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# Large deviations and Wschebor's theorems

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## Abstract

We revisit Wschebor's theorems on small increments for processes with scaling and stationary properties and deduce large deviation principles.

**Keywords:** Brownian motion, stable processes, scaling properties, strong theorems, large deviations

**MSC 2010:** 0J65, 60G51, 60F15, 60F10

## 1 Introduction : Wschebor's theorem and beyond

In 1992, Mario Wschebor [22] proved the following remarkable property of the linear Brownian motion ( $W(t), t \geq 0; W(0) = 0$ ). If  $\lambda$  is the Lebesgue measure on  $[0, 1]$ , then, almost surely, for every  $x \in \mathbb{R}$  and every  $t \in [0, 1]$ :

$$(1) \quad \lim_{\varepsilon \rightarrow 0} \lambda \left\{ s \leq t : \frac{W(s + \varepsilon) - W(s)}{\sqrt{\varepsilon}} \leq x \right\} = t\Phi(x),$$

where  $\Phi(x) = \mathcal{N}((-\infty, x])$  and  $\mathcal{N}$  is the standard normal distribution. It is a sort of law of large numbers (LLN) for the random measure defined as

$$\mu_\varepsilon(A) = \lambda \left\{ s \in [0, 1] : \frac{W(s + \varepsilon) - W(s)}{\sqrt{\varepsilon}} \in A \right\},$$

that a.s. weakly converges towards  $\mathcal{N}$ .

This result was generalized shortly after by Wschebor [23] for Lévy processes and by Azaïs and Wschebor [1] for random processes with stationary increments and other processes. Moreover, also in [22] and [23], the result was shown for mollified processes as follows.

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For  $\psi \in BV$  (the set of bounded variation functions on  $\mathbb{R}$ ), let

$$\psi^\varepsilon(t) = \frac{1}{\varepsilon} \psi \left( \frac{t}{\varepsilon} \right)$$

denote the rescaled version of  $\psi$  and for  $X$  a measurable function, set  $X_\psi^\varepsilon = X \star \psi^\varepsilon$ , i.e.

$$(2) \quad X_\psi^\varepsilon(t) := \int \psi^\varepsilon(t-s) X(s) ds,$$

and

$$(3) \quad \dot{X}_\psi^\varepsilon := \int X(s) d\psi^\varepsilon(t-s).$$

The result reads, when  $X = W$  and  $\psi \in BV \cap L^2$ ,

$$(4) \quad \lambda\{s \leq t : \mathcal{W}_\psi^\varepsilon(s) \leq x\} \rightarrow t\Phi(x/\|\psi\|_2) \quad (a.s.),$$

with

$$(5) \quad \mathcal{W}_\psi^\varepsilon(s) := \sqrt{\varepsilon} \dot{W}_\psi^\varepsilon(s).$$

Notice that when  $\psi = \psi_1 := 1_{[-1,0]}$ ,

$$\dot{X}_\psi^\varepsilon(t) = \varepsilon^{-1} (X(t+\varepsilon) - X(t)) \quad , \quad \mathcal{W}_\psi^\varepsilon := \varepsilon^{-1/2} (W(\cdot + \varepsilon) - W(\cdot)),$$

and we recover (1).

In subsequent articles the a.s. result was extended to obtain a stable central limit theorem (CLT). For instance, let us consider a real even function  $g$  such that  $\mathbb{E}[g^2(N)] < \infty$ , for  $N \sim \mathcal{N}$  (a typical example is  $g(x) = |x|^p - \mathbb{E}[|N|^p]$ ). Defining for a bounded and continuous function  $f$  the family

$$\mathcal{Z}_\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^1 f(W(s)) (g(\mathcal{W}_\psi^\varepsilon(s)) - \mathbb{E}g(N)) ds,$$

we observe first that the convergence in (4) implies that

$$\lim_\varepsilon \int_0^1 f(W(s)) g(\mathcal{W}_\psi^\varepsilon(s)) ds = \mathbb{E}g(N/\|\psi\|_2) \int_0^1 f(W(s)) ds \quad (a.s.).$$

Moreover, for  $\mathcal{S}$  the convergence stable of measures we have

$$(6) \quad \lim_\varepsilon \mathcal{Z}_\varepsilon \stackrel{\mathcal{S}}{=} \sigma(g) \int_0^1 f(W(s)) dB(s),$$

where  $B(s)$  is another Brownian motion independent of  $W$  and  $\sigma$  is an explicit positive constant. Let us point out that if we take  $f = 1$  and by integrating on the interval  $[0, t]$  the above result turns into a functional CLT:

$$(7) \quad \lim_\varepsilon \left( \frac{1}{\sqrt{\varepsilon}} \int_0^t (g(\mathcal{W}_\psi^\varepsilon(s)) - \mathbb{E}g(N)) ds, t \in [0, 1] \right) = (\sigma(g)B(t), t \in [0, 1]),$$

in distribution.

The result in (6) was obtained in [4]. Since then, such type of matters were generalized to: diffusions, fractional Brownian motion (fBm), stationary increments Gaussian processes, Lévy processes, etc. A very complete review with a large number of references can be found in [24]. More recently, in 2008, Marcus and Rosen in [18] have studied the convergence of the  $L^p$  norm (this is  $g(x) = |x|^p$  in (7)) of the increments of stationary Gaussian processes. In the cited article the authors closed the problem in a somewhat definitive form. In another article ([19]) they said that their proofs were initially based on Wschebor's method, but afterwards they changed and looking for a more general and broadly used procedure.

When we are faced with a LLN-type result (a.k.a. convergence of a family of random objects to a deterministic one), it is nowadays natural to ask for a possible large deviation principle (LDP). Let us give some notations. If  $\Sigma = \mathbb{R}, \mathbb{R}^+ \times \mathbb{R}$  or  $[0, 1] \times \mathbb{R}$ , we denote by  $\mathcal{M}^+(\Sigma)$  and  $\mathcal{M}^r(\Sigma)$  the set of Borel measures on  $\Sigma$  positive and having total mass  $r$ , respectively.

If  $Z$  is a measurable function from  $\mathbb{R}^+$  to  $\mathbb{R}$ , let  $M_Z \in \mathcal{M}^+(\mathbb{R}^+ \times \mathbb{R})$  be defined by

$$(8) \quad M_Z(I \times A) = \lambda\{s \in I : Z(s) \in A\},$$

for every Borel subset  $I \times A$  of  $\mathbb{R}^+ \times \mathbb{R}$ . The first marginal of  $M_Z$  is  $\lambda$ . The second marginal  $\mu_Z$  is defined either by its action on a Borel set  $A$

$$(9) \quad \mu_Z(A) = M_Z([0, 1] \times A) = \lambda\{s \in [0, 1] : Z \in A\}$$

or, by its action on a test function  $f \in \mathcal{C}_b(\mathbb{R})$  (set of bounded continuous functions on  $\mathbb{R}$ )

$$\int_{\mathbb{R}} f(x) d\mu_Z(x) = \int_0^1 f(Z(t)) dt,$$

so that  $\mu_Z$  is the occupation measure

$$\mu_Z = \int_0^1 \delta_Z dt.$$

In this framework, we can consider (1) and (4) as laws of large numbers (LLN):

$$M_{\mathcal{W}_1^\varepsilon} \Rightarrow \lambda \times \mathcal{N} \quad (a.s.)$$

where  $\Rightarrow$  stands for the weak convergence. Since the Brownian motion  $W$  is self-similar (Property P1) and has stationary increments (P2), it is possible to reduce the problem about  $\mu_{\mathcal{W}_1^\varepsilon}$  ( $\varepsilon \rightarrow 0$ ) to a problem of an occupation measure in large time ( $T := \varepsilon^{-1} \rightarrow \infty$ ) for a process  $Y$  independent of  $\varepsilon$ . This new process is stationary and ergodic. Moreover the independence of increments of  $W$  (P3) and its self-similarity induces a 1-dependence for  $Y$ , which allows to apply a criterion of Chiyonobu and Kusuoka [8] to get an LDP.

Actually, as the crucial properties (P1, P2, P3) are shared by  $\alpha$ -stable Lévy processes, we state the LDP in this last framework. This is the content of Section 3 with an extension to random

measures built with mollifiers. Previously a basic lemma on equalities in law is stated in Section 2.

The fBM with Hurst index  $H \neq 1/2$  shares also properties (P1, P2) but not (P3) with the above processes. Nevertheless, since it is Gaussian, with an explicit spectral density, we prove the LDP for  $(\mu_\varepsilon)$  under specific conditions on the mollifier, thanks to a criterion of [6]. This is the content of Section 4. In Section 5 we state an LDP for the space-time measure defined in (8) when  $Z$  is one of the above processes and in Section 6, we state a result for some “process level” empirical measure. At last, in Section 7 we study discrete versions of Wschebor’s theorem.

Among the issues not addressed here, we may quote: increments for Gaussian random fields in  $\mathbb{R}^d$  and multi-parameter indexed processes.

Let us notice that except in a specific case in Section 4.3.2, we cannot give an explicit expression for the rate function. Moreover if one would be able to prove that the rate function is strictly convex and its minimum is reached at  $\lambda \times \mathcal{N}$ , this would give an alternate proof of Wschebor’s results.

## 2 General framework

Recall that a real-valued process  $\{X(t), t \in \mathbb{R}\}$

- has stationary increments if

$$\{X(t+h) - X(h), t \in \mathbb{R}\} \stackrel{(d)}{=} \{X(t) - X(0), t \in \mathbb{R}\},$$

- is self-similar with index  $H > 0$  if

$$\forall a > 0 \{X(at), t \in \mathbb{R}\} \stackrel{(d)}{=} \{a^H X(t), t \in \mathbb{R}\}.$$

If  $X$  is a self-similar process with index  $H$  we set, if  $\psi \in BV$

$$(10) \quad \mathcal{X}_\psi^\varepsilon = \varepsilon^{1-H} \dot{X}_\psi^\varepsilon,$$

where  $\dot{X}_\psi^\varepsilon$  is defined as in (3) by

$$(11) \quad \dot{X}_\psi^\varepsilon = \int X(s) d\psi^\varepsilon(t-s) = \frac{1}{\varepsilon} \int X(t-\varepsilon u) d\psi(u).$$

In particular

$$(12) \quad \mathcal{X}_\psi^1(t) = \int X(s) d\psi(t-s).$$

The following lemma is the key for our study.

