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Transverse generalized metrics and 2d sigma models

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Abstract

We reformulate the compatibility condition between a generalized metric and a small (non-maximal rank) Dirac structure in an exact Courant algebroid found in the context of the gauging of strings and formulated by means of two connections in purely Dirac-geometric terms. The resulting notion, a transverse generalized metric, is also what is needed for the dynamics on the reduced phase space of a string theory.

1. Let E be an exact Courant algebroid over M , characterized by the class $[H] \in H_{dR}^3(M)$ [11,12] and let $V \subset E$ be a *generalized metric*, i.e. a positive definite, rank $n = \dim M$ subbundle of E .¹ These data are equivalent to the choice of a Riemannian metric g and a representative closed 3-form H on M (since there is a unique splitting of $\rho: E \rightarrow TM$ such that V can be written as the graph of a symmetric 2-tensor). They are also the data needed on the target space for the definition of a standard sigma model with Wess-Zumino term. Choose a *small Dirac structure*, i.e. an involutive, isotropic $C^\infty(M)$ -submodule \mathcal{D} of $\Gamma(E)$. In this note we only consider regular \mathcal{D} 's, i.e. those of the form $\mathcal{D} = \Gamma(D)$ for some sub-vector bundle $D \subset E$.

We call a rank n subbundle $W \subset E$ a pre- D -transverse generalized metric if $D \subset W \subset D^\perp$ and $\langle w, w \rangle > 0$ for every $w \in W$ with $w \notin D$. This becomes a D -transverse generalized metric, or simply a *transverse generalized metric*, if in addition the invariance property

$$[\Gamma(D), \Gamma(W)] \subset \Gamma(W) \tag{1}$$

holds true.

2. If D is such that $\rho|_D: D \rightarrow TM$ is injective (in which case we call D *projectable*), then a D -transverse generalized metric is equivalent to a Riemannian metric and a closed 3-form, both on the space of leaves of the resulting foliation $F := \rho(D) \subset TM$. In more detail, we have:

Proposition 1. *Suppose that the leaves of the foliation $F = \rho(D)$ generated by a projectable small Dirac structure D are the fibers of a surjective submersion $\pi: M \rightarrow Q$. If $W \subset E$ is a D -transverse generalized metric, then there is a unique splitting $E \cong (T \oplus T^*)M$ such that the resulting 3-form is of the form π^*H_Q and W is the graph of π^*g_Q , where, respectively, H_Q and g_Q are a closed 3-form and a Riemannian metric on Q .*

Proof. There is a unique splitting identifying E with $(T \oplus T^*)M$ such that W is the graph of a (degenerate) symmetric bilinear form h on TM . Using this splitting, one has $D = F = \ker h$.

The condition $[\Gamma(D), \Gamma(W)] \subset \Gamma(W)$ means

$$[(X, 0), (u, h(u, \cdot))] = ([X, u], (\mathcal{L}_X h)(u, \cdot) + h([X, u], \cdot) + H(X, u, \cdot)) \in \Gamma(W)$$

¹See also, e.g., the first part of Section 3 in [6] for a concise review of the required notions or Sections 2 and 3 of [7] for a review with further details.

for every $X \in \Gamma(F)$ and every vector field u , and thus

$$\mathcal{L}_X h = 0, \quad \iota_X H = 0.$$

Together with $\iota_X h = 0$, $dH = 0$, and the semi-positivity requirement on W , these two equations imply that h and H are the pullback of a Riemannian metric g_Q and a closed 3-form H_Q , respectively. \square

3. If (M, g) is a Riemannian manifold and $\pi: M \rightarrow Q$ a submersion, then π is called a Riemannian submersion iff g descends to a Riemannian metric on Q . This means the following: for any $m \in M$ we decompose $T_m M$ orthogonally to $T_m^{\parallel} M \oplus T_m^{\perp} M$, where $T_m^{\parallel} M := \ker d_m \pi$ is the subspace tangent to the fiber of π . Define h as the unique symmetric bilinear form on $T_m M$ which agrees with g on $T_m^{\perp} M$ and which gives zero when one of the vectors is in $T_m^{\parallel} M$. Then π is a Riemannian submersion iff $h = \pi^* g_Q$ for some Riemannian metric g_Q on Q .

