we have that

$$u_i(b; a) = u(b)$$

for any  $b < a$  and hence,  $u_i(\min C^i(\omega_t^i); \max C^i(\omega_t^i)) = u_i(\min C^i(\omega_t^i))$ . ■

### Proof of Proposition 7:

As noted above, Axiom 2 implies Koopmans' Postulates 3 and 3'. Monotonicity implies Postulates 2 and 5. Postulate 1 is implied by a combination of Axiom 2.4 and Uniform Continuity. The proof in Koopmans (1960) shows that these conditions imply the desired result. ■

### Proof of Proposition 8:

Combining Propositions 6 and 7, and noting as above that Koopmans' Monotonicity is implied by Axiom 2 in Gul and Pesendorfer (2014) and our Axiom 2, we obtain the representation:

$$V_i(c^i) = v_i(c^i(\sigma_0)) + \sum_{t=1}^{\infty} \sum_{\omega_t^i \in \Omega_t^i} \beta_i^t v_i(u_i^{-1}(\pi^i(\omega_t^i) u_i(\min\{C^i(\omega_t^i)\})))$$

It remains thus to show that the function  $v_i$  can be chosen to coincide with  $u_i$ . To do so, normalize both  $u_i$  and  $v_i$  so that  $u_i(\tilde{l}) = v_i(\tilde{l}) = 0$  and  $u_i(\tilde{m}) = v_i(\tilde{m}) = 1$ .

Next note that

$$\left( \underbrace{\tilde{l} \dots \tilde{l}}_{t-1}; \underbrace{\tilde{m}_E \tilde{l}}_t; \tilde{l} \dots \tilde{l} \dots \right) \sim \left( \tilde{m}_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right)$$

implies

$$\pi^i(E) = \sum_k \beta_i^{\tau_k}$$

and hence, Axiom 5 requires that for any  $\delta = \sum_k \beta_i^{\tau_k}$  for some sequence  $(\tau_k)_k$ , and any  $z \in [\tilde{l}; \tilde{m}]$ ,

$$u_i^{-1}[\delta u_i(z)] = v_i^{-1}[\delta v_i(z)].$$

Let then  $y = u_i^{-1}[\delta u_i(z)] = v_i^{-1}[\delta v_i(z)]$ . We then have

$$\frac{u_i(y)}{u_i(z)} = \frac{v_i(y)}{v_i(z)} = \delta$$

for all  $z$  and all  $y$  as above. This, in turn implies that for  $z = 1$ ,  $y_0 = u_i^{-1}(\delta) = v_i^{-1}(\delta)$ ,  $u_i(y_0) = v_i(y_0)$ . Setting  $y_1 = u_i^{-1}(\delta y_0) = v_i^{-1}(\delta y_0)$  gives  $u_i(y_2) = v_i(y_2)$  and proceeding by induction, we obtain the equality of the functions  $u_i$  and  $v_i$  for all  $z$  in the set  $Z$  defined recursively as follows

$$\begin{aligned} Z_0 &= \left\{ z \in [\tilde{l}; \tilde{m}] \mid \text{there is a sequence } (\tau_k)_k \text{ such that } \left( z; \tilde{l} \dots \tilde{l} \dots \right) \sim \left( \tilde{m}_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right) \right\} \\ Z_1 &= \left\{ z \in [\tilde{l}; \tilde{m}] \mid \text{there is a sequence } (\tau_k)_k \text{ and } z' \in Z_0 \text{ such that } \left( z; \tilde{l} \dots \tilde{l} \dots \right) \sim \left( z'_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right) \right\} \\ &\dots \\ Z_K &= \left\{ z \in [\tilde{l}; \tilde{m}] \mid \text{there is a sequence } (\tau_k)_k \text{ and } z' \in Z_{K-1} \text{ such that } \left( z; \tilde{l} \dots \tilde{l} \dots \right) \sim \left( z'_{(\tau_k)_k}; \tilde{l}_{-(\tau_k)_k} \right) \right\} \\ &\dots \\ Z &= \bigcup_{\kappa=0}^{\infty} Z_{\kappa} \end{aligned}$$

Note, furthermore, that differently from  $u_i$ ,  $v_i$  is only unique up to an affine-linear transformation on  $Z$  and we have shown  $u_i(z) = v_i(z)$  for all  $z \in Z$ .

The continuity of the functions  $u_i$  and  $v_i$  ensures that  $Z$  is closed. Consider an outcome  $y \notin Z$ . It follows that  $y \in (\underline{y}; \bar{y}) \subset [\tilde{l}; \tilde{m}]$  such that  $u_i(z) = v_i(z)$  for

$z \in \{\underline{y}; \bar{y}\}$ . Furthermore, any monotonic uniformly continuous transformation of  $v_i$  on  $(\underline{y}; \bar{y})$ , including  $u_i$ , represents the same preferences. Hence, we can set  $v_i(y) = u_i(y)$  for all  $y \in (\underline{y}; \bar{y})$ .

This gives us the desired representation. ■

To simplify the proofs of the following results, we state and prove the following:

**Lemma 25** *Consider two agents  $i$  and  $j$  such that  $j$  is fully aware and  $i$  is partially aware. In equilibrium, for any path  $\omega^i$ , if  $i$ 's consumption is measurable w.r.t.  $\Omega^i$ ,*

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\left( \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i) \right)} \\ &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \left( \sum_{w^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right). \end{aligned}$$

**Proof of Lemma 25:**

We will use the analogue of the Blume and Easley (2006) decomposition. The f.o.c.'s of two agents,  $i$ , who is partially aware and  $j$ , who is fully aware give:

$$\frac{u'_i(c^i(\sigma_0))}{\beta_i u'_i(c^i(\omega_{T+1}^i)) \pi^i(\omega_{T+1}^i)} = \frac{u'_j(c^j(\sigma_0))}{\beta_j \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1})}.$$

Hence,

$$\frac{u'_i(c^i(\sigma_0))}{\beta_i u'_i(c^i(\omega_{T+1}^i)) \pi^i(\omega_{T+1}^i)} = \frac{u'_j(c^j(\sigma_0))}{\beta_j \pi^j(\omega_{T+1}^i) \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)}, \quad (9)$$

which reduces to:

$$\begin{aligned} \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)} &= \frac{\beta_j \pi^j(\omega_{T+1}^i) u'_i(c^i(\sigma_0))}{\beta_i \pi^i(\omega_{T+1}^i) u'_j(c^j(\sigma_0))} \\ &= \frac{u'_i(c^i(\sigma_0))}{u'_j(c^j(\sigma_0))} \prod_{t=1}^{T+1} \frac{\beta_j \pi^j(w_t^i)}{\beta_i \pi^i(w_t^i)} \end{aligned}$$

and we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)} &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} \ln \frac{\pi^j(w_t^i)}{\pi^i(w_t^i)} \\ &\quad + \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\sigma_0))}{u'_j(c^j(\sigma_0))}. \end{aligned}$$



Since  $u'_i(c^i(\sigma_0))$  and  $u'_j(c^j(\sigma_0))$  are finite, the third term on the r.h.s. converges to 0, furthermore,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)} \\ &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} (\ln \pi^j(w_t^i) - \ln \pi(w_t^i)) + \\ & \quad + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} (\ln \pi(w_t^i) - \ln \pi^i(w_t^i)). \end{aligned}$$

Since  $\ln \frac{\pi^j(w_t^i)}{\pi(w_t^i)}$  and  $\ln \frac{\pi(w_t^i)}{\pi^i(w_t^i)}$  are i.i.d and are equal in expectations to the relative entropy of  $i$ 's and  $j$ 's beliefs with respect to the truth, we obtain that a.s.,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^i} u'_j(c^j(\tilde{\sigma}_{T+1})) \pi^j(\tilde{\sigma}_{T+1} | \omega_{T+1}^i)} \\ &= \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \left( \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right). \end{aligned}$$

**Proof of Proposition 16:**

Note that when the initial endowment of the economy is measurable w.r.t.  $\Omega^i = \Omega^j$ ,  $i$  and  $j$ 's consumption in equilibrium will be measurable w.r.t.  $\Omega^i$ . Indeed, suppose that there is a node  $\omega_t^i$  such that  $c^i(\sigma_t) < c^i(\sigma'_t)$  for some  $\sigma_t, \sigma'_t \in \omega_t^i$ . It then follows that  $p(\sigma'_t) = 0$ . Indeed, otherwise  $i$  can increase his overall utility by selling  $c^i(\sigma'_t) - c^i(\sigma_t)$  units at  $\sigma'_t$  and using the strictly positive revenue to increase strictly his consumption (and hence, his utility) at a set of nodes  $\omega_t^i$ .