**Lemma 2.1.** *Assume that  $X$  is self-similar with index  $H$ . For fixed  $\varepsilon$  and  $\psi \in BV$ , we have*

$$(13) \quad (\mathcal{X}_\psi^\varepsilon(t), t \in \mathbb{R}) \stackrel{(d)}{=} (\mathcal{X}_\psi^1(t\varepsilon^{-1}), t \in \mathbb{R})$$

$$(14) \quad \mu_{\mathcal{X}_\psi^\varepsilon} \stackrel{(d)}{=} \varepsilon \int_0^{\varepsilon^{-1}} \delta_{\mathcal{X}_\psi^1(t)} dt.$$

Moreover, if  $X$  has stationary increments, then  $\mathcal{X}_\psi^1$  is stationary.

**Proof:** It is straightforward. First,

$$(15) \quad \dot{X}_\psi^\varepsilon(t) = \varepsilon^{-1} \int X(t - \varepsilon u) d\psi(u) \stackrel{(d)}{=} \varepsilon^{H-1} \int X\left(\frac{t}{\varepsilon} - u\right) d\psi(u),$$

where the last equality comes from self-similarity and holds as a process in  $t \in \mathbb{R}$ . This yields (13), and then

$$(16) \quad \mu_{\mathcal{X}_\psi^\varepsilon} = \int_0^1 \delta_{\mathcal{X}_\psi^\varepsilon(t)} dt = \varepsilon \int_0^{1/\varepsilon} \delta_{\mathcal{X}_\psi^\varepsilon(\varepsilon\tau)} d\tau \stackrel{(d)}{=} \varepsilon \int_0^{1/\varepsilon} \delta_{\mathcal{X}_\psi^1(\tau)} d\tau. \quad \square$$

We give now a definition which will set the framework for the processes studied in the sequel. Recall that the  $\tau$ -topology on  $\mathcal{M}^1(\mathbb{R})$  is the topology induced by the space of bounded measurable functions on  $\mathbb{R}$ . It is stronger than the weak topology which is induced by  $\mathcal{C}_b(\mathbb{R})$ .

**Definition 2.2.** *Let  $\mathcal{F}$  be a subset of the set  $BV$  of bounded variation function from  $\mathbb{R}$  in  $\mathbb{R}$ . We say that a process  $X$  with stationary increments and self-similar with index  $H$  has the  $(LDP_w, \mathcal{F}, H)$  (resp.  $(LDP_\tau, \mathcal{F}, H)$ ) property if the process  $\mathcal{X}_\psi^1$  is well defined and if for every  $\psi \in \mathcal{F}$ , the family  $(\mu_{\mathcal{X}_\psi^\varepsilon})$  satisfies the LDP in  $\mathcal{M}^1(\mathbb{R})$  equipped with the weak topology (resp. the  $\tau$ -topology), in the scale  $\varepsilon^{-1}$ , with good rate function*

$$(17) \quad \Lambda_\psi^*(\mu) = \sup_{f \in \mathcal{C}_b(\mathbb{R})} \int f d\mu - \Lambda_\psi(f),$$

(the Legendre dual of  $\Lambda_\psi$ ) where for  $f \in \mathcal{C}_b(\mathbb{R})$ ,

$$(18) \quad \Lambda_\psi(f) = \lim_{T \rightarrow \infty} T^{-1} \log \mathbb{E} \exp \int_0^T f(\mathcal{X}_\psi^1(t)) dt,$$

in particular, the above limit exists.

Roughly speaking, this means that the probability of seeing  $\mu_{\mathcal{X}_\psi^\varepsilon}$  close to  $\mu$  for a small  $\varepsilon$  is asymptotically  $e^{-\Lambda_\psi^*(\mu)/\varepsilon}$ .

### 3 The $\alpha$ -stable Lévy process

Let  $\alpha \in (0, 2]$  fixed. The  $\alpha$ -stable Lévy process  $(S(t), t \geq 0; S(0) = 0)$  has independent and stationary increments and is  $1/\alpha$ -self-similar. If  $\psi \in BV$  is compactly supported, we set

$$\mathcal{S}_\psi^\varepsilon(t) := \varepsilon^{1-1/\alpha} \int S(s) d\psi_\varepsilon(t-s)$$

and as in (8-9), we may build the measures  $M_{\mathcal{S}_\psi^\varepsilon}$  and  $\mu_{\mathcal{S}_\psi^\varepsilon}$ . In [1], Theorem 3.1, it is proved that a.s.

$$M_{\mathcal{S}_\psi^\varepsilon} \Rightarrow \lambda \times \Sigma_\alpha \text{ (a.s.)}$$

where  $\Sigma_\alpha$  is the law of  $\|\psi\|_\alpha S(1)$ .

**Proposition 3.1.** *If  $\mathcal{F}$  is the set of bounded variation functions with compact support, then the  $\alpha$ -stable Lévy process has the  $(LDP_\tau, \mathcal{F}, 1/\alpha)$  property.*

**Proof:** We apply Lemma 2.1 with  $X = S$  and  $H = 1/\alpha$ .

Assume that the support of  $\psi$  is included in  $[a, b]$ . Since  $S$  has independent and stationary increments, the process  $\mathcal{S}_\psi^1$  is stationary and  $(b-a)$ -dependent. This last property means that  $\sigma(\mathcal{S}_\psi^1(u), u \in A)$  and  $\sigma(\mathcal{S}_\psi^1(u), u \in B)$  are independent as soon as the distance between  $A$  and  $B$  is greater than 1. The process  $(\mathcal{S}_\psi^1)$  satisfies the condition (T) in [8] (see also condition (S) in [7] p. 558). Then the family  $(\mu_{\mathcal{S}_\psi^\varepsilon})$  satisfies the LDP and the other conclusions hold.  $\square$

**Remark 3.2.** *When  $\alpha = 2$  we recover the Brownian case. In particular, when  $\psi = \psi_1$*

$$(19) \quad \mathcal{S}_\psi^1(u) = W(u+1) - W(u), \quad u \in \mathbb{R}.$$

*This process is called often Slepian process; it is Gaussian, stationary and 1-dependent.*

## 4 The fractional Brownian motion

### 4.1 General statement

We treat now the case of self-similar Gaussian processes with stationary increments, i.e. fractional Brownian motion (fBm in short). The fBm with Hurst parameter  $H \in [0, 1)$  is the Gaussian process  $(B_H(t), t \in \mathbb{R})$  with covariance

$$\mathbb{E}B_H(t)B_H(s) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |t-s|^{2H}),$$

It has a chaotic (or harmonizable) representation of  $B_H$  (see [20] Prop. 7.2.8)

$$(20) \quad B_H(t) = \frac{1}{C_H} \int_{\mathbb{R}} (e^{i\lambda t} - 1) |\lambda|^{-H-\frac{1}{2}} d\mathbf{W}(\lambda)$$

where  $\mathbf{W}$  is a complex Brownian motion and

$$C_H^2 = \frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}.$$

This process has stationary increments and is self-similar of index  $H$ . When  $H = 1/2$  we recover the Brownian motion, and it is the only case where the increments are independent.

When  $\psi \in BV$  with compact support, the LLN can be formulated as :

$$(21) \quad M_{\lambda^\varepsilon} \Rightarrow \lambda \times \mathcal{N}(\cdot, \sigma_\psi) \quad (a.s.),$$

where

$$\sigma_\psi^2 = -\frac{1}{2} \iint |u-v|^{2H} d\psi(u)d\psi(v),$$

(see [1]).

Our result on large deviations is the following. In Fourier analysis we adopt the following notation: when  $f, g \in L^1(\mathbb{R})$

$$\hat{f}(\theta) = \int e^{it\theta} f(t)dt, \quad \check{g}(\gamma) = \frac{1}{2\pi} \int e^{-i\gamma x} g(x)dx.$$

**Proposition 4.1.** *Denote*

$$\mathcal{G} := \{\psi \in BV \cap L^1\}$$

$$\mathcal{G}_H = \{\psi \in L^1 : \exists \lim_{\lambda \rightarrow 0} |\hat{\psi}(\lambda)| |\lambda|^{\frac{1}{2}-H}\}, \quad (0 < H < 1).$$

*The process  $B_H$  has the  $(LDP_w, \mathcal{F}, H)$  property if one of the following conditions are satisfied:*

$$(22) \quad H \leq 1/2 \quad \text{and} \quad \mathcal{F} = \mathcal{G},$$

$$(23) \quad 1/2 < H < 1 \quad \text{and} \quad \mathcal{F} = \mathcal{G} \cap \mathcal{G}_H.$$

Particular cases are examined in Section 4.3.

**Remark 4.2.** *If we define*

$$\mathcal{G}_0 = \{\psi \in L^1 : \int \psi(t)dt = 0 \text{ and } \int |t\psi(t)|dt < \infty\}$$

*then it holds that*

$$(24) \quad \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_H.$$

Proof of Proposition 4.1:

We apply Lemma 2.1 with  $X = B_H$ . But now, for lack of independence, we will work with the spectral density and apply Theorem 2.1 in [6], which ensures the LDP as soon as the spectral density is in  $\mathcal{C}_0(\mathbb{R})$ .



