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Abstract

We axiomatize in continuous logic for metric structures σ-finite W^*-probability spaces and preduals of von Neumann algebras jointly with a weak-* dense C^*-algebra of its dual. This corresponds respectively to the Ocneanu ultrapower and the Groh ultrapower of (σ-finite in the first case) von Neumann algebras. We give various axiomatizability results corresponding to recent results of Ando and Haagerup including axiomatizability of III_λ factors for $0 < \lambda \leq 1$ fixed and their preduals. We also strengthen the concrete Groh theory to an axiomatization result for preduals of von Neumann algebras in the language of tracial matrix-ordered operator spaces, a natural language for preduals of dual operator systems. We give an application to the isomorphism of ultrapowers of factors of type III and II_∞ for different ultrafilters.

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1. Introduction

The model theory of metric structures (see [2]) was recently applied to analyse ultrapowers of C^*-algebras and tracial von Neumann algebras [12, 13, 14]. We refer to [16] for a survey and to [19, 11, 4] for more recent developments in the von Neumann algebraic context. From an operator algebraic viewpoint, the relations between the various ultraproducts of von Neumann algebras (or σ-finite von Neumann algebras) was recently clarified in [1]. It is thus expected that continuous model theoretic tools would enhance the study of those ultraproducts beyond the tracial case.

This is the goal of this paper to give axiomatizations in continuous model theory of various classes of von Neumann algebras and recover as model theoretic ultraproducts the two main ultrapowers: the Groh ultrapower $\prod^\omega M$ of a von Neumann algebra M and the Ocneanu ultrapower $(M, \varphi)^\omega$ of a σ-finite W^*-probability space (M, φ), namely a pair of a σ-finite von Neumann algebra and a faithful normal state φ. In the first case, our model theory will rather be a model theory of the class of preduals of von Neumann algebras, giving an axiomatization of these preduals in continuous model theory. This is not surprising since the Groh ultraproduct is by definition the dual of the metric ultraproduct of preduals. All our axiomatizations will be in the continuous model theory setting for operator algebras from [13], a multidomain variant of the first order (metric) continuous logic from [2]. We want to mention that Ilijas Farah and Bradd Hart have checked in an unpublished work that general von Neumann algebras form a "compact abstract theory" in the sense of [3]. It is crucial for our purposes to consider σ-finite von Neumann algebras where a faithful normal state is available to get an axiomatization in the better behaved metric setting with more convenient syntactic counterpart of the semantic. However, for preduals of von Neumann algebras, we do axiomatize without any σ-finiteness assumption (a property that wouldn't be axiomatizable in our language for preduals anyway).

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Let us point out that our axiomatizations will be often explicit, but sometimes, as for preduals of von Neumann algebras (theorem 3.3), we will obtain the existence of an axiomatization in some explicit language by using a standard model-theoretic result [2, Prop 5.14] and proving only stability by ultraproduct and ultraroot of a class of models. In this case, the stability by ultraproducts for ultrafilters on $\mathbb{N}$ is always contained in [1] (sometimes with another language) but we give a model theoretic proof of the general ultraproduct case. Of course, determining an appropriate language where we can show both stability by ultraproduct and ultraroot is the key new contribution in getting such an axiomatizability result. This enables us to prove axiomatization results for the natural classes found to be stable by countable ultraproducts in [1], such as III_λ factors for a fixed $0 < \lambda \leq 1$ (cf. Theorem 2.10). Note that since the model theoretic result on axiomatizability is based on Keisler-Shelah theorem characterizing elementary equivalence, considering ultraproducts based on ultrafilters on $\mathbb{N}$ is a priori not enough even in the separable case (cf. [14, Question after Rmk 4.2]). Our extra-work with uncountable ultraproducts is thus necessary to use the available model theoretic results.

We also want to emphasize that, even though it is inspired from [1], we give an alternative construction of the Ocneanu ultraproduct that does not use the relation to the Groh ultraproduct to prove we have a von Neumann algebra structure. We use their results only to identify our ultraproduct with the usual Ocneanu ultraproduct.

Of course, having a model theoretic axiomatization enables to use interesting continuous model theoretic tools to study ultraproducts. Ultraproducts are essential tools in both operator algebras and model theory. Classifying them remains a problem of central interest. For instance, only recently it was discovered an infinite family of II_1 factors that have non-isomorphic ultrapowers [4]. For the type III case the results are even more scarce. Let us give, as a motivation for our study, a first consequence in the next theorem concerning isomorphism of factorial ultrapowers. We leave for further investigation the study of stability of (σ-finite) von Neumann algebras parallel to [12].

Theorem 1.1. *Let M a von Neumann algebra with separable predual and φ a faithful normal state on M.*

1. *If the Continuum Hypothesis holds, then for any nonprincipal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, we have isomorphisms of the Groh ultrapowers $\prod^{\mathcal{U}} M \simeq \prod^{\mathcal{V}} M$ and the Ocneanu ultrapowers $(M,\varphi)^{\mathcal{U}} \simeq (M,\varphi)^{\mathcal{V}}$.*
2. *If the Continuum Hypothesis fails and M is a factor which is not of type III_0, then M is not of type I if and only if there exist nonprincipal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ such that $(M,\varphi)^{\mathcal{U}} \not\simeq (M,\varphi)^{\mathcal{V}}$ as von Neumann algebras.*

We give the proof in the next-to-last section 6. It uses in an intrinsically linked way various strong model theoretic results (most notably from [15]) available thanks to our various axiomatizations and general structure theory of factors well-known to operator algebraists. The reader familiar with the finite case and/or some structure theory of type III factors can probably read it right away without reading all the axiomatization details of the general case. The second point partially generalizes [12, Th 4.7] (in the factor case). We conjecture that it is also valid for type III_0 factors and probably for non-factors. The study of the non-factorial case would require a separate investigation of commutation of ultrapowers and central decompositions. We leave this for a future study. Since ultraproducts of type III_0 factors are usually not factors, we point out that we even stick here to factorial ultrapowers, at least in the second statement. We also emphasize that, even though the Ocneanu theory is a theory for W^*-probability spaces, the above non-isomorphism for ultrapowers is at the von Neumann algebraic level. This may be surprising since the Ocneanu theory strongly depends on the state φ we put on M. However, the general structure theory of factors and axiomatizability of certain kinds of states (such as periodic states in the III_λ case with $0 < \lambda < 1$) enable us to remove this dependence from the final statement.

Let us now summarize the content of this paper. More details on the main ideas and results are given at the beginning of each section. In addition to the introduction, the paper has 6 sections.

Section 2 is mainly concerned with producing a theory corresponding to the Ocneanu ultraproduct. Of course, the main problem in the non-tracial case is to deal with Tomita-Takesaki modular theory. Since the modular group is well behaved for this ultraproduct [1, Th 4.1], it is natural to include it in the theory. However, in Subsection 2.4, suggested by questions of Ilijas Farah and Itaï Ben Yaacov, we give a more

minimal language where an axiomatization of σ-finite W^*-probability spaces is available, at the cost of loosing the universal explicit axiomatization. This language does not contain the modular group.

Section 3 shows the axiomatizability of preduals of von Neumann algebras in a natural language giving an ultraproduct corresponding to the Groh ultraproduct after taking duals. Most notably, it uses various operator system techniques.

In order to study the Ocneanu theory, one needs a relation between the Groh and Ocneanu theories. To make our theory easier to read, we start with an axiomatization of standard forms in subsection 3.3. This is a natural expansion of Groh's theory. The theory linking Groh and Ocneanu theories is then suggested by the corresponding relations of ultraproducts in [1] and written down explicitly in section 4. That's why we call it the Ando-Haagerup theory.

Finally, to get the lack of stability by ultrapowers of type III_0 factors, one needs a stability property of their discrete decomposition. We obtain it in the case of non-countable ultraproducts in section 5 (the countable case is again contained in [1]).

Section 6 contains the proof of Theorem 1.1. Section 7 is an appendix dealing with the elementary model theory of Lipschitz curves and their Riemann integrals. Its purpose is mostly to improve the readability of the main text. Since at least eight of our axioms depend on Riemann integrals related to spectral theory of the modular group, it was worth having a general framework enabling to write down those integral formulas explicitly in our axiomatizations.

Let us finish by pointing out that, following the operator algebraic tradition, ω will always be a non-principal ultrafilter on a set I (maybe uncountable).

2. The Ocneanu Theory for σ-finite von Neumann algebras

This section is mainly concerned with producing a theory corresponding to the Ocneanu ultraproduct.

Subsection 2.1 proves elementary lemmas needed to identify the language and several crucial properties of the theory that will enable us to characterize most of the pieces of this language (including the modular group) in first order continuous logic. Its goal is to give operator algebraic background for model theorists while showing to operator algebraists the way to look for useful operator algebraic results for our axiomatization purposes. Note that the KMS condition does not seem easy to express in this way and we have to prefer explicit integral formulas for unbounded operators. This makes the axiomatization much trickier than the one of tracial von Neumann algebras or C^*-algebras. In the non-tracial case, the choice of the topology turns out to be crucial. Since a model theoretic ultraproduct is always a quotient of bounded sequences, we have to consider a topology so that the Ocneanu ultraproduct will be given by a quotient of a set of bounded sequences and not some multiplier algebra as in the original definition. The inspiration comes from such a quotient description in [1, Proposition 3.14]. Instead of the *-strong topology, we use the topology for which a net $x_n \to 0$ if $x_n = y_n + z_n$ with y_n converging strongly to 0 and z_n^* converging strongly to 0. The second critical problem is the identification of the product which is not uniformly continuous for neither the strong topology nor the topology described above on the unit ball of M. Usually, in modular theory, this lack of uniform continuity is circumvented by restriction to the so-called spectral algebras. But unfortunately, they are not in general stable by ultraproducts, so that we cannot include them in our first order theory. Instead of including these algebras in the theory, we follow a more concrete approach, nicely commuting with ultraproducts, to produce elements in those algebras. In that respect, we follow Ando and Haagerup and use Fejer maps F_N^φ obtained by integrating Fejer's kernel with the modular group. Hence, we will only include in the theory smeared products $m_{N,M}(x,y) = F_N^\varphi(x).F_M^\varphi(y)$. Those smeared products are uniformly continuous, as wanted. Combined with standard estimates on spectral algebras recalled in lemma 2.1, this will give all the necessary pieces of data for the theory: the state, the adjoint, the metric, the modular group and the smeared products. However, to obtain a universal axiomatization, we also use various other data we will detail in due time. With these preliminaries at hand, we can write down our axiomatization in subsection 2.2. In the proof of theorem 2.7, we produce a von Neumann algebra from a model of this theory. This uses a GNS construction starting from an algebra generated by the various $F_N^\varphi(x)$ between which the product is already defined. This especially does not use any relation to the Groh ultraproduct or any other

non-σ-finite von Neumann algebra. Of course, the theory reduces to the tracial theory from [13] when the state is a trace, all the supplementary data being trivial, for instance all the smeared products $m_{N,M}$ are the usual multiplication map.

Subsection 2.3 then gives various supplementary axiomatization results for natural classes of von Neumann algebras in theorem 2.10. The reader should notice that the non-explicit axiomatization for type III_λ factors, $\lambda \in (0,1]$ fixed, uses for its proof various generalisations of results of [1] obtained later in sections 4 and 5. We also advertise in Rmk 2.12 various non-axiomatizability results in our language for W^*-probability spaces straightforwardly deduced from results in [1].

Subsection 2.4, suggested by questions of Ilijas Farah and Itaï Ben Yaacov, gives a more minimal language where an axiomatization of σ-finite W^*-probability spaces is available, at the cost of loosing the universal explicit axiomatization. The main point is to check a definability in the sense of [2] of the modular group in a minimal enough language using some more technical (but standard) spectral theory.

2.1. Setting and preliminaries

We endow σ-finite von Neumann algebras with a fixed faithful normal state φ which will have one sort U with domains of quantification $D_n = D_n(U)$ for the operator norm ball of radius n centred around 0. We will also write $D_n(M)$ for this operator norm ball of a von Neumann algebra M, or any subspace. The metric d will be related to φ below in a way reducing to the usual L^2 norm when φ is a trace, thus reducing to [13] in this special case. We refer to this paper for necessary continuous model theory background. By Ocneanu theory, we mean that we want a theory such that model theoretic ultraproduct recovers the Ocneanu ultraproduct.

Let us recall several norms related to φ:

$$||x||_\varphi^2 = \varphi(x^*x), \quad ||x||_\varphi^\# = \sqrt{||x||_\varphi^2 + ||x^*||_\varphi^2}$$

so that it is well-known that $||.||_\varphi$ defines the strong operator topology and $||.||_\varphi^\#$ the strong-* operator topology on the unit ball of M in the σ-finite case with φ faithful. In the non-tracial case, these results are of course not obvious and will be explained later, they depend on the modular theory.

For our purposes, we introduce the following important norm:

$$||x||_\varphi^* = \inf_{y \in M} \left[\sqrt{\varphi(y^*y) + \varphi((x-y)(x-y)^*)} \right]. \tag{2.1}$$

We want to take $d(x,y) = ||x-y||_\varphi^*$. The motivation for this is the description of the Ocneanu ultraproduct as a quotient vector space in [1, Proposition 3.14].

Thus we have to check that all common operations are uniformly continuous, and to specify their modulus of continuity. Note first that

$$||x^*||_\varphi^* = ||x||_\varphi^*,$$

and $|\varphi(x)| \leq |\varphi(y) + \varphi(x-y)| \leq \sqrt{2}\sqrt{\varphi(y^*y) + \varphi((x-y)(x-y)^*)}$ for any y by Cauchy-Schwarz and thus

$$|\varphi(x)| \leq \sqrt{2}||x||_\varphi^*.$$

Unfortunately, there is no uniform continuity bound for product. To deal with that, we will use modular theory. We refer to [30] for general results or [1] for some more specific properties.

Only recall that if ξ_φ denotes the GNS vector for φ, $S_\varphi^0(x\xi_\varphi) = x^*\xi_\varphi$ defines a densely defined closable operator with closure S_φ such that $\Delta_\varphi = S_\varphi^* S_\varphi$ and the polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$. J_φ is called the modular conjugation operator and Δ_φ the modular operator. The modular automorphism group is then defined by:

$$\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$$

Tomita's fundamental Theorem states that σ_t^φ leave M invariant and even defines a one parameter automorphism group of M. Note that we may drop the index φ in σ_t^φ and we will do so systematically when

we think of it as a map σ_t in the theory we will introduce in the next subsection. Moreover, σ_t^φ preserves adjoint and $\varphi \circ \sigma_t^\varphi = \varphi$. We will need Arveson's spectral theory (see [30, Section XI.1]). For $f \in L^1(\mathbb{R})$ and $x \in M$, one can define:

$$\sigma_f^\varphi(x) := \int_{\mathbb{R}} dt f(t) \sigma_t^\varphi(x) \in M, \quad \hat{f}(y) = \int_{\mathbb{R}} dt f(t) e^{ity},$$

so that we have a relation between our Fourier transform and functional calculus:

$$\sigma_f^\varphi(x)\xi_\varphi = \hat{f}(\ln(\Delta))(x\xi_\varphi).$$

Note that $\sigma_f^\varphi \circ \sigma_g^\varphi = \sigma_g^\varphi \circ \sigma_f^\varphi$. Then the spectrum of x is better understood by describing its complement as support usually is:

$$[\mathrm{Spec}_{\sigma^\varphi}(x)]^c = \{t \in \mathbb{R} \ : \ \exists f \in L^1(\mathbb{R}),\ \hat{f}(t) \neq 0 \text{ and } \sigma_f(x) = 0\}.$$

Conversely, from [30, Lemma XI.1.3], if $x \neq 0$, $\mathrm{Spec}_{\sigma^\varphi}(x) \neq \emptyset$ and if $\mathrm{supp}(\hat{f}) \subset [\mathrm{Spec}_{\sigma^\varphi}(x)]^c$ then $\sigma_f(x) = 0$. The crucial definition for us is the spectral subspace of a subset $E \subset \mathbb{R}$:

$$M(\sigma^\varphi, E) = \{x \in M : \mathrm{Spec}_{\sigma^\varphi}(x) \subset E\}.$$

$M(\sigma^\varphi, \{0\})$ is called the centralizer of φ and from [30, Corol XI.1.8], we have

$$M(\sigma^\varphi, E)^* = M(\sigma^\varphi, -E); \quad M(\sigma^\varphi, E)M(\sigma^\varphi, F) \subset M(\sigma^\varphi, \overline{E+F}). \tag{2.2}$$

We will also use the Arveson spectra defined by its complement:

$$\begin{aligned}[\mathrm{Sp}(\sigma^\varphi)]^c &= \{t \in \mathbb{R} \ : \ \exists f \in L^1(\mathbb{R}),\ \hat{f}(t) \neq 0 \text{ and } \sigma_f = 0\} \\ &= \{t \in \mathbb{R} \ : \ \exists \epsilon > 0,\ M(\sigma^\varphi, [t-\epsilon, t+\epsilon]) = \{0\}\}.\end{aligned}$$

The following result is deduced from an old result of Haagerup (cf the proof of [1, lemma 4.13]).

Lemma 2.1. *For any $a > 0$, $x \in M(\sigma^\varphi, [-a, a]), y \in M$, we have, with $C_a = 2e^a + e^{a/2}$:*

$$||(xy)^*||_\varphi \leq C_a ||x||\ ||y^*||_\varphi, \quad ||xy||_\varphi^\# \leq C_a ||x||\ ||y||_\varphi^\#, \quad ||xy||_\varphi^* \leq C_a ||x||\ ||y||_\varphi^*.$$

Thus product will be uniformly continuous on balls of $M(\sigma^\varphi, [-a, a])$. We could try taking those balls as domain of quantification $E_{a,n}, a, n-1 \in \mathbb{N}$ of another sort V, but they are in general not stable by ultraproduct and cannot be included in the theory.

We also record the following useful spectral theory result and deduce the modular theory formula for our distance. We will need it in our next lemma and we will use it crucially later.

Lemma 2.2. *If $g_s(t) = \frac{2e^{-ist}}{e^{\pi t} + e^{-\pi t}}$ then $||g_s||_{L^1(\mathbb{R})} = 1$ and*

$$\sigma_{g_s}^\varphi(x)\xi_\varphi = 2e^{s/2}\Delta^{1/2}(\Delta + e^s)^{-1}(x\xi_\varphi),$$

and, if we call $G_s^\varphi = \sigma_{g_s}^\varphi$ we have the equality, for any $x \in M$:

$$2||x||_\varphi^* = ||G_0^\varphi(x)||_\varphi^\#. \tag{2.3}$$

Proof : This first equality is [30, Lemma VI.1.21]. A completeness argument and a computation shows

$$\begin{aligned}&(||x||_\varphi^*)^2 \\ &= \inf_{y \in M} \left[||y\xi_\varphi||^2 + ||\Delta^{1/2} x \xi_\varphi||^2 + ||\Delta^{1/2}(y\xi_\varphi)||^2 - 2\Re\langle \Delta^{1/2}(y\xi_\varphi), \Delta^{1/2}(x\xi_\varphi)\rangle \right] \\ &= \inf_{\eta \in D(\Delta^{1/2})} \left[||\eta||^2 + ||\Delta^{1/2} x \xi_\varphi||^2 + ||\Delta^{1/2}(\eta)||^2 - 2\Re\langle \Delta^{1/2}(\eta), \Delta^{1/2}(x\xi_\varphi)\rangle \right].\end{aligned} \tag{2.4}$$

By Lax-Milgram lemma (see e.g. [5, Corol V.8] with $a(u,v) = \langle (1+\Delta)u, v\rangle$), the infimum is easily reached at $\eta = \eta_0 = \Delta(1+\Delta)^{-1}(x\xi_\varphi)$. Indeed, the minimization problem is equivalent to finding the inf of $a(\eta,\eta) - 2\Re[a(\eta,\eta_0)] + a(\eta_0,\eta_0) = a(\eta-\eta_0, \eta-\eta_0)$ which is obviously minimal at $\eta = \eta_0$.