If in the economy, there is a consumer  $k$  for whom  $\sigma_t$  and  $\sigma'_t$

Since the endowment of the economy is measurable w.r.t.  $\Omega^i$ , it further follows that there is at least one consumer  $k$  for whom  $c^k(\sigma_t) > c^k(\sigma'_t)$ . Note that this would be suboptimal for  $k$ , unless,  $c^k(\sigma''_t) \leq c^k(\sigma'_t) < c^k(\sigma_t)$  for some  $\sigma''_t \in \omega_t^k, \sigma''_t \neq \sigma'_t$ . Then  $p(\sigma_t) = 0$ , as above. And hence, if  $i$ 's consumption is optimal, then there is a  $\sigma'''_t$  such that

$$c^i(\sigma'''_t) < c^i(\sigma_t) < c^i(\sigma'_t)$$

and a  $k'$  such that

$$c^{k'}(\sigma'''_t) > c^{k'}(\sigma_t)$$

which, in turn, can only be optimal  $p(\sigma'''_t) = 0$ . Proceeding by induction, we conclude that if the aggregate endowment is measurable w.r.t.  $\Omega^i$ , then  $c^i(\sigma_t) < c^i(\sigma'_t)$  for some  $\sigma_t, \sigma'_t \in \omega_t^i$  can only obtain iff  $p(\sigma_t) = 0$  for all  $\sigma_t \in \omega_t^i$ . Recall, however, that  $i$  places a strictly positive probability on  $\omega_t^i$

and his utility is strictly increasing in  $\min C^i(\omega_t^i)$ . Hence, if  $p(\sigma_t) = 0$  for all  $\sigma_t \in \omega_t^i$ ,  $i$  will demand an infinite amount of consumption at each  $\sigma_t \in \omega_t^i$ , which is inconsistent with an equilibrium.

Since the same argument applies to  $j$ , we can use Lemma 25 and replacing the original state space  $\Omega$  by the common partition,  $\Omega^i$ , we obtain that for two such agents,  $i$  and  $j$ , with  $\beta^i > \beta^j$ :

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}))}{u'_j(c^j(\omega_{T+1}))} = \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} < 0$$

and hence  $u'_j(c^j(\omega_{T+1})) \rightarrow \infty$ , or  $c^j(\omega_{T+1}) \rightarrow 0$ .

When beliefs differ,

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}))}{u'_j(c^j(\omega_{T+1}))} = \lim_{T \rightarrow \infty} \ln \frac{\beta_j}{\beta_i} + \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=1}^{T+1} \ln \frac{\pi^j(w_t)}{\pi^i(w_t)}$$

and both beliefs and the actual distribution over states of the world are i.i.d.,

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}))}{u'_j(c^j(\omega_{T+1}))} = \ln \frac{\beta_j}{\beta_i} + \left( \sum_{w \in W^i} \pi(w) \ln \frac{\pi(w)}{\pi^i(w)} - \sum_{w \in W^i} \pi(w) \ln \frac{\pi(w)}{\pi^j(w)} \right).$$

Hence, for equal discount factors, the agent whose beliefs with respect to the common partition are closer to the truth survives, while the other vanishes. When both discount factors and beliefs differ, we conclude that lower discount factors can be offset by having beliefs closer to the truth and vice versa. In particular, if

$$\ln \frac{\beta_j}{\beta_i} + \left( \sum_w \pi(w) \ln \frac{\pi(w)}{\pi^i(w)} - \sum_w \pi(w) \ln \frac{\pi(w)}{\pi^j(w)} \right) < 0,$$

$u'_j(c^j(\omega_{T+1})) \rightarrow \infty$ , or  $c^j(\omega_{T+1}) \rightarrow 0$  and  $j$  vanishes.

Finally noting that for agents who form awareness-biased beliefs,

$$\sum_{w \in W} \pi(w) \ln \frac{\pi(w)}{\pi^i(w)} = \sum_{w \in W} \pi(w) \ln \frac{\pi(w)}{\pi(w \cap S^i | S^i)}$$

completes the proof.

**Proof of Proposition 17:**

First note that w.l.o.g., we can identify the finest partition of an agent in the economy with the state space  $\Omega$ . Order the agents from 1 to  $n$  w.r.t. the coarseness of their partitions with 1 being the agent with the finest partition. Then, we can set  $\Omega^1 = \Omega$ . Since the total endowment of the economy is measurable w.r.t. this partition, so will be the consumption of agent 1. Furthermore, since agent 1 assigns strictly positive probability to each node  $\sigma_t \in \Omega$ ,  $p(\sigma_t) > 0$  will hold for all  $\sigma_t$ . Finally, since all partitions  $\Omega^2 \dots \Omega^n$  are coarser than that of 1, we conclude that each agent's equilibrium consumption will be measurable w.r.t. his own partition, see the proof of Proposition 16.

**Lemma 26** *The agent with the finest partition, agent 1, cannot be the only one to survive on a path  $\sigma \in \Omega$ .*

**Proof of Lemma 26:**

Take a path  $\sigma_t$  and assume that on this path, only agent 1 survives, whereas all agents with coarser partitions,  $i > 1$  vanish. Measurability of consumption implies that any agent  $i > 1$  also vanishes on the set of paths  $\omega^i$  such that  $\sigma \in \omega^i$ , whereas agent 1 survives on the set of paths  $\omega^2$  such that  $\sigma \in \omega^2$  for the agent with the second finest partition. Then, for agents 1 and 2, (9) implies:

$$\frac{u'_2(c^2(\omega_{T+1}^2))}{\sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^2} u'_1(c^1(\tilde{\sigma}_{T+1})) \pi^1(\tilde{\sigma}_{T+1} | \omega_{T+1}^2)} = \frac{u'_2(c^2(\sigma_0)) \beta_1 \pi^1(\omega_{T+1}^2)}{u'_1(c^1(\sigma_0)) \beta_2 \pi^2(\omega_{T+1}^2)}$$

Note that the first ratio on the r.h.s. is finite (marginal utilities in period 0). Since  $\beta_1 = \beta_2$ , the second ratio is 1. And since both 1 and 2 have identical beliefs on  $\omega^2$ , the third ratio is also 1. Hence, the expression on the r.h.s. remains bounded, whereas the fact that 2 vanishes on  $\omega^2$ , whereas 1 survives on all  $\sigma \in \omega^2$ , imply that the denominator on the l.h.s. remains bounded, whereas the numerator converges to infinity, a contradiction to the equality.

It follows that at least one agent,  $i > 1$  has to survive on  $\sigma$ , and thus on  $\omega^i$  such that  $\sigma \in \omega^i$ .