Using (20), the process  $\mathcal{X}_\psi^1$  may be written as

$$(25) \quad \mathcal{X}_\psi^1(t) = \int B_H(t-s) d\psi(s) = C_H^{-1} \iint (e^{i\lambda(t-s)} - 1) |\lambda|^{-H-\frac{1}{2}} d\mathbf{W}(\lambda) d\psi(s).$$

Now, when  $\psi \in \mathcal{G}$  it holds that  $\lim_{|t| \rightarrow \infty} |\psi(t)| = 0$  and by integration by parts,

$$(26) \quad \begin{aligned} \int e^{i\lambda s} d\psi(s) &= -i\lambda \hat{\psi}(\lambda) \\ \int (e^{i\lambda(t-s)} - 1) d\psi(s) &= i\lambda e^{i\lambda t} \hat{\psi}(-\lambda) \end{aligned}$$

so that

$$(27) \quad \mathcal{X}_\psi^1(t) = iC_H^{-1} \int e^{it\lambda} \hat{\psi}(-\lambda) \frac{\lambda}{|\lambda|^{H+\frac{1}{2}}} d\mathbf{W}(\lambda).$$

The spectral density of the stationary process  $\mathcal{X}_\psi^1$  is then

$$(28) \quad \ell_H(\lambda) = C_H^{-2} \left| \hat{\psi}(\lambda) \right|^2 |\lambda|^{1-2H}.$$

Let us check conditions on  $H$  and  $\psi$  so that  $\ell_H \in \mathcal{C}_0(\mathbb{R})$ .

- From (26) it holds that  $|\hat{\psi}(\lambda)| = o(|\lambda|^{-1})$  and then, for all  $0 < H < 1$

$$(29) \quad \lim_{|\lambda| \rightarrow \infty} |\ell_H(\lambda)| = 0.$$

- If  $\psi \in L^1$ ,  $\hat{\psi} \in \mathcal{C}_0$  (Riemann-Lebesgue) and  $\ell_H$  is even and continuous on  $(0, \infty)$ .
- For  $H \leq 1/2$ , the continuity of  $\ell_H$  at 0 is obvious.

For  $H > 1/2$ , this continuity is ensured by the assumption  $\psi \in \mathcal{G}_H$ . □

Proof of Remark 4.2:

To prove (24), note that  $\mathcal{G}_1 \subset \mathcal{G}_H$  is obvious and that, if  $\psi \in \mathcal{G}_0$ ,

$$\begin{aligned} \hat{\psi}(0) &= \int \psi(x) dx = 0 \\ |\hat{\psi}(\lambda)| &= \left| \int (e^{i\lambda x} - 1) \psi(x) dx \right| \leq |\lambda| \int |x\psi(x)| dx, \end{aligned}$$

so that  $\psi \in \mathcal{G}_1$ . □

## 4.2 Contraction

Since the mapping  $\mu \mapsto \int |x|^p d\mu(x)$  is not continuous, we cannot obtain an LDP for the moments of  $\mu_{\mathcal{X}_\psi^\varepsilon}$  by invoking the contraction principle (Th. 4.2.1 in [12]). Nevertheless, in the case of the fBm, the Gaussian stationary character of the process allows to conclude. It is a direct application of Corollary 2.1 in [6].

**Proposition 4.3.** *If either  $H \leq 1/2$  and  $\psi \in \mathcal{G}$  or  $H > 1/2$  and  $\psi \in \mathcal{G} \cap \mathcal{G}_H$ , then the family  $(\int_0^1 |\mathcal{X}_\psi^\varepsilon(t)|^2 dt)$ , where  $X = B_H$ , satisfies the LDP, in the scale  $\varepsilon^{-1}$  with good rate function*

$$I_\psi(x) = \sup_{-\infty < y < 1/(4\pi M)} \{xy - L(y)\},$$

where

$$L(y) = -\frac{1}{4\pi} \int \log(1 - 4\pi y \ell_H(s)) ds$$

and

$$M = \sup_\lambda \ell_H(\lambda).$$

More generally, for  $0 \leq p \leq 2$ , the family  $(\int_0^1 |\mathcal{X}_\psi^\varepsilon(t)|^p dt)$  satisfies the LDP at scale  $\varepsilon$  with a convex rate function.

## 4.3 Particular cases

### 4.3.1 Two basic mollifiers

1) As seen before, the function  $\psi_1 = 1_{[-1,0]}$  is the most popular. It allows to study the first order increments  $X(t + \varepsilon) - X(t)$ . It belongs to  $\mathcal{G}$  but since

$$|\hat{\psi}_1(\lambda)| = \frac{|\sin(\lambda/2)|}{|\lambda/2|}$$

it does not belong to  $\mathcal{G}_H$  for  $H > 1/2$ .

For  $H = 1/2$ , we recover the Brownian motion and replace the notation  $\mathcal{X}$  by  $\mathcal{W}$ . The process  $\mathcal{W}_{\psi_1}^1$  is the Slepian process (19) with covariance

$$r(t) = (1 - |t|)^+,$$

and spectral density:

$$\check{r}(\lambda) = \frac{1}{2\pi} \left( \frac{\sin \frac{\lambda}{2}}{\frac{\lambda}{2}} \right)^2.$$

As it is said above since  $\check{r}$  is  $\mathcal{C}_0$ , the occupation measure satisfies a LDP in the weak topology in the scale  $\varepsilon^{-1}$ . in the scale  $\varepsilon^{-1}$ . This argument could have been used to prove the LDP, instead of the argument in Section 2 (but for the weak topology and not the  $\tau$ -topology). Notice that

although  $\tilde{r}$  is differentiable, we could not apply Theorem 5.18 in Chiyonobu and Kusuoka [8], since the condition (5.19) therein is violated in  $x \in 2\pi\mathbb{Z}$ .

2) Another interesting function is

$$\psi_2 = \frac{1}{2} (1_{[-1,0]} - 1_{[0,1]})$$

which yields

$$(30) \quad \dot{X}_{\psi_2}^\varepsilon(t) = \frac{X(t+\varepsilon) - 2X(t) + X(t-\varepsilon)}{2\varepsilon}.$$

Since

$$|\hat{\psi}_2(\lambda)| = \frac{\sin^2(\lambda/2)}{|\lambda/2|},$$

we see that  $\psi_2 \in \mathcal{G} \cap \mathcal{G}_H$  for every  $H \in (0, 1)$  and then  $(\mu_{\mathcal{X}_{\psi_2}^\varepsilon})$  satisfies the LDP.

In (30) we are faced with second order increments of the process  $X$ . These increments are linked with the behavior of the second derivative of  $X^\varepsilon$  when it exists. Let us consider  $\psi$  smooth enough so that  $X_\psi^\varepsilon$ , defined in (2), has a second derivative. For instance, let  $\psi \in \mathcal{G}$  and such that  $\psi' \in \mathcal{G}$ . Then the function  $X_\psi^\varepsilon$  is twice differentiable and

$$\ddot{X}_\psi^\varepsilon(t) = \varepsilon^{-2} \int X(t - \varepsilon s) d\psi'(s) = \varepsilon^{-1} \dot{X}_{\psi'}^\varepsilon(t).$$

Now,  $\psi' \in \mathcal{G}_H$  since

$$|\widehat{\psi}'(\lambda)| |\lambda|^{\frac{1}{2}-H} = |\widehat{\psi}(\lambda)| |\lambda|^{\frac{3}{2}-H} \rightarrow 0$$

as  $\lambda \rightarrow 0$ .

Since  $\mathcal{X}_{\psi'}^\varepsilon = \varepsilon^{2-H} \ddot{X}_\psi^\varepsilon$ , we conclude that for every  $H \in (0, 1)$ , the family  $(\mu_{\varepsilon^{2-H} \ddot{X}_\psi^\varepsilon})$  satisfies the LDP in the scale  $\varepsilon^{-1}$  and good rate function  $\Lambda_{\psi'}^*$ . The choice

$$\psi(t) = \frac{1}{2} (1 - |t|)^+$$

allows to recover  $\psi' = \psi_2$  and the second order increments.

### 4.3.2 Looking for an explicit rate function

It is not easy to find examples of explicit rate functions for the occupation measures of the above stationary processes  $\mathcal{X}_\psi^1$ , since in general the limiting cumulant generating function  $\Lambda$  is not explicit. A particularly nice situation in the Gaussian case will occur if the process is also Markovian, i.e. if  $\mathcal{X}_\psi^1$  is the Ornstein-Uhlenbeck (OU) process. Indeed, for the OU, the rate function for the LDP of the occupation measure is given by the Donsker-Varadhan theory ([21] ex. 8.28) :

$$\Lambda^*(\mu) = \frac{1}{2} \int_{\mathbb{R}} |g'(x)|^2 d\mathcal{N}(x)$$

if  $d\mu = g^2 d\mathcal{N}$ . The goal is then to find a mollifier  $\psi$  such that  $\mathcal{X}_\psi^1$  is distributed as OU.

To begin with, let us assume that the underlying process is Brownian, which implies that  $\mathcal{W}_\psi^1$  is again Gaussian and stationary, with spectral density (cf. (28)):

$$\check{r}(\lambda) = \frac{1}{2\pi} |\hat{\psi}(\lambda)|^2.$$

For OU, the covariance and spectral density are, respectively

$$r(t) = e^{-|t|}, \quad \check{r}(\lambda) = \frac{1}{\pi(1 + \lambda^2)}.$$

To solve the equation

$$\mathcal{X}_\psi^1 \stackrel{(d)}{=} OU$$

turns out to solve

$$(31) \quad |\hat{\psi}(\lambda)|^2 = \frac{2}{1 + \lambda^2}.$$

We present two answers.

1) Let us choose

$$\hat{\psi}(\lambda) = \frac{\sqrt{2}}{1 - i\lambda}, \quad \psi(x) = \sqrt{2}e^{-x}1_{[0,\infty)}(x),$$

and then, the formula (5) becomes

$$\mathcal{W}_\psi^1(t) = \sqrt{2} \int_{-\infty}^t e^{-(t-s)} dW_s$$

which is the classical representation of the stationary OU as a stochastic integral ([20] p.138).