Since $x\xi_\varphi - \Delta(1+\Delta)^{-1}(x\xi_\varphi) = (1+\Delta)^{-1}(x\xi_\varphi)$ we obtain by an easy computation:

$$(||x||_\varphi^*)^2 = ||\Delta(1+\Delta)^{-1}(x\xi_\varphi)||^2 + ||\Delta^{1/2}(1+\Delta)^{-1}(x\xi_\varphi)||^2 = ||\Delta^{1/2}(1+\Delta)^{-1/2}(x\xi_\varphi)||^2. \qquad (2.5)$$

On the other hand, a similar easy computation gives

$$\begin{aligned}\frac{(||G_0^\varphi(x)||_\varphi^\#)^2}{4} &= ||\Delta^{1/2}(\Delta+1)^{-1}(x\xi_\varphi)||^2 + ||\Delta(\Delta+1)^{-1}(x\xi_\varphi)||^2 \\ &= ||\Delta^{1/2}(1+\Delta)^{-1/2}(x\xi_\varphi)||^2.\end{aligned}$$

■

We will also use the next completeness result:

Lemma 2.3. *The operator norm unit balls $D_1(M)$ and $D_1(M(\sigma^\varphi,[-K,K]))$, for $K \geq 0$, are complete for d.*

Proof : Fix (a_n) a Cauchy net with $a_n \in D_1(M)$, then take a decomposition $a_n = b_n + c_n, b_n, c_n \in M$, b_n Cauchy for $||.||_\varphi$ and c_n Cauchy for $||(.)^*||_\varphi$ as is possible from the definition of d based on (2.1). From the proof of lemma 2.2, one can take $||b_n||_\varphi \leq 2||x||, ||(c_n)^*||_\varphi \leq 2||x||$. From the completeness of $L^2(M,\varphi)$ there is $b, c^* \in L^2(M,\varphi)$ such that $||b_n - b||_\varphi \to 0, ||c_n^* - c^*||_\varphi \to 0$. Let us call $a = b + c$ which is also the weak-* limit of $a_n = b_n + c_n$, which is thus in the operator norm unit ball $D_1(M)$. Finally using the alternative infimum describing d in formula (2.4) (and $b_n - b = a_n - a - (c_n - c) \in D(\Delta^{1/2})$):

$$d(a_n - a, 0)^2 \leq ||b_n - b||_\varphi^2 + ||c_n^* - c^*||_\varphi^2 \to 0,$$

thus $D_1(M)$ is indeed complete. Take any $g \in L^1(\mathbb{R})$ with $\mathrm{supp}(\hat{g}) \subset [-K,K]^c$, then if $a_n \in M(\sigma^\varphi,[-K,K])$ as above, $\sigma_g^\varphi(a_n) = 0$ and, using formula (2.3), one gets

$$2||\sigma_g^\varphi(a_n - a)||_\varphi^* = ||\sigma_g^\varphi G_0^\varphi(a_n - a)||_\varphi^\# \leq ||G_0^\varphi(a_n - a)||_\varphi^\# = ||(a_n - a)||_\varphi^* \to 0.$$

Thus we deduce $\sigma_g^\varphi(a) = 0$ and, since this is for all g as above, $a \in M(\sigma^\varphi,[-K,K])$. ■

From lemma 2.2, we also deduce an explicit (uniform) continuity bound for the modular group in the distance d. We could deduce from this an explicit bound giving an approximation formula for certain σ_f^φ that we could use in our axiomatization. Instead, we will use our general appendix and see the function $(f\sigma(x))_t = f(t)\sigma_t(x)$ integrated in the definition of $\sigma_f^\varphi(x)$ in a extra sort $Lip_{|f|}(\mathbb{R},U)$ for Lipschitz curves with a domination condition enabling to write in our theory the integral formula $\sigma_f^\varphi(x) = \int_{\mathbb{R}}(f\sigma(x))$. This Riemann integral and the extra sort $Lip_{|f|}(\mathbb{R},U)$ is shown to be easily axiomatizable in the appendix : section 7. Let us just point out that the norm giving domains of quantification $D_K(Lip_{|f|}(\mathbb{R},U))$ is given in (7.1) and the distance on those balls for our U is $d_{Lip}(f,g) = \sup_{s\in\mathbb{R}} ||f(s) - g(s)||_\varphi^*$.

Lemma 2.4. *For any $x \in M, t > 0$ we have $||\sigma_t^\varphi(x) - x||_\varphi^* \leq 2t||x||_\varphi^\#$ and $||\sigma_t^\varphi(x)||_\varphi^* \leq ||f||_{L^1(\mathbb{R})}||x||_\varphi^*$.*

As a consequence, for $x \in M$ and $f \in L^1(\mathbb{R}) \cap C_b^1(\mathbb{R})$, if $|f| \leq cw$ for a weight w, then $f\sigma$ belongs to $D_K(Lip_w(\mathbb{R},U))$ for $K = \lceil \max(c||x||, ||f||_\infty||x||, ||f'||_\infty\ ||x||_\varphi^* + 2||f||_\infty||x||_\varphi^\#)\rceil$. We also have the Lipschitz bound as a map:

$$d_{Lip}(f\sigma(x), f\sigma(y)) \leq ||f||_\infty||x - y||_\varphi^*.$$

We will be especially interested in this result for Fejer's kernel $f_m : \mathbb{R} \to \mathbb{R}, m > 0$ defined by

$$f_m(t) = \frac{m}{2\pi}1_{\{t=0\}} + 1_{\{t\neq 0\}}\frac{1-\cos(mt)}{\pi m t^2} \geq 0$$

of Fourier transform $\widehat{f_m}(t) = \max(0, 1 - |t|/m)$. Note that $||f_m||_1 = 1$. We will write

$$F_m^\varphi = \sigma_{f_m}^\varphi.$$

Note also that $||f_m'||_\infty \leq \frac{m^2}{\pi}, ||f_m||_\infty = \frac{m}{\pi}$. More generally, we will write $f_{m,l}(t) = f_m(t)e^{itl}$ the variant with translated Fourier transform, $F_{m,l}^\varphi = \sigma_{f_{m,l}}^\varphi$ and $||f_{m,l}'||_\infty \leq \frac{m^2}{\pi} + \frac{lm}{2\pi}$. It will be convenient to use throughout a common weight as domination $w(t) = \frac{1}{1+t^2}$. Then $|f_{m,l}(t)| \leq c_m w(t)$ with $c_m = \frac{2}{\pi m} + \frac{m}{2\pi}$. We will also use those estimates for $g_s(t) = \frac{2e^{-ist}}{e^{\pi t} + e^{-\pi t}}$ introduced before with $||g_s||_\infty = 1, ||g_s'||_\infty \leq \pi + s, |g_s(t)| \leq 2e^{-\pi|t|} \leq 2w(t)$.

PROOF : From (2.5), bounding $||\sigma_t^\varphi(x) - x||_\varphi^*$ corresponds to bounding:

$$\begin{aligned} ||\Delta^{1/2}(1+\Delta)^{-1/2}([\sigma_t^\varphi(x) - x]\xi_\varphi)||^2 &= \langle \frac{|\Delta^{it} - 1|^2}{\Delta + \Delta^{-1}}(\Delta^2 + 1)(1+\Delta)^{-1}x\xi_\varphi, x\xi_\varphi\rangle \\ &\leq (2t)^2\langle (1+\Delta)x\xi_\varphi, x\xi_\varphi\rangle, \end{aligned}$$

where we used spectral theory and the elementary bound on $\mathbb{R}^2$, $\frac{|e^{ith}-1|^2}{e^h + e^{-h}} \leq t^2h^2e^{-|h|} \leq (2t)^2e^{-2}$ and well-known bounds $||(1+\Delta)^{-1}|| \leq 1, ||\Delta(1+\Delta)^{-1}|| \leq 1$. The last bound is nothing but the expected one.

For the second estimate, we only need to notice $||f\sigma(x))_t|| \leq |f(t)|||x||$ and the similar bound for $||.||_\varphi^*$ which will define the metric. To identify K, we use the following Lipschitz estimate:

$$||(f\sigma(x))_t - f\sigma(x))_s||_\varphi^* \leq |t-s|\Big(||f'||_\infty\ ||x||_\varphi^* + 2||f||_\infty||x||_\varphi^\#\Big).$$

■

We will also need a way to identify the spectral algebras. This of course works in a more general setting of covariant systems $(M, \mathbb{R}, \sigma)$ over $\mathbb{R}$ (cf. e.g. [30, section XI.1]). This is standard and we only include a proof for the reader's convenience since we will use this quite often.

Lemma 2.5. *Let $x \in M$ and $K \in I\!N$. Then $x \in M(\sigma, [-K, K])$ if and only if for any $L \geq 2K, L \in I\!N$ $\sigma_{f_{K,\pm L}}(x) = 0$.*

PROOF : For $x \in M(\sigma, [-K, K])$, note that $[-K, K] \cap \text{supp}(\widehat{f_{K-\epsilon,\pm L}}) = [-K, K] \cap [-K + \epsilon \mp L, K - \epsilon \mp L]) = \emptyset$ thus by [30, lemma XI.1.3] $\sigma_{f_{K-\epsilon,\pm L}}(x) = 0$. The limit $\epsilon \to 0$ concludes since $||f_{K-\epsilon,\pm L} - f_{K,\pm L}||_1 \to 0$ and thus $||\sigma_{f_{K-\epsilon,\pm L}}(x) - \sigma_{f_{K,\pm L}}(x)|| \to 0$. Conversely, note that $\widehat{f_{K,\pm L}}(x) = \tau_{\pm L}\widehat{f_K}(x) = \widehat{f_K}(x \mp L)$ is the translation of $\widehat{f_K}$ by $\pm L$ which is non-zero in $]-K \pm L, K \pm L[$. Thus, if $\sigma_{f_{K,\pm L}}(x) = 0$ then $[\text{Spec}_{\sigma^\varphi}(x)]^c \supset]-K \pm L, K \pm L[$. Those sets cover $[-K, K]^c$ thus any point outside $[-K, K]^c$ is not in the spectrum of x. ■

We will finally need another standard fact of spectral theory to compute the form $\mathscr{E}_\alpha(x, y) = \langle \Delta^\alpha(x\xi_\varphi), (y\xi_\varphi)\rangle$ in the case $\alpha = 1/3, 2/3, 1$. We will use crucially that $M \subset D(\Delta^{1/2})$ in the case above. We give explicit bounds for maps with values in $Lip_w(\mathbb{R}, \mathbb{C}) \subset Lip_w(\mathbb{R}, U)$ for our weight $w(t) = \frac{1}{1+t^2}$. We will often use $e^{-\alpha|t|} \leq \max(1, \alpha^{-2})w(t)$ for $\alpha > 0$.

Lemma 2.6. *For any positive closed densely defined operator as Δ, and $0 < \alpha < 1, \epsilon > 0$, we have*

$$(\Delta + \epsilon)^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty s^{-\alpha}(\Delta + s + \epsilon)^{-1}ds. \tag{2.6}$$

If moreover $M \subset D(\Delta^{1/2})$, and if we write $G_s = 2e^{s/2}\Delta^{1/2}(\Delta + e^s)^{-1}$, then for $\alpha \in]0, 1/2[$ and any $x, y \in M$, $\beta \in [0, 1 - \alpha[$ (or if $\alpha + \beta = 1$, for $x, y \in D(\Delta^{3/4})$) we have

$$\mathscr{E}_{\alpha+\beta}(x, y) = \frac{\cos(\alpha\pi)}{2\pi}\int_{-\infty}^\infty dt e^{\alpha t}\mathscr{E}_\beta(G_t(x), y). \tag{2.7}$$

In particular, if we define $(\mathscr{E}_\beta G^\alpha(x,y))_t = e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)$, *then we have* $\mathscr{E}_\beta G^\alpha(x,y) \in D_K(Lip_w(I\!R,\mathbb{C}))$ *for* $K = \left\lceil 2\max(1,(\alpha+\hat\beta)^{-2},(\delta-\alpha)^{-2},2(e^2-1))||x||_\varphi^\#||y||_\varphi^\# \right\rceil$ *and* $\hat\beta = \min(1/2,\beta), \delta = \min(1/2,1-\beta)$, *if* $\alpha+\beta<1, \alpha<1/2$ *and the function is bilipschitz in its arguments on bounded sets.*

And for $\alpha+\beta=1$, *we have the same with*

$$K = \left\lceil 2\max(1,(\alpha+1/2)^{-2},(1/2-\alpha)^{-2},2(e^2-1))\sqrt{||x||_\varphi^2+||\Delta^{3/4}(x\xi_\varphi)||_\varphi^2}\sqrt{||y||_\varphi^2+||\Delta^{3/4}(y\xi_\varphi)||_\varphi^2}\right\rceil.$$

Proof : The first result is well-known, see e.g. [24, Rmk V.3.50], the integral is absolutely converging in bounded operators since $||s^{-\alpha}(\Delta+s+\epsilon)^{-1}|| \le s^{-\alpha}(s+\epsilon)^{-1}$. The end of the proof is inspired by the standard arguments used to define fractional powers of unbounded operators as quadratic forms (see e.g. Kato's book [24]). We first show the integrals are absolutely converging. Note that $||(\Delta+e^s)^{-1}|| \le e^{-s}, ||\Delta(\Delta+e^s)^{-1}|| = ||1-e^s(\Delta+e^s)^{-1}|| \le 1$, (and for $\beta\le 1/2$, we have $||\Delta^\beta(y\xi_\varphi)||^2 \le ||\Delta^{1/2}(y\xi_\varphi)||^2+||y\xi_\varphi||^2$ since $\Delta^{2\beta}\le 1+\Delta$.) . Note also that by definition $|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| = 2e^{(\alpha+1/2)t}|\langle\Delta^{\beta+1/2}(\Delta+e^t)^{-1}(x\xi_\varphi),(y\xi_\varphi)\rangle|$ so that one gets if $t>0$

$$|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| \le 2e^{(\alpha+\beta-1)t}\|\Delta^{1/2}(x\xi_\varphi)\|\|\Delta^{1/2}(y\xi_\varphi)\| \text{ if } 1-\alpha>\beta>1/2,$$

$$|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| \le 2e^{(\alpha-1/2)t}\|\Delta^{1/2}(x\xi_\varphi)\|\|\Delta^{\beta}(y\xi_\varphi)\| \quad \text{if } \beta\le 1/2,$$

and if $t<0$:

$$|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| \le 2e^{(\alpha+1/2)t}\|\Delta^{\beta-1/2}(x\xi_\varphi)\|\|y\xi_\varphi\| \quad \text{if } \beta>1/2,$$

$$|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| \le 2e^{(\alpha+1/2)t}e^{(\beta-1/2)t}\|(x\xi_\varphi)\|\|(y\xi_\varphi)\| \quad \text{if } \beta\le 1/2.$$

Altogether, in the case $\alpha+\beta<1$ we have the following estimate:

$$|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| \le 2\max(e^{-(\alpha+\hat\beta)|t|},e^{-(\delta-\alpha)|t|})||x||_\varphi^\#||y||_\varphi^\#$$

This gives the expected integrability in all cases but the case $\alpha+\beta=1$ in which case we use a bound similar to the second bound for $t>0, 1-\alpha=\beta>1/2$ (since for $\beta\le 1$, we have $||\Delta^{\beta/2+1/4}(y\xi_\varphi)||^2 \le ||\Delta^{3/4}(y\xi_\varphi)||^2+||y\xi_\varphi||^2$ using $\Delta^{\beta+1/2}\le 1+\Delta^{3/2}$):

$$|e^{\alpha t}\mathscr{E}_\beta(G_t(x),y)| \le 2e^{(\alpha-1/2)t}\|\Delta^{\beta/2+1/4}(x\xi_\varphi)\|\|\Delta^{\beta/2+1/4}(y\xi_\varphi)\|$$

Applying (2.6) to $\gamma = 1/2-\alpha$ instead of α and with a change of variable $s=e^u$, one gets

$$(\Delta+\epsilon)^{\alpha-1/2} = \frac{\cos(\alpha\pi)}{\pi}\int_{-\infty}^{\infty} du e^{(\alpha+1/2)u}(\Delta+e^u+\epsilon)^{-1}$$

so that, first on the domain of $\Delta^{1/4+\beta/2}$ and then on M by density and with $h(u,\epsilon)=\ln(e^u+\epsilon)$, we obtain:

$$\mathscr{E}_{\beta+1/2}((\Delta+\epsilon)^{\alpha-1/2}x,y) = \frac{\cos(\alpha\pi)}{\pi}\int_{-\infty}^{\infty} du e^{(\alpha+1/2)u}e^{-h(u,\epsilon)/2}\mathscr{E}_\beta(G_{h(u,\epsilon)}(x\xi_\varphi),(y\xi_\varphi))$$

By dominated convergence theorem with bounds similar to those above (and for $a,\epsilon\ge 0$, $(e^{-t}\epsilon+1)^{-a}\le 1$), we obtain the result at the limit $\epsilon\to 0$.

It remains to check the expected Lipschitz bounds in the parameter. By the resolvent equation we have

$$\begin{aligned} e^{\alpha t}G_t(x) - e^{\alpha s}G_s(x) &= 2e^{(1/2+\alpha)t}\Delta^{1/2}(\Delta+e^t)^{-1}(1-e^{t-s})e^s(\Delta+e^s)^{-1}(x) \\ &\quad + 2(e^{(1/2+\alpha)(t-s)}-1)e^{(1/2+\alpha)s}\Delta^{1/2}(\Delta+e^s)^{-1}(x) \end{aligned}$$

so that for $t\ge s$ (note $\alpha+1/2\le 1$)

$$\begin{aligned} |e^{\alpha t}\mathscr{E}_\beta(G_t(x),y) - e^{\alpha s}\mathscr{E}_\beta(G_s(x),y)| &\le |(1-e^{(t-s)})|\ \left(|e^{\alpha t}\mathscr{E}_\beta(G_t(e^s(\Delta+e^s)^{-1}(x),y)| + |e^{\alpha s}\mathscr{E}_\beta(G_s(x),y)|\right) \\ &\le (e^2-1)|t-s|4||x||_\varphi^\#||y||_\varphi^\#, \end{aligned}$$

where the second inequality is valid for $|t-s|\le 2, \alpha+\beta<1$ but the overall bound is also valid for $|t-s|>2$ as a direct consequence of the triangle inequality since $1/2\le e^2-1$. The case $\alpha+\beta=1$ is similar. ∎

2.2. Axiomatization

Recall we have one sort U with domains of quantification $D_n = D_n(U)$ for the operator norm ball of radius n of M and a second sort $ILip_w(\mathbb{R}, U)$ for Lipschitz curves with value in M dominated by multiples of $w(t) = \frac{1}{1+t^2}$ from section 7. As usual $\mathbb{N}^*$ denotes the set of strictly positive natural numbers. The language will be composed of

- The constant 0 which will be in D_1.
- For every $\lambda \in \mathbb{C}$ a unary function symbol also denoted λ to be interpreted as scalar multiplication. For simplicity we shall write λx instead of $\lambda(x)$.
- A unary function symbol $*$ for involution on U, leaving stable all domains.
- A binary function symbol $+ : D_n \times D_m \to D_{n+m}$. and for $K, L \in \mathbb{N}^*$ $m_{(K,L)} : D_n \times D_m \to D_{nm}$ (interpreted as $F_K^{\varphi}(.)F_L^{\varphi}(.)$ with modulus of continuity as obtained in lemma 2.1).
- The constant 1 in D_1.
- Two unary relation symbols φ_r and φ_i for the real and imaginary parts of the state φ, on U. We will often just write φ and assume that the expression can be decomposed into the real and imaginary parts.
- For each $t \in \mathbb{Q}$, unary function symbols $\sigma_t : D_n \to D_n$ (for the modular group), $G_t : D_n \to D_n$ (for G_t^{φ}) and for $(m, l) \in \mathbb{Q}^2, m > 0$ $F_{m,l} : D_n \to D_n, F_{N,0} = F_N$ (for Fejer's map $F_{m,l}^{\varphi}$).$H_K = (K+1)F_{K+1} - KF_K$.
- A function symbol $\tau_{p,\lambda,N}$ (meaning $p(\sum_{i=1}^n \lambda_i F_{N_i}(x))$) for every $*$-polynomial in one variable p, any $N = (N_1, ..., N_n) \in (\mathbb{N}^*)^n$, any $\lambda = (\lambda_1, ...\lambda_n) \in (\mathbb{Q} \cap [0, 1])^n \sum \lambda_i = 1$. If we write

  $$m(n, p) = \lceil \sup\{||p(a)||, a \in C, C\ C^* - \text{algebra and } ||a|| \leq n\} \rceil,$$

  then we require $\tau_{p,\lambda,N} : D_n \to D_{m(n,p)}$ and we want it to have same modulus of continuity as $p\cdot(\sum_{i=1}^n \lambda_i F_{N_i}(.))$ (we will use this as notation for products obtained with maps $m_{(N,K)}$ in a way we will explain below).
- For each $\alpha \in \mathbb{Q} \cap [0, 1[, N, M \in \mathbb{N}^*$ binary relation symbols $\mathscr{E}_{\alpha,N,M}$ on D_n (as for φ above, formally decomposed into imaginary and real part and meaning $\mathscr{E}_{\alpha}(F_N^{\varphi}(.), F_M^{\varphi}(.))$.
- For $w(x) = \frac{1}{1+x^2}$, $\mathcal{L}(ILip_w(\mathbb{R}, U))$ from section 7 (case $\alpha = 1$), with its function symbols such as $\int_{\mathbb{R}}$ and evaluations $\delta_t, t \in \mathbb{Q}$. Curves valued function symbols

  $$g_s\sigma : D_n(U) \to D_{nK_s}(ILip_w(\mathbb{R}, U)), s \in \mathbb{Q}, \quad f_{m,l}\sigma : D_n(U) \to D_{nK_{m,l}}(ILip_w(\mathbb{R}, U)), (m, l) \in \mathbb{Q}^2, m > 0$$

  and

  $$\mathscr{E}_{\beta,N,M}G^{\alpha} : D_n(U) \times D_{\nu}(U) \to D_{n\nu K_{N,M}^{\alpha,\beta}}(ILip_w(\mathbb{R}, U)), \alpha+\beta \leq 1, \alpha < 1/2, \alpha, \beta \in \mathbb{Q} \cap [0, 1[, N, M, n, \nu \in \mathbb{N}^*$$

  $$\text{with}: K_s = \left\lceil (\pi+s+2\sqrt{2}) \right\rceil, K_{N,M}^{\alpha,1-\alpha} = \left\lceil 2\max(1, (\alpha+1/2)^{-2}, (1/2-\alpha)^{-2}, 2(e^2-1))\sqrt{1+e^{3N/2}}\sqrt{1+e^{3M/2}} \right\rceil,$$

  $$K_{m,l} = \left\lceil \max((\frac{2}{\pi m} + \frac{m}{2\pi}), (\frac{m^2}{\pi} + \frac{ml}{2\pi})\ + 2\sqrt{2}\frac{m}{\pi}) \right\rceil, K_{N,M}^{\alpha,\beta} = \left\lceil 4\max(1, (\alpha+\hat{\beta})^{-2}, (\delta-\alpha)^{-2}, 2(e^2-1)) \right\rceil$$

  if $\alpha + \beta < 1$ with $\hat{\beta} = \min(1/2, \beta), \delta = \min(1/2, 1-\beta)$.The modulus of uniform continuity are also those determined in lemmas 2.4 and 2.6.