**Lemma 27** *For any  $\omega_t^n \in \Omega^n$*

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n) \leq \max_{i \in \{1 \dots n\}} u'_i(m) \frac{u'_1(c^1(\sigma_0))}{u'_i(c^i(\sigma_0))} \quad (10)$$

**Proof of Lemma 27:**

Take a path  $\sigma \in \Omega$  and note that for every  $\sigma_t$ , there is an  $i(\sigma_t) \in I$  such that  $c_i(\sigma_t) \geq m$ . It follows that for any  $\sigma_t \in \Omega$  and the corresponding  $i(\sigma_t)$  and  $\omega^{i(\sigma_t)}$  such that  $\sigma \in \omega^{i(\sigma_t)}$ , we have:

$$\frac{\sum_{\tilde{\sigma}_t \subseteq \omega_t^{i(\sigma_t)}} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^{i(\sigma_t)})}{u'_{i(\sigma_t)}(c^{i(\sigma_t)}(\omega_t^{i(\sigma_t)}))} = \frac{u'_1(c^1(\sigma_0))}{u'_{i(\sigma_t)}(c^{i(\sigma_t)}(\sigma_0))} \frac{\beta_{i(\sigma_t)} \pi^{i(\sigma_t)}(\omega_t^{i(\sigma_t)})}{\beta_1 \pi^1(\omega_t^{i(\sigma_t)})}$$

and since all agents have identical discount factors and beliefs,

$$\sum_{\tilde{\sigma}_t \subseteq \omega_t^{i(\sigma_t)}} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^{i(\sigma_t)}) \leq u'_{i(\sigma_t)}(m) \frac{u'_1(c^1(\sigma_0))}{u'_{i(\sigma_t)}(c^{i(\sigma_t)}(\sigma_0))}$$

It follows that for any  $\sigma_t$  and  $\omega_t^{i(\sigma_t)} \in \Omega^{i(\sigma_t)}$ ,

$$\sum_{\tilde{\sigma}_t \subseteq \omega_t^{i(\sigma_t)}} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^{i(\sigma_t)}) \leq \max_{i \in \{1 \dots n\}} u'_i(m) \frac{u'_1(c^1(\sigma_0))}{u'_i(c^i(\sigma_0))} := P$$

Consider  $\omega_t^n \in \Omega^n$  and note that we can partition  $\omega_t^n$  into at most  $n - 1$  subsets  $x_k(\omega_t^n) \subseteq \Omega_t$  such that on  $\sigma_t \in x_k(\omega_t^n)$  agent  $k$  is the agent with the coarsest partition and consumption  $c^k(\sigma_t) \geq m$ . Note furthermore that  $x_k(\omega_t^n) \subseteq \Omega_t^k$ . It follows that for every  $\omega_t^k \in x_k(\omega_t^n)$ ,

$$\sum_{\tilde{\sigma}_t \subseteq \omega_t^k} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^k) \leq P$$

and thus,

$$\sum_{\omega_t^k \in x_k(\omega_t^n)} \pi^1(\omega_t^k | x_k(\omega_t^n)) \sum_{\tilde{\sigma}_t \subseteq \omega_t^k} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^k) \leq P$$

Finally, since the number of the elements of the partition  $(x_k(\omega_t^n))_{k \in \{1 \dots n\}}$  is finite, we have:

$$\begin{aligned} & \sum_{k=1}^n \pi^1(x_k(\omega_t^n) | \omega_t^n) \sum_{\tilde{\sigma}_t \subseteq x_k(\omega_t^n)} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | x_k(\omega_t^n)) \\ &= \sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n) \leq P \end{aligned}$$

for any  $\omega_t^n \in \Omega^n$ , and thus,

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n) \leq P$$

proving the statement of the Lemma.

**Lemma 28** *On any  $\omega^n \in \Omega^n$ , agent  $n$  survives. Furthermore, the consumption of agent  $n$  on each path is uniformly bounded below.*

**Proof of Lemma 28:**

Consider agents 1 and  $n$ . We have that:

$$\frac{\sum_{\tilde{\sigma}_t \subseteq \omega_t^n} u'_1(c^1(\tilde{\sigma}_t)) \pi^1(\tilde{\sigma}_t | \omega_t^n)}{u'_n(c^n(\omega_t^n))} = \frac{u'_1(c^1(\sigma_0))}{u'_n(c^n(\sigma_0))}$$

and since by (10) the numerator on the l.h.s. remains bounded by  $P$ , we have that

$$\limsup_{t \rightarrow \infty} u'_n(c^n(\omega_t^n)) \leq P \frac{u'_n(c^n(\sigma_0))}{u'_1(c^1(\sigma_0))} \quad (11)$$

and hence,  $n$  survives on any  $\omega_t^n$ . Furthermore, (11) implies

$$\liminf_{t \rightarrow \infty} c^n(\omega_t^n) \geq u_n'^{-1} \left( P \frac{u'_n(c^n(\sigma_0))}{u'_1(c^1(\sigma_0))} \right) > 0.$$

**Lemma 29** For any  $j \in \{2 \dots n - 1\}$  and any  $\omega_t^n \in \Omega^n$

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\omega}_t^j \subseteq \omega_t^n} u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) \pi^j \left( \tilde{\omega}_t^j \mid \omega_t^n \right) \leq P \frac{u'_j \left( c^j \left( \sigma_0 \right) \right)}{u'_1 \left( c^1 \left( \sigma_0 \right) \right)} \quad (12)$$

**Proof of Lemma 29:**

The result follows by combining the fact that since all agents have identical beliefs and discount factors, for any  $j$  and any  $\omega_t^n$

$$\frac{\sum_{\tilde{\omega}_t^j \subseteq \omega_t^n} u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) \pi^j \left( \tilde{\omega}_t^j \mid \omega_t^n \right)}{u'_n \left( c^n \left( \omega_t^n \right) \right)} = \frac{u'_j \left( c^j \left( \sigma_0 \right) \right)}{u'_n \left( c^{i(\sigma_0)} \left( \sigma_0 \right) \right)}$$

and  $u'_n$  is bounded as in (11).

**Lemma 30** Agent  $j \in \{1 \dots n - 1\}$  survives  $\pi$ -a.s. on  $\Omega$ .

**Proof of Lemma 30:**

Suppose in a manner of contradiction that there exists a set  $\tilde{\Omega}^j \subseteq \Omega$  with positive measure w.r.t.  $\pi$ , on which 1 vanishes a.s. Then, on this set we would have:

$$\limsup_{t \rightarrow \infty} u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) = \infty \text{ a.s. on } \tilde{\Omega}^j$$

and hence,

$$\limsup_{t \rightarrow \infty} \sum_{\tilde{\omega}_t^j \in \tilde{\Omega}^j} \pi \left( \tilde{\omega}_t^j \right) u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) = \infty$$

Now let  $\tilde{\Omega}^n$  denote the smallest measurable event on  $\Omega^n$  such that for every  $\sigma \in \tilde{\Omega}$ , there is an  $\omega^n \in \tilde{\Omega}^n$  with  $\sigma \in \omega^n$ . We then have that if 1 were to vanish a.s. on  $\tilde{\Omega}$ , then

$$\limsup_{t \rightarrow \infty} \sum_{\omega_t^n \in \tilde{\Omega}^n} \pi \left( \omega_t^n \right) \sum_{\substack{\tilde{\omega}_t^j \in \omega_t^n \\ \tilde{\omega}_t^j \in \tilde{\Omega}^j}} \pi \left( \tilde{\omega}_t^j \mid \omega_t^n \right) u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) = \infty$$

Since,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \sum_{\omega_t^n \in \tilde{\Omega}^n} \pi \left( \omega_t^n \right) \sum_{\substack{\tilde{\omega}_t^j \in \omega_t^n \\ \tilde{\omega}_t^j \in \tilde{\Omega}^j}} \pi \left( \tilde{\omega}_t^j \mid \omega_t^n \right) u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) = \\ & \leq \limsup_{T \rightarrow \infty} \sum_{\omega_t^n \in \tilde{\Omega}^n} \pi \left( \omega_t^n \right) \sum_{\substack{\tilde{\omega}_t^j \in \omega_t^n \\ \tilde{\omega}_t^j \in \tilde{\Omega}^j}} \pi^j \left( \tilde{\omega}_t^j \mid \omega_t^n \right) u'_j \left( c^j \left( \tilde{\omega}_t^j \right) \right) \end{aligned}$$

where the inequality follows from the definition of  $\tilde{\Omega}^j$  and the fact that  $j$ 's beliefs satisfy

$$\pi^j \left( \tilde{\omega}_t^j \mid \omega_t^n \right) = \pi \left( \tilde{\omega}_t^j \mid \omega_t^n \right)$$

and since (10) and (12) hold, we obtain a contradiction for any  $j \in \{1 \dots n - 1\}$ . It follows that  $j$   $\pi$ -a.s. survives on  $\Omega^j$  and thus, by measurability on  $\Omega$ .