2) Let us choose  $\psi$  such that

$$\hat{\psi}(\lambda) = \frac{\sqrt{2}}{\sqrt{1 + \lambda^2}}$$

This is equivalent to say

$$\psi(x) = \frac{1}{2\pi} \int e^{-ix\lambda} \frac{\sqrt{2}}{\sqrt{1 + \lambda^2}} d\lambda$$

i.e.

$$\psi(x) = \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{\cos(x\lambda)}{\sqrt{1 + \lambda^2}} d\lambda = \frac{\sqrt{2}}{\pi} K_0(x),$$

where  $K_0$  is the MacDonald (or modified Bessel) function (see [10] p.369 or [13] formula 17 p.9). This function can be expressed also as

$$K_0(x) = \sqrt{\pi}e^{-x}\Psi(1/2, 1; 2x),$$

where  $\Psi$  is the confluent hypergeometric function (see [14] p. 265).

Let us now extend the study to the fBm. Looking for a kernel  $\psi$  leading to the OU process, (28) leads to the equation

$$|\hat{\psi}(\lambda)|^2 = C_H^2 \frac{|\lambda|^{2H-1}}{\pi(1+\lambda^2)},$$

hence, for instance if  $\psi$  is even,

$$\hat{\psi}(\lambda) = C_H \frac{|\lambda|^{H-\frac{1}{2}}}{\sqrt{\pi(1+\lambda^2)}}.$$

For  $H < 1/2$ , we did not find a closed expression for the kernel

$$(32) \quad \psi(x) = \frac{C_H}{\pi} \int_0^\infty \cos(\lambda x) \frac{|\lambda|^{H-\frac{1}{2}}}{\sqrt{\pi(1+\lambda^2)}} d\lambda$$

in the literature.

For  $H > 1/2$ , this function is not continuous in 0, so it cannot be the Fourier transform of an integrable kernel. We have proved

**Proposition 4.4.** *When  $H \leq 1/2$  and  $\psi$  is given by (32), the family  $(\mu_{\mathcal{H}_\psi^\varepsilon})$  satisfies the LDP, in the scale  $\varepsilon^{-1}$  with good rate function*

$$\Lambda^*(\mu) = \frac{1}{2} \int_{\mathbb{R}} |g'(x)|^2 d\mathcal{N}(x)$$

if  $d\mu = g^2 d\mathcal{N}$ .

**Remark 4.5.** *In this case,  $\Lambda^*$  has a unique minimum at  $\mu = \mathcal{N}$  which allows to recover Wschebor's result on a.s. convergence.*

## 5 A space-time LDP

We will state a complete LDP for some of our models, i.e. an LDP for  $(M_{\mathcal{X}_\psi^\varepsilon})$ , whenever  $(\mu_{\mathcal{X}_\psi^\varepsilon})$  satisfies the LDP. Following the notations of Dembo and Zajic in [11] we denote by  $\mathcal{AC}_0$  the set of maps  $\nu : [0, 1] \rightarrow \mathcal{M}^+(\mathbb{R})$  such that

- $\nu$  is absolutely continuous with respect to the variation norm,

- $\nu(0) = 0$  and  $\nu(t) - \nu(s) \in \mathcal{M}^{t-s}(\mathbb{R})$  for all  $t > s \geq 0$ ,
- for almost every  $t \in [0, 1]$ ,  $\nu(t)$  possesses a weak derivative.

(This last point means that  $\nu(t + \eta) - \nu(t)/\eta$  has a limit as  $\eta \rightarrow 0$  - denoted by  $\dot{\nu}(t)$ - in  $\mathcal{M}^+(\mathbb{R})$  equipped with the topology of weak convergence).

Let  $F$

$$(33) \quad \begin{aligned} \mathcal{M}^+([0, 1] \times \mathbb{R}) &\rightarrow D([0, 1]; \mathcal{M}^+(\mathbb{R})) \\ M &\mapsto (t \mapsto F(M)(t) = M([0, t], \cdot)) \end{aligned}$$

or in other words  $F(M)(t)$  is the positive measure on  $\mathbb{R}$  defined by its action on  $\varphi \in \mathcal{C}_b$ :

$$\langle F(M)(t), \varphi \rangle = \langle M, 1_{[0, t]} \times \varphi \rangle.$$

Here  $D([0, 1]; \cdot)$  is the set of càd-làg functions, equipped with the supremum norm topology. At last, let  $\mathcal{E}$  be the image of  $F$ .

**Theorem 5.1.** *When the process  $X$  is the  $\alpha$ -stable Lévy process and  $\psi \in BV$  has a compact support, the family  $(M_{\mathcal{X}_\psi^\varepsilon})$  satisfies the LDP in  $\mathcal{M}^1([0, 1] \times \mathbb{R})$  equipped with the weak topology, in the scale  $\varepsilon^{-1}$  with the good rate function*

$$(34) \quad \Lambda^*(M) = \begin{cases} \int_0^1 \Lambda_\psi^*(\dot{\gamma}(t)) dt & \text{if } \gamma := F(M) \in \mathcal{AC}_0, \\ \infty & \text{otherwise.} \end{cases}$$

**Proof:** As in the above sections, it is actually a problem of large deviations in large time. For the sake of simplicity, set

$$Y = \mathcal{X}_\psi^1$$

and  $T = \varepsilon^{-1}$ . Using Lemma 2.1, the problem reduces to the study of the family  $(M_{Y(\cdot T)})$ . First, we study the corresponding distribution functions.

Actually, we have

$$(35) \quad F(M_{Y(\cdot T)})(t) = \int_0^t \delta_{Y(sT)} ds = T^{-1} \int_0^{tT} \delta_{Y(s)} ds =: H_T(t).$$

In a first step we will prove that the family  $(H_T)$  satisfies the LDP, then in a second step we will transfer this property to  $M_{Y(\cdot T)}$ .

First step : We follow the method of Dembo-Zajic [11]. We begin with a reduction to their “discrete time” method by introducing

$$\xi_k = \int_{k-1}^k \delta_{Y_s} ds, (k \geq 1) \quad \text{and} \quad S_T(t) = \sum_1^{\lfloor tT \rfloor} \xi_k.$$

It holds that

$$(36) \quad T^{-1} \int_0^{tT} \delta_{Y(s)} ds - T^{-1} S_T(t) = T^{-1} \int_{\lfloor tT \rfloor}^{tT} \delta_{Y(s)} ds$$

and this difference has a total variation norm less than  $T^{-1}$ , so that the families  $(T^{-1}S_T)$  and  $(H_T)$  are exponentially equivalent (Def. 4.2.10 in [12]).

The sequence  $\xi_k$  is 1-dependent, hence satisfies condition (S) in [11] p.22 which implies, by Th. 4 in the same paper that  $(T^{-1}S_T)$  satisfies the LDP in  $D([0, 1]; \mathcal{M}^+(\mathbb{R}))$  provided with the uniform norm topology, with the convex good rate function

$$(37) \quad I(\nu) = \int_0^1 \Lambda_\psi^*(\dot{\nu}(t)) dt$$

when  $\nu \in \mathcal{AC}_0$  and  $\infty$  otherwise.

We conclude, owing to Th. 4.2.13 in [12], that  $(H_T)$  satisfies the same LDP.

Second step : We have now to carry this LDP to  $(M_{Y(\cdot T)})$  (see (35)). For every  $T > 0$ ,  $H_T \in \mathcal{E} \subset \overline{D}([0, 1]; \mathcal{M}^+(\mathbb{R}))$ . We saw that the effective domain of  $I$  is included in  $\mathcal{E}$ . So, by Lemma 4.1.5 in Dembo-Zeitouni [12],  $(H_T)$  satisfies the same LDP in  $E$  equipped with the (uniform) induced topology. Now,  $F$  is bijective from  $\mathcal{M}^1([0, 1] \times \mathbb{R})$  to  $\mathcal{E}$ . Let us prove that  $F^{-1}$  is continuous from  $\mathcal{E}$  (equipped with the uniform topology) to  $\mathcal{M}^1([0, 1] \times \mathbb{R})$  equipped with the weak topology.

For  $f : [0, 1] \rightarrow \mathbb{R}$ , let

$$(38) \quad \|f\|_{BL} = \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

$$(39) \quad d_{BL}(\mu, \nu) = \sup_{f: \|f\|_{BL} \leq 1} \left| \int f d\mu - \int f d\nu \right|$$

The space  $\mathcal{M}^+(\mathbb{R})$  is a Polish space when equipped with the topology induced by  $d_{BL}$ , compatible with the weak topology.

It is known that  $M_n \rightarrow M \in \mathcal{M}^1([0, 1] \times \mathbb{R})$  weakly as soon as

$$(40) \quad M_n(1_{[0,t]} \otimes f) \rightarrow M(1_{[0,t]} \otimes f)$$

for every  $t \in [0, 1]$  and every  $f$  such that  $\|f\|_{BL} < \infty$ . But, for such  $t, f$  we have

$$(41) \quad \sup_t |M_n(1_{[0,t]} \otimes f) - M(1_{[0,t]} \otimes f)| \leq d_{BL}(F(M_n), F(M))$$

which implies that  $F^{-1}$  is continuous from  $\mathcal{E}$  to  $\mathcal{M}^1([0, 1] \times \mathbb{R})$ .