H_K is inspired by the so-called *De la vallée poussin Kernel* $h_K = (K+1)f_{K+1} - Kf_K$ with Fourier transform

$$\widehat{h_K}(t) = \max(0, \min(K+1-|t|, 1)).$$

which is equal to 1 on $[-K, K]$ and with support in $[-K-1, K+1]$. Thus H_K^φ is identity on $M(\sigma^\varphi, [-K+\epsilon, K-\epsilon]), \epsilon > 0$ and we will use them in axiom (2) to get substitutes to and consequences of equation (2.2) and associativity for smeared products. Indeed, since $F_K^\varphi(x)$ is an element of $M(\sigma^\varphi, [-K, K])$ we expect $H_{K+1}^\varphi(F_K^\varphi(x)) = F_K^\varphi(x), H_{K+L+1}^\varphi(m_{(K,L)}(x,y)) = m_{(K,L)}(x,y)$ this will be our first substitute. Moreover, in order to write down the associativity equation $F_K(x).(F_L(y).F_M(z)) = (F_K(x).F_L(y)).F_M(z)$, one needs a product equal to $F_K(x).y$ built only from $m_{(K,L)}$'s.

With that purpose in mind, we will use as a shorter notation for what is supposed to be $H_{K+1}^\varphi(a).H_{L+1}^\varphi(b)$ namely:

$$\begin{aligned} M_{(K,L)}(a,b) = {} & (K+1)(L+1)m_{(K+1,L+1)}(a,b) + (K+2)(L+2)m_{(K+2,L+2)}(a,b) \\ & - (K+1)(L+2)m_{(K+1,L+2)}(a,b) - (K+2)(L+1)m_{(K+2,L+1)}(a,b). \end{aligned}$$

The previous remarks enable us to write down

$$F_K(x).(F_L(y).F_M(z)) = H_{K+1}(F_K(x)).H_{L+M+1}(m_{(L,M)}(y,z)) = M_{(K,L+M)}(F_K(x), m_{(L,M)}(y,z)).$$

This is the case of monomials of degree 3 in the next equation. With this notation we can define for a monomial: $p = x^{\epsilon_1}...x^{\epsilon_k}, \epsilon_i \in \{1, *\}$ the expression used above for $\lambda_i \in [0,1]$ and then extend by linearity to a more general polynomial

$$\begin{aligned} p^{\cdot}(\sum_{i=1}^n \lambda_i F_{N_i}(x)) = {} & \sum_{i_1,\dots,i_k=1}^n \lambda_{i_1}...\lambda_{i_k} M_{(N_{i_1}, N_{i_2}+\dots+N_{i_k})}(F_{N_{i_1}}(x^{\epsilon_1}), \cdots \\ & M_{(N_{i_{k-2}}, N_{i_{k-1}}+N_{i_k})}(F_{N_{i_{k-2}}}(x^{\epsilon_{k-2}}), m_{(N_{i_{k-1}}, N_{i_k})}(x^{\epsilon_{k-1}}, x^{\epsilon_k})\cdots). \end{aligned}$$

This will correspond in the step 2 of the next theorem to a product in a well-defined associative product.

Our models will thus be models of σ-finite von Neumann algebras having such a faithful normal state [30, Prop 3.19] or more precisely of σ-finite W^* probability spaces since the theory will depend on the state φ in a non-trivial way.

As in [13], we now write down axioms satisfied by any σ-finite W^* probability space (either obvious or coming from the preliminary subsection 2.1):

(1) $x + (y+z) = (x+y) + z, x + 0 = x, x + (-x) = 0$ (where $-x$ is the scalar -1 acting on x), $x + y = y + x, \lambda(\mu x) = (\lambda\mu)x, \lambda(x+y) = \lambda x + \lambda y, (\lambda+\mu)x = \lambda x + \mu x$, $1(x) = x$.

(2) For $K, L, K_1, K_2, K_3 \in \mathbb{N}^*$ $\lambda m_{(K,L)}(x,y) + m_{(K,L)}(x,z) = m_{(K,L)}(x, \lambda y + z)$,

$$m_{(K_1,L)}(F_{K_2}(x), y) = m_{(K_2,L)}(F_{K_1}(x), y),$$

$$H_{K+L}(F_K(x)) = F_K(x),$$

$$H_{K_1+K_2+L}(m_{(K_1,K_2)}(x_1,x_2)) = m_{(K_1,K_2)}(x_1,x_2),$$

$$\begin{aligned} & (K_1+K_2+2)m_{(K_1+K_2+2,K_3)}(m_{(K_1,K_2)}(x_1,x_2),x_3) - (K_1+K_2+1)m_{(K_1+K_2+1,K_3)}(m_{(K_1,K_2)}(x_1,x_2),x_3) \\ & = (K_3+K_2+2)m_{(K_1,K_2+2+K_3)}(x_1, m_{(K_2,K_3)}(x_2,x_3)) - (K_3+K_2+1)m_{(K_1,K_2+1+K_3)}(x_1, m_{(K_2,K_3)}(x_2,x_3)). \end{aligned}$$

(3) $(x^*)^* = x, (x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda}x^*$.

(4) For $K, L \in \mathbb{N}^*$, $[m_{(K,L)}(x,y)]^* = m_{(L,K)}(y^*, x^*)$, $F_N(x^*) = [F_N(x)]^*$. $d_U(x,0) = d_U(x^*,0)$.

(5) $d_U(x,y) = d_U(x-y,0)$, we write $||x||_\varphi^* = d_U(x,0)$.

(6) For 1 the constant symbol $1 \in D_1$, $F_N(1) = 1$, $m_{(K,N)}(1,x) = F_N(x) = m_{(N,K)}(x,1)$.

(7) $\varphi(x+y) = \varphi(x) + \varphi(y)$.

(8) $\varphi(x^*) = \overline{\varphi(x)}$, $\varphi(\lambda x) = \lambda\varphi(x)$, $\varphi(1) = 1$.
(9) $\max(0, -\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{(K_i,K_j)}(x_i^*, x_j)) = 0$.
(10) For every $n, N, m, K, K_1, ..., K_n \in \mathbb{N}^*$,

$$\sup_{a\in D_N(U)} \sup_{(x_1,...,x_n)\in(D_m(U))^n} \max[0, -N^2 \sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{(K_i,K_j)}(x_i^*, x_j)) + \sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(M_{(K+K_i,K+K_j)}([m_{(K,K_j)}(a, x_i)]^*, m_{(K,K_j)}(a, x_j)))] = 0$$

(11) $\tau_{p,\lambda,N}(x) = p^{\cdot}(\sum_{i=1}^n \lambda_i F_{N_i}(x))$ for every $*$-polynomial p in one variable x, $\lambda_i \in \mathbb{Q}\cap[0,1]$ with $\sum_{i=1}^n \lambda_i = 1$, $N = (N_1, ..., N_n) \in (\mathbb{N}^*)^n$.
(12) For $K, K_i, m, l \in (\mathbb{N}^*)$,

$$\sup_{x\in D_m} \sup_{y_i\in D_l} \max(0, \left\| F_K(x) + \sum_{i=1}^n m_{(K_i,K)}(y_i, x) \right\|_\varphi^* - 3me^K \left\| 1 + \sum_{i=1}^n F_{K_i}(y_i) \right\|_\varphi^*) = 0.$$

These axioms are really similar to [13]. Axiom (11) mimics the trick they found to identify the operator norm unit ball in a universal axiomatization (rather than a $\forall\exists$ one). We will often call inequality a statement such as (10) or (12) which is equivalent to an obvious inequality. We now need to specify the metric to coincide with the Ocneanu ultraproduct and to deal with the modular theory. We of course find our inspiration in our previous section and require first the modular group relations (including the continuity obtained in lemma 2.4):

(13) $\sigma_t(\sigma_s(x)) = \sigma_{t+s}(x), \sigma_t(\lambda x + y) = \lambda\sigma_t(x) + \sigma_t(y), \sigma_0(x) = x, \sigma_t(x^*) = (\sigma_t(x))^*, \varphi(\sigma_t(x)) = \varphi(x)$.
(14) $\sigma_t([m_{(K,L)}(x, y)]) = [m_{(K,L)}(\sigma_t(x), \sigma_t(y))]$, $\sigma_t(F_N(x)) = F_N(\sigma_t(x))$.
(15) For every $n \in \mathbb{N}$,

$$\sup_{x\in D_n} \max(0, d_U(\sigma_t(x), x) - 4tn) = 0.$$

We also need the relations between σ_t, G_t, F_m from lemma 2.2 and lemma 2.4:

(16) $Th(ILip_w(\mathbb{R}, U))$ from section 7 which defines $\int_{\mathbb{R}}$. For $s, t \in \mathbb{Q}, m \in \mathbb{N}, x \in D_m(U)$

$$\delta_t(g_s\sigma(x)) = \frac{2e^{-ist}}{e^{\pi t} + e^{-\pi t}}\sigma_t(x), \qquad G_s(x) = \int_{\mathbb{R}} g_s\sigma(x).$$

(17) For $\lambda_i \in \mathbb{C}, K_i \in \mathbb{N}^*$

$$4(||\sum_{i=1}^n \lambda_i F_{K_i}(x_i)||_\varphi^*)^2 = \sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{(K_i,K_j)}(G_0(x_i)^*, G_0(x_j))) + \varphi(m_{(K_j,K_i)}(G_0(x_j), G_0(x_i)^*)).$$

(18) For $N \in \mathbb{Q}\cap]0,\infty[, t \in \mathbb{Q}^*, l \in \mathbb{Q}, m \in \mathbb{N}^*, x \in D_m(U)$, $\delta_0(f_{N,l}\sigma(x)) = \frac{N}{2\pi}x$

$$\delta_t(f_{N,l}\sigma(x)) = e^{ilt}\frac{1 - \cos(Nt)}{\pi Nt^2}\sigma_t(x), \qquad F_{N,l}(x) = \int_{\mathbb{R}} f_{N,l}\sigma(x).$$

We finally have the relations defining our forms from lemma 2.6

(19) $\mathscr{E}_{0,K,L}(x, y) = \varphi(m_{(K,L)}(x^*, y))$ and for $\alpha, \beta \in \mathbb{Q}\cap[0, 1[, 0 < \alpha < 1/2, \alpha + \beta < 1$, $m, K, L, n \in \mathbb{N}^*, t \in \mathbb{Q}, (x, y) \in D_m^2(U)$

$$\delta_t(\mathscr{E}_{\beta,K,L}G^\alpha(x, y)) = e^{\alpha t}\mathscr{E}_{\beta,K,L}(G_t(x), y), \qquad \mathscr{E}_{\alpha+\beta,K,L}(x, y) = \frac{\cos(\alpha\pi)}{2\pi}\int_{\mathbb{R}} \mathscr{E}_{\beta,K,L}G^\alpha(x, y).$$

(20) For $\alpha, \beta \in \mathbb{Q} \cap [0,1[, 0 < \alpha < 1/2, \mu = 1/2 - \alpha, m, K, L, n \in \mathbb{N}^*, L \leq K$, for $\alpha + \beta = 1$, we have the same formula for $\mathscr{E}_{\beta,K,L} G^\alpha(x,y)$ and:

$$\varphi(m_{(L,K)}(y, x^*)) = \frac{\cos(\alpha\pi)}{2\pi} \int_{\mathbb{R}} \mathscr{E}_{\beta,K,L} G^\alpha(x,y).$$

Recall that a (structure) model of a theory will be a metric space with each domain of quantification (for us balls) complete in the metric and with all the symbols having the specified uniform continuity functions. Recall that an axiomatization of a category $\mathcal{C}$ will be as in [13], a functor $\mathcal{M}$ from $\mathcal{C}$ to models of the theory T such that $\mathcal{M}(A)$ is determined up to isomorphism for any $A \in \mathcal{C}$, for any model M of T there is $A \in \mathcal{C}$ such that M is isomorphic to $\mathcal{M}(A)$ and for every $A, B \in \mathcal{C}$, there is a bijection $Hom_{\mathcal{C}}(A,B) \simeq Hom(\mathcal{M}(A), \mathcal{M}(B))$.

The category of (σ-finite) W^* probability spaces may not have the most expected morphisms. We will consider as morphism only those state preserving $*$-homomorphisms having an image admitting a state preserving conditional expectation. Recall that by σ-finite W^* probability spaces, we mean a pair (M, φ) of a σ-finite von Neumann algebra M having a fixed faithful normal state φ. Since we put in the structure the modular group and we want our morphism to correspond to model-theoretic morphisms and thus commute with the modular group, the image of a morphism will thus be left invariant by the modular group of the target state, and by a result of Takesaki (cf e.g. [30, Th IX.4.2]) this is equivalent to the existence of such a conditional expectation.

We are ready to obtain our axiomatization with a compatibility with ultraproducts.

Theorem 2.7. *The class of σ-finite W^* probability spaces (with morphisms as described above) is axiomatizable by the Ocneanu theory $T_{\sigma W^*}$, theory consisting of axioms (1)-(20) above. Moreover, if (M_n, φ_n) are W^* probability spaces of this type, then for any non-principal ultrafilter ω, the model of the Ocneanu ultraproduct of [1] is given by the model-theoretic ultraproduct:*

$$\mathcal{M}((M_n, \varphi_n)^\omega) = [\mathcal{M}(M_n, \varphi_n)]^\omega.$$

Remark 2.8. *The model theoretic ultraproduct thus gives a construction of a von Neumann algebra for which the ultraproduct of modular groups is the modular group of the ultraproduct state. This is the same result as in [1, Th 4.1] without any use of the Groh-Raynaud ultraproduct. However, in order to check that this von Neumann algebra structure coincides with the Ocneanu ultraproduct of [1], we will use all their results even those using the Groh-Raynaud ultraproduct. Our construction does not really provide a new proof that the standard Ocneanu construction is a von Neumann algebra with the right modular theory. But it provides an alternative root to the same object and the proof of its main properties without using any non-σ finite von Neumann algebra. Note also that of course (1)-(20) means all the axioms from (1) to (20). We would write (1),(20) for singling out the two axioms (1) and (20).*

Remark 2.9. *Let us comment on our language from a model theoretic viewpoint. First, it is easy to see that all the data depends only on (M, φ), the pair of a von Neumann algebra and a faithful normal state. Indeed, it is well known that this is the case for σ_t^φ that is fixed by t, φ and all the other data has been defined from the modular group, the product, adjoint and the state in the previous subsection. More precisely, the modular group is the standard way to encode the unbounded operator Δ^φ which is defined only at Hilbert space level on $L^2(M, \varphi)$ (as unbounded operator) and thus does not fit well with the model theoretic setting. This unbounded operator can only appear as a sesquilinear form giving the relation $\mathscr{E}_1(x,y) = \langle \Delta^\varphi(x\xi_\varphi), y\xi_\varphi \rangle$. As already mentioned in the introduction of section 2, we won't use the KMS condition to check that the automorphism group we put in the theory is indeed the modular group, we will rather use spectral theory. Explicit formulas already appeared in the previous section and are better suited for model theory. In the step 4 of the proof below, we will compute the form corresponding to the modular group given as relation on M^2: $Q_t(x,y) = \varphi(\sigma_t(x^*)y)$. By density of M in $L^2(M, \varphi)$, this determines uniquely σ_t and its Hilbert space extension Δ_σ^{it}. It is computed from and determines the generator Δ_σ by functional calculus. We will then compute the form $q_\sigma(x,y) = \langle \Delta_\sigma(x\xi_\varphi), y\xi_\varphi \rangle$ (smeared by some F_K). We thus deduce that it has the expected*

value $q_\sigma = \mathscr{E}_1$ so that the (exponentiated) generator $\Delta_\sigma = \Delta^\varphi$. Hence, the modular group that encodes it in the theory will also be equal to its expected value $\sigma_t = \sigma_t^\varphi$. We will see in subsection 2.4 *that all this data (including the modular group) is not strictly speaking necessary in a theory of σ-finite W^* probability spaces. It enables us to get a universal axiomatization (some of the data, as the forms $\mathscr{E}_\alpha$, are even only here to get short and readable enough axioms). In order to remove this main extra piece of data, the modular group, we will need some more technical but standard spectral theory. Since having an explicit universal axiomatization is interesting in its own right, we thus postpone the quest of minimality in the language to this supplementary subsection* 2.4.

Proof :

We already noticed that any (M, φ) gives a model of $T_{\sigma W^*}$ using the lemmas of the previous section. Using Takesaki's theorem [30, Th IX.4.2], a (state preserving) *-homomorphism of von Neumann algebras having a state preserving conditional expectation on its range gives a map between the corresponding models preserving the structure and a fortiori vice versa. Let us say a supplementary word on that for the reader's convenience. By injectivity of a (state preserving) *-homomorphism of von Neumann algebras, saying that such a morphism preserves the modular group boils down to the basic case where this morphism is the embedding of a subalgebra. A modular group of an algebra leaves invariant this algebra. Hence, the modular group of a subalgebra N of (M, φ) computed with the restricted state can be the restricted modular group only if the subalgebra is left invariant by the huge modular group. By [30, Th IX.4.2], this invariance is actually equivalent to the existence of the stated conditional expectation. For, the restriction of the modular group in this case is an automorphism group and satisfies the KMS modular condition with respect to the restricted state (thus the uniqueness in e.g. [30, Th VIII.1.2] concludes). We explained in the previous section how the extra-data is computed from the modular group and thus commutation of the morphism and this data is deduced from the one with the modular group. For the converse, the only piece of data that a structure preserving morphism does not preserve by definition is the product, since it only preserves smeared products, but the limiting description of the product obtained below from smeared products gives the homomorphism property for the product too. Since the structure preserving morphism preserves the modular group, the converse in [30, Th IX.4.2] gives existence of a state preserving conditional expectation on its image. For our choice of category, we have thus checked the bijection $Hom_{\mathcal{C}}(A, B) \simeq Hom(\mathcal{M}(A), \mathcal{M}(B))$, as expected for a nice axiomatization.