We reformulate this in a language adapted to our purposes: Take V and W to be the graphs of g and h inside the standard Courant algebroid $E_0 := (T \oplus T^*)M$, respectively, and let $D := \{(u, 0) \in TM \oplus T^*M \mid \pi_* u = 0\}$. One verifies easily that then $W = D \oplus (D^{\perp} \cap V)$. As we saw above, the relation $h = \pi^* g_Q$ for some g_Q is equivalent to W being a D -transverse generalized metric.

If one, more generally, has a foliation on a Riemannian manifold (M, g) where there is no good quotient manifold Q but where g descends to the local spaces of leaves (i.e. to a transverse Riemannian metric), one talks of a (regular) Riemannian foliation [10]. This motivates the following definition:

A triple (E, V, D) —of an exact Courant algebroid E , a generalized metric $V \subset E$, and a small Dirac structure $D \subset E$ —is a *Dirac-Riemannian foliation* iff $V_D := D \oplus (D^{\perp} \cap V)$ is a D -transverse generalized metric.

4. In their simplest setting, sigma models are variational problems defined on the space of maps $X: \Sigma \rightarrow M$ from a d -dimensional, Lorentzian signature pseudo-Riemannian manifold (Σ, γ) to a Riemannian manifold (M, g) . In the traditional setting of gauging such a theory, one needs to be given a group G acting isometrically on (M, g) . The procedure is devised in such a way that when the group acts freely and properly on M , then the gauged sigma model is equivalent to the ungauged one with target manifold M/G .

The gauging is achieved by coupling to $\text{Lie}(G)$ -valued 1-forms A on Σ in such a way that two maps X_1 and X_2 which differ from one another only by the application of a Σ -dependent group element are related by a symmetry of the action functional depending on X and A . If the original model is twisted by a closed $d + 1$ -form H , called a Wess-Zumino term, then gauging in this way requires that H has an equivariantly closed extension [4, 5].

While every isometric G -action equips (M, g) with an—in general only singular—Riemannian foliation, where the leaves are given by the G -orbits, not every such a foliation results from a G -action. According to [8, 9]², it is not necessary to restrict gauging to eventual isometries of g , it is sufficient that (M, g) defines (a somewhat controlled form of) a singular Riemannian foliation. While the general theory of such gaugings is not worked out yet in the presence of a Wess-Zumino twist for arbitrary dimensions d , it was done so in [3] for $d = 2$. Two-dimensional sigma models are somewhat particular since the Hodge dual of a 1-form A is again a 1-form, yielding additional options for the gauging, and they are in general intimately related to Dirac geometry, see, e.g., [1, 11].

²See also Section 2 of [3] for a possible definition of such a generalized notion of gauging.

5. Let $V \subset E$ be a generalized metric and $D \subset E$ a small Dirac structure. Let us use the (unique) splitting $E \cong (T \oplus T^*)M$ that turns V into the graph of a Riemann metric. The resulting inclusion $D \rightarrow (T \oplus T^*)M$ then gives us a section $(\rho_D, \alpha_D) \in \Gamma(D^* \otimes (T \oplus T^*)M)$. Let us set $\bar{\rho}_D := \iota_{\rho_D}g \in \Gamma(D^* \otimes T^*M)$.

In [3] it was shown that a two-dimensional sigma model with the above data on the target space can be *gauged* with respect to D , if D can be equipped with two connections $\nabla^\pm := \nabla \pm \phi$, $\phi \in \Omega^1(\text{End}(D))$, such that (M, g, H, D) or (E, V, D) satisfy the following compatibility conditions:

$$\text{Sym}(\bar{\nabla}\bar{\rho}_D - \phi^*(\alpha_D)) = 0, \quad (2)$$

$$\text{Alt}(\bar{\nabla}\alpha_D - \phi^*(\bar{\rho}_D) - \frac{1}{2}\iota_{\rho_D}H) = 0, \quad (3)$$

where $\bar{\nabla}$ is the extension of ∇ to T^*M by means of the Levi-Civita connection of g , Sym and Alt denote the symmetrization and antisymmetrization projections in $T^*M \otimes T^*M$, respectively, and $\phi^* \in \Omega^1(\text{End}(D^*))$ is the 1-form valued map dual to ϕ .