By the contraction principle (Th. 4.2.1 in [12]) we deduce that  $M_{Y(\cdot T)}$  satisfies the LDP in  $\mathcal{M}^1([0, 1] \times \mathbb{R})$  with good rate function  $J(M) = I(F(M))$ , wherer  $I$  is given by (37).  $\square$

## 6 “Level process” study

In the study of strong convergence problems such as the a.s. CLT (see [16] and [17]), an interesting problem is the LDP of empirical measures at the level of processes. If we restrict us to the Brownian case to simplify, the corresponding problem could be the behavior of

$$\int_0^1 \delta_{\left\{ \frac{W(s+\varepsilon) - W(s)}{\sqrt{\varepsilon}}, s \geq t \right\}} dt.$$

Here we do not see clearly the interest of such a study for the Wschebor’s theorem. Nevertheless, it seems natural to consider the family  $(\xi_t^\varepsilon, t \geq 0)$  of shifted processes

$$(42) \quad \xi_t^\varepsilon : s \mapsto \frac{W(t + \varepsilon s) - W(t)}{\sqrt{\varepsilon}} \in \mathcal{C}([0, 1]),$$

and the following empirical measure

$$(43) \quad \mathcal{L}_\varepsilon := \int_0^1 \delta_{\xi_t^\varepsilon} dt.$$

By the scaling invariance, for every  $\varepsilon > 0$ ,

$$(44) \quad (\xi_{\varepsilon t}^\varepsilon, t \geq 0) \stackrel{(d)}{=} (\xi_t^1, t \geq 0),$$

and then

$$(45) \quad \mathcal{L}_\varepsilon = \int_0^1 \delta_{\xi_t^\varepsilon} dt \stackrel{(d)}{=} \tilde{\mathcal{L}}_\varepsilon := \varepsilon \int_0^{\varepsilon^{-1}} \delta_{\xi_t^1} dt.$$

Since we have

$$(46) \quad \xi_t^1 = (W(t + s) - W(t), s \in [0, 1]),$$

the process  $(\xi_t^1, t \geq 0)$  will be called the the *meta-Slepian* process in the sequel. For every  $t$ , the distribution of  $\xi_t^1$  is the Wiener measure  $\mathbb{W}$  on  $\mathcal{C}([0, 1])$ .

The meta-Slepian process is clearly stationary and 1-dependent. Since it is ergodic, the Birkhoff theorem tells us that, almost surely when  $\varepsilon \rightarrow 0$ ,  $\tilde{\mathcal{L}}_\varepsilon$  converges weakly to  $\mathbb{W}$ . From the equality in distribution (45) we deduce that  $(\mathcal{L}_\varepsilon)$  converges in distribution to the same limit. But this limit is deterministic, hence the convergence of  $(\mathcal{L}_\varepsilon)$  holds in probability. We just proved:

**Theorem 6.1.** *When  $\varepsilon \rightarrow 0$ , the family of random probability measures  $(\mathcal{L}_\varepsilon)$  on  $\mathcal{C}([0, 1])$  converges in probability weakly to the Wiener measure  $\mathbb{W}$  on  $\mathcal{C}([0, 1])$ .*

The problem of almost sure convergence raises some difficulties. We have obtained on the one hand a partial almost sure fidi convergence (which is no more that a multimensional extension of Wschebor theorem) and on the other hand an almost sure convergence when we plug  $\mathcal{C}([0, 1])$  into the Hilbert space  $L^2([0, 1])$ , equipped with its norm.

To this last purpose, if  $\mu$  is a measure on  $\mathcal{C}([0, 1])$ , we will denote by  $\mu^L$  its extension to  $L^2([0, 1])$ , i.e. that for every  $B$  Borel set of  $L^2([0, 1])$ ,

$$\mu^L(B) = \mu(B \cap \mathcal{C}([0, 1])).$$



**Theorem 6.2.** 1. For every integer  $d$  and every  $t_1, \dots, t_d \in [0, 1]$ , almost surely when  $\varepsilon \rightarrow 0$ , the family  $(\mathcal{L}_\varepsilon \pi_{t_1, \dots, t_d}^{-1})$  of random probability measures on  $\mathbb{R}^d$  converges weakly to  $\mathbb{W} \pi_{t_1, \dots, t_d}^{-1}$  on  $\mathcal{C}([0, 1])$ , where  $\pi_{t_1, \dots, t_d}$  be the projection  $: f \in \mathcal{C}([0, 1]) \mapsto f(t_1), \dots, f(t_d)$ .

2. When  $\varepsilon \rightarrow 0$ , the family of random probability measures  $(\mathcal{L}_\varepsilon^L)$  on  $L^2([0, 1])$  converges weakly almost surely to the Wiener measure  $\mathbb{W}^L$  on  $L^2([0, 1])$ .

We failed to prove a (full) almost sure fidi convergence, i.e. in 1. to state that “almost surely, for every  $t_1, \dots, t_d \dots$ ”. Moreover we do not know if an almost sure convergence at the level of processes is true.

For the proof, we need the following lemma, which is straightforward owing to the properties of stationarity and 1-dependence.

**Lemma 6.3.** If  $F$  is a bounded differentiable function with bounded derivative from  $\mathcal{C}([0, 1])$  (resp.  $L^2([0, 1])$ ) to  $\mathbb{R}$ . Then

$$(47) \quad a.s. \lim_{\varepsilon \rightarrow 0} \int_0^1 F(\xi_t^\varepsilon) dt = \int_{\mathcal{C}([0,1])} F(\xi) \mathbb{W}(d\xi).$$

Proof of Lemma 6.3:

It is along the lines of [1]. We first claim a quadratic convergence as follows. By Fubini and stationarity

$$\mathbb{E} \left( \int_0^1 F(\xi_t^\varepsilon) dt \right) = \int_0^1 \mathbb{E} F(\xi_t^\varepsilon) dt = \int_{\mathcal{C}([0,1])} F(\xi) \mathbb{W}(d\xi),$$

and by Fubini and 1-dependence,

$$(48) \quad \text{Var} \left( \int_0^1 F(\xi_t^\varepsilon) dt \right) = \int \int_{|t-s| < 2\varepsilon} \text{Cov}(F(\xi_t^\varepsilon), F(\xi_s^\varepsilon)) dt ds \leq 4\varepsilon \|F\|_\infty.$$

The Borel-Cantelli lemma implies a.s. convergence of  $\int_0^1 F(\xi_t^\varepsilon) dt$  along any sequence  $(\varepsilon_n)$  such that  $\sum_n \varepsilon_n < \infty$ .

To go on, take  $\varepsilon_{n+1} < \varepsilon < \varepsilon_n$  and notice that

$$(49) \quad \left| \int_0^1 F(\xi_t^\varepsilon) - F(\xi_t^{\varepsilon_n}) dt \right| \leq \|F'\|_\infty \sup_{t,u} |\xi_t^\varepsilon(u) - \xi_t^{\varepsilon_n}(u)|.$$

Now we use the properties of Brownian paths. On the interval  $[0, 2]$  the Brownian motion satisfies a.s. a Holder condition with exponent  $\beta < 1/2$ , so that we can define the a.s. finite random variable

$$(50) \quad M := 2 \sup_{u,v \in [0,2]} \frac{|W(u) - W(v)|}{|v - u|^\beta}.$$

So,

$$\begin{aligned}
\sup_{s \in [0,1]} |\xi_t^\varepsilon(s) - \xi_t^{\varepsilon_n}(s)| &\leq \frac{M}{2} \frac{(\varepsilon_n - \varepsilon)^\beta}{\varepsilon^{1/2}} + \frac{M}{2} (\varepsilon_n)^\beta (\varepsilon^{-1/2} - (\varepsilon_n)^{-1/2}) \\
&= \frac{M}{2} \frac{(\varepsilon_n)^\beta}{\varepsilon^{1/2}} \left[ \left(1 - \frac{\varepsilon}{\varepsilon_n}\right)^\beta + \left(1 - \sqrt{\frac{\varepsilon}{\varepsilon_n}}\right) \right] \\
(51) \qquad \qquad \qquad &\leq M \frac{\varepsilon_n^\beta - \varepsilon^\beta}{\varepsilon^{1/2}} \leq M \frac{\varepsilon_n^\beta - \varepsilon_{n+1}^\beta}{\varepsilon_{n+1}^{1/2}}.
\end{aligned}$$

The choice of  $\varepsilon_n = n^{-a}$  with  $a > 1$  and  $\beta \in \left(\frac{a}{2(a+1)}, \frac{1}{2}\right)$  ensures that the right hand side of (51), hence of (49) tends to 0 a.s., which ends the proof.  $\square$

Proof of Theorem 6.2

1. The (random) characteristic functional of the (random) probability measure  $\mathcal{L}_\varepsilon$  on  $[0, 1]$  equipped with the Borel  $\sigma$ -field and the Lebesgue measure is a function from the dual space of  $\mathcal{C}([0, 1])$ , i.e.  $\mathcal{M}([0, 1])$  to  $\mathbb{C}$  defined by

$$\hat{G}_\varepsilon : \rho \mapsto \int_0^1 \exp\{i \int \xi_t^\varepsilon(s) \rho(ds)\} dt.$$

Actually,  $\hat{G}_\varepsilon(\rho) = \int F(\xi_t^\varepsilon) dt$  with  $F(\xi) = \exp\{i \int \xi(u) \rho(du)\}$ . This function fulfils the conditions of Lemma 6.3.

We have then, for every  $\rho$ ,

$$(52) \qquad a.s. \lim \hat{G}_\varepsilon(\rho) = \hat{G}(\rho) \quad := \int_{\mathcal{C}([0,1])} \exp\{i \int_0^1 \xi(s) \rho(ds)\} \mathbb{W}(d\xi)$$

$$(53) \qquad \qquad \qquad = \exp\left(-\frac{1}{2} \int_0^1 (\rho([u, 1])^2 du)\right).$$

Let fix  $d$  and  $t_1, \dots, t_d$ . For every  $\mathbf{a} := (a_1, \dots, a_d) \in \mathbb{R}^d$ , let us consider the measure  $\rho_{\mathbf{a}} = \sum_1^d a_k \delta_{t_k}$  and the following event

$$A(\mathbf{a}) := \left\{ \lim_{\varepsilon \rightarrow 0} \hat{G}_\varepsilon(\rho_{\mathbf{a}}) = \hat{G}(\rho_{\mathbf{a}}) \right\}.$$

The above analysis tells us that  $\mathbb{P}(A(\mathbf{a})) = 1$  for every  $\mathbf{a}$ . By a classical argument using Fubini's theorem we deduce that almost surely, for almost every  $\mathbf{a} \in \mathbb{R}^d$

$$(54) \qquad \lim_{\varepsilon \rightarrow 0} \int_0^1 \exp\left(i \sum_1^d a_k \xi_t^\varepsilon(t_k)\right) dt = \hat{G}\left(\sum_1^d a_k \delta_{t_k}\right).$$

By a slight adaptation of the Lévy's continuity theorem (which is detailed in Appendix), we conclude that  $(\mathcal{L}_\varepsilon \pi_{t_1, \dots, t_d}^{-1})$  converges weakly to the good limit.