Assume M satisfies $T_{\sigma W^*}$. We want to see that in the sort U, the set M gives a W^* probability space, having the expected modular theory and balls. This is the most technical part and we divide this into several steps.

Step 1: First properties of modular theoretic maps.

From (13)-(15) σ_t is a continuous one parameter group of linear state preserving maps on M. We can extend by continuity for d, σ_t to $t \in \mathbb{R}$.

Based on section 7, $\int_{\mathbb{R}}$ has the expected meaning. We thus deduce from (16),(18) that for any $x \in D_n(M), N \in \mathbb{Q}^*, N > 0, l \in \mathbb{Q}$

$$G_s(x) = \int_{\mathbb{R}} dt \frac{2e^{-ist}}{e^{\pi t} + e^{-\pi t}} \sigma_t(x) \in D_n(M), \quad F_{N,l}(x) = \int_{\mathbb{R}} dt f_{N,l}(t) \sigma_t(x) \in D_n(M). \tag{2.8}$$

By dominated convergence theorem, one also deduces G_s is strongly continuous in s for d and extends with the same formula to $G_s, s \in \mathbb{R}$.

We will first use only the case $F_N = F_{N,0}, N \in \mathbb{N}^*$. Let us show that

$$A := \mathrm{Vect}\{F_N(x), x \in M, N \in \mathbb{N}^*\}$$

is dense in M for d and that even $A \cap D_n(M)$ is dense in $D_n(M)$.

Note that $x = \int_{\mathbb{R}} dt f_N(t) x$ so that $F_N(x) - x = \int_{\mathbb{R}} dt f_N(t)(\sigma_t(x) - x)$ and thus

$$d(F_N(x), x) \leq \int_{\mathbb{R}} dt f_N(t) d(\sigma_t(x), x).$$

Since f_m is an approximation of a Dirac mass δ_0 and $d(\sigma_t(x), x)$ is continuous with value 0 at 0, one deduces $d(F_N(x), x) \to_{N\to\infty} 0$ proving the expected density. Note also that the equalities $\sigma_s(F_N(x)) = F_N(\sigma_s(x)), G_s(F_N(x)) = F_N(G_s(x)), x \in M$ are consequences of the above integral formulas (2.8) and (13) which implies $\sigma_t\sigma_s = \sigma_s\sigma_t$.

Step 2: Building the algebra structure on A and its faithful representation.

We first build the algebra structure on $A = \text{Vect}\{F_N(x), x \in M, N \in \mathbb{N}^*\}$, which is already stable by adjoint and modular group by (4),(14).

We want to extend by bilinearity the product

$$[F_K(x)].[F_L(y)] = m_{(K,L)}(x, y) = H_{K+L+1}(m_{(K,L)}(x, y)) \in A$$

by (2). If $\mathcal{A}$ is the abstract direct sum of $\mathbb{N}^*$ copies of M with x_n the n-th copy of $x \in M$ there is a map of $f : \mathcal{A} \to A$ sending $x_N \mapsto F_N(x)$. Obviously the product $\mathcal{A} \times \mathcal{A} \to A$ defined by $x_K.y_L = m_{(K,L)}(x, y)$ is well defined and it suffices to see it vanishes on $Ker(f) \times \mathcal{A} + \mathcal{A} \times Ker(f)$. Axiom (12) (and a symmetric variant insured using adjoints and (4)) exactly guaranties this, since $f(a) = 0$ iff $||f(a)||^*_\varphi = 0$.

We can now see that A is in this way a $*$-algebra.

Indeed the last part of the axiom (2) can be rewritten

$$[H_{K_1+K_2+1}(m_{(K_1,K_2)}(x_1, x_2))].F_{K_3}(x_3) = F_{K_1}(x_1).[H_{K_3+K_2+1}(m_{(K_2,K_3)}(x_2, x_3))]$$

But using the definition and the second relation in (2) this is nothing but the associativity relation

$$([F_{K_1}(x_1)].[F_{K_2}(x_2)])[F_{K_3}(x_3)] = [F_{K_1}(x_1)].([F_{K_2}(x_2)].[F_{K_3}(x_3)]).$$

The $*$ algebra relation $([F_{K_1}(x_1)].[F_{K_2}(x_2)])^* = [F_{K_2}(x_2)]^*.[F_{K_1}(x_1)]^*$ is obtained from the first part of (4). Note that similarly, the expression appearing in (10) can be interpreted using the product as:

$$\begin{aligned} M_{(K+K_i,K+K_j)}([m_{(K,K_j)}(a, x_i)]^*, m_{(K,K_j)}(a, x_j)) \\ = ([F_K(a)].[F_{K_i}(x_i)])^*.([F_K(a)].[F_{K_j}(x_j)]). \end{aligned}$$

Moreover, from (7)-(9) A is a complex pre-Hilbert space with inner product given by $\langle y, x\rangle = \varphi(y^*x)$. Left multiplication by $a \in A$ is a linear operator on A and axiom (10) guarantees that for $x \in D_n(M)$, $F_N(x)$ is bounded of norm less than n. The operation $*$ is the adjoint because for all x and y we have

$$\langle ax, y\rangle = \varphi((ax)^*y) = \varphi(x^*a^*y) = \langle x, a^*y\rangle.$$

Thus A is represented (by left multiplication) as a *-algebra of Hilbert space (bounded) operators (on the completion $L^2(A, \varphi)$ with cyclic vector ξ_φ).

We now want to check that the representation is faithful in showing that for any $x \in A$:

$$4d_U(x, 0)^2 \leq ||x\xi_\varphi||^2 + ||x^*\xi_\varphi||^2. \tag{17b}$$

We use (17). We first note that using the kernel for G_0 is a probability and Cauchy-Schwarz inequality in

$L^2(A,\varphi)$:

$$\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{K_i,K_j}(G_0((x_i)^*),G_0(x_j)))$$
$$= \int_{\mathbb{R}} dt \frac{2}{e^{\pi t}+e^{-\pi t}} \int_{\mathbb{R}} ds \frac{2}{e^{\pi s}+e^{-\pi s}} \sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{K_i,K_j}(\sigma_t((x_i)^*),\sigma_s(x_j)))$$
$$\leq \sup_{s,t\in\mathbb{R}} |\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{K_i,K_j}(\sigma_t((x_i)^*),\sigma_s(x_j)))| = \sup_{s,t\in\mathbb{R}} \left|\left\langle \sum_{i=1}^n \lambda_i F_{K_i}(\sigma_t(x_i)), \sum_{i=1}^n \lambda_i F_{K_i}(\sigma_s(x_i))\right\rangle_{L^2(A,\varphi)}\right|$$
$$\leq \sup_{s,t\in\mathbb{R}} \left\| \sum_{i=1}^n \lambda_i F_{K_i}(\sigma_t(x_i))\right\| \left\| \sum_{i=1}^n \lambda_i F_{K_i}(\sigma_s(x_i))\right\|$$
$$= \sup_{t\in\mathbb{R}} |\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{K_i,K_j}(\sigma_t((x_i)^*),\sigma_t(x_j)))|^{1/2} \times \sup_{s\in\mathbb{R}} |\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{K_i,K_j}(\sigma_s((x_i)^*),\sigma_s(x_j)))|^{1/2}$$
$$= |\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(m_{K_i,K_j}((x_i)^*,x_j))| = \|[\sum_{i=1}^n \lambda_i F_{K_i}(x_i)]\xi_\varphi\|^2$$

where we finally used (13)-(14) in the form $\varphi(m_{K,L}(\sigma_s(x),\sigma_s(y)) = \varphi(\sigma_s(m_{K,L}(x,y))) = \varphi(m_{K,L}(x,y))$. Thus (17b) now follows immediately from (17). Faithfulness also follows, since if the operator a is 0, we have $||a\xi_\varphi|| = ||a^*\xi_\varphi|| = 0$ and thus $d(a,0) = 0$, i.e $a = 0$ in $A \subset M$ since M is a metric space.

Step 3: Obtaining the von Neumann algebra structure on M.

We have now to use the modular theory to be able to represent M (and not only the dense A) on $L^2(A,\varphi)$. Note that (12) gives a Lipschitz bound $||a.F_N(x)||_\varphi^* \leq C_{N,||x||}||a||_\varphi^*$ valid for any $a \in A$. Thus, by the uniform continuity bound for φ, we have:

$$|\langle F_K(y)\xi_\varphi, a.F_N(x)\xi_\varphi\rangle| \leq \sqrt{2}||F_K(y)^*.(a.F_N(x))|| \leq \sqrt{2}C_{K,||y||}C_{N,||x||}||a||_\varphi^*.$$

Using boundedness of the action of A and density of $A \subset L^2(A,\varphi)$, for any $x \in A$, the map $a \mapsto ax\xi_\varphi$ is uniformly continuous as a map $(A\cap D_n(M),d) \to L^2(A,\varphi)$ when the target space is equipped with the weak topology (uniform structure). Thus, since closed balls in $L^2(A,\varphi)$ are complete and Hausdorff for the weak topology and we have checked in the first step that $A \cap D_n(M) \subset D_n(M)$, it follows (e.g. [25, §5.4.(4)]) that the map extends to a (uniformly) continuous map $D_n(M) \to L^2(A,\varphi)$. This gives an action of M on $L^2(A,\varphi)$ by bounded operators, $D_n(M)$ acting by operators of norm less than n (this is used to extend the action from $A\xi_\varphi$ to $L^2(A,\varphi)$).

Note that $(F_N(x)-x)\xi_\varphi, (F_N(x)^*-x^*)\xi_\varphi \to 0$ weakly (from the metric to weak continuity), thus passing to convex combination, a usual consequence of Hahn-Banach theorem says a net of convex combinations of $(F_N(x)\xi_\varphi, F_N(x)^*\xi_\varphi, F_N(x))$ converges to $(x\xi_\varphi, x^*\xi_\varphi, x)$ in $L^2(A,\varphi)^2 \times M$. Applying the above inequality (17b) to this net, one gets for any $x \in M$:

$$4d_U(x,0)^2 \leq ||x\xi_\varphi||^2 + ||x^*\xi_\varphi||^2. \tag{2.9}$$

We also call $U_n(x)$ the net above of convex combination of $F_N(x)$ that we can even assume to converge in the $*$-strong operator topology to x (replacing ξ_φ by $x\xi_\varphi, x \in A$).

The restriction to balls of the map $i : (M,d) \to B(L^2(A,\varphi))$ is thus continuous when the target space is equipped with the weak operator topology. From the inequality (2.9), this gives a faithful action. Let us see that $i(M) = A''$ so that the image $i(M)$ will be a von Neumann algebra (isomorphic to M and will especially induce a product extending the one of A). By density of A and the previously stated continuity for i, the image $i(M)$ is included in the weak closure A''. Conversely, by Kaplansky density theorem, take a net $i(a_n) \to a \in A''$ $||a|| \leq m$, $a_n \in D_m(M) \cap A$ which converges in the strong-* operator topology. From

the inequality (2.9), $d_U(a_n,a_m)^2 \leq ||i(a_n-a_m)\xi_\varphi||^2 + ||i(a_n-a_m)^*\xi_\varphi||^2$ so that a_n is a Cauchy net for the metric d in $D_m(M)$, thus by completeness of the balls of the model, it converges to $A \in D_m(M)$ and it remains to check $i(A)=a$. But by the metric to weak operator topology convergence $i(a_n) \to i(A)$ in the weak operator topology, and since $i(a_n) \to a$ in this separated topology, this concludes.

We thus have a von Neumann algebra structure on M with $j : A \to M$ a state preserving $*$-homomorphism with dense range. Since for $x \in D_m(M)$, there is a net $U_n(x) \in D_m(M) \cap A$ (of convex combinations of $F_N(x)$ as above) converging $*$-strongly to x so that any commutative polynomial $p(U_N(x))$ has the right norm by (11) and tends to $p(x)$, we deduce that it has the right operator norm. Thus arguing as in [13, p 486] before their Proposition 3.2, one gets $D_m(M)$ is the operator-norm ball of radius m in M.

Step 4: Identifying the modular theory $\sigma_t = \sigma_t^\varphi$.

We now want to identify $\sigma_t = \sigma_t^\varphi$. First note that σ_t is an automorphism of M by extension of the property for A and that $t \mapsto \sigma_t(x)$ is weakly continuous for $x \in M$ from the continuity in metric, and since it is bounded, it is also σ-weakly continuous (see e.g. [30, lemma II.2.5]) and thus $(M, \mathbb{R}, \sigma)$ defines a covariant system by [30, proposition X.1.2].

Note that σ_t induces a one parameter group of isometries on $L^2(A,\varphi)$, which is strongly continuous from the continuity on A extended by density, we will write Δ^{it} this semigroup, with Δ an unbounded non-singular densely defined operator on $L^2(A,\varphi)$. Then $\Delta^{it}(x\xi_\varphi) = \sigma_t(x)\xi_\varphi$ for $x \in A$.

Note $\widehat{h_{R+1}}$ is continuous compactly supported in $[-R-2, R+2]$ equal to 1 on $[-R-1, R+1]$ and we have $h_{R+1} \in L^1(\mathbb{R})$. If $x \in M(\sigma, [-R,R])$, we know that $\sigma_{h_{R+1}}(x) = x$ by [30, lemma XI.1.3] and $x\xi_\varphi = \int_{-\infty}^{\infty} dt h_R(t)\Delta^{it}(x\xi_\varphi) = [\widehat{h_R}(\ln(\Delta))](x\xi_\varphi)$ by spectral calculus (since $\int_{-K}^{K} dt h_R(t)e^{itx}$ converges uniformly on $\mathbb{R}$ to $\widehat{h_R}$ and $\int_{-K}^{K} dt h_R(t)\Delta^{it}$ (as uniform limit of Riemann sums) thus converges in norm when $K \to \infty$ to $[\widehat{h_R}(\ln(\Delta))]$ by [28, p326 and Th 13.30] since the unbounded selfadjoint case is developed from the unitary case). Thus by functional calculus again $||\Delta\alpha(\alpha+\Delta)^{-1}(x\xi_\varphi)|| = ||\Delta\alpha(\alpha+\Delta)^{-1}\widehat{h_{R+1}}(\ln(\Delta))(x\xi_\varphi)|| \leq e^{(R+2)}||(x\xi_\varphi)||$ and since this is independent of α one gets $x\xi_\varphi \in D(\Delta)$ and at the limit $\alpha \to \infty$: $||\Delta(x\xi_\varphi)|| \leq e^{(R+2)}||(x\xi_\varphi)||$ for any $x \in M(\sigma,[-R,R])$.

Note that the argument above on spectral algebras for σ suffices to get $A\xi_\varphi \subset D(\Delta)$. Indeed, it suffices to use the equation $\sigma_{h_{R+1}}(x) = H_{R+1}(x) = x$ from axiom (2) for $x = F_R(y)$.

As in lemma 2.2, one deduces $G_s(x)\xi_\varphi = e^{s/2}2\Delta^{1/2}(\Delta+e^s)^{-1}(x\xi_\varphi)$. Let us write for $\beta \in]0,1[$ $\mathscr{E}_\beta(x,y) = \langle \Delta^\beta x\xi_\varphi, y\xi_\varphi\rangle$ and note that $\mathscr{E}_0(F_K(x),F_L(y)) = \mathscr{E}_{0,K,L}(x,y)$. (19) gives for $x,y \in M$, $0<\alpha<1/2, \alpha+\beta<1$:

$$\mathscr{E}_{\alpha+\beta,K,L}(x,y) = \frac{\cos(\alpha\pi)}{2\pi}\int_{-\infty}^{\infty} dt e^{\alpha t}\mathscr{E}_{\beta,K,L}(G_t(x),y).$$

For $\beta = 0$, we have

$$\mathscr{E}_{0,K,L}(G_s(x),y) = \langle e^{s/2}2\Delta^{1/2}(\Delta+e^s)^{-1}(F_K(x)\xi_\varphi), F_L(y)\xi_\varphi\rangle$$

and from the formula above and from the same formula for Δ, one deduces, $\mathscr{E}_{\alpha,K,L}(x,y) = \langle \Delta^\alpha(F_K(x)\xi_\varphi), (F_L(y)\xi_\varphi)\rangle$, $\alpha < 1/2$. Replacing $\beta = 0$ by any $\beta < 1/2$ one thus gets for any $\alpha+\beta<1$,

$$\mathscr{E}_{\alpha+\beta,K,L}(x,y) = \langle \Delta^{\alpha+\beta}(F_K(x)\xi_\varphi), (F_L(y)\xi_\varphi)\rangle.$$

Applying now similarly (20) one gets for $\alpha+\beta=1, x,y \in A$

$$\varphi(F_L(y)[F_K(x)]^*) = \frac{\cos(\alpha\pi)}{2\pi}\int_{-\infty}^{\infty} dt e^{\alpha t}\mathscr{E}_\beta(G_t^\varphi(x),y) = \langle \Delta(F_K(x)\xi_\varphi), (F_L(y)\xi_\varphi)\rangle.$$

Thus by sesquilinearity, we even have for any $x,y \in A$:

$$\varphi(yx^*) = \langle \Delta(x\xi_\varphi), (y\xi_\varphi)\rangle. \tag{2.10}$$

Since $\Delta^{1/2}$ is a closed bounded operator, taking $x_n \to x$, (*-strongly) $x_n \in A, x \in M$ a bounded net one deduces from the formula above that

$$||\Delta^{1/2}(x_n\xi_\varphi)|| \leq ||x_n||$$

is bounded thus $(\Delta^{1/2}(x_n\xi_\varphi))$ has a weakly converging subnet and a normwise converging convex combination so that by closability, $x\xi_\varphi \in D(\Delta^{1/2}) \supset M\xi_\varphi$ and the equality (2.10) is thus extended to M in the form

$$\varphi(yx^*) = \langle y^*\xi_\varphi, x^*\xi_\varphi\rangle = \langle \Delta^{1/2}(x\xi_\varphi), \Delta^{1/2}(y\xi_\varphi)\rangle, \quad x, y \in M. \tag{2.11}$$

But using again the equality in (17) now extended to M (by strong-* density and continuity on $L^2(M,\varphi)$ of $\Delta^{1/2}(1+\Delta)^{-1} = \Delta^{1/2}(1+\Delta)^{-1/2}(1+\Delta)^{-1/2})$

$$\begin{aligned} 4d_U(x,0)^2 &= ||G_0(x)\xi_\varphi||^2 + ||G_0(x)^*\xi_\varphi||^2 \\ &= ||\Delta^{1/2}(1+\Delta)^{-1}(x\xi_\varphi)||^2 + ||\Delta(1+\Delta)^{-1}(x\xi_\varphi)||^2 \\ &\leq 2||x\xi_\varphi||^2. \end{aligned}$$

Thus this implies that φ is faithful on M and thus has a modular group and the relation (2.11) implies $\Delta = \Delta_\varphi$ and thus $\sigma_t = \sigma_t^\varphi$. The previous computations show that, possibly with the exception of $m_{(K,L)}$, all data of our model coincides to the one arising from (M,φ). However for $m_{(K,L)}$ we give a separate argument.

From the definition of step 3, the product $F_K(x).F_L(y)$ computed in M is obtained by seeing $F_K(x)$ as a limit of a convex combination $\sum_i \lambda_i F_{O_i}(F_K(x))$ for O_i large enough. Hence it is a limit of the following convex combinations that we rewrite from an equality in (2):

$$\sum_{i,j} \lambda_i\mu_j m_{(O_i,P_j)}(F_K(x), F_L(y)) = \sum_i \lambda_i m_{O_i,L}(F_K(x), \sum_j \mu_j F_{P_j}(y)).$$

Now, from the inequality in (12), one gets:

$$\begin{aligned} ||\sum_i \lambda_i m_{(O_i,L)}(F_K(x), \sum_j \mu_j F_{P_j}(y)) - m_{(K,L)}(x, \sum_j \mu_j F_{P_j}(y))||_\varphi^* \\ \leq e^L||y||||\sum_i \lambda_i F_{O_i}(F_K(x)) - F_K(x)||_\varphi^* \end{aligned}$$

and this net has been chosen so that this tends to 0. Similarly, we have:

$$m_{(K,L)}(x, \sum_j \mu_j F_{P_j}(y)) \to m_{(K,L)}(x,y)$$

and thus, one obtains $F_K(x).F_L(y) = m_{(K,L)}(x,y)$. We have proved our first axiomatization result.