In this case the variational problem can be gauged by extending the fields from maps $X: \Sigma \rightarrow M$ to vector bundle morphisms $a: T\Sigma \rightarrow D$ —thus adding gauge field 1-forms $A \in \Omega^1(\Sigma, X^*D)$. Using the canonical splitting given by the above data, the independent field A gives rise to its projections $A_{TM} \in \Omega^1(\Sigma, X^*TM)$ and $A_{T^*M} \in \Omega^1(\Sigma, X^*T^*M)$ to TM and T^*M , respectively. The gauged variational problem is then described symbolically by [3, 6]

$$S[a] = \int_{\Sigma} \frac{1}{2} \|dX - A_{TM}\|^2 + \langle A_{T^*M} \hat{\wedge} dX - \frac{1}{2}A_{TM} \rangle + \int H, \quad (4)$$

where for every $\nu \in \Omega^1(\Sigma, X^*TM)$ one has $\|\nu\|^2 \equiv (X^*g)(\nu \hat{\wedge} *\nu)$ with $*$ denoting the Hodge dual associated to g —symbolically, since S is not really a functional due to the Wess-Zumino term, while it still defines a unique variational problem for the field a in the standard manner.

Remark 2. While the definition of the variational problem of (4)—its Euler Lagrange equations *and* its gauge equivalence of solutions—does not require the knowledge of connections ∇^\pm satisfying the Equations (2) and (3), the off-shell gauge symmetries of a properly defined (possibly multi-valued) functional (4) do [3]. Thus this applies also to an eventual quantization of S .

Remark 3. In [3] the gauging is described by gauge fields taking values in an almost Lie algebroid L , where L maps into a possibly singular small Dirac structure $\mathcal{D} \subset \Gamma(E)$. Here we displayed the simplified situation with $L = D$ only. While the more general situation is more complicated to describe, it is evidently more flexible and may have advantages even when $\mathcal{D} = \Gamma(D)$: consider, for example, a metric g on a maximally symmetric target manifold M . Then the rank of the isometry Lie algebra \mathfrak{g} is $\frac{n(n+1)}{2} > n$ and choosing for L the corresponding action Lie algebroid, $L = M \times \mathfrak{g}$, provides a simpler description of the gauge theory than the small Dirac structure it maps to.

6. It is one of the main purposes of the current letter to reformulate the conditions (2) and (3) in terms which are intrinsic to Courant algebroids. This is essentially achieved by means of the following statement.

Theorem 4. *A triple (E, V, D) is a Dirac-Riemannian foliation iff there exist connections ∇^\pm such that the compatibility conditions (2) and (3) are satisfied.*

Proof. We use the canonical splitting to identify E with $(T \oplus T^*)M$. We then have

$$V_+ := V = \{(u, \iota_u g) := u_+ \mid u \in TM\}, \quad V_- := V^\perp = \{(u, -\iota_u g) := u_- \mid u \in TM\}$$

where we implicitly defined maps $u \mapsto u_\pm$ from TM to V_\pm . Consider in addition the vector bundle maps $\pi_\pm: D \rightarrow T^*M, (X, \alpha) \mapsto \alpha \pm \iota_X g$. These are isomorphisms between $D_\pm := \pi_\pm(D) \subset T^*M$ and D , since $\iota_X(\alpha \pm \iota_X g) = \pm \|X\|^2$ vanishes only for $X = 0$. We have

$$D^\perp \cap V_\pm = \{u_\pm \mid u \in \text{Ann}(D_\pm)\}.$$

As $V_D^\perp = D \oplus (D^\perp \cap V_-)$ and $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$, Condition (1) for $V_D \equiv D \oplus (D^\perp \cap V_+)$ can be restated as that for every $(X, \alpha) \in \Gamma(D)$:

$$\langle [(X, \alpha), u_+], v_- \rangle = 0 \quad \text{whenever } u_+ \text{ and } v_- \text{ are in } D^\perp, \quad (5)$$

i.e. whenever u is annihilated by D_+ and v by D_- . On the other hand, one computes

$$[(X, \alpha), u_+] = (\mathcal{L}_X u, \iota_{(\mathcal{L}_X u)} g + \iota_u (\mathcal{L}_X g - d\alpha + \iota_X H)).$$

giving

$$\langle [(X, \alpha), u_+], v_- \rangle = (\mathcal{L}_X g + i_X H - d\alpha)(u, v).$$