2. We will use a method coming from [15] p. 46<sup>1</sup>. It consists in checking Billingsley's criterion on intersection of balls ([5] p.18) and approximating indicators by smooth functions. Let us give details for only one ball to shorten the proof.

For  $\delta \in (0, 1)$ , define

$$(55) \quad \phi_\delta(t) = \mathbf{1}_{(-\infty, 1]}(t) + \mathbf{1}_{[1, (1+\delta)^2]}(t) \frac{1}{C} \int_0^{\frac{((1+\delta)^2-t)}{(2\delta+\delta^2)}} e^{-\frac{1}{s(1-s)}} ds ,$$

where

$$C = \int_0^1 e^{-\frac{1}{s(1-s)}} ds .$$

The function  $\phi_\delta$  has a bounded support and it is continuous and  $\|\phi_\delta\|_\infty = 1$ . Now we consider  $\psi_\delta : L^2([0, 1]) \rightarrow \mathbb{R}$  defined by

$$\psi_\delta(\xi) = \phi_\delta(\|\xi\|^2).$$

This function is  $\mathcal{C}^\infty$  and has all its derivatives bounded. For every  $\xi_c \in L^2([0, 1])$ ,  $r > 0$ ,  $\delta \in (0, r)$  we have the nesting

$$(56) \quad \mathbf{1}_{B(\xi_c; r-\delta)}(\xi) \leq \psi_{\frac{\delta}{r-\delta}} \left( \frac{\xi - \xi_c}{r - \delta} \right) \leq \mathbf{1}_{B(\xi_c; r)}(\xi) \leq \psi_{\frac{\delta}{r}} \left( \frac{\xi - \xi_c}{r} \right) \leq \mathbf{1}_{B(\xi_c; r+\delta)}(\xi) .$$

Take a sequence  $\delta_n \rightarrow 0$ .

Let us remind that the measure  $\mathcal{L}_\varepsilon^L$  is random. We did not write explicitly the item  $W$  for simplicity, although it is present in (42).

For every test function  $F$  as in Lemma 6.3, we have a null set  $N_F$  such that for  $W \notin N_F$

$$(57) \quad \int_{L^2([0, 1])} F(\xi) \mathcal{L}_\varepsilon^L(d\xi) \rightarrow \int_{\mathcal{C}([0, 1])} F(\xi) \mathbb{W}(d\xi) .$$

Let  $(g_k)_{k \geq 1}$  be a countable dense set in  $L^2([0, 1])$ , and for  $q \in \mathbb{Q}$ ,

$$F_{n,k,q}^-(\xi) = \psi_{\delta_n/(q-\delta_n)} \left( \frac{\xi - g_k}{q - \delta_n} \right), F_{n,k,q}^+(\xi) = \psi_{\delta_n/q} \left( \frac{\xi - g_k}{q} \right)$$

and

$$N = \bigcup_{n,k,q} \left( N_{F_{n,k,q}^-} \cup N_{F_{n,k,q}^+} \right) .$$

Take  $W \notin N$ . Assume that the ball  $B(\xi_c; r)$  is given. Take  $\gamma > 0$ , then by density one can find  $k \geq 1$  and  $q \in \mathbb{Q}^+$  such that

$$(58) \quad \|\xi_c - g_k\| \leq \gamma, |r - q| \leq \gamma .$$

---

<sup>1</sup>It is used there to prove that in Hilbert spaces, convergence in the Zolotarev metric implies weak convergence.

By (56) we have

$$(59) \quad \mathcal{L}_\varepsilon^L(B(\xi_c; r)) \leq \int \psi_{\delta_n/r} \left( \frac{\xi - \xi_c}{r} \right) \mathcal{L}_\varepsilon^L(d\xi).$$

Besides, by (58) and by differentiability, there exists  $C_n > 0$  such that

$$(60) \quad \psi_{\delta_n/r} \left( \frac{\xi - \xi_c}{r} \right) \leq F_{n,k,q}^+(\xi) + C_n \gamma.$$

Now, by (57),

$$(61) \quad \lim_\varepsilon \int_{L^2([0,1])} F_{n,k,q}^+(\xi) \mathcal{L}_\varepsilon^L(d\xi) = \int_{\mathcal{C}([0,1])} F_{n,k,q}^+(\xi) \mathbb{W}(d\xi)$$

By (56) again

$$(62) \quad \int_{\mathcal{C}([0,1])} F_{n,k,q}^+(\xi) \mathbb{W}(d\xi) \leq \mathbb{W}(B(g_k, q + \delta_n)).$$

So far, we have obtained

$$(63) \quad \limsup_\varepsilon \mathcal{L}_\varepsilon^L(B(\xi_c; r)) \leq \mathbb{W}(B(g_k, q + \delta_n)) + C_n \gamma.$$

It remains, in the right hand side, to let  $\gamma \rightarrow 0$  (hence  $g_k \rightarrow \xi_c$  and  $q \rightarrow r$ ), and then  $n \rightarrow \infty$  to get

$$(64) \quad \limsup_\varepsilon \mathcal{L}_\varepsilon^L(B(\xi_c; r)) \leq \mathbb{W}(B(\xi_c, r))$$

With the same line of reasoning, using the other part of (56) we can obtain

$$(65) \quad \liminf_\varepsilon \mathcal{L}_\varepsilon^L(B(\xi_c; r)) \geq \mathbb{W}(B(\xi_c, r)),$$

which ends the proof for one ball.

A similar proof can be made for functions approximating intersection of balls as in Theorem 2.2 of [15] and as a consequence the a.s. weak convergence follows.  $\square$

Eventually, we have the LDP as in Proposition 3.1. Recall that  $(\xi_t^1)$  is the meta-Slepian process defined in (46). We omit the proof since it is the same as in the scalar case.

**Proposition 6.4.** *The family  $(\mathcal{L}_\varepsilon)$  satisfies the LDP in  $\mathcal{M}_1(\mathcal{C}([0, 1]))$  equipped with the weak topology, in the scale  $\varepsilon^{-1}$  with good rate function*

$$(66) \quad \Lambda^*(\mathcal{L}) = \sup_{F \in \mathcal{C}_b(\mathcal{C}([0,1]))} \int_{\mathcal{C}([0,1])} F(\xi) \mathcal{L}(d\xi) - \Lambda(F),$$

(the Legendre dual of  $\Lambda$ ) where for every  $F \in \mathcal{C}_b(\mathcal{C}([0, 1]))$ ,

$$(67) \quad \Lambda(F) = \lim_{T \rightarrow \infty} T^{-1} \log \mathbb{E} \int_0^T F(\xi_t^1) dt.$$

## 7 Discrete versions

For a possible discrete version of Wschebor's theorem and associated LDP, we can consider a continuous process observed at times  $(k/n)$  where  $k \leq n$  with lag  $r$ . On this basis, there are two points of view. When  $r$  is fixed, there are already results on a.s. convergence of empirical measures of increments of fBm ([2]) and we explain which LDP holds. When  $r$  depends on  $n$  with  $r_n \rightarrow \infty$  and  $r_n/n \rightarrow 0$ , we are actually changing  $t$  in  $k/n$  and  $\varepsilon$  in  $r_n/n$  in the above sections. We state convergence (Prop. 7.1) and LDP (Prop. 7.2) under specific conditions.

All the LDPs mentioned take place in  $\mathcal{M}^1(\mathbb{R})$  equipped with the weak convergence.

### 7.1 Fixed lag

In [2], beyond the Wschebor's theorem, there are results of a.s. convergence of empirical statistics on the increments of fBm. The authors defined p. 39 the second order increments as

$$\Delta_n B_H(i) = \frac{n^H}{\sigma_{2H}} \left[ B_H \left( \frac{i+2}{n} \right) - 2B_H \left( \frac{i}{n} \right) + B_H \left( \frac{i}{n} \right) \right].$$

and claimed that as  $n \rightarrow \infty$

$$(68) \quad \frac{1}{n-1} \sum_0^{n-2} \delta_{\Delta_n B_H(i)} \Rightarrow \mathcal{N} \quad (a.s.),$$

(Th. 3.1 p.44 in [2]). Moreover, in a space-time extension, they proved that

$$(69) \quad \frac{1}{n-1} \sum_0^{n-2} \delta_{\frac{i}{n}, \Delta_n B_H(i)} \Rightarrow \lambda \otimes \mathcal{N} \quad (a.s.),$$

(Th. 4.1 in [3]).

Let us restrict for the moment to the case  $H = 1/2$ . The empirical distribution of (68) has the same distribution as

$$\frac{1}{n-1} \sum_0^{n-2} \delta_{2^{-1/2}(X_{i+2}-X_{i+1})}$$

where the  $X_i$  are independent and  $\mathcal{N}$  distributed. We can deduce the LDP (in the scale  $n$ ) from the LDP for the 2-empirical measure by contraction. If  $i$  is the mapping

$$(70) \quad \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R} \\ (x_1, x_2) & \mapsto & (x_2 - x_1)/\sqrt{2} \end{array}$$

the rate function is

$$(71) \quad I(\nu) = \inf \{ I_2(\mu); \mu \circ i^{-1} = \nu \},$$

where  $I_2$  is the rate function of the 2-empirical distribution (see [12] Th. 6.5.12).