Step 5: Identifying the ultraproduct.

First, with our original formula for $||x||_\varphi^*$ equivalent to the one in (17) from lemma 2.2 a bounded sequence (x_n) with $d(x_n,0) \to_{n\to\omega} 0$ can be decomposed in $x_n = y_n + z_n$ with $||y_n||_\varphi \to_{n\to\omega} 0, ||z_n^*||_\varphi \to_{n\to\omega} 0$, i.e. with the notation of [1] $(y_n) \in \mathcal{L}_\omega, (z_n) \in \mathcal{L}_\omega^*$. But by their proposition 3.14, $(M_n,\varphi_n)^\omega = \ell^\infty(\mathbb{N}, M_n)/(\mathcal{L}_\omega + \mathcal{L}_\omega^*)$ as vector spaces thus this is the same set as the model theoretic ultraproduct for our structure.

$(\varphi_n)^\omega$ is then defined in the same way and we have to see that the canonical map $\mathcal{M}$, defined during the previous steps, give a state preserving $*$-homomorphism. The only non obvious part is the identification of the product. But if one uses all the results of [1, Th 4.1, lemma 4.13,4.14], it is obvious that all our ultraproduct data is the data taken in their ultraproduct, even the multiplication map $m_{(K,L)}(x_n,y_n) = F_K(x_n).F_L(y_n)$ since $(F_K(x_n))) \in \mathcal{M}^\omega$ and the product is defined as the sequence of products on those sequences. Thus the von Neumann algebra structure has to coincide with the model theoretic one, since we have just checked the model of the Ocneanu ultraproduct (which is known to be a W^*-probablity space) is indeed the model theoretic ultraproduct of models (and since we have checked in our previous steps that the model determines the von Neumann algebra structure). ∎

2.3. More axiomatization results

We now consider extra properties enabling to axiomatize explicitly interesting classes. We fix a closed discrete set $\Gamma \subset \mathbb{R}$ that will contain $Sp(\sigma^\varphi) \subset \Gamma$. The next axiom says σ^φ has the appropriate spectrum with respect to Γ.

(21) For $N \in \mathbb{Q} \cap]0, \infty[, l \in \mathbb{Q}^*, m \in \mathbb{N}^*$ with $]l - N, l + N[\subset \Gamma^c$

$$\sup_{x \in D_m} d_U(F_{N,l}(x)), 0) = 0.$$

Up to now, all axioms form a $\forall$-axiomatizable theory.

Our last axioms in the case $Sp(\sigma^\varphi) = \log(\lambda)\mathbb{Z}, \lambda \in]0, 1[$ will enable to identify a III_λ-factor with a periodic state. We will use notations similar to those in [13] in order to use their characterization of II_1 factors on the centralizer:

$$\xi_K(x) = \sqrt{\varphi(M_{(K,K)}(x^*, x)) - |\varphi(x)|^2}$$

$$Com_K(a, b) = [m_{(K,K)}(b, a) - m_{(K,K)}(a, b)],$$

$$\eta_K(x) = \sup_{y \in D_1} \varphi\left[M_{(2K,2K)}([Com_K(x, y)]^*, [Com_K(x, y)])\right],$$

$$Proj_K(a) = [m_{(K,K)}(a, a^*) - m_{(2K,2K)}(m_{(K,K)}(a, a^*), [m_{(K,K)}(a, a^*)]^*)],$$

Then our next two axioms are expressed as follows:

(22) For $N \in \mathbb{Q} \cap]0, \infty[, l \in \mathbb{Q}^*, m \in \mathbb{N}^*$ with $]l - N, l + N[\cap \Gamma = \{n \log(\lambda)\}, n \in \mathbb{N}^*$ and $n \log(\lambda) \in \Gamma \cap]l - N/2, l + N/2[$

$$\inf_{x \in D_1} \max\left[|d_U(M_{(\lceil |l|+N \rceil + 1, \lceil |l|+N \rceil + 1)}(x^*, x), 1), d_U(2F_{N,l}(x) - F_{N/2,l}(x), x)\right] = 0.$$

(23) For $N \in \mathbb{Q} \cap]0, \infty[, N < |\log(\lambda)|, m \in \mathbb{N}^*, K \geq \lceil N \rceil$

$$\sup_{x \in D_1} \max(0, \xi_K(F_N(x)) - \eta_K(F_N(x))) = 0,$$

$$\inf_{x \in D_1} (\varphi(m_{(4K,4K)}((Proj_K(F_N(x)))^*, Proj_K(F_N(x))))$$
$$+ |\varphi(M_{(K,K)}(F_N(x), F_N(x^*))) - 1/\pi|) = 0.$$

We will see that a result of [1] will state that II_1-factors are not axiomatizable in our language for σ-finite von Neumann algebras. Of course, tracial W^*-probability spaces are axiomatizable in requiring φ to be a trace. Similarly, even though II_∞ factors won't be axiomatizable, we may be interested in having a canonical model in choosing a specific state $\varphi = tr \otimes \varphi|_{B(H)}$ on $M = N \otimes B(H)$, with N a tracial von Neumann algebra. This is the purpose of the next axiom that uses extra constant symbols $w_{i,j} \in D_1, i, j \in \mathbb{N}$ for a matrix unit and also that $\psi = \varphi|_{B(H)}$ is given by $\psi(w_{j,j}) = 2^{-j-1}$, and $\psi(w_{j,k}) = 0$ for $j \neq k$. We will say φ is a geometric state (for this matrix unit). Note that this implies (e.g. by [30, Th 2.11]) that $\sigma_t^\varphi(w_{j,k}) = 2^{(k-j)it} w_{j,k}$ and thus $w_{k,j} \in M(\sigma^\varphi, \{(j - k) \ln(2)\})$.

(24) $\varphi(w_{j,j}) = 2^{-j-1}$, $\varphi(w_{j,k}) = 0$, $w_{k,j}^* = w_{j,k}$, $H_{\lceil |j-k| \ln(2) \rceil + 1}(w_{j,k}) = w_{j,k}$,

$$M_{(\lceil |j-k| \ln(2) \rceil, \lceil |l-m| \ln(2) \rceil)}(w_{j,k}, w_{l,m}) = \delta_{k,l} w_{l,m},$$

(25) $H_1(w_{0,0} F_N(x) w_{0,0}) = w_{0,0} F_N(x) w_{0,0}$, with

$$w_{k,k} F_N(x) w_{j,j} := M_{(N,0)}(M_{(0,N)}(w_{k,k}, F_N(x)), w_{j,j})$$

$$\varphi(M_{(0,0)}(w_{0,0} x w_{0,0}, w_{0,0} y w_{0,0})) = \varphi(M_{(0,0)}(w_{0,0} y w_{0,0}, w_{0,0} x w_{0,0})).$$

(26)
$$\sup_{x\in D_1} \max(0, \xi_1(w_{0,0}xw_{0,0}) - \eta_1(w_{0,0}xw_{0,0})) = 0,$$

$$\inf_{x\in D_1} (\varphi(m_{(4K,4K)}((Proj_1(w_{0,0}xw_{0,0})^*, Proj_1(w_{0,0}xw_{0,0})))$$
$$+ |\varphi(M_{(1,1)}(w_{0,0}xw_{0,0}, w_{0,0}x^*w_{0,0})) - 1/\pi|) = 0.$$

(27) $\varphi(M_{(0,0)}(w_{0,0}xw_{0,0}, w_{0,0}yw_{0,0})) = 2\varphi(w_{0,0}xw_{0,0})\varphi(w_{0,0}yw_{0,0})$

Theorem 2.10. *The following classes are also axiomatizable in the same language as σ-finite W^*-probability spaces:*

(i) σ-finite W^* probability spaces with $Sp(\sigma^\varphi) \subset \Gamma$ for a discrete set Γ, by the theory $T_{\sigma W^*}(Sp \subset \Gamma)$ consisting of axioms (1)-(21).
(ii) III_λ factors with a periodic state for z fixed $0 < \lambda < 1$, by the theory $T_{\sigma III_\lambda}$ consisting of axioms (1)-(23) with $\Gamma = \log(\lambda)\mathbb{Z}$.
(iii) III_λ factors with some faithful state for a fixed $0 < \lambda \leq 1$.

In appropriate expansions of this language described above, the following classes are axiomatizable :

(iv) σ-finite W^* probability spaces of the form $N \otimes B(H)$ for N tracial and with φ a geometric state, by $T_{\sigma W^* geom}$ consisting of $T_{\sigma W^*}$ and (24)-(25).
(v) σ-finite II_∞ factors with φ a geometric state, by $T_{\sigma W^* II_\infty geom}$ consisting of $T_{\sigma W^*}$ and (24)-(26).
(vi) σ-finite type I_∞ factors with φ a geometric state, by $T_{\sigma W^* I_\infty geom}$ consisting of $T_{\sigma W^* geom}$ and (27).

Remark 2.11. *The reader should note that $T_{\sigma W^*}(Sp \subset \Gamma), T_{\sigma W^* geom}, T_{\sigma W^* I_\infty geom}$ are universal theories while $T_{\sigma W^* II_\infty geom}, T_{\sigma III_\lambda}$ are $\forall\exists$ theories as it was the case for the theory of tracial W^* probability spaces which are II_1 factors.*

Proof : Condition (21) says explicitly $Sp(\sigma_\varphi) \subset \Gamma$ in saying $M(\sigma, E) = \{0\}$ for any closed set $E \subset]l - N, l + N[\subset \Gamma^c$.

For a III_λ factor with a periodic state Takesaki's theorem [29, Th 1.27], there is an isometry u (thus $u^*u = 1$) in $M(\sigma^\varphi, \{\log(\lambda)\})$. By discreteness of Γ for $\gamma = \log(\lambda) \in \Gamma$, one can find l, N as in (22). Since $2\widehat{f_{N,l}} - \widehat{f_{N/2,L}}$ is supported in $[l - N, l + N]$ and equal to 1 on $[l - N/2, l + N/2]$ thus a neighbourhood of $\{\gamma\}$ so that $2F_{N,l}(u) - F_{N/2,L}(u) = u$ and similarly, by [30, lemma XI.1.3] again, $M_{(\lceil |l|+N\rceil+1, \lceil |l|+N\rceil+1)}(u^*, u) = u^*u = 1$ and thus (22) is satisfied.

Moreover, by the same theorem of Takesaki, the centralizer is a II_1 factor. Since the state is lacunary (as soon as (21) holds), the argument in [22, lemma 2.3] gives that F_N with N as in (23) is the state preserving projection on the centralizer. (23) thus says that the centralizer satisfies the axioms (16)-(17) in [13] and thus is not only a tracial von Neumann algebra but a II_1 factor. Thus it is indeed satisfied if M is a III_λ factor with a periodic state.

Conversely, assuming not only (21) but also (23), we know that the centralizer is a II_1 factor and one argues in the spirit of [1, Rmk 6.12], the center of M is in the centralizer, thus in the center of the centralizer, thus M is a factor. Then by general results, the log of Connes' S-invariant is $\log(S(M) - \{0\}) = Sp(\sigma^\varphi)$. We then claim that (22) implies that $Sp(\sigma^\varphi) = \log(\lambda)\mathbb{Z}$, which, by definition, means that M is a type III_λ factor and φ a periodic state. Note that in our case $\Gamma = \log(\lambda)\mathbb{Z}$, we already know $Sp(\sigma^\varphi) \subset \log(\lambda)\mathbb{Z}$ by (21). Indeed, assume $M(\sigma^\varphi, \{log(\lambda)\}) = \emptyset$. Thus for every x, we have $F_{N,l}(x) = F_{N/2}(x, l) = 0$ and since (22) holds, one can find $x = x(\epsilon)$ with $||x||^*_\varphi \leq \epsilon$ and $||(M_{(\lceil |l|+N\rceil+1, \lceil |l|+N\rceil+1)}(x^*, x) - 1||^*_\varphi \leq \epsilon$. But $||M_{(K,L)}(x^*, x)||^*_\varphi \leq 4(K + 1)(L + 1)C_{K+1}||x||\ ||x||^*_\varphi$ which is as small as one wants if ϵ small enough, contradicting $|\ ||(M_{(\lceil |l|+N\rceil+1, \lceil |l|+N\rceil+1)}(x^*, x)||^*_\varphi - 1| \leq \epsilon$. Thus (22) implies $M(\sigma^\varphi, \{log(\lambda)\}) \neq \emptyset$. The same reasoning applies to $M(\sigma^\varphi, \{n\ log(\lambda)\}) \neq \emptyset$ for $n \in \mathbb{N}$ and then by adjoint for $n \in \mathbb{Z}$.

This concludes the axiomatization of III_λ factors with periodic states.

For II_∞ factors (algebras) with geometric state, we have checked that a geometric state satisfies (24) and conversely, $w_{i,j} \in A$ and is thus a matrix unit (for $\sum_i w_{i,i} = 1$ since $w_{i,i}$ are orthogonal projections, $||1 - \sum_{i \leq n} w_{i,i}||_\varphi^\# = \varphi(1 - \sum_{i=0}^n w_{i,i}) = 2^{-(n+1)} \to 0$. This gives the strong convergence of $\sum_{i \leq n} w_{i,i}$ to 1). Moreover if $e = w_{0,0}$, φ restricted to eMe is tracial by (25) and for instance [30, Prop IV.1.8] gives $M \simeq eMe \otimes B(H)$. Using [13], axiom (26) then says eMe is a II_1 factor. (27) says $eMe = \mathbb{C}$ since $2\varphi(w_{0,0}.w_{0,0})$ is a state on eMe with e as unit.

Finally, the axiomatizability statement for III_λ factors, $\lambda \in]0,1]$ will follow from [2, Prop 5.14]. We will only check that variants of the results of [1] implies their stability by ultraproducts and ultraroots. Note that actually, it suffices to check stability by ultraroots for countably incomplete ultrafilters (since one can always take an ultrafilter of that type in Keisler-Shelah theorem). We thus assume our ultrafilters are countably incomplete until the end of the proof.

The stability by ultraproducts is [1, Th 6.11] and our slight extension corollary 4.2 below (in the non-countable case). Recall for a fixed state $(M, \varphi)^\omega = M^\omega$ does not depend on φ. First, if M^ω is a III_λ-factor, the center of M is included in the center of M^ω thus M is a factor. The ultrapower of a type I_n, I_∞, II_1 factor is of the same type (see e.g. [1, Prop 6.1] for references, for type I_∞ factors one can use our axiomatization $T_{\sigma W^* I_\infty geom}$ with a geometric state), the ultrapower of a type III_λ, $\lambda \in]0,1]$ is of the same type as recalled, and using [1, Th 6.18], the ultrapower of a type III_0 factor is never a factor (see corollary 5.2 for the uncountable variant in case of countably incomplete ultrafilters). Thus it only remains to exclude M to be a II_∞ factor (which is explained in the separable case in [1, Prop 6.1]). But if $M = N \otimes B(H)$ for H separable and N a II_1-factor, since the ultrapower does not depend on the state, we can realize the ultrapower with the geometric state so that M satisfies $T_{\sigma W^* II_\infty geom}$, thus so does its ultrapower, and thus it would be a II_∞ factor too. ∎

Remark 2.12. *[Non-Axiomatizability results of various classes]*

Using [1, Prop 6.3, Th 6.18], one sees that factors, III_0 factors are not stable by ultrapowers, and II_1-factors are not stable by ultraproducts, thus none of these classes are axiomatizable in the language for σ-finite factors. However, the proof above shows that every other classes of factors namely $I_n, n \in \mathbb{N}^* \cup \{\infty\}, II_1, II_\infty$ are local (i.e. stable by ultrapowers and ultraroots).

Corollary 2.13. *The theory of $B(H)$ described above as $T_{\sigma W^* I_\infty geom}$, in the language of σ-finite W^* probability spaces with added constants for a matrix unit, is ω-categorical and admits quantifier elimination.*

Proof : The theory has a unique model $B(H)$ and it is separable, implying a fortiori ω-categoricity. We can apply the quantifier elimination test [2, Prop 13.6]. Indeed, for any substructure $M \subset N$ in this language with N a model of $T_{\sigma W^* I_\infty geom}$, since the constant of the matrix units are in the language and generate a d dense space, one deduces $M = N$. Hence the extension requirement is trivial. ∎

2.4. An approximately minimal language enabling axiomatization of σ-finite W^*-probability spaces.

A really natural question, first asked to us by Itaï Ben Yaacov, concerns the minimal language enabling to carry out the previous axiomatization. At first sight, we thought that the modular group would be one piece of this language, and obtaining this axiomatization (even an explicit one) was easy but unreadable. But, in trying to answer another really natural question communicated to us by Ilijas Farah about the definability of the modular group, we realized that even this piece of data was not necessary. We summarize those results in this subsection.

We consider the following countable language $\mathcal{L}_\nu$. We still consider one sort U with domains of quantification $D_n = D_n(U)$. They are still expected to be interpreted as the operator norm unit balls $D_n(M)$. Since we aim at minimality, we don't take Lipschitz curve sorts from section 7 in this subsection. In this section we call $T_{\sigma W^*}$ a variant of our previous theory with only one sort U (and hence no sorts for Lipschitz curves). To obtain this variant, we replace any axiom involving Riemann integrals such as (16),(18),(19),(20) by the explicit definition of those Riemann integrals used to define them in section 7. This theory was actually the original theory written in the arxiv preprints having lead to that article and can be easily deduced from our previous subsections and section 7.

- The constant 0 which will be in D_1.
- For every $\lambda \in \mathbb{Q}[i]$ a unary function symbol also denoted λ to be interpreted as a scalar multiplication. For simplicity we shall write λx instead of $\lambda(x)$.
- A unary function symbol $*$ for involution on U, leaving stable all domains.
- A binary function symbol $+ : D_n \times D_m \to D_{n+m}$. and for $K, L \geq \nu$, $m_{(K,L)} : D_n \times D_m \to D_{nm}$ (interpreted as $F_K^{\varphi}(.)F_L^{\varphi}(.)$ with modulus of continuity as obtained in lemma 2.1).
- The constant 1 in D_1.
- Two unary relation symbols φ_r and φ_i for the real and imaginary parts of the state φ, on U.

Using the (obvious multisorted/multidomain variant of the) definition 9.27 of [2], we want to obtain the following result for the theory $T^{\nu}_{\sigma W^*}$ composed of logical consequences of $T_{\sigma W^*}$ in the language $\mathcal{L}_\nu$, i.e. its so-called restriction to $\mathcal{L}_\nu$.

Theorem 2.14. *For any $\nu \in \mathbb{N}$, $T_{\sigma W^*}$ is an extension by definitions of $T^{\nu}_{\sigma W^*}$.*

Proof : By definition, $T_{\sigma W^*}$ is a conservative extension of $T^{\nu}_{\sigma W^*}$. It thus suffices to check that any supplementary logical symbol is definable. The main part is to prove the modular group σ_t is defined in $T_{\sigma W^*}$ over $\mathcal{L}_\nu$.

Step 1: Definability of σ_t in $T_{\sigma W^*}$ over $\mathcal{L}_\nu$.

This will use several spectral theory results we didn't recall yet. We will thus explain this in several sub-steps. To explain the general idea, we want to define Δ^{it} as a quadratic form, dominated by the form corresponding to the metric with generator $\Delta(1+\Delta)^{-1}$. We will define $(u+\Delta)^{it} = e^{it\ln(u+\Delta)}$. Since $\ln(u+\Delta)$ is bounded below, one can use the standard Hille-Yosida theory of semigroups to define this from the resolvants $(\beta + \ln(u+\Delta))^{-1}$. They can be themselves produced from the semigroup at positive times $(u+\Delta)^{-t}$ and this semigroup can be defined by composition and integral formulas for fractional powers of $(u+\Delta)^{-1}$. Thus, in reverse order, we will start by defining those as forms and check all the steps can be written as Riemann integrals with explicit Lispchitz constants and bounds on our domains of quantification in order to get the final definability. The starting point is an infimum formula to define $(u+\Delta)^{-1}$ in a way similar to our first definition of the metric. However, we should obtain an explicit domain of quantification where our infimum will be reached.