**Proof of Propositions 18:**

Suppose that agent  $i$  has the coarser partition. By the proof of Proposition 17,  $i$ 's consumption will be measurable w.r.t.  $\Omega^i$ . We can then use the proof of Lemma 25 by replacing the original state space by the finer of the two partitions,  $\Omega^j$ . We conclude that

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_i(c^i(\omega_{T+1}^i))}{\left( \sum_{\tilde{\omega}_{T+1}^j \in \omega_{T+1}^i} u'_j(c^j(\tilde{\omega}_{T+1}^j)) \pi^j(\tilde{\omega}_{T+1}^j | \omega_{T+1}^i) \right)} \quad (13) \\ &= \ln \frac{\beta_j}{\beta_i} + \left( \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right). \end{aligned}$$

Since consumption is bounded, and hence,  $u'_j(c^j(\tilde{\omega}_{T+1}^j))$  cannot become 0, and since the sum of the probabilities in the denominator is 1, if

$$\ln \frac{\beta_j}{\beta_i} + \left( \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} - \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right) > 0,$$

$i$  vanishes a.s. Finally applying (5) gives the desired result.

**Proof of Proposition 21:**

Let  $(c; p)$  be an equilibrium of the economy. Suppose that for some  $i \in I$ ,  $\sigma_t, \sigma'_t \in \omega_t^i$ ,  $c^i(\sigma_t) > c^i(\sigma'_t)$ . Obviously, this implies that  $p(\sigma_t) = 0$  and  $\sigma_t, \sigma'_t \in \omega_t^j$  for all agents  $j \in I$ . Then consider the allocation  $\tilde{c}$ , which differs from  $c$  only in that consumption at  $\sigma_t$  is reallocated from agent  $i$  to agent  $\tilde{k}$ , the agent with the finest partition, whose unawareness is relevant in the limit. Formally,

$$\begin{aligned} \tilde{c}^i(\sigma_t) &= \min \{ C^i(\omega_t^i) \} \\ \tilde{c}^{\tilde{k}}(\sigma_t) &= c^{\tilde{k}}(\sigma_t) + c^i(\sigma_t) - \tilde{c}^i(\sigma_t) \\ \tilde{c}^j(\tilde{\sigma}_t) &= c^j(\tilde{\sigma}_t), \end{aligned}$$

otherwise. Obviously,  $(\tilde{c}; p)$  is also an equilibrium of the economy. Hence, for each equilibrium of the economy, we can construct an equilibrium with the same price system, in which the consumption streams of each of the consumers but agent  $k$  are measurable w.r.t. his awareness partition. Furthermore, it is easy to see that the f.o.c. for this equilibrium are exactly the same as those in an economy, in which both the initial endowments and the equilibrium allocations of the consumers are given by:  $e^i = \tilde{c}^i$ ,  $e^{\tilde{k}}(\sigma_t) = \min \tilde{c}^{\tilde{k}}(\omega_t^{\tilde{k}})$  for  $\sigma_t \in \omega_t^{\tilde{k}}$ . We will now show that this type of equilibria satisfy the statement of the Proposition.

**Lemma 31** *Suppose that the unawareness given by the partition  $\Omega^{\tilde{k}}$  is relevant. A.s., at least one of the agents  $1 \dots (\tilde{k} - 1)$  survives.*

**Proof of Lemma 31:**

Assume that a.s., only agents with indices  $k \geq \tilde{k}$  survive. For a given path  $\sigma$ , let  $i \geq \tilde{k}$  be the minimal index of an agent who survives on  $\sigma$ . Let  $\sigma \in \omega^i \in \Omega^i$ . Clearly, if  $i$  survives on  $\sigma$ , he also survives on any  $\sigma' \in \omega^i$ . According to Definition 19, there are  $s$  and  $s' \in w^{\tilde{k}} \subseteq w^i$  and  $\epsilon > 0$  such that for any  $\sigma, \sigma' \in \omega^{\tilde{k}} \subseteq \omega^i$ ,

$$\limsup_{t \rightarrow \infty} [e(\sigma_t; s) - e(\sigma'_t; s')] > \epsilon.$$

Since for every  $k > i$ ,  $k$ 's consumption is measurable with respect to  $\omega^i$  and since the partitions are nested, we have that for every  $t$ , every  $\sigma_t \in \omega_t^i$  and  $s' \in w_{t+1}^i$ ,

$$\sum_{k \geq i} c^k(\omega_{t+1}^i) \leq e(\sigma_t; s'), \quad (14)$$

and hence, for every  $\sigma \in \omega^i$ , on which states  $s$  and  $s'$  occur infinitely often,

$$\begin{aligned} \limsup_{T \rightarrow \infty} \sum_{j < i} c^j(\sigma_T) &= \limsup_{T \rightarrow \infty} \left[ e(\sigma_T) - \sum_{k \geq i} c^k(\sigma_T) \right] \\ &\geq \limsup_{T \rightarrow \infty} [e(\sigma_T) - \min e(\sigma_{T-1}; s')] \\ &\geq \limsup_{T \rightarrow \infty} [e(\sigma_{T-1}; s) - e(\sigma_{T-1}; s')] > \epsilon. \end{aligned}$$

Since  $s$  occurs infinitely often on almost every path  $\sigma$ , and since there is a finite number of agents in the economy, and since the unawareness of all agents with indices larger than  $\tilde{k} - 1$  is relevant in the limit, a.s., at least one of the agents  $j \leq \tilde{k} - 1$  survives on  $\sigma$ .

**Lemma 32** *Suppose that there are no two agents  $i$  and  $j$  such that  $\pi^i = \pi^j$  and  $\beta_i = \beta_j$ . Then, all agents  $i < \tilde{k} - 1$  a.s. vanish, whereas all agents  $i \geq \tilde{k} - 1$  a.s. survive.*

**Proof of Lemma 32:**

First consider agents 1 and 2. Note that since in the limit, the total consumption of the economy is measurable w.r.t. the partition defined by  $\Omega^{\tilde{k}-1}$ , in the limit, agent 1's consumption has to be measurable w.r.t. the partition of agent 2,  $\Omega^2$ , i.e.

$$\lim_{t \rightarrow \infty} |c^1(\sigma_t) - c^1(\sigma'_t)| = 0 \text{ if } \sigma, \sigma' \in \omega^2 \text{ for some } \omega^2 \in \Omega^2.$$

Suppose first that agent 2 survives on a path  $\sigma \in \Omega$  and thus, by measura-

bility of consumption on  $\omega^2 \in \Omega^2$  such that  $\sigma \in \omega^2$ . It follows that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_2(c^2(\omega_{T+1}^2))}{\left( \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^2} u'_1(c^1(\tilde{\sigma}_{T+1})) \pi^k(\tilde{\sigma}_{T+1} | \omega_{T+1}^2) \right)} = \\
&= \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_2(c^2(\omega_{T+1}^2))}{u'_1(c^1(\omega_{T+1}^2))} = \\
&= \ln \frac{\beta_1}{\beta_2} + \left( \sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^2(w^2)} - \sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^1(w^2)} \right) < 0.
\end{aligned}$$

Since, by assumption, agent 2 survives on  $\omega^2$ , it follows that the numerator in the  $\ln$  term is bounded. If the denominator were similarly bounded, the expression on the second line would converge to 0, in contradiction to the fact that the third line is strictly negative. We thus have to conclude that  $\lim_{T \rightarrow \infty} \sup u'_1(c^1(\omega_{T+1}^2)) = \infty$  and thus,  $\lim_{T \rightarrow \infty} \inf c^1(\omega_{T+1}^2) = 0$ , i.e., 1 vanishes on  $\sigma$  and on  $\omega^2$ .