In the same vein, we could study the LDP for the empirical measure

$$\frac{1}{n-r} \sum_0^{n-r-1} \delta_{\frac{W(k+r)-W(k)}{\sqrt{r}}}$$

which looks like  $\mathcal{W}_1^r$ . When this lag  $r$  is fixed, the scale is  $n$  and the rate function is obtained also by contraction ( $r = 1$  is just Sanov's theorem).

This point of view could be developed also for the fBm using stationarity instead of independence.

## 7.2 Unbounded lag

Let  $(X_i)$  be a sequence of i.i.d. random variables and  $(S_i)$  the process of partial sums. Let  $(r_n)$  be a sequence of positive integers such that  $\lim_n r_n = \infty$ , and assume that

$$(72) \quad \varepsilon_n := \frac{r_n}{n} \searrow 0.$$

Set

$$(73) \quad V_k^n := \frac{S_{k+r_n} - S_k}{\sqrt{r_n}}, \quad m_n = \frac{1}{n} \sum_1^n \delta_{V_k^n}.$$

The next propositions state some extensions of Wschebor's theorem and give the associated LDPs. The a.s. convergence is obtained only in the Gaussian case under an additional condition. It seems difficult to find a general method.

**Proposition 7.1.** 1. If  $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = 1$ , then

$$(74) \quad m_n \Rightarrow \mathcal{N} \quad (\text{in probability}).$$

2. If  $X_1 \sim \mathcal{N}$  and if  $(\varepsilon_n)$  is such that there exists  $\delta \in (0, 1/2)$  and a subsequence  $(n_k)$  satisfying

$$(75) \quad \sum_k \varepsilon_{n_k} < \infty \text{ and } \varepsilon_{n_k} = \varepsilon_{n_{k+1}} + o(\varepsilon_{n_{k+1}}^{1+\delta}),$$

it holds that

$$(76) \quad m_n \Rightarrow \mathcal{N} \quad (\text{a.s.}).$$

**Proposition 7.2.** 1. Assume that  $X_1 \sim \mathcal{N}$ . If  $\lim_n \varepsilon_n n^{1/2} = \infty$ , then  $(m_n)$  satisfies the LDP in the scale  $\varepsilon_n^{-1}$  with rate function given in (17-18) where  $\psi = \Psi_1$ .

2. Assume that  $X_1$  has all its moments finite and satisfies  $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = 1$  and that

$$(77) \quad 0 < \liminf_n \varepsilon_n \log n \leq \limsup_n \varepsilon_n \log n < \infty.$$

Then  $(m_n)$  satisfies the LDP in the scale  $\varepsilon_n^{-1}$  with rate function given in (17-18) where  $\psi = \Psi_1$ .

**Remark 7.3.** Two examples of  $(r_n)$  satisfying the assumptions of Prop. 7.1 2. are of interest, particularly in relation to the LDP of Prop. 7.2. The first one is  $r_n = \lfloor n^\gamma \rfloor$  with  $\gamma \in (0, 1)$  (hence  $\varepsilon_n \sim n^{\gamma-1}$ ), for which we can choose  $n_k = \lfloor k^{a(1-\gamma)} \rfloor$  with  $a > 1$ . The second one is  $r_n = \lfloor n/\log n \rfloor$  (hence  $\varepsilon_n \sim (\log n)^{-1}$ ), for which we can choose  $n_k = \lfloor e^{k^2} \rfloor$ .

**Proof of Prop. 7.1:** We use the method of the above Lemma 6.3 inspired by [1]. For a bounded continuous test function  $f$

$$\mathbb{E} \int f dm_n = \mathbb{E} f \left( \frac{S_{r_n}}{\sqrt{r_n}} \right) \rightarrow \int f d\mathcal{N}$$

thanks to the CLT. Moreover

$$\text{Var} \left( \int f dm_n \right) = \frac{1}{n^2} \sum_{|j-k| \leq r_n} \text{Cov} \left( F \left( \frac{S_{j+r_n} - S_j}{\sqrt{r_n}} \right), F \left( \frac{S_{k+r_n} - S_k}{\sqrt{r_n}} \right) \right) \leq \frac{2r_n}{n} \|f\|_\infty.$$

This gives the convergence in probability.

In the Gaussian case, it is possible to repeat the end of the proof of Lemma 6.3. Under our assumption, we see that for any  $\beta \in (0, 1/2)$

$$\frac{\varepsilon_{n_k}^\beta - \varepsilon_{n_{k+1}}^\beta}{\varepsilon_{n_{k+1}}^{1/2}} = o \left( \varepsilon_{n_{k+1}}^{\delta+\beta-\frac{1}{2}} \right),$$

which implies that it is enough to choose  $\beta \in (\frac{1}{2} - \delta, \frac{1}{2})$ . □

**Proof of Prop. 7.2:** 1) If  $X_1 \sim \mathcal{N}$ , then

$$(V_k^n, k = 1, \dots, n) \stackrel{(d)}{=} \left( (\varepsilon_n)^{-1/2} \left( W \left( \frac{k}{n} + \varepsilon_n \right) - W \left( \frac{k}{n} \right) \right), k = 1, \dots, n \right)$$

and then it is natural to consider  $m_n$  as a Riemann sum. We have now to compare  $m_n$  with

$$\mu_{\mathcal{W}_1^{\varepsilon_n}} = \int_0^1 \delta_{\varepsilon_n^{-1/2}(W(t+\varepsilon_n)-W(t))} dt.$$

It is known that  $d_{BL}(\mu, \nu)$  given by (39) is a convex function of  $(\mu, \nu)$  so that :

$$\begin{aligned} d_{BL}(m_n, \mu_{\mathcal{W}_1^{\varepsilon_n}}) &\leq \int_0^1 d_{BL}(\delta_{\varepsilon_n^{-1/2}(W(t+\varepsilon_n)-W(t))}, \delta_{V_{\lfloor nt \rfloor}^n}) dt \\ &\leq \varepsilon_n^{-1/2} \int_0^1 \left| W(t+\varepsilon_n) - W(t) - W \left( \frac{\lfloor nt \rfloor}{n} + \varepsilon_n \right) + W \left( \frac{\lfloor nt \rfloor}{n} \right) \right| dt \\ &\leq 2(\varepsilon_n)^{-1/2} \sup_{|t-s| \leq 1/n} |W(t) - W(s)| \end{aligned}$$

hence

$$\mathbb{P}(d_{BL}(m_n, \mu_{\mathcal{W}_1^{\varepsilon_n}}) > \delta) \leq \mathbb{P} \left( \sup_{|t-s| \leq 1/n} |W(t) - W(s)| > \frac{\delta(\varepsilon_n)^{1/2}}{2} \right) \leq 2 \exp -n \frac{\delta^2 \varepsilon_n}{4}.$$

If  $\lim_n \varepsilon_n n^{1/2} = \infty$  we conclude that

$$\lim_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(d_{BL}(m_n, \mu_{\mathcal{W}_1^{\varepsilon_n}}) > \delta) = -\infty,$$

which means that  $(m_n)$  and  $(\mu_{\mathcal{W}_1^{\varepsilon_n}})$  are exponentially equivalent in the scale  $\varepsilon_n^{-1}$  (Def. 4.2.10 in [12]).

Now, from our Prop. 3.1 or 4.1,  $(\mu_{\mathcal{W}_1^{\varepsilon_n}})$  satisfies the LDP in the scale  $\varepsilon_n^{-1}$ . Consequently, from Th. 4.2.13 of [12], the family  $(m_n)$  satisfies the LDP at the same scale with the same rate function.

2) Let us go to the case when  $X_1$  is not normal. We use the Skorokhod representation, as in [16] or in [17] (see also [9] Th. 2.1.1 p.88).

When  $(X_i)$  is a sequence of independent (real) random variables such that  $\mathbb{E}X_1 = 0$  and  $\mathbb{E}X_1^2 = 1$ , there exists a probability space supporting a Brownian motion  $(B(t); 0 \leq t < \infty)$  and an increasing sequence  $(\tau_i)$  of stopping times such that

- $(\tau_{i+1} - \tau_i)$  are i.i.d., with  $\mathbb{E}\tau_1 = 1$
- $(B(\tau_{i+1}) - B(\tau_i))$  are independent and distributed as  $X_1$ ,

Moreover, if  $\mathbb{E}X_1^{2q} < \infty$ , then  $\mathbb{E}\tau_1^q < \infty$ .

We have

$$S_{j+r} - S_j \stackrel{(d)}{=} B(\tau_{j+r}) - B(\tau_j)$$

so that

$$(78) \quad m_n \stackrel{(d)}{=} \tilde{m}_n := \frac{1}{n} \sum_1^n \delta_{\tilde{V}_k^n} \text{ with } \tilde{V}_k^n = \frac{B(\tau_{k+r_n}) - B(\tau_k)}{\sqrt{r_n}}.$$

We will compare these quantities with

$$(79) \quad \pi_n = \frac{1}{n} \sum_1^n \delta_{U_k^n} \text{ with } U_k^n := \frac{B(k+r_n) - B(k)}{\sqrt{r_n}},$$

which fall into the regime of the above part of the proof. We will prove that the sequences  $(\tilde{m}_n)$  and  $(\pi_n)$  are exponentially equivalent.

Again by convexity of  $d_{BL}$ , we have

$$(80) \quad \begin{aligned} d_{BL}(\tilde{m}_n, \pi_n) &\leq \sum_1^n \frac{1}{n} d_{BL}(\delta_{\tilde{V}_k^n}, \delta_{U_k^n}) \\ &\leq \frac{1}{\sqrt{r_n}} \left( \sup_{k \leq n} |B(\tau_{k+r_n}) - B(k+r_n)| + \sup_{k \leq n} |B(\tau_k) - B(k)| \right) \end{aligned}$$

Our proof will be complete if we show that for all  $\delta > 0$

$$(81) \quad \lim_n \frac{r_n}{n} \log \mathbb{P} \left( \max_{k \leq n+r_n} |B(\tau_k) - B(k)| > \delta \sqrt{r_n} \right) = -\infty.$$



We will apply three times the following known result.