We first show that the relation

$$\psi_{K,L,u}(x,y) = \varphi(F_K(y^*).\Delta(u+\Delta)^{-1}(F_L(x))) \text{ for } u \in \mathbb{Q}, u > 0, K, L \geq \nu$$

is defined in $T_{\sigma W^*}$ over $\mathcal{L}_\nu$ (It is even equivalent to a formula of the language in any model). Formally, this formula in $\mathcal{L}$ has to be understood as using the form for $\Delta^{1/2}$ and $G_{\ln(u)}$ related to $\Delta^{1/2}(u+\Delta)^{-1}$ namely

$$\psi_{K,L,u}(x,y) := \frac{1}{2\sqrt{u}}\mathscr{E}_{1/2,K,L}(y, G_{ln(u)}(x)).$$

First note that by definition and polarization:

$$\begin{aligned}\psi_{K,L,u}(x,y) &= \langle \Delta^{1/2}(u+\Delta)^{-1/2}(F_K(y)\xi_\varphi), \Delta^{1/2}(u+\Delta)^{-1/2}(F_L(x)\xi_\varphi)\rangle \\ &= \frac{1}{4}\sum_{k=0}^{3}(-i)^k \psi'_{K,L,u}(i^k x, y),\end{aligned}$$

with

$$\begin{aligned}\psi'_{K,L,u}(x,y) &:= ||\Delta^{1/2}(u+\Delta)^{-1/2}[(F_K(y)+F_L(x))\xi_\varphi]||^2 \\ &= ||\Delta(u+\Delta)^{-1}[(F_K(y)+F_L(x))\xi_\varphi]||^2 \\ &+ u||\Delta^{1/2}(u+\Delta)^{-1}[(F_K(y)+F_L(x))\xi_\varphi]||^2.\end{aligned}$$

Now if $x, y \in D_1(M)$, $X = F_K(y) + F_L(x)$, one sees as in lemma 2.2 that

$$\psi'_{K,L,u}(x,y) = \inf_{z \in M} \left[\varphi(z^*z) + \frac{1}{u}\varphi((X-z)(X-z)^*) \right]$$
$$= \inf_{z=2F_{2K+2L+2}(z)-F_{K+L+1}(z) \in D_{m(K,L,u)}(M)} \left[\varphi(z^*z) + \frac{1}{u}\varphi((X-z)(X-z)^*) \right]$$

since the first infimum, in any W^* probability space, is reached at $z = \Delta(u+\Delta)^{-1}(X) = X - u(u+\Delta)^{-1}(X) \in D_{m(K,L,u)}(M)$ with $m(K,L,u) = \frac{3(e^K+e^L)}{2\sqrt{u}}$ since $||\Delta^{1/2}(F_K(y))||_M \leq 3e^K$ (using again the proof of [1, lemma 4.13]) and the bound on G_s in lemma 2.2 as operator on M. And the equality with the second infimum follows since we also have the infimum reached in $M(\sigma^\varphi, [-K-L, K+L])$ explaining the identity $z = 2F_{2K+2L+2}(z) - F_{K+L+1}(z)$. Thus we also obtain the explicit formula in $\mathcal{L}_\nu$. Indeed recall that $\tilde{m}$ is explicitly defined from m by

$$\tilde{m}_{(N,N)}(x,y) = 4m_{(2N,2N)}(x,y) + m_{(N,N)}(x,y) - 2m_{(2N,N)}(x,y) - 2m_{(N,2N)}(x,y). \tag{2.12}$$

Let us write for short $F_{L,K}(x,y) := m_{(L,L)}(x,1) + m_{(K,K)}(y,1)$. We claim that we have already obtained :

$$\psi'_{K,L,u}(x,y) = \inf_{Z \in D_{m(K,L,u)}(M)} \left[\varphi(\tilde{m}_{(K+L+1,K+L+1)}(Z^*, Z)) \right.$$
$$\left. + \frac{1}{u}\varphi(\tilde{m}_{(K+L+1,K+L+1)}((F_{L,K}(x,y) - Z), (F_{L,K}(x,y) - Z)^*) \right].$$

Indeed, this last infimum corresponds to the first infimum above for elements of the form $z = 2F_{2K+2L+2}(Z) - F_{K+L+1}(Z)$ and is thus a priori in between the two infima we have already showed equal.

We thus have our starting point to apply spectral theory and make definable the modular group. Since the various spectral theory maps are only defined at L^2 level, we only consider for a while either the corresponding forms and show that they give definable relations, or their composition with F_K for which we can get operator norm estimates. The intermediate formulas below won't be part of the language $\mathcal{L}$ but we will show they are definable universally in any W^* probability space.

First, we note that $\psi_{K,L,u_1,...,u_n}(x,y) = \varphi(F_K(y^*).\Delta(u_1+\Delta)^{-1}...(u_n+\Delta)^{-1}(F_L(x)))$ for $u_1 \neq ... \neq u_n \in \mathbb{Q}, u_i > 0, K, L \geq \nu$ are definable in $T_{\sigma W^*}$ over $\mathcal{L}_\nu$ by using the resolvent relations $(u_1+\Delta)^{-1}(u_2+\Delta)^{-1} = \frac{1}{u_2-u_1}[(u_1+\Delta)^{-1} - (u_2+\Delta)^{-1}]$ iteratively. Finally for general $u_1, ..., u_n$ the definability comes from (the same equation implying with standard bounds on operator norms of resolvents of positive operators):

$$T_{\sigma W^*} \models$$
$$\sup_{(x,y) \in D_m^2} |\psi_{K,L,u_1,...,u_i,...,u_n}(x,y) - \psi_{K,L,u_1,...,u_i+\epsilon,...,u_n}(x,y)| \leq \frac{\epsilon m^2}{u_1...u_n(u_i+\epsilon)}.$$

Then we want to show the definability of a map corresponding in any model to

$$A_{u,K} := (u+\Delta)^{-1}F_K, u \in \mathbb{Q}, u > 0, K \geq \nu.$$

First note that $A_{u,K} = \frac{1}{2\sqrt{u}}\Delta^{-1/2}G_{ln(u)}F_K$ and thus maps $D_m(M) \to D_{\lceil \frac{3me^K}{2\sqrt{u}} \rceil}(M)$. This uses again the proof of [1, lemma 4.13]. But even more, we have the relation

$$A_{u,K} = u^{-1}F_K - u^{-1}\Delta(u+\Delta)^{-1}F_K = u^{-1}F_K - \frac{1}{2\sqrt{u}u}\Delta^{1/2}G_{ln(u)}F_K : D_m(M) \to D_{\frac{m}{u} + \frac{3me^K}{2\sqrt{u}u}}(M) \tag{2.13}$$

Unlike the previous estimates this can be effectively used to show norm convergence for some integrals. Thus we want to estimate uniformly over models $||A_{u,K}(x) - y||_\varphi^*$ and thus since

$$\sup_{(x,y) \in D_m^2} | \, ||A_{u,K}(x) - y||_\varphi^* - ||A_{u,K}(x) - F_L(y)||_\varphi^* \, | \leq \sup_{y \in D_m} ||y - F_L(y)||_\varphi^* \to_{L \to \infty} 0,$$

(from the integral definition, the fact that Fejer's kernel is a positive mollifier and lemma 2.4) it suffices to check definability, for L large enough, $u \neq 1$, of

$$(||A_{u,K}(x) - F_L(y)||_\varphi^*)^2 = ||\Delta^{1/2}(1+\Delta)^{-1/2}(A_{u,K}(x) - F_L(y))\xi_\varphi||^2$$
$$= \Re\langle 2F_L(y) + (u+\Delta)^{-1}F_K(x), \Delta(1+\Delta)^{-1}(u+\Delta)^{-1}F_K(x)\rangle + (||F_L(y)||_\varphi^*)^2$$
$$= 2\Re\psi_{L,K,1,u}(x,y) + \psi_{K,K,1,u,u}(x,x) + (||F_L(y)||_\varphi^*)^2.$$

Thus our previous computation gives the definability of $A_{u,K}$ for every $u \in \mathbb{Q}, u > 0$ over $\mathcal{L}_\nu$.

We then check the same kind of definability for

$$B_{u,t,K} = (u+\Delta)^{-t}F_K, u, t \in \mathbb{Q}, u, t > 0, K \geq \nu, t < 1/2$$

One uses the relation in lemma 2.6, $B_{u,t,K} = \frac{\sin(t\pi)}{\pi}\int_0^\infty dv A_{u+v,K}v^{-t}$. From the bound (2.13) for A, one deduces the first bound for $y \in D_m$:

$$||B_{u,t,K}(y)|| \leq \frac{\sin(t\pi)}{\pi} m \int_0^\infty dv(\frac{v^{-t}}{u+v} + \frac{3e^K v^{-t}}{2(u+v)\sqrt{u+v}})$$
$$\leq m(\frac{u^{-t}}{(1-t)} + \frac{3t(4-t)e^K u^{-1/2-t}}{2(1+2t)(1-t)})$$
$$\leq m 8 e^K u^{-1/2-t} \text{ for } u < 1, t < 1/2.$$

The second inequality is obtained by cutting the integral at u and using standard bounds such as $\frac{\sin(t\pi)}{\pi} \leq t$.

This gives a domain of value for $B_{u,t,K}$ and the definability is easy since the integral is a Riemann integral with uniform Lipschitz bound in v for $A_{u+v,K}$ coming again from the resolvent equation. We then extend $B_{u,t,K} = (2B_{u,t/n,2K} - B_{u,t/n,K})^{\circ(n-1)} \circ B_{u,t/n,K}$ for other $n/2 > t > 0$ which is thus definable by composition and has also the same formula as the original $B_{u,t,K}$ in any model. Note also (the last inequality for $u < 1, t < n/2$) $||B_{u,t,K}(y)|| \leq 8^n 3^{n-1} e^{2(n-1)K+K} u^{-n/2-t}\|y\|$.

For $\beta \in \mathbb{Q}$, large enough, we then show similarly the definability of

$$C_{u,\beta,K} := (\beta + \ln(u+\Delta))^{-1}F_K = \int_0^\infty dv e^{-\beta v} B_{u,v,K},$$

since for $u < 1, y \in D_m(M)$ we have the bound:

$$||C_{u,\beta,K}(y)|| \leq m\sum_{k=0}^\infty \int_{(k-1)/2}^{k/2} dv e^{-(\beta+\ln(u))v} 8^k .3^{k-1} e^{2(k-1)K+K} u^{-k/2}$$
$$\leq \frac{e^{(\beta+\ln(u))/2}}{3(\beta+\ln(u))(1-e^{-1})} e^{-K} m.$$

Here, we use the geometric series $\sum_k Ce^{-k}$ together with the estimate $\beta + 2\ln(u) > 4K + 2\ln(24) + 2$. The definability is again easy by Riemann integration. Indeed, for $v > t$, we have the Lipschitz bound deduced from spectral theory:

$$\|(B_{u,v,K} - B_{u,t,K})(y)\|_\varphi^* \leq u^{-t}|v-t|\|\ln(u+\Delta)F_K(y)\|_\varphi \leq u^{-t}|v-t|\ln(u+e^K)\|y\|.$$

We now want to exponentiate (the map in front of F_K) in

$$\beta\ln(u+\Delta)C_{u,\beta,K} = \beta(F_K - \beta C_{u,\beta,K})$$

of course in a definable way to get for $t \in \mathbb{Q}$, $E_{u,\beta,t,K} := e^{it\beta\ln(u+\Delta)(\beta+\ln(u+\Delta))^{-1}}F_K$, in any model, we have the formula (using that all the maps leave stable the spectral algebras with spectrum $[-K, K]$, reached by

the first F_K, so that all the $2F_{2(K+1)} - F_{K+1}$ appearing can be replaced by identity):

$$E_{u,\beta,t,K} = F_K + \sum_{k=0}^{\infty} \frac{(it)^k}{k!}[2\beta(F_{2K+2} - \beta C_{u,\beta,2K+2}) - \beta(F_{K+1} - \beta C_{u,\beta,K+1})]^{\circ k} \circ [\beta(F_K - \beta C_{u,\beta,K})].$$

From this, for β large enough as above, it is easy to get the domain of value of the map and the definability over $\mathcal{L}_\nu$. We now want to make $\beta \to \infty, u \to 0, K \to \infty$.

For this final step, we won't need anymore any composition of maps, thus we come back to definition as forms. From the above definability, one gets the same for the relation (from the equality in any model):

$$\begin{aligned} F_{K,L,u,\beta,t}(x,y) &:= \varphi(F_L(y^*).\Delta(1+\Delta)^{-1}E_{u,\beta,t,K}(x)) \\ &= 2\psi_{L,2K,1}(E_{u,\beta,t,K}(x),y) - \psi_{L,K,1}(E_{u,\beta,t,K}(x),y). \end{aligned}$$

Now, we consider the potential limit for $\beta \to \infty$:

$$F_{K,L,u,t}(x,y) = \varphi(F_L(y^*).\Delta(1+\Delta)^{-1}e^{it\ln(u+\Delta)}F_K(x)).$$

One can use Duhamel's formula to get:

$$\begin{aligned} &F_{K,L,u,t}(x,y) - F_{K,L,u,\beta,t}(x,y) = \int_0^t ds[\varphi(F_L(y^* \\ &.\Delta(1+\Delta)^{-1}e^{i(t-s)\beta\ln(u+\Delta)(\beta+\ln(u+\Delta))^{-1}}\ln(u+\Delta)(1-\beta(\beta+\ln(u+\Delta))^{-1})e^{is\ln(u+\Delta)}F_K(x))]. \end{aligned}$$

Now since $\beta\ln(u+\Delta)(\beta+\ln(u+\Delta))^{-1}, \ln(u+\Delta)$ are self-adjoint, one gets:

$$\begin{aligned} &|F_{K,L,u,t}(x,y) - F_{K,L,u,\beta,t}(x,y)| \\ &\leq \frac{t}{\beta}\|\Delta(1+\Delta)^{-1}\ln(u+\Delta)F_L(y)\|_\varphi\|\ln(u+\Delta)\beta(\beta+\ln(u+\Delta))^{-1}F_K(x)\|_\varphi \\ &\leq \frac{t}{\beta}\ln(u+e^L)\ln(u+e^K)\|y\|_\varphi\|x\|_\varphi. \end{aligned}$$

One thus deduces from the limit $\beta \to \infty$, the definability of $F_{K,L,u,t}$. Similarly we show the definability over $\mathcal{L}_\nu$ of

$$F_{K,L,t}(x,y) = \varphi(F_L(y^*).\Delta(1+\Delta)^{-1}\sigma_t^\varphi(F_K(x)))$$

by using Duhamel's formula again to get the inequality

$$|F_{K,L,u,t}(x,y) - F_{K,L,t}(x,y)| \leq \frac{tu}{u+e^{-K}}\|y\|_\varphi\|x\|_\varphi.$$

Finally, we make $K, L \to \infty$ to get the interesting limit $F_t(x,y) = \varphi(y^*.\Delta(1+\Delta)^{-1}\sigma_t^\varphi(x))$ in noting that in any model, by Schwarz inequality:

$$\begin{aligned} &|F_t(x,y) - F_{K,L,t}(x,y)| \\ &\leq \sqrt{\varphi([F_L(y)-y]^*.\Delta(1+\Delta)^{-1}[F_L(y)-y]))\varphi(\sigma_t^\varphi(x))^*.\Delta(1+\Delta)^{-1}\sigma_t^\varphi(x))}. \\ &+ \sqrt{\varphi([F_L(y)]^*.\Delta(1+\Delta)^{-1}[F_L(y)]))\varphi(\sigma_t^\varphi(F_K(x)-x))^*.\Delta(1+\Delta)^{-1}\sigma_t^\varphi(F_K(x)-x))} \\ &\leq \|F_L(y)-y\|_\varphi^*\|x\|_\varphi^* + \|F_K(x)-x\|_\varphi^*\|y\|_\varphi^* \end{aligned}$$

with the last inequality coming from the identification in the proof of lemma 2.2 again. One deduces the uniform convergence in x, y, uniformly over models, as $K, L \to \infty$ from lemma 2.4.

Finally, we are ready to prove the expected definability of σ_t in $T_{\sigma W^*}$ over $\mathcal{L}_\nu$. For, we need by definition to check the definability of $d(\sigma_t(x),y)^2 = \|\Delta^{1/2}(1+\Delta)^{-1/2}(\sigma_t(x)-y)\xi_\phi\|^2$ by the proof of lemma 2.2. But by commutation of σ_t with Δ and invariance of φ, we have the alternative formula:

$$d(\sigma_t(x),y)^2 = \Re(F_0(x,x)+F_0(y,y)-2F_t(x,y)),$$

And this gives definability by our previous work.

Step 2: Definability of other maps from $\mathcal{L}$ in $T_{\sigma W^*}$ over $\mathcal{L}_\nu$. The inequality in axiom (16) then gives the axiomatization of G_t from the one of σ_t (using the triangular inequality to make $d(G_t(x),y)$ close to the corresponding distance function with G_t replaced by Riemann sums), similarly for $F_{N,l}$ with (18), $\mathscr{E}_{\alpha,K,L}$ with (19). $\tau_{p,\lambda,N}$ is obvious by axiom (11).

The definability of $\lambda(.)$ for $\lambda \in \mathbb{C}-\mathbb{Q}[i]$ is of course easy by density.

Finally, it remains to check the definability of $m_{(K,L)}$ for $K,L \in \mathbb{N}^*$ (if we don't have $K,L \geq \nu$). From the formula $m_{(K,L)}(x,y) = \tilde{m}_{(O,P)}(F_K(x),F_L(y))$ for $O \geq \max(K+1,\nu), P \geq \max(L+1,\nu)$ and with the notation of (2.12), it suffices to show definability of F_K (for $K<\nu$) and this is a special case of $F_{m,l}$ above. This concludes the definability of all supplementary maps. ∎

Corollary 2.15. *For any $\nu \in \mathbb{N}$, σ-finite W^* probability spaces are axiomatizable in the language $\mathcal{L}_\nu$ (by the not so explicit theory $T^\nu_{\sigma W^*}$).*

Proof : Since $T_{\sigma W^*}$ axiomatizes σ-finite W^* probability spaces, $T^\nu_{\sigma W^*}$ is exactly the theory considered in the already used consequence of Keisler-Shelah theorem [2, Prop 5.14], namely the set of all $\mathcal{L}_\nu$-conditions satisfied in all σ-finite W^* probability spaces. We thus check that this class of models is stable by ultraproducts and ultraroot. Of course, this class of spaces is stable by ultraproducts since the language is a restriction of the previous one. Conversely, if (M,φ) is a $\mathcal{L}_\nu$-structure, such that $(M,\varphi)^\omega$ is a σ-finite W^*-probability space, $(M,\varphi)^\omega$ has a unique expansion to a model $[(M,\varphi)^\omega]_{\mathcal{L}}$, by [2, Corol 9.31].

By definition, the result of extension by definitions implies that each symbol in $\mathcal{L}$ not in $\mathcal{L}_\nu$ gives a definable constant, function or predicate in $(M,\varphi)^\omega$ as an $\mathcal{L}_\nu$-structure in the sense of [2, Def 9.1]. Thus by [2, Prop 9.7,9.25] and the theorem of ultraproducts, one deduces from the elementary embedding $(M,\varphi) \preceq (M,\varphi)^\omega$ as $\mathcal{L}_\nu$ theory, the elementary embedding $(M,\varphi)_{\mathcal{L}} \preceq [(M,\varphi)^\omega]_{\mathcal{L}}$ for the restriction to (M,φ) of the above extensions (note that for functions one uses [2, Prop 9.25] for the restriction to be well-defined). In particular, $(M,\varphi)_{\mathcal{L}}$ is a model of $T_{\sigma W^*}$ thus a σ-finite W^*-probability space as expected. ∎

Remark 2.16. *It is not difficult to see (in combining (16), (19) and (20) in a formula corresponding to a discretization of a sixfold integral to get the form $\mathscr{E}_1$, with one integral to express it in terms of $\mathscr{E}_{2/3}$, one to express this one in terms of $\mathscr{E}_{1/3}$ and another one in terms of $\mathscr{E}_0$ and 3 more to express each G_s in those formulas) that σ-finite W^* probability spaces are axiomatizable by a $\forall\exists$-theory in the language $\mathcal{L}_\nu$ with all modular group $\sigma_t, t \in \mathbb{Q}$ added. If we even add $\tau_{p,\lambda,n}$ in the language (say with $N_i \geq \nu$ in $\mathbb{N}$) one can even get back the universal axiomatization.*

This suggests the following questions (The author rather conjectures a negative answer).