Next consider the case, in which 2 vanishes on  $\sigma$  and thus on  $\omega^2$ . Then,

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left( \sum_{\tilde{\sigma}_{T+1} \in \omega_{T+1}^2} u'_1(c^1(\tilde{\sigma}_{T+1})) \pi^1(\tilde{\sigma}_{T+1} | \omega_{T+1}^2) \right)}{u'_2(c^2(\omega_{T+1}^2))} = \\
&= \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_1(c^1(\omega_{T+1}^2))}{u'_2(c^2(\omega_{T+1}^2))} = \\
&= \ln \frac{\beta_2}{\beta_1} + \left( \sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^1(w^2)} - \sum_{w^2 \in W^2} \pi(w^2) \ln \frac{\pi(w^2)}{\pi^2(w^2)} \right) > 0.
\end{aligned}$$

By assumption,  $\lim_{T \rightarrow \infty} \inf c^2(\omega_{T+1}^2) = 0$  and thus,  $\lim_{T \rightarrow \infty} \sup u'_2(c^2(\omega_{T+1}^2)) = \infty$ . Hence, if  $u'_1$  were bounded, the term

$$\lim_{T \rightarrow \infty} \ln \frac{u'_1(c^1(\omega_{T+1}^2))}{u'_2(c^2(\omega_{T+1}^2))} = -\infty$$

and thus the second line would be less or equal to 0 in contradiction to the third line being strictly positive.  $\lim_{T \rightarrow \infty} \sup u'_1(c^1(\omega_{T+1}^2)) = \infty$  and thus,  $\lim_{T \rightarrow \infty} \inf c^1(\omega_{T+1}^2) = 0$ , i.e., 1 vanishes on  $\sigma$  and on  $\omega^2$ .

Hence, regardless of whether agent 2 survives or not, agent 1 vanishes a.s.

We now proceed by induction. Suppose that we have shown that all agents  $1 \dots k-1 < \tilde{k}-2$  a.s. vanish. In the discussion that follows, restrict attention to the set of paths with measure 1, on which all agents  $1 \dots k-1$  vanish. Consider agents  $k$  and  $k+1$ . Note that since all agents  $1 \dots k-1$  vanish, and since the unawareness defined by partition  $\Omega^k$  ( $k < \tilde{k}-2$ ) is not relevant in the limit, the consumption of agent  $k$  has to be measurable w.r.t. to the partition of agent  $k+1$ ,  $\Omega^{k+1}$ .

Suppose first that agent  $k+1$  survives on a set of paths  $\omega^k \in \Omega^k$  and thus, by measurability of consumption on  $\omega^{k+1} \in \Omega^{k+1}$  such that  $\omega^k \subseteq \omega^{k+1}$ . It follows



that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))}{\left( \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \right)} = \\
& \left( = \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))}{u'_k(c^k(\omega_{T+1}^{k+1}))} = \right) \\
& = \ln \frac{\beta_k}{\beta_{k+1}} + \left( \sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^{k+1}(w^{k+1})} - \sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^k(w^{k+1})} \right) < 0.
\end{aligned}$$

Since, by assumption, agent  $k+1$  survives on  $\omega^2$ , it follows that the numerator in the  $\ln$  term is bounded. If the denominator were similarly bounded, the expression on the second line would converge to 0, in contradiction to the fact that the second (third) line is strictly negative. We thus have to conclude that

$$\limsup_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) = \infty$$

Next consider the case, in which  $k+1$  vanishes on  $\omega^k$  and thus on  $\omega^{k+1}$ . Then,

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left( \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \right)}{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))} = \\
& \left( = \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{u'_k(c^k(\omega_{T+1}^{k+1}))}{u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1}))} = \right) \\
& = \ln \frac{\beta_{k+1}}{\beta_k} + \left( \sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^k(w^{k+1})} - \sum_{w^{k+1} \in W^{k+1}} \pi(w^{k+1}) \ln \frac{\pi(w^{k+1})}{\pi^{k+1}(w^{k+1})} \right) > 0.
\end{aligned}$$

By assumption,  $\lim_{T \rightarrow \infty} \inf c^{k+1}(\omega_{T+1}^{k+1}) = 0$  and thus,  $\lim_{T \rightarrow \infty} \sup u'_{k+1}(c^{k+1}(\omega_{T+1}^{k+1})) = \infty$ . Hence, if the term

$$\left( \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \right)$$

were bounded, the term

$$\lim_{T \rightarrow \infty} \ln \frac{\sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1})}{u'_2(c^2(\omega_{T+1}^2))} = -\infty$$

and thus the first line would be less or equal to 0 in contradiction to the third line being strictly positive. We thus conclude that

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k(c^k(\tilde{\omega}_{T+1}^k)) \pi^k(\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) = \infty$$

Hence, whether  $k + 1$  survives or vanishes,  $\pi$ -a.s. on  $\omega^{k+1}$ ,

$$\lim_{T \rightarrow \infty} u'_k (c^k (\tilde{\omega}_{T+1}^k)) \pi^k (\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) = \infty$$

Furthermore,  $\pi$ -a.s.,  $\lim_{T \rightarrow \infty} |c^k (\tilde{\omega}_T^k) - c^k (\tilde{\omega}'_T^k)| = 0$  if  $\tilde{\omega}^k$  and  $\tilde{\omega}'^k \in \omega^{k+1}$  for some  $\omega^{k+1} \in \tilde{\Omega}^{k+1}$ .

Consider a set  $\tilde{\Omega}^{k+1}$  with a strictly positive measure w.r.t.  $\pi$  and note that:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} \pi (\omega_{T+1}^{k+1}) \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k (c^k (\tilde{\omega}_{T+1}^k)) \pi^k (\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \\ & \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} u'_k (c^k (\tilde{\omega}_{T+1}^k)) \pi (\tilde{\omega}_{T+1}^k) \frac{\pi^k (\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1})}{\pi (\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1})} \\ & = \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} u'_k (c^k (\tilde{\omega}_{T+1}^{k+1})) \sum_{\tilde{\omega}_{T+1}^k \subseteq \omega_{T+1}^{k+1}} \pi (\tilde{\omega}_{T+1}^{k+1}) \pi^k (\tilde{\omega}_{T+1}^k | \omega_{T+1}^{k+1}) \\ & = \lim_{T \rightarrow \infty} \sum_{\omega_{T+1}^{k+1} \in \tilde{\Omega}^{k+1}} u'_k (c^k (\tilde{\omega}_{T+1}^{k+1})) \pi (\tilde{\omega}_{T+1}^{k+1}) \\ & = \infty \end{aligned}$$

Since  $\tilde{\Omega}^{k+1}$  is an arbitrary set with strictly positive measure, we conclude that  $\pi$ -a.s.

$$\lim_{T \rightarrow \infty} u'_k (c^k (\tilde{\omega}_{T+1}^{k+1})) \rightarrow \infty$$

and hence,  $k$  vanishes.

It follows that if all agents  $1 \dots \tilde{k} - 1$  have strictly ordered survival indices then all agents  $1 \dots \tilde{k} - 2$  a.s. vanish and agent  $\tilde{k} - 1$  a.s. survives.

Note that on any path  $\sigma$ , on which state  $s$  occurs infinitely often, and hence, on any  $\omega^{\tilde{k}-1}$  such that  $\sigma \in \omega^{\tilde{k}-1}$ , the consumption of agent  $\tilde{k} - 1$  satisfies  $\lim_{t \rightarrow \infty} \sup c^{\tilde{k}-1} (\omega_t^{\tilde{k}-1}) > \epsilon$  and hence, his marginal utility on such a path is bounded above:

$$\liminf_{t \rightarrow \infty} u'_{\tilde{k}-1} (c^{\tilde{k}-1} (\omega_t^{\tilde{k}-1})) < u'_{\tilde{k}-1} (\epsilon)$$

$\pi$ -a.s., but also, by Assumption 3,  $\pi^{\tilde{k}-1}$ -a.s.