If  $(\xi_i)$  are i.i.d. centered with  $\mathbb{E}(\xi_1^{2p}) < \infty$  for some  $p \geq 1$ , then there exists a universal constant  $C > 0$  such that for all integers  $n \geq 1$

$$(82) \quad \mathbb{E}(\xi_1 + \dots + \xi_n)^{2p} \leq C(2p)! \mathbb{E}(\xi_1^{2p}) n^p,$$

(cf. [17] Lemma 8 or [16] Lemma 2.9).

Actually, for  $\alpha \in (0, 1)$  and  $k \leq r_n^\alpha$ , with Markov inequality and (82)

$$(83) \quad \mathbb{P}(|B(\tau_k)| > \delta\sqrt{r_n}) \leq C(2p)! \delta^{-2p} r_n^{-p} \mathbb{E}((X'_1)^{2p}) k^p \leq C(2p)! \delta^{-2p} \mathbb{E}((X'_1)^{2p}) r_n^{(\alpha-1)p},$$

and for the same reasons

$$(84) \quad \mathbb{P}(B(k) > \delta\sqrt{r_n}) \leq C(2p)! \mathbb{E}(N^{2p}) \delta^{2p} r_n^{(\alpha-1)p}.$$

Now, for  $k \geq r_n^\alpha$ , and  $\beta > 1/2$

$$\mathbb{P}(|\tau_k - k| \geq k^\beta) \leq C(2p)! \mathbb{E}((\tau_1 - 1)^{2p}) k^{p(1-2\beta)} \leq C(2p)! \mathbb{E}((\tau_1 - 1)^{2p}) r_n^{\alpha p(1-2\beta)}.$$

Besides,

$$\begin{aligned} \mathbb{P}(|B(\tau_k) - B(k)| \geq 2\delta\sqrt{r_n}, |\tau_k - k| \leq k^\beta) &\leq \mathbb{P}\left(\sup_{|t-k| \leq k^\beta} |B(t) - B(k)| > 2\delta\sqrt{r_n}\right) \\ &\leq 2\mathbb{P}\left(\sup_{t \in (0, k^\beta)} |B(t) - B(k)| > 2\delta\sqrt{r_n}\right) \leq 4e^{-2\delta^2 r_n k^{-\beta}}, \end{aligned}$$

which, for  $k \leq n + r_n < 2n$ , yields

$$(85) \quad \mathbb{P}(|B_{\tau_k} - B_k| \geq 2\delta\sqrt{r_n}, |\tau_k - k| \leq k^\beta) \leq 4e^{-2^{1-\beta} \delta^2 r_n n^{-\beta}}.$$

Gathering (83-84-85), we obtain, by the union bound,

$$(86) \quad \mathbb{P}\left(\max_{k \leq n+r_n} |B(\tau_k) - B(k)| > 2\delta\sqrt{r_n}\right) \leq C_p \left(\delta^{2p} r_n^{1+(\alpha-1)p} + n r_n^{\alpha(1-2\beta)p}\right) + 8ne^{-2^{1-\beta} \delta^2 r_n n^{-\beta}},$$

where the constant  $C_p > 0$  depends on  $p$  and on the distribution of  $X'_1$ .

Choosing  $\beta > 1/2$  and  $r_n$  such that

$$(87) \quad \liminf_n \frac{r_n}{n} \log r_n > 0, \quad \limsup_n \frac{r_n}{n} \log n < \infty, \quad \liminf_n \frac{r_n^2}{n^{1+\beta}} > 0,$$

we will ensure that for every  $p > 0$

$$(88) \quad \lim \frac{r_n}{n} \log \mathbb{P}\left(\max_{k \leq n+r_n} |B(\tau_k) - B(k)| > \delta\sqrt{r_n}\right) \leq -Cp$$

where  $C$  is a constant independent of  $p$ , which will prove (81).

Now, the set of sufficient conditions (87) is equivalent to the condition:

$$0 < \liminf_n \frac{r_n}{n} \log n \leq \limsup_n \frac{r_n}{n} \log n < \infty,$$

which is exactly (77). □

## 8 Appendix

The extension of Lévy's continuity theorem, already invoked in [1] is the following. It is probably well known, but since we do not know any reference, we give its proof for the convenience of the reader.

**Lemma 8.1.** *Let  $\nu_n, \nu$  be probability measures on  $\mathbb{R}^d$  with characteristic functions  $\varphi_n, \varphi$ . A sufficient condition for  $\nu_n \Rightarrow \nu$  is that  $\varphi_n(\mathbf{a}) \rightarrow \varphi(\mathbf{a})$  for almost every  $\mathbf{a} \in \mathbb{R}^d$ .*

We follow the classical proof as given for instance in Billingsley [5] Theorem 26.3. Since the limiting point is determined, we have just to prove the tightness. Considering the compact set  $(-M, M]^d \subset \mathbb{R}^d$ , we have by the union bound

$$\nu_n(K^c) \leq \nu_n^k(|x| > M)$$

where  $\nu_n^k, k = 1, \dots, d$  are the marginals of  $\nu_n$ . So the problem can be reduced to  $d = 1$ . The basic inequality

$$\nu_n^k(|x| > M) \leq \frac{M}{2} \int_{-2/M}^{2/M} (1 - \nu_n^k(a)) da.$$

Since  $\varphi$  is continuous at 0 with  $\varphi(0) = 1$ , there is for positive  $\eta$  some  $M$  such that

$$\frac{M}{2} \int_{-2/M}^{2/M} (1 - \nu^k(a)) da < \eta$$

Since  $\varphi_n$  converges to  $\varphi$  almost everywhere, and since the integrand is bounded by 2, the dominated convergence theorem gives

$$\lim_n \int_{-2/M}^{2/M} (1 - \nu_n^k(a)) da = \int_{-2/M}^{2/M} (1 - \nu^k(a)) da,$$

so that there exists  $n_0$  such that

$$\frac{M}{2} \int_{-2/M}^{2/M} (1 - \nu_n^k(a)) da < \frac{\eta}{2}$$

for  $n \geq n_0$ . Ending is routine. □

## References

- [1] J-M. Azaïs and M. Wschebor. Almost sure oscillation of certain random processes. *Bernoulli*, pages 257–270, 1996.
- [2] C. Berzin, A. Latour, and J. León. *Inference on the Hurst parameter and the variance of diffusions driven by fractional Brownian motion*, volume 216 of *Lecture Notes in Statistics*. Springer, 2014.

- [3] C. Berzin, A. Latour, and J. León. Variance estimator for fractional diffusions with variance and drift depending on time. *Electron. J. Stat.*, 9(1):926–1016, 2015.
- [4] C. Berzin and J. León. Weak convergence of the integrated number of level crossings to the local time for Wiener processes. *Theory Probab. Appl.*, 42:568–579, 1997.
- [5] P. Billingsley. Probability and measure. 2nd edn., 1986.
- [6] W. Bryc and A. Dembo. On large deviations of empirical measures for stationary Gaussian processes. *Stochastic Processes Appl.*, 58(1):23–34, 1995.
- [7] W. Bryc and A. Dembo. Large deviations and strong mixing. *Ann. Inst. Henri Poincaré Probab. Stat.*, 32:549–569, 1996.
- [8] T. Chiyonobu and S. Kusuoka. The large deviation principle for hypermixing processes. *Probab. Theory Related Fields*, 78(4):627–649, 1988.
- [9] M. Csörgö and P. Révész. *Strong approximations in probability and statistics*. Academic Press, 2014.
- [10] B. Davies. *Integral transforms and their applications*, volume 25 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1985.
- [11] A. Dembo and T. Zajic. Large deviations: from empirical mean and measure to partial sums process. *Stochastic Processes Appl.*, 57(2):191–224, 1995.
- [12] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.
- [13] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of integral transforms. Vol. I*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. Based, in part, on notes left by Harry Bateman.
- [14] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Higher transcendental functions. Vol. I*. Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, With a preface by Mina Rees, With a foreword by E. C. Watson, Reprint of the 1953 original.
- [15] E. Giné and J. León. On the central limit theorem in hilbert space. *Stochastica*, 4(1):43–71, 1980.
- [16] M.K. Heck. The principle of large deviations for the almost everywhere central limit theorem. *Stochastic Process Appl.*, 76:61–75, 1998.
- [17] P. March and T. Seppäläinen. Large deviations from the almost everywhere central limit theorem. *J. Theoret. Probab.*, 10(4):935–965, 1997.

- [18] M. Marcus and J. Rosen. CLT for  $L^p$  moduli of continuity of Gaussian processes. *Stochastic Process. Appl.*, 118(7):1107–1135, 2008.
- [19] M. Marcus and J. Rosen.  $L^p$  moduli of continuity of Gaussian processes and local times of symmetric Lévy processes. *Ann. Probab.*, 36(2):594–622, 2008.
- [20] G. Samorodnitsky and M. Taqqu. *Non-Gaussian Stable Processes: Stochastic Models with Infinite Variance*. Chapman et Hall, London, 1994.
- [21] D.W Stroock. *An introduction to the theory of large deviations*. Springer Science & Business Media, 2012.
- [22] M. Wschebor. Sur les accroissements du processus de Wiener. *C. R. Math. Acad. Sci. Paris*, 315(12):1293–1296, 1992.
- [23] M. Wschebor. Almost sure weak convergence of the increments of Lévy processes. *Stochastic Processes Appl.*, 55(2):253–270, 1995.
- [24] M. Wschebor. Smoothing and occupation measures of stochastic processes. *Ann. Fac. Sci. Toulouse Math. (6)*, 15(1):125–156, 2006.