Question 2.17. *Does there exist a $\forall\exists$-axiomatization of σ-finite W^* probability spaces in the language $\mathcal{L}_\nu$? Does there exist a universal axiomatization (maybe with $\tau_{p,\lambda,n}$ added)?*

3. The Groh theory for preduals of von Neumann algebras

Our next goal is to show the axiomatizability of preduals of von Neumann algebras in a natural language giving an ultraproduct corresponding to the Groh ultraproduct after taking duals. Unfortunately, neither Groh's construction nor the related construction of Raynaud (of ultraproducts of standard forms) considered instead in [1] gives an insight in a possible language for such a theory. Looking at Groh's construction rather

suggests a theory for a pair of a predual X of a von Neumann algebra and a weak-* dense C^*-subalgebra of X^*. And of course, the C^*-algebra structure is strongly used to show X^* is a von Neumann algebra. We thus look for an implicit axiomatization. The strategy is to use Groh's idea to obtain stability by ultraproducts. But to identify the language, we look for a natural language containing enough information to obtain stability by ultraroot. This right language becomes clear in the proof of theorem 3.3. As should be expected from the commutative case, it contains the structure of the predual X as an operator space, some orders (as in the commutative case) but in each $M_n(X)$ and a dualization of a unit in X^* which is usually called the Haagerup trace. This theory is chosen in order to use a result of Choi and Effros giving a C^*-algebra structure to an operator system, image of a completely positive unital projection on a subspace of a C^*-algebra. In our case the known C^*-algebra will be an ultrapower of the subspace we want to put a C^*-algebra structure on. The language is thus natural for a predual of a dual operator system. Then, once identified the theory with a language rich enough to obtain stability by ultraroot, we prove (and this is our starting point and the content of theorem 3.2) that Groh's expansion with a C^*-algebra can be used to strengthen his result and give stability by ultraproduct in this stronger but still quite natural language. Note once again that section 3 leaves open the question of an explicit theory for preduals of von Neumann algebras. The theory given by the model theoretic axiomatizability result gathers all continuous model theoretic formulas satisfied by all von Neumann algebra preduals in our language for preduals of dual operator systems. It would be interesting if a more explicit theory could be found in following the suggestion of Effros-Ruan [9, p 303] and could be based on looking at the subspaces completely isometric to trace class operators. However our paper gives a first answer to the related question of finding some axiomatization of preduals using operator space/system theory and not involving the dual C^*-algebra structure (even in the form of a coproduct).

3.1. The explicit theory for preduals jointly with a weak- dense subalgebra of its dual*

We will use the notion of matrix-ordered operator space in the sense of [31], that includes duals of C^*-algebras (and preduals of W^*-algebras). We call language for matrix-ordered operator spaces the union of the language for operator spaces and operator systems in [18] without any unit symbol (and thus without their map h_n containing a unit) but with an extra $\oplus$ operation on positive cones. Matrix-ordered operator spaces are easily axiomatized in this language if we remove Werner's axiom (M_0) (saying that a sum of positive is 0 if and only if each is 0). We will recover this axiom later, we only keep the axiom saying that diagonal direct sum of positive are positive (M_1) and conjugation by matrices keeps positivity (M_2). More precisely we also add in any cone sort, for any polynomial in one variable positive on $\mathbb{R}_+$, the function τ_p with an axiom saying $\tau_p(x) = p(d(x,0))x$ with domains $\tau_p : D_n \to D_{\lceil \sup_{t\in[0,n]} |p(t)t| \rceil}$ so that approximating $x/\max(1, d(x,0))$ by such polynomials, one obtains as in [13] that the norm unit ball is indeed D_n.

We call tracial matrix-ordered operator spaces X a matrix-ordered operator space with a completely positive linear functional tr such that for $\phi \in M_n(X)$ positive, $||tr(\phi)||_{M_n(\mathbb{C})} = ||\phi||_{M_n(X)}$. For instance, by Stinespring's theorem, the predual of a von Neumann algebra is such a space with Haagerup's trace: $tr(\phi) = \phi(1)$. Note that, since $M_n(X)$ is a metric space, any $x, y \in M_n(X)$ positive then $tr(x+y)$ positive and is thus 0 only if $tr(x) = tr(y) = 0$ and by the norm relation $x = y = 0$. In particular, one recovers for free the axiom (M_0) so that tracial matrix-ordered operator spaces are easily (explicitly) axiomatizable.

Groh's construction of a von Neumann algebra ultrapower in [20, Prop 2.2], as dual of a Banach space ultrapower of preduals, suggests that preduals of von Neumann algebras should form an axiomatizable class. Moreover, the proof of the construction of the ultrapower gives a theory for a pair (M, X) of a C^*-algebra and a predual of a von Neumann algebra with $M \subset X^*$ weak-* dense. We will thus explicitly axiomatize the theory of these pairs.

We now explain the following theory but we consider X as a tracial matrix-ordered operator space to get later our stronger axiomatizability result for preduals alone.

We consider a theory with five (families of) sorts: $M_n(V)$, each with the language of C^*-algebras from [13] with domains of quantification $D_m(M_n(V))$, with also the language of operator spaces (we will call this the language of C^*-algebras as operator spaces and in the corresponding theory the identification of the product with the matrix product will be written) and the sorts $(M_{(p,q)}(\mathcal{S}), \mathcal{C}_n, M_{m,n}(\mathbb{C}), \mathbb{R}_{\geq 0})$ with the

language of tracial matrix-ordered operator spaces explained above, with domains of quantification written $D_m(.)$ with dot replaced by the sort. We add the following data:

- Unary function symbols $|.| : D_m(M_{n,n}(\mathcal{S})) \to D_m(\mathcal{C}_n)$.
- Binary functions symbols:

$$m^l_{N,n} : (D_l(M_N(V))) \times D_m(M_n(S)) \to D_{lm}(M_{Nn}(S)),$$

$$m^r_{n,N} : D_m(M_n(S)) \times (D_l(M_N(V))) \to D_{lm}(M_{Nn}(S)),$$

  interpreted as module (tensorial at matrix level) action of $M_N(U)$ on $M_n(\mathcal{S}))$ and $m^i_1 := m^i_{1,1}$.
- a binary function symbol $B : D_m(M_n(S)) \times D_l(M_n(V)) \to D_{n^2ml}(\mathbb{C})$ (see axiom (30) for its meaning)
- Unary function symbols $\pi^{(n)}_{ij} : D_m(M_{2n}(S)) \to D_m(M_n(S)), i, j \in \{1, 2\}$ (meaning a shorter notation for block projections), and $\exp i\Re : D_m(V) \to D_1(V), \arctan \Re : D_m(V) \to D_2(V), r : D_m(V) \to D_1(V)$ (meaning $\exp i\Re(a) = exp(i(a + a^*)/2), \arctan \Re(a) = \arctan((a + a^*)/2), r(a) = \frac{1}{1+(a+a^*)^2/4}$).
- Unary function symbols $tr_n : D_m(M_n(S)) \to D_m(M_n(\mathbb{C}))$ (for $tr \otimes id_n$).

Some maps as $\exp i\Re, arctan\Re$ are not necessary, but will be used effectively to produce convenient unitaries. We will define them by classical series valid in any C^*-algebra (Euler's series converging on $\mathbb{R}$ uniformly on segments will be convenient for $arctan(t) = \frac{t}{1+t^2} \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)!} \left(\frac{t^2}{1+t^2}\right)^n$).

Before stating any axiom, we have to make explicit our uniform continuity bounds. But first, let us recall some preliminary background and notation from operator spaces and operator systems we will need. We refer to [9] for more details. Since we see the predual X as an operator space, we have to characterize the norm on $M_n(X)$. The easiest way is to see $M_n(X) \hookrightarrow CB(X^*, M_n(\mathbb{C}))$ as a space of completely bounded maps on a C^*-algebra X^*. This embedding is realized as follows: $B(\varphi, .) = \hat{\varphi}$ is the map defined later in (30) or in [6, p 16] which converts $M_n(X) = NCB(X^*, M_n(\mathbb{C}))$ (considered as normal completely bounded maps) into forms in $(M_n(X^*))^*$. To a map $\varphi \in CB(X^*, M_n(\mathbb{C}))$ it associates a form $\hat{\varphi} \in (M_n(X^*))^*$ defined by its evaluations on a matrix $f = (f_{ij}) \in M_n(X^*)$ by:

$$\widehat{\varphi}(f) = \sum_{i,j=1}^{n} (\varphi(f_{ij}))_{i,j}. \tag{3.1}$$

In order to define the completely bounded norm on $CB(X^*, M_n(\mathbb{C}))$ without using a supremum over a countable set, we need an alternative formula as an infimum. In that way the norm we want to characterize will be bounded by a usual lower estimate given by the usual supremum definition of the completely bounded norm, and an upper estimate obtained in relaxing the infimum alternative definition. This alternative definition is what is called the decomposition norm of a completely bounded map $\varphi : A \to B$ between C^*-algebras and uses crucially complete positivity (c.p. will be an abbreviation for "completely positive"). It is defined [9, p 95] as:

$$||\varphi||_{dec} = \inf \Big\{ \max(||\psi_1||_{cb}, ||\psi_2||_{cb}) \mid \psi_i : A \to B \text{ c.p. such that } \begin{pmatrix} \psi_1 & \varphi \\ \varphi^* & \psi_2 \end{pmatrix} \text{ c.p.} \Big\}. \tag{3.2}$$

Finally, we also need to recall that the Banach space dual to $M_n(\mathbb{C})$ with operator norm is a space of trace class operators TC_n (matrices with trace class norm). We will sometimes use two other important operator spaces: the column and row Hilbert spaces C_n, R_n (see [9]). They are related by a tensor product formula $TC_n \simeq R_n \otimes_h C_n$ (involving the so-called Haagerup tensor product of operator spaces). We are now ready to compute our uniform continuity bounds

Note that for $\phi, \varphi \in M_n(X)$ we use the definition for $\hat{\phi}$ given in (3.1):

$$\begin{aligned}
&\| |\varphi| - |\phi| \|_{cb} \\
&= \sup_{k, \|a\|_{M_k(X^*)} \leq 1} \|(|\varphi| - |\phi|)(a)\|_{M_{kn}(\mathbb{C})} \\
&= \sup_{T \in D_1(TC_{nk})} |B(i_n(|\varphi|) - i_n(|\phi|), (\sum_{j,J=1}^{k} T_{(i,j),(I,J)} a_{j,J})_{iI})| \\
&\leq \| \, |\widehat{\varphi}| - |\widehat{\phi}| \, \| \sup_{k, ||a||_{M_k(X^*)} \leq 1} \sup_{T \in D_1(TC_{nk})} \|(\sum_{j,J=1}^{k} T_{(i,j),(I,J)} a_{j,J})_{iI}\|_{M_n(M)} \\
&\leq c_n \left(2\sqrt{\|\widehat{\varphi}\| \|\widehat{\varphi} - \widehat{\phi}\|} + \|\widehat{\varphi} - \widehat{\phi}\| \right)
\end{aligned}$$

with the last inequality coming e.g. from the proof of [30, Prop III.4.10] and with $c_n = \sup_{||a||_{TC_n \widehat{\otimes} M} \leq 1} ||a||_{M_n(M)}$ is a universal constant[2] used thanks to the bound[3]

$$\begin{aligned}
||(\sum_{j,J=1}^{k} T_{(i,j),(I,J)} a_{j,J})_{iI}||_{M_n(M)} &\leq c_n ||(\sum_{j,J=1}^{k} T_{(i,j),(I,J)} a_{j,J})_{iI}||_{TC_n \widehat{\otimes} M} \\
&\leq c_n ||T||_{TC_n \widehat{\otimes} TC_k} ||a||_{M_k(M)}.
\end{aligned}$$

Using then $||\widehat{\varphi}|| \leq n||\varphi||_{cb}$ one thus gets an explicit uniform continuity function for $|.|$ that we take in the language setting. This was the only non-obvious uniformly continuous map.

We consider the following axioms:

(28) For $x, y \in D_l(M_N(V)), z, t \in D_m(M_n(S)), \lambda \in \mathbb{C}$, any $N, n \in \mathbb{N}^*$

$$m^l_{N,n}(\lambda x + y, z) = \lambda m^l_{N,n}(x, z) + m^l_{N,n}(, z),$$

$$m^l_{n,N}(x, \lambda z + t) = \lambda m^l_{n,N}(x, z) + m^l_{n,N}(x, t),$$

$$m^r_{n,N}(\lambda z + t, x) = \lambda m^r_{n,N}(z, x) + m^r_{n,N}(t, x), m^r_{n,N}(z, \lambda x + y) = \lambda m^r_{n,N}(z, x) + m^r_{n,N}(z, y),$$

and $m^l_{1,n}(1, z) = z = m^r_{n,1}(z, 1)$,

$$\pi^{(i,j),(I,J)}_{nN}(m^l_{N,n}(x, z)) = m^l_1(\pi^{iI}_N(x), \pi^{jJ}_n(z))$$

$$\pi^{(i,j),(I,J)}_{nN}(m^r_{n,N}(z, y)) = m^r_1(\pi^{iI}_n(z), \pi^{jJ}_N(y))$$

$$m^l_1(xy, z) = m^l_1(x, m^l_1(y, z)), m^r_1(z, xy) = m^r_1(m^r_1(z, x), y),$$

$$m^l_1(x, m^r_1(z, y)) = m^r_1(m^l_1(x, z), y),$$

$$\sup_{a \in D_m(V)} \max(d(r(a)[1 + (a + a^*)^2/4], 1), d([1 + (a + a^*)^2/4] r(a), 1)) = 0,$$

[2] this is quite standard it is finite since for instance from [9, section 9.3] in terms of row and column Hilbert spaces: $TC_n \widehat{\otimes} M \simeq R_n \otimes_h M \otimes_h C_n \to C_n \otimes_h M \otimes_h R_n \simeq M_n(M)$ coming from identity CB maps $R_n \to C_n$, $C_n \to R_n$ so that it is even known that $c_n \leq n$

[3] coming from the canonical completely contractive map $(TC_n \widehat{\otimes} TC_k) \widehat{\otimes} (M_k \otimes_{min} M) \simeq TC_n \widehat{\otimes} (TC_k \widehat{\otimes} (M_k \otimes_{min} M)) \to TC_n \widehat{\otimes} ((TC_k \widehat{\otimes} M_k) \otimes_{min} M)) \to TC_n \widehat{\otimes} M$ using in the middle the shuffle map $TC_k \widehat{\otimes} (M_k \otimes_{min} M)) \simeq R_k \otimes_h (M_k \otimes_{min} M) \otimes_h C_k \to (R_k \otimes_h M_k \otimes_h C_k) \otimes_{min} M$ from [26, Th 5.15]

$$\sup_{a\in D_m(V)} \max(0, d(\exp i\Re(a), \sum_{k=0}^{N} \frac{i^k}{2^k k!}(a+a^*)^k) - exp(m) + \sum_{k=0}^{N} \frac{m^k}{k!}) = 0,$$

$$\sup_{a\in D_m(V)} \max(0, d(\arctan \Re(a), \frac{a+a^*}{2} r(a) \sum_{n=0}^{N} \frac{(n!)^2 4^n}{(2n+1)!} \left(\frac{(a+a^*)^2}{4} r(a)\right)^n$$
$$- (1+m^2)\arctan(m) + m\sum_{n=0}^{N} \frac{(n!)^2 4^n}{(2n+1)!}\left(\frac{m^2}{1+m^2}\right)^n) = 0.$$

(29) $tr(m_1^l(x,z)) = tr(m_1^r(z,x))$ and

$$\sup_{x\in D_n(V)} |d(x,0) - \sup_{y\in D_1(S)} |tr(m_1^l(x,y))| \,| = 0,$$

$$\sup_{x\in D_n(\mathcal{S})} |d(x,0) - \sup_{y\in D_1(V)} |tr(m_1^r(x,y))| \,| = 0,$$

(30) For $x \in C_n, y \in M_n(S), a \in D_l(M_n(V))$, $|i_n(x)| = x$,

$$B(y,a) = \sum_{i,j=1}^{n} tr(m_1^l(\pi_n^{ij}(a), \pi_n^{ij}(y))),$$

$$\sup_{y\in D_l(M_n(S))} \sup_{a\in D_k(M_n(V))} \max\left[0, |B(y,a)|^2 - \sup_{b\in D_1(M_n(V))} |B(y,b)|\ B(i_n(|y|), aa^*)\right],$$

$$\sup_{y\in D_l(M_n(S)} \left| \sup_{b\in D_1(M_n(V))} |B(y,b)| - \sup_{b\in D_1(M_n(V))} |B(i_n(|y|),b)| \right| = 0,$$

(31) Axioms for $tr_n, \pi_{ij}^{(n)}$ and the following formulas are equal to 0:

$$\sup_{y\in D_l(M_n(\mathcal{S}))} \left| d(y,0) - \inf_{x\in D_{5l}(\mathcal{C}_{2n})} \left[\max(||tr_n(\pi_{11}^{(n)}(x))||, ||tr_n(\pi_{22}^{(n)}(x)||) + d(y, \pi_{12}^{(n)}(x))\right] \right|,$$

$$\sup_{y\in D_l(M_n(\mathcal{S}))} \sup_{a\in D_k(M_N(V))} \max\left(0, ||(tr_{nN}(m_{N,n}^l(a,y)))||_{M_{nN}(\mathbb{C})} - d(y,0)d(a,0)\right).$$

We will also consider a last axiom depending on a parameter $d \in [0,2]$ for the diameter of state space to recover the results of [1, Section 6.2] in our continuous logic setting.

(32) For any $m \in \mathbb{N}^*$, and write for short $f_r(a) = \exp i\Re(2\arctan\Re(a))$

$$\left| \sup_{x,y\in D_1(\mathcal{C}_1)} \inf_{a\in D_m(V)} d(i_1(x)tr(i_1(y)), m_1^l(f_r(a), m_1^r(i_1(y)tr(i_1(x)), [f_r(a)]^*)) - d \right|$$
$$\leq 4\sin(2\arctan(m))$$

For any $N \in \mathbb{N}^*$,

$$\inf_{(x_1,\ldots,x_N)\in(D_1(V))^N}$$
$$\max_{i=1,\ldots,N} \max\left(d(x_i x_i^*, (x_i x_i^*)^2), |d(x_i x_i^*, 0) - 1|), \max_{j\neq i} d(0, x_i x_i^* x_j x_j^*)\right) = 0.$$

Let us now explain how a pair (M, X) of a weak-* dense unital C^* subalgebra M in a von Neumann algebra X^* with predual X gives rise to a model of this theory. Of course m_1^l, m_1^r are induced by the usual actions of X^* on X for $x, y \in X^*, \varphi \in X$, $m_1^l(x,\varphi)(y) = \varphi(yx)$ and then m_n^l is defined entry-wise. $B(\varphi, .) = \hat{\varphi}$ is the map defined in (30) and already recalled in (3.1).Being a CP map i.e. in the cone C_n is then equivalent to have $\hat{\varphi}$ positive as a functional by their proposition 1.5.14. The axiom (30) then states that $\widehat{|\varphi|} = |\hat{\varphi}|$ in the sense of absolute values in $(M_n(X^*))_*$ characterized by [30, Prop III.4.6]. By strong-* density of M in X^* the defining relation is only checked on M and since $\hat{}$ is a bijection $|.|$ is defined uniquely and has indeed its range in completely positive maps and all of (30) is satisfied. (28),(29) are then obvious and (31) comes from the identification of the completely bounded norm with the decomposable norm (see e.g. [9, lemma 5.4.3]). Indeed, we have from this lemma

$$d(y,0) = ||y||_{cb} \leq ||y - x||_{cb} + ||x||_{dec} = ||y - x||_{cb} + \max(||\psi_1||, ||\psi_2||)$$

when $\begin{pmatrix} \psi_1 & x \\ x^* & \psi_2 \end{pmatrix}$ is completely positive i.e. in $\mathcal{C}_{2n}$. Thus taking the infimum

$$d(y,0) \leq \inf_{x \in (\mathcal{C}_{2n})} \Big[\max(||tr_n(\pi_{11}^{(n)}(x))||, ||tr_n(\pi_{22}^{(n)}(x)||) + d(y, \pi_{12}^{(n)}(x))\Big]$$

and since $||y||_{cb} = ||y||_{dec}$ the infimum is reached when $\pi_{12}^{(n)}(x) = y$ and we can take $||\psi_i|| = ||y||$ giving, in particular, the equality stated in (31). The second part states $||y||_{cb} \leq d(y,0)$ which is satisfied when we have equality.