**Lemma 33** For any  $n > j \geq \tilde{k} - 1$ ,  $\lim_{t \rightarrow \infty} |c^j (\sigma_t) - \bar{c}^j (\sigma_t)| = 0$   $\pi$ -a.s. on  $\Omega$ , where for  $\sigma_t \in \omega_t^j \subseteq \omega_t^{j+1}$ ,

$$\bar{c}^j (\sigma_t) = \min_{\sigma_t \in \omega_t^j} e (\sigma_t) - \min_{\tilde{\omega}_t^j \subseteq \omega_t^{j+1}} e (\tilde{\omega}_t^j)$$

Hence,  $j$  survives a.s. if his awareness is relevant in the limit w.r.t. those of  $j + 1$  and vanishes a.s. if it is irrelevant in the limit w.r.t. those of  $j + 1$ .

**Proof of Lemma 33:**

Consider first agent  $\bar{k} - 1$ .  $\pi$ -a.s.

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left( \sum_{\tilde{\omega}_{T+1}^{\bar{k}-1} \in \omega_{T+1}^j} u'_{\bar{k}-1} \left( c^{\bar{k}-1} \left( \tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left( \tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^{\bar{k}} \right) \right)}{u'_{\bar{k}} \left( c^{\bar{k}} \left( \omega_{T+1}^{\bar{k}} \right) \right)} = \\ & = \ln \frac{\beta_{\bar{k}}}{\beta_{\bar{k}-1}} + \left( \sum_{w^{\bar{k}} \in W^{\bar{k}}} \pi \left( w^{\bar{k}} \right) \ln \frac{\pi \left( w^{\bar{k}} \right)}{\pi^{\bar{k}-1} \left( w^{\bar{k}} \right)} - \sum_{w^{\bar{k}} \in W^{\bar{k}}} \pi \left( w^{\bar{k}} \right) \ln \frac{\pi \left( w^{\bar{k}} \right)}{\pi^{\bar{k}} \left( w^{\bar{k}} \right)} \right) > 0. \end{aligned}$$

and hence,

$$\lim_{t \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{\bar{k}-1} \subseteq \omega_{T+1}^j} u'_{\bar{k}-1} \left( c^{\bar{k}-1} \left( \tilde{\omega}_{T+1}^{\bar{k}-1} \right) \right) \pi^{\bar{k}-1} \left( \tilde{\omega}_{T+1}^{\bar{k}-1} \mid \omega_{T+1}^j \right) = \infty$$

In an economy, in which all agents  $1.. \bar{k} - 2$  vanish, only agent  $\bar{k} - 1$  can consume the part of the initial endowment, which is not measurable relative to  $\bar{k}$ 's (and all subsequent agents') partitions. Hence, in the limit, at each  $\omega^{\bar{k}-1} \subseteq \omega^{\bar{k}}$ , we have:

$$\lim_{t \rightarrow \infty} \inf c_t^{\bar{k}-1} \left( \omega_t^{\bar{k}-1} \right) - \bar{c}_t^{\bar{k}-1} \left( \omega_t^{\bar{k}-1} \right) \geq 0,$$

where

$$\bar{c}_t^{\bar{k}-1} \left( \omega_t^{\bar{k}-1} \right) = e \left( \omega_t^{\bar{k}-1} \right) - \min_{\tilde{\omega}_t^{\bar{k}-1} \subseteq \omega_t^{\bar{k}}} e \left( \tilde{\omega}_t^{\bar{k}-1} \right)$$

For a given  $\omega_t^{\bar{k}}$  denote by

$$\Omega_t^{\bar{k}-1} \left( \omega_t^{\bar{k}} \right) = \arg \min_{\tilde{\omega}_t^{\bar{k}-1} \subseteq \omega_t^{\bar{k}}} e \left( \tilde{\omega}_t^{\bar{k}-1} \right)$$

Similarly, for agent  $j > \bar{k} - 1$ , define

$$\bar{c}_t^j \left( \omega_t^j \right) = \min_{\sigma_t \in \omega_t^j} e \left( \sigma_t \right) - \min_{\tilde{\omega}_t^j \subseteq \omega_t^{j+1}} e \left( \tilde{\omega}_t^j \right)$$

and

$$\Omega_t^j \left( \omega_t^{j+1} \right) = \arg \min_{\tilde{\omega}_t^j \subseteq \omega_t^{j+1}} e \left( \tilde{\omega}_t^j \right)$$

Intuitively,  $\bar{c}^j$  is the minimal consumption that has to be attributed to  $j$  provided that the consumption plans of all consumers  $i < j$  satisfy  $\lim_{t \rightarrow \infty} c^i \left( \sigma_t \right) = \lim_{t \rightarrow \infty} \bar{c}^i \left( \sigma_t \right)$ . Note as well that on any of the sets  $\Omega_t^j \left( \omega_t^{j+1} \right)$ , the consumption of all consumers  $k > j$  is constant across  $\sigma_t \in \Omega_t^j \left( \omega_t^{j+1} \right)$  in the limit and since total endowment is constant in those states, we have that the consumption of  $j$  also has to be constant across  $\sigma_t \in \Omega_t^j \left( \omega_t^{j+1} \right)$  in the limit.

Note that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^j} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \\
&= \lim_{T \rightarrow \infty} \left[ \sum_{\tilde{\omega}_{T+1}^{\tilde{k}-1} \in \Omega_{T+1}^{\tilde{k}-1}(\omega_{T+1}^{\tilde{k}})} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \right. \\
&\quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^{\tilde{k}} \\ \tilde{\omega}_{T+1}^{\tilde{k}-1} \notin \Omega_{T+1}^{\tilde{k}-1}(\omega_{T+1}^{\tilde{k}})}} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \right] \\
&= \lim_{T \rightarrow \infty} \left[ u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}} \right) \right) \pi^{\tilde{k}-1} \left( \Omega_{T+1}^{\tilde{k}-1} \left( \omega_{T+1}^{\tilde{k}} \right) \right) \right. \\
&\quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^{\tilde{k}} \\ \tilde{\omega}_{T+1}^{\tilde{k}-1} \notin \Omega_{T+1}^{\tilde{k}-1}(\omega_{T+1}^{\tilde{k}})}} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \right]
\end{aligned}$$

where we write  $c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}} \right)$  to express the fact that in the limit, on  $\Omega_{T+1}^{\tilde{k}-1} \left( \omega_{T+1}^{\tilde{k}} \right)$ ,  $\tilde{k} - 1$ 's consumption is constant. Note further that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^{\tilde{k}} \\ \tilde{\omega}_{T+1}^{\tilde{k}-1} \notin \Omega_{T+1}^{\tilde{k}-1}(\omega_{T+1}^{\tilde{k}})}} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \\
&\leq \lim_{T \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^{\tilde{k}} \\ \tilde{\omega}_{T+1}^{\tilde{k}-1} \notin \Omega_{T+1}^{\tilde{k}-1}(\omega_{T+1}^{\tilde{k}})}} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \\
&= \lim_{T \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^{\tilde{k}} \\ \tilde{\omega}_{T+1}^{\tilde{k}-1} \notin \Omega_{T+1}^{\tilde{k}-1}(\omega_{T+1}^{\tilde{k}})}} u'_{\tilde{k}-1} \left( e \left( \omega_{T+1}^{\tilde{k}-1} \right) - \min_{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^{\tilde{k}}} e \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^{\tilde{k}} \right) \\
&\leq u'_{\tilde{k}-1}(\varepsilon)
\end{aligned}$$

where the last inequality follows from the fact that  $\tilde{k}$ 's unawareness is relevant in the limit.