This shows

Lemma 5. $V_D \equiv D \oplus (D^\perp \cap V)$ is a D -transverse generalized metric, iff one has for all $(X, \alpha) \in \Gamma(D)$

$$\mathcal{L}_X g + i_X H - d\alpha \in \Gamma(D_+ \otimes T^*M + T^*M \otimes D_-). \quad (6)$$

Denote by $e_a := (X_a, \alpha_a)$ a local basis of D and let $\beta_a^\pm = \pi_\pm(e_a)$ be the induced bases in D_\pm . On a local level, Condition (6), and thus V_D to be a transverse generalized metric, is equivalent to the existence of locally defined coefficient 1-forms $(\omega^\pm)_a^b$ such that³

$$\mathcal{L}_{X_a} g + \iota_{X_a} H - d\alpha_a = \beta_b^+ \otimes (\omega^+)_a^b - (\omega^-)_a^b \otimes \beta_b^-.$$

This now is verified to be the local form of the equations (2) and (3), with $(\omega^\pm)_a^b$ being the connection coefficients of ∇^\pm in the chosen basis, $\nabla^\pm e_a = (\omega^\pm)_a^b \otimes e_b$. The global existence of the connections then follows by a standard argument using a partition of unity. \square

Corollary 6. *The sigma model associated to (M, g, H) or (E, V) on a pseudo-Riemannian 2-manifold (Σ, γ) can be gauged along a (possibly singular) foliation $\mathcal{T} \subset \Gamma(TM)$, if there exists a small Dirac structure D covering \mathcal{T} , $\rho(\Gamma(D)) = \mathcal{T}$, which makes (E, V, D) a Dirac-Riemannian foliation.*

7. An exact Courant algebroid $E \rightarrow M$ gives rise to an infinite-dimensional symplectic manifold \mathcal{M} which is the phase space of 2-dimensional sigma models: If a splitting of $E = (T \oplus T^*)M$ is chosen, giving rise to a closed 3-form $H \in \Omega^3(M)$, then we have $\mathcal{M} = T^*(LM)$ with the standard symplectic 2-form modified by the transgression of H . More naturally, \mathcal{M} is the space of all vector

³For the special case $H = dB$ together with $D = \text{graph}(-B)$, i.e. $\alpha = -\iota_X B$ in the description of D above, the left-hand side of the following equation becomes simply $\mathcal{L}_{X_a}(g + B)$, thus reproducing Equation (2.21) in [2].

bundle maps $f: TS^1 \rightarrow E$, covering the base map $f_0: S^1 \rightarrow M$, such that $\rho \circ f: TS \rightarrow TM$ agrees with the tangent map of f_0 .

A generalized metric $V \subset E$ then defines a function \mathcal{H}_V on \mathcal{M} which is the Hamiltonian of the corresponding two-dimensional sigma model:

$$\mathcal{H}_V(f) = \frac{1}{2} \int_{S^1} \langle f(\partial_\sigma), R_V f(\partial_\sigma) \rangle d\sigma,$$

where σ is the coordinate on S^1 and $R_V: E \rightarrow E$ is the reflection with respect to V .

A small Dirac structure $D \subset E$ defines the Lie algebra $\mathfrak{g}_D := C^\infty(S^1) \otimes \Gamma(D)$ together with a Lie algebra map $\mu^*: \mathfrak{g}_D \rightarrow C^\infty(\mathcal{M})$ given by [1]

$$(\mu^*(s))(f) = \int_{S^1} \langle s, f \rangle.$$

The reduced phase space $\mathcal{M} // \mathfrak{g}_D$ is composed of maps $f: TS^1 \rightarrow D^\perp$, i.e. f 's in \mathcal{M} satisfying $(\mu^*(s))(f) = 0$ for all $s \in \mathfrak{g}_D$, modulo the action of \mathfrak{g}_D . A D -transverse generalized metric $W \subset E$ is then precisely what is needed to provide a Hamiltonian on the reduced phase space $\mathcal{M} // \mathfrak{g}_D$. Define

$$\mathcal{H}_W([f]) = \frac{1}{2} \int_{S^1} \langle Pf(\partial_\sigma), R_W Pf(\partial_\sigma) \rangle d\sigma,$$

where P denotes the natural projection from D^\perp to D^\perp/D , and $R_W: D^\perp/D \rightarrow D^\perp/D$ is the orthogonal reflection with respect to W/D which corresponds to the transverse generalized metric W .

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