The second equation in (32) states that there are N orthogonal non-zero projections and thus axiomatizes infinite dimensionality. Let us also interpret the first equation in axiom (32).

Lemma 3.1. *For a pair (M, X) as above, the first equation in (32) is equivalent to the equality on the state space diameter of X^*: $d(X^*) = d$.*

Proof : Note that, in our formula, we only look at unitaries $u = \exp i\Re(2\arctan\Re(a))$ so that for $x, y \in D_1(X)$:

$$\begin{aligned}
&\delta_m(x,y) \\
&:= \inf_{a \in D_m(M)} d(i_1(x), m_1^l(\exp i\Re(2\arctan\Re(a)), m_1^r(i_1(y), \exp i\Re(-2\arctan\Re(a))))) \\
&\geq \delta(x,y) := \inf_{u \in D_1(X^*), uu^* = 1 = u^*u} d(i_1(x), m_1^l(u, m_1^r(i_1(y), u^*)))
\end{aligned}$$

But, since by functional calculus, any unitary u is close to a unitary $u_m = \exp i\Re(2\arctan\Re(a_m))$ with $a_m \in D_m(X^*)$ and explicitly with $||u - u_m|| \leq 2\sin(2\arctan(m))$, one gets using Kaplansky density theorem for the first equality and $f(m) = 4\sin(2\arctan(m))$:

$$\begin{aligned}
\delta_m(x,y) &= \inf_{a \in D_m(X^*)} d(x, \exp i\Re(2\arctan\Re(a)).y.\exp i\Re(-2\arctan\Re(a)))) \\
&\leq \delta(x,y) + f(m).
\end{aligned}$$

One thus obtains that

$$\begin{aligned}
\Big| &\sup_{x,y \in D_1(\mathcal{C}_1)} \delta_m(i_1(x)tr(i_1(y)), i_1(y)tr(i_1(x))) \\
&- \sup_{x,y \in D_1(\mathcal{C}_1)} \delta(i_1(x)tr(i_1(y)), i_1(y)tr(i_1(x)))\Big| \leq f(m).
\end{aligned}$$

Finally, it is easy to see that the state space diameter of X^* is

$$\sup_{x,y \in D_1(\mathcal{C}_1)} \delta(i_1(x)tr(i_1(y)), i_1(y)tr(i_1(x))) = \sup_{x,y \in D_1(\mathcal{C}_1), tr(x) = tr(y) = 1} \delta(i_1(x), i_1(y))$$

and since $f(m) \to_{m\to\infty} 0$, (32) is equivalent to the state space diameter having value d. ∎

We are now ready to prove our axiomatization result. We consider the category whose objects are pairs (M, X) of a weak-* dense C^*-algebra M of X^* and a predual of a von Neumann algebra X, and morphism are any pair of a $*$-homomorphism $M_1 \to M_2$ extending to a weak-* continuous $*$-homomorphism $X_1^* \to X_2^*$ onto a subalgebra having a normal conditional expectation from X_2^* which is the dual map of a completely positive tracial map of preduals $X_1 \to X_2$.

Theorem 3.2. *The class of couples (M, X) of a weak-* dense C^*-algebra M of X^* and a predual of a von Neumann algebra X is axiomatizable by the theory $T_{C^*,(W^*)_*}$ consisting of axioms of C^*-algebras as operator space for M, tracial matrix-ordered operator spaces for X and (28)-(31). The subclass with X such that X^* is a type III_λ-factor for a fixed $0 < \lambda \leq 1$ is axiomatizable by the theory $T_{C^*,(III_\lambda)_*}$ consisting of $T_{C^*,(W^*)_*}$ and (32) with $d = 2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$.*

Proof : We explained how a couple gives a model of $T_{C^*,(W^*)_*}$. Moreover, a model homomorphism $(i, j) : (M_1, X_1) \to (M_2, X_2)$, then $j^* : X_2^* \to X_1^*$ is a completely positive unital contraction and when restricted to M_2 satisfies for all $\phi \in X_2, c \in M_1$ and $j(c.\phi) = i(c).j(\phi)$, so that for $d \in X_2^*$

$$\begin{aligned} tr(j^*(d.i(c)).\phi) &= tr(d.i(c).j(\phi)) \\ &= tr(dj(m(c,\phi))) \\ &= tr((j^*(d).c).\phi) \end{aligned}$$

and thus, since ϕ arbitrary in X_1 , $j^*(d.i(c)) = j^*(d).c$ in X_1 for $c \in M_1$ and similarly $j^*(i(c).d) = c.j^*(d)$.

Consider N the von Neumann algebra generated by $i(M_1) \subset X_2^*$ the restriction of j^* satisfies $j^*(i(x).i(y)) = x.(j^*(i(y))) = (j^*(i(x))).(j^*(i(y)))$ so that j^* is a weak-* continuous $*$-homomorphism on N. Its image is a von Neumann subalgebra of X_1^* containing $j^*(i(M_1)) = M_1$ thus X_1^* so that $j^*|_N : N \simeq X_1^*$ and its inverse α is thus a von Neumann algebra isomorphism extending i and $\alpha \circ j^*$ is a completely positive weak-* continuous projection $X_2^* \to N \simeq X_1^*$. Similarly, consider any $*$-homomorphism $i : M_1 \to M_2$ extending to a weak-* continuous $*$-homomorphism $X_1^* \to X_2^*$ onto a subalgebra having a normal conditional expectation from $X_2^* \to X_1^*$ with predual map j. It gives rise to a model morphism since j^* is completely positive tracial and the conditional expectation property implies $j^*(i(c).d.i(c')) = c.j^*(d).c'$ which implies $j(c.\phi.c') = i(c).j(\phi).i(c')$, $c, c' \in M_1, \phi \in X_1$. The remaining structure is easily preserved by (i, j), for instance $j(|\phi|)$ satisfies the characterizing properties of $|j(\phi)|$ in (30).

Conversely, let (M, X) the underlying sets for such a model, so that $M_n(M)$ are matrix C^*-algebras over M with their norm and operator norm balls as domains of quantification and X is tracial matrix-ordered operator space with its norm balls as domains of quantification. (29) gives a duality pairing and thus isometric embeddings $M \to X^*, X \to M^*$. M^* is the predual of the von Neumann algebra M^{**} and using (28), for $x \in X, a, b \in M$ the duality pairing

$$\langle ab, x\rangle = tr(m_1^l(ab, x)) = \langle a, m_1^l(b, x)\rangle$$

and since a is arbitrary we can replace it by density by $a \in M^{**}$ so that computed in M^*, the product $b.x = m_1^l(b, x)$ and similarly $x.b = m_1^r(x, b)$. Since X is complete (as any model, which has complete balls, and for which metric balls are domains of quantifications) it is closed in M^* and both a right and left ideal, thus [30, Th III.2.7], $X = M^*e$ for a central projection in M^{**} and thus $X^* = M^{**}e$ is a von Neumann subalgebra of M^{**}. Note that the previously isometric inclusion $M \subset X^*$ is a $*$-algebra homomorphism by definition of the $*$-algebra structure on X^* induced from M^{**}. Indeed we have using (28) the relations for $a \in M, x \in X$:

$$\langle a^*, x\rangle = tr(m_1^l(a^*, x)) = \overline{\langle a, x^*\rangle}$$

defines $x^* \in M^*$ and reading backwards the formula define $a^* \in M^{**}$ thus in X^*. Moreover the canonical map $M \to M^{**} \to M^{**}e$ is clearly a $*$ algebra homomorphism since e central projection, and for $x \in X = M^*e$, $\langle ae, x\rangle = \langle a, xe\rangle = \langle a, x\rangle$ so that the above map coincides with the original map $M \to X^*$ we defined, which is thus as expected an algebra homomorphism. Note also that if $a \in X^* \subset M^{**}$ we have by Goldstine

lemma a net $a_n \to a$ weak-* in M^{**} with $a_n \in M_{||a||}$, thus weak-* in X^* since this topology is weaker. Thus M is a weak-* dense subalgebra of X^* as expected.

Let us finally show that the tracial matrix ordered structure of X, as well as the extra structure of the language, is the one induced as predual of X^*. This is the "new" (but standard) part with respect to Groh's proof. The map B is defined unambiguously by (30). If we write $B(\varphi, .) = \hat{\varphi}$ as before the map defined in (30) or [6, p 16] that converts $M_n(X) = NCB(X^*, M_n(\mathbb{C}))$ into maps in $(M_n(X^*))^*$. Then the (in)equalities implied by (30) extend by strong-* density from $a \in M$ to $a \in X^*$ so that, by the characterization of [30, Prop III.4.6] $\widehat{i_n(|\varphi|)} = |\widehat{\varphi}|$ with the second absolute value computed in $(M_n(X^*))^*$. This determines $|\varphi|$ as the value we expected and moreover, by the bijection in [6, Prop 1.5.14], $i_n(|\varphi|)$ is always in $NCP(X^*, M_n(\mathbb{C}))$. Thus since the image of $i_n(|.|)$ is exactly $i_n(\mathcal{C}_n)$, by the first relation in (30), one deduces that $i_n(\mathcal{C}_n)$, is exactly the set of completely positive maps and thus the expected positive cone of X induced by the von Neumann algebra X^*. The fact that tr is also the expected Haagerup trace is obvious. It remains to use (31) to identify the norms on $M_n(\mathcal{S})$. The last formula implies the inequality for $y \in M_n(\mathcal{S})$, $||y||_{cb} \leq d(y, 0)$. Moreover, from the equation $||tr_n(\phi)||_{M_n(\mathbb{C})} = ||\phi||_{M_n(X)}$ in the theory of tracial matrix-normed operator spaces for $\phi \in \mathcal{C}_n$ we have $||\phi||_{cb} = d(y, 0)$ in this case. Finally, we have from the first identity for $y \in D_l(M_n((S)))$ (thus with $||y||_{cb} \leq l$)

$$d(y,0) = \inf_{x \in D_{5l}(\mathcal{C}_{2n})} \Big[\max(||tr_n(\pi_{11}^{(n)}(x))||, ||tr_n(\pi_{22}^{(n)}(x))||) + d(y, \pi_{12}^{(n)}(x)) \Big]$$
$$\leq \inf_{x \in \mathcal{C}_{2n}, ||x||_{cb} \leq 5l, \pi_{12}^{(n)}(x) = y} \Big[\max(||tr_n(\pi_{11}^{(n)}(x))||, ||tr_n(\pi_{22}^{(n)}(x))||) \Big] = ||y||_{dec}$$

with the last equality coming from identification with C_{2n} as the right cone of CP maps (and of d on positive elements) and thus by [9, lemma 5.4.3], since $||y||_{dec} = ||y||_{cb}$ this equals $d(y, 0)$ and this concludes. Among the extra structure m^r, m^l are determined by the fourth and fifth equation in (28) since m_1^l, m_1^r is already determined, $\exp i\Re, \arctan \Re, r$ are determined in the 3 last equations in (28), and $tr_n, \pi_{ij}^{(n)}$ are determined implicitly in (31).

Since the state space diameter is 2 as soon we don't have a factor, the computation recalled in [1, Theorem 6.6] characterizes III_λ factors by infinite dimensionality and $d = 2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$. This concludes the second axiomatization. ∎

3.2. The abstract axiomatization result for preduals of von Neumann algebras

We now obtain a more abstract result.

Theorem 3.3. *The class of preduals of von Neumann algebras is axiomatizable in the language of tracial matrix-ordered operator spaces and so are the classes of preduals of III_λ-factors for each fixed $0 < \lambda \leq 1$.*

Proof : We use the model theoretic result [2, Prop 5.14]. Thus, it suffices to check those classes are stable by ultraproducts and ultraroots.

First, by [1, Th 3.24] Banach space ultraproducts of preduals is a predual (the result is due to Groh for ultrapowers). We can actually deduce this result and the general ultraproduct case even in the sense of tracial matrix-ordered operator spaces from the previous theorem 3.2. Indeed, for any X_n preduals, we associate a couple (X_n^*, X_n) which gives a model $\mathcal{M}(X_n^*, X_n)$ of $T_{C^*,(W^*)_*}$. And so is the model-theoretic ultraproduct $\mathcal{M}(X_n^*, X_n)^\omega$ which has as second space the ultraproduct $(X_n)^\omega$ as tracial matrix-ordered operator space. This implies this is indeed a predual of a von Neumann algebra with the corresponding structure of tracial matrix-ordered operator space. The III_λ-factor case is similar.

It remains to check the class is stable by ultraroot. Let (X, tr) be a tracial matrix-ordered operator space such that $((X^\omega)^*, 1 = tr^\omega)$ is a von Neumann algebra. Recall that by definition $M_n(X^\omega) = (M_n(X))^\omega$ (first as Banach space and then as operator space) and the ultraproduct of positive cones is the cone of the ultraproduct. Obviously we have a completely isometric completely positive injection $i : X^* \hookrightarrow (X^\omega)^*$ given by

$$[i(\varphi)]((x_n)_\omega) = \lim_{n \to \omega} \varphi(x_n).$$

Note that by definition $i(tr) = 1$. Moreover, we have a completely positive map $E : (X^\omega)^* \to X^*$ dual to the canonical injection and $E \circ i = id$. Thus $P = i \circ E : (X^\omega)^* \to (X^\omega)^*$ is a completely positive projection with image $i(X^*) \simeq X^*$. Note also $E(1) = E(i(tr)) = tr$ so that $P(1) = 1$. By the result of [8] (see also [10]), the image of P, i.e. $P((X^\omega)^*) = i(X^*) \simeq X^*$ becomes a C^*-algebra for the product $P(x).P(y) = P(xy) = P(P(x)P(y))$. Thus X^* is a C^*-algebra which is a dual of a Banach space, this is thus a von Neumann algebra. Note that φ is positive in the C^* structure if and only if there exists h such that $\varphi = P(hh^*)$. This implies for $x_n = x \in X$ positive

$$\varphi(x) = P(hh^*)(x) = \lim_{n\to\omega}[E(hh^*)](x) = [E(hh^*)](x) = (hh^*)(x_n) \geq 0$$

since (x_n) is by definition positive in the ultraproduct thus positive on positive elements of the C^*-algebra dual to this ultraproduct. Thus φ is a positive element of X^* for the duality (using also matrix variants to obtain complete positivity). Conversely, if φ is such a positive element, then, for $(x_n) \in M_k(X^\omega)$ consisting of positive elements $i(\varphi)(x_n) = \lim_{n\to\omega} \varphi(x_n)$ is positive, thus $i(\varphi)$ is positive in $(X^\omega)^*$ thus of the form hh^* and $i(\varphi) = P(i(\varphi)) = P(hh^*)$ is positive in our C^*-algebra structure on $i(X^*) \simeq X^*$. Thus i from X^* with dual matrix-ordered norm structure and with unit $1 = tr$ to $i(X^*)$ is a unital complete order isomorphism (complete since the reasoning above applies also on $M_n(X^*)$). Thus one deduces that the order structure and trace on X are those as predual of the von Neumann algebra X^*, as expected. Thus (X^*, X) satisfies $T_{C^*,(W^*)_*}$. If its ultraproduct satisfies $T_{C^*,(III_\lambda)_*}$ as this is the case if X^ω is the predual of a III_λ factor, so does (X^*, X) and thus we also get stability by ultraroot of preduals of III_λ factors for $\lambda \in]0, 1]$. ∎

3.3. The Groh-Haagerup-Raynaud theory for standard forms

We write down in this short subsection the model theory for standard forms implicit in [1] and strongly based on the original works of [21, 27]. We thus call Groh-Haagerup-Raynaud theory the resulting theory. We add to the previous language for Groh theory the language of complex Hilbert spaces, namely the (complex variant) of [2, section 15] and the following supplementary data:

- Unary function symbols $\Pi_P : D_m(\mathcal{H}) \to D_m(\mathcal{H}), J : D_m(\mathcal{H}) \to D_m(\mathcal{H})$ for the projection onto the positive cone P of the standard form and the modular operator.
- Unary function symbols $\xi : D_{m^2}(\mathcal{C}_1) \to D_m(\mathcal{H})$, $\omega : D_m(\mathcal{H}) \to D_{m^2}(\mathcal{C}_1)$, for the two inverse bijections between normal states and the positive cone of the standard form. (we may also write $i_1\omega$ as ω)
- Binary function symbols $\omega_2 : D_m(\mathcal{H})^2 \to D_{m^2}(\mathcal{S})$ the sesquilinear form of the previous quadratic map.
- Binary functions symbols:
$$\pi : D_l(\mathcal{V}) \times D_m(\mathcal{H}) \to D_{ml}(\mathcal{H})$$
interpreted as an action.

The expected uniform continuity modulus are obvious (Π_P, J, ω_2, π are contractions) or are well-known for ξ, ω by Powers-Störmer inequality (see e.g. [21, lemma 2.10] or [30, Th IX.1.2 (iv)]).

$$||\xi(\phi) - \xi(\psi)|| \leq \sqrt{||\phi - \psi||}$$

$$||\omega(h) - \omega(k)|| \leq ||h + k||||h - k|| \leq 2m||h - k||$$

if $h, k \in D_m(H)$. We need few more axioms:

(33) π bilinear, ω_2 sesquilinear , $J = J^2$ antilinear.

$$\pi(a, \pi(b, h)) = \pi(ab, h),\ \pi(1, h) = h,\ \langle\pi(a, h), k\rangle = \langle h, \pi(a^*, k)\rangle,$$

$$\xi(\omega(\Pi_P(h))) = \Pi_P(h), \xi(\psi) = \Pi_P(\xi(\psi)), \omega(\xi(\psi)) = \psi,\ \omega_2(h, h) = i_1(\omega(h)),$$

$$tr(m_1^l(a, \omega_2(\xi, \eta))) = \langle\pi(a, \xi), \eta\rangle$$

(34) $\Pi_P(\Pi_P(h)) = h$, $\forall \lambda > 0, \Pi_P(\lambda h) = \lambda \Pi_P(h)$, $\Pi_P(\Pi_P(h) + \Pi_P(k)) = \Pi_P(h) + \Pi_P(k)$, $J(\Pi_P(h)) = \Pi_P(h)$,

$$\sup_{(x,y)\in(D_m(\mathcal{H}))^2} \max(0, -\Re\langle \Pi_P(x), \Pi_P(y)\rangle) = 0,$$

$$\sup_{(x,y)\in(D_m(\mathcal{H}))^2} \max(0, \Re\langle x - \Pi_P(x), \Pi_P(y) - \Pi_P(x)\rangle) = 0.$$

(35)

$$\Pi_P(\pi(a, J\pi(a, J(\Pi_P(h))))) = \pi(a, J\pi(a, J(\Pi_P(h)))),$$

$$\sup_{x\in D_m(\mathcal{H})} d(\Pi_P(x + J(x)) - x - J(x), \Pi_P(\Pi_P(x + J(x)) - x - J(x))) = 0,$$

$$\sup_{x\in D_m(\mathcal{H})} \sup_{(a,b)\in D_m(V)} d(J\pi(b, J\pi(a, x)), \pi(a, J\pi(b, Jx)))) = 0,$$

$$\sup_{(x_1,\ldots,x_n,y_1,\ldots,y_n)\in D_m(\mathcal{H})^{2n}} \max(0, \inf_{a\in D_1(V)} \sum_{i=1}^{n} ||J\pi(a, Jx_i) - y_i||^2$$
$$- \sup_{b\in D_1(M_n(V))} \left(\sum_{