Since

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{\tilde{k}-1} \subseteq \omega_{T+1}^j} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \right) \right) \pi^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}-1} \mid \omega_{T+1}^j \right) = \infty,$$

we thus have that:

$$\lim_{T \rightarrow \infty} u'_{\tilde{k}-1} \left( c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}} \right) \right) \pi^{\tilde{k}-1} \left( \Omega_{T+1}^{\tilde{k}-1} \left( \omega_{T+1}^{\tilde{k}} \right) \right) = \infty$$

and hence

$$c^{\tilde{k}-1} \left( \left( \tilde{\omega}_{T+1}^{\tilde{k}} \right) \right) \rightarrow 0.$$

We thus conclude that

$$\lim_{T \rightarrow \infty} \left| c^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}} \right) - \bar{c}^{\tilde{k}-1} \left( \tilde{\omega}_{T+1}^{\tilde{k}} \right) \right| = 0$$

We can thus repeat the argument by induction for any agent with an index greater than  $\tilde{k} - 1$ . Indeed, suppose that for all agents  $j \in \{\tilde{k} - 1 \dots k\}$  we have shown that  $\lim_{T \rightarrow \infty} |c_j(\sigma_t) - \bar{c}_j(\sigma_t)| = 0$ . Hence, in the limit, only agent  $k + 1$  can consume the part of the initial endowment,  $e - \sum_{j=\tilde{k}-1}^k \bar{c}_j$ , which is not measurable relative to  $k + 2$ 's (and all subsequent agents') partitions. Hence,

$$\liminf_{t \rightarrow \infty} c_t^{k+1}(\omega_t^{k+1}) - \bar{c}_t^{k+1}(\omega_t^{k+1}) \geq 0$$

where

$$\begin{aligned} \bar{c}_t^{k+1}(\omega_t^{k+1}) &= \min_{\sigma_t \in \omega_t^{k+1}} e(\sigma_t) - \min_{\tilde{\omega}_t^{k+1} \subseteq \omega_t^{k+2}} e(\tilde{\omega}_t^{k+1}) \\ &= e(\omega_t^{k+1}) - \sum_{j=\tilde{k}-1}^k \bar{c}_j(\omega_t^{k+1}) - \min_{\tilde{\omega}_t^{k+1} \subseteq \omega_t^{k+2}} e(\tilde{\omega}_t^{k+1}) \end{aligned}$$

Since  $k + 1$ 's survival index is strictly smaller than that of  $k + 2$ , we have that a.s.

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{T+1} \ln \frac{\left( \sum_{\tilde{\omega}_{T+1}^{k+1} \in \omega_{T+1}^j} u'_{k+1} \left( c^{k+1} \left( \tilde{\omega}_{T+1}^{k+1} \right) \right) \pi^{k+1} \left( \tilde{\omega}_{T+1}^{k+1} \mid \omega_{T+1}^{k+2} \right) \right)}{u'_{k+2} \left( c^{k+2} \left( \omega_{T+1}^{k+2} \right) \right)} = \\ &= \ln \frac{\beta_{k+2}}{\beta_{k+1}} + \left( \sum_{w^{k+2} \in W^{k+2}} \pi(w^{k+2}) \ln \frac{\pi(w^{k+2})}{\pi^{k+1}(w^{k+2})} - \sum_{w^{k+2} \in W^{k+2}} \pi(w^{k+2}) \ln \frac{\pi(w^{k+2})}{\pi^{k+2}(w^{k+2})} \right) > 0. \end{aligned}$$

and hence,

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{k+1} \in \omega_{T+1}^{k+2}} u'_{k+1} \left( c^{k+1} \left( \tilde{\omega}_{T+1}^{k+1} \right) \right) \pi^{k+1} \left( \tilde{\omega}_{T+1}^{k+1} \mid \omega_{T+1}^{k+2} \right) = \infty.$$

Suppose first that on all  $\omega_{T+1}^{k+2}$ ,  $\lim_{t \rightarrow \infty} \bar{c}_{k+1} = 0$ , i.e.,  $k + 2$ 's financial constraint is irrelevant in the limit w.r.t. that of  $k + 1$ . Hence, in the limit,  $k + 1$ 's consumption will be measurable w.r.t.  $\Omega^{k+2}$  and thus,

$$\lim_{t \rightarrow \infty} u'_{k+1} \left( c^{k+1} \left( \omega_{T+1}^{k+2} \right) \right) = \infty$$

implies  $\lim_{T \rightarrow \infty} c^{k+1}(\omega_{T+1}^{k+2}) = \bar{c}^k(\omega_{T+1}^{k+2}) = 0$  and  $k+1$  a.s. vanishes.

Suppose now that on all  $\omega_{T+1}^{k+2}$ ,  $\lim_{t \rightarrow \infty} \bar{c}_{k+1} > \epsilon$ , i.e.,  $k+2$ 's financial constraint is relevant in the limit w.r.t. that of  $k+1$ . Then,

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2}} u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\
&= \lim_{T \rightarrow \infty} \inf \left[ \sum_{\tilde{\omega}_{T+1}^{k+1} \in \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})} u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \right. \\
&\quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \right] \\
&= \lim_{T \rightarrow \infty} \inf \left[ u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+2})) \pi^{k+1}(\Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})) \right. \\
&\quad \left. + \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \right]
\end{aligned}$$

where we write  $c^{k+1}(\tilde{\omega}_{T+1}^{k+2})$  to express the fact that in the limit, on  $\Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})$ ,  $k+1$ 's consumption is constant. Note further that

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1}(c^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\
&\leq \lim_{t \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1}(\bar{c}^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\
&= \lim_{t \rightarrow \infty} \inf \sum_{\substack{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2} \\ \tilde{\omega}_{T+1}^{k+1} \notin \Omega_{T+1}^{k+1}(\omega_{T+1}^{k+2})}} u'_{k+1} \left( e(\omega_{T+1}^{k+1}) - \sum_{j=\bar{k}-1}^k \bar{c}_j(\omega_{T+1}^{k+1}) - \min_{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2}} e(\tilde{\omega}_{T+1}^{k+1}) \right) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) \\
&\leq u'_{\bar{k}-1}(\epsilon)
\end{aligned}$$

where the last inequality follows from the assumption that  $k+2$ 's unawareness is relevant in the limit w.r.t. that of  $k+1$ .

Since

$$\lim_{T \rightarrow \infty} \sum_{\tilde{\omega}_{T+1}^{k+1} \subseteq \omega_{T+1}^{k+2}} u'_{k+1}(\bar{c}^{k+1}(\tilde{\omega}_{T+1}^{k+1})) \pi^{k+1}(\tilde{\omega}_{T+1}^{k+1} | \omega_{T+1}^{k+2}) = \infty,$$

we thus have that:

$$\lim_{T \rightarrow \infty} u'_{k+1} (c^{k+1} (\tilde{\omega}_{T+1}^{k+2})) \pi^{k+1} (\Omega_{T+1}^{k+1} (\omega_{T+1}^{k+2})) = \infty$$

and hence on  $\Omega_{T+1}^{k+1} (\omega_{T+1}^{k+2})$ ,

$$c^{k+1} (\tilde{\omega}_{T+1}^{k+2}) \rightarrow 0$$

since  $c^{k+1} (\tilde{\omega}_{T+1}^{k+1}) - \bar{c}^{k+1} (\tilde{\omega}_{T+1}^{k+1}) \rightarrow_{T \rightarrow \infty} 0$ , we conclude that:

$$\lim_{T \rightarrow \infty} |c^{k+1} (\sigma_T) - \bar{c}^{k+1} (\sigma_T)| = 0.$$

It follows that  $k+1$  a.s. survives if his unawareness is relevant in the limit w.r.t. those of  $k+2$ .

**Lemma 34** *Agent  $n$  a.s. survives.*

**Proof of Lemma 34:**

We have shown above that for agents  $1 \dots \tilde{k} - 2$ , consumption converges to 0 a.s., whereas for agents  $j \in \{\tilde{k} - 1 \dots n - 1\}$ , consumption converges a.s. to  $\bar{c}^j$ . It follows that for  $\sigma_t \in \omega_t^n$ ,  $\pi$ -a.s.,

$$\lim_{t \rightarrow \infty} c^n (\sigma_t) = e (\sigma_t) - \sum_{j=\tilde{k}-1}^{n-1} \bar{c}^j (\sigma_t) = \min_{\tilde{\sigma}_t \in \omega_t^n} e (\tilde{\sigma}_t) \geq m$$

and hence,  $n$  a.s. survives.

**Proof of Proposition 22:**

Since the initial endowment of every agent specifies a consumption of at least  $m$  at every node  $\sigma_t$ , and since  $L \leq m$ , each agent  $i$  can afford to buy a consumption stream which avoids unforeseen unfavorable surprises, regardless of the price sequence. Since agents exhibit strict aversion to unfavorable surprises, in equilibrium, each partially aware agent will purchase a consumption stream not subject to unfavorable surprises. Hence, partially aware agents will not disappear as long as they are strictly averse to unforeseen unfavorable surprises. Hence, the equilibrium will be identical to that in an economy, in which we restrict partially aware agents to consumption bundles that satisfy  $c^i (\sigma_t) \geq L$  for all  $\sigma_t \in \Omega$ . Since the initial endowment of the economy satisfies this restriction, and since preferences on the so-restricted consumption sets are convex and continuous, all the conditions of Bewley's (1972) theorem are satisfied. Hence, an equilibrium of the economy exists. Furthermore, just as in the proof of Proposition 21, for each equilibrium, we can redistribute the equilibrium allocation and obtain a new equilibrium, in which the partially aware agents hold consumption streams which are measurable relative to their respective partitions  $\otimes^i$ . It is obvious that none of the partially aware agents vanish (since their consumption is bounded below by  $L$  at each  $\sigma_t$ ). It is also obvious that with aggregate uncertainty in the limit, as long as prices are positive, the total demand of the partially aware agents will not sum up to the total demand of the economy and hence, a fully aware consumer will consume the remaining quantity of the consumption good.

## 9 References

Ahn, D., Ergin, H. (2010). "Framing Contingencies", *Econometrica* 78: 655-695.

Bewley, T. (1972). "Existence of equilibria in economies with infinitely many commodities", *Journal of Economic Theory* 4: 514-540.

Blume, L., Easley, D. (2006). "If You Are So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets", *Econometrica* 74: 929-966.

Borges, B., Goldstein, D. G., Ortmann, A., Gigerenzer, G. (1999). "Can Ignorance Beat the Stock Market" in Gigerenzer, G., Todd, P., ABC Research Group (eds.) *Simple Heuristics That Make Us Smart*, Oxford University Press, Oxford, pp. 59-72.

Borovicka, J. (2014). "Survival And Long-Run Dynamics with Heterogeneous Beliefs Under Recursive Preferences", mimeo, New York University.

Chichilnisky, G. (2010). "The Foundations of Statistics with Black Swans", *Mathematical Social Sciences* 54: 184-192.

Condie, S. (2008). "Living with Ambiguity: Prices and Survival When Investors Have Heterogeneous Preferences for Ambiguity", *Economic Theory* 36: 81-108.

Coury, T., Scubba, E. (2012). "Belief Heterogeneity and Survival in Incomplete Markets", *Economic Theory* 49: 37-58.

Da Silva, C. (2011). "On Asymptotic Behavior of Economies with Complete Markets: The Role of Ambiguity Aversion", Instituto Nacional de Matematica Pura e Aplicada, Rio de Janeiro, Brasil, mimeo.

Dietrich, F. (2017). "Savage's Theorem under Changing Awareness", *Journal of Economic Theory*, forthc.

Forster, M. (1999). "How Do Simple Rules "Fit to Reality" in an Imperfect World", *Minds and Machines* 9: 543-564.

Easley, D., Yang, L. (2014). "Loss Aversion, Survival and Asset Prices", mimeo, Cornell University.

Eichberger, J., Guerdjikova, A. (2018). "Do Markets Prove Pessimists Right", *International Economic Review*, forthc.

Gigerenzer, G. (2007). *Gut Feelings: The Intelligence of the Unconscious*, Penguin Books.

Gilboa, I., Postlewaite, A., Schmeidler, D. (2004). "Rationality of Belief Or Why Bayesianism is Neither Necessary Nor Sufficient for Rationality," PIER Working Paper Archive 04-011, Penn Institute for Economic Research, Department of Economics, University of Pennsylvania.

Goetzmann, W. N., Kumar, A. (2008). "Equity Portfolio Diversification", *Review of Finance* 12: 433-463.

Goldstein, D. G., Gigerenzer, G. (1996). "Models of Ecological Rationality: The Recognition Heuristics", *Psychological Review* 109: 75-90.

Grant, S., Quiggin, J. (2013a). "Inductive Reasoning About Unawareness", *Economic Theory* 54: 717-755.



- Grant, S., Quiggin, J., (2013b). "Bounded Awareness, Heuristics and the Precautionary Principle," *Journal of Economic Behavior and Organization* 93: 17-31.
- Grant, S., Quiggin, J. (2015). "A Preference Model for Choice Subject to Surprise", *Theory and Decision* 79: 167-180.
- Guerdjikova, A., Sciubba, E. (2015). "Survival with Ambiguity", *Journal of Economic Theory*.155: 50–94.
- Guerdjikova, A., Quiggin, J. (2016). "Market Selection with Differential Unawareness: The Case of Coarsening", University of Cergy-Pontoise, mimeo.
- Gul, F., Pesendorfer, W. (2014). "Expected Uncertain Utility Theory", *Econometrica* 82: 1-39.
- Halpern, J., Rego, L. (2009). "Reasoning about knowledge of unawareness", *Games and Economic Behavior* 67: 503-25.
- Heifetz, A., Meier, M., Schipper, B. (2006), "Interactive Unawareness", *Journal of Economic Theory* 130: 78-94.
- Karni, E., Vierø, M.-L. (2013) "“Reverse Bayesianism”: A Choice-Based Theory of Growing Awareness", *American Economic Review* 103: 2790—2810.
- Karni, E., Vierø, M.-L. (2014), "Awareness of unawareness: A theory of decision-making in the face of ignorance", Queen’s Economics Department Working Paper No. 1322.
- Kogan, L., Ross, St., Wang, J., Westerfield, M. M. (2006). "The Price Impact and Survival of Irrational Traders", *Journal of Finance* 61: 195-229.
- Kogan, L., Ross, St., Wang, J., Westerfield, M. M. (2011). "Market Selection", AFA 2009 San Francisco Meetings Paper.
- Koopmans, T. C. (1960). "Stationary Ordinal Utility and Impatience", *Econometrica* 28: 287-309.
- Mehra, R., Prescott, E. (1985). "The Equity Premium: A Puzzle", *Journal of Monetary Economics* 10: 335-359.
- Modica, S., Rustichini, A., Tallon, J.-M. (1998). "Unawareness and Bankruptcy: A General Equilibrium Model", *Economic Theory* 12: 259-292.
- Orlik, A., Veldkamp, L. (2015). "Understanding Uncertainty Shocks and the Role of Black Swans", mimeo, NYU Stern School of Business.
- Sandroni, A. (2000). "Do Markets Favor Agents Able to Make Accurate Predictions?", *Econometrica* 68: 1303-1334.
- Schipper, B. (2018). The Unawareness Bibliography, <http://www.econ.ucdavis.edu/faculty/schipper/unaw.l>
- Taleb, N. N. (2007). *The Black Swan: The Impact of the Highly Improbable*, Random House.
- Weil, Ph. (1989). "The Equity Premium Puzzle and the Risk-Free Rate Puzzle", *Journal of Monetary Economics* 24: 410-421.
- Yan, H. (2008). "Natural Selection in Financial Markets: Does It Work?", *Management Science* 54: 1935-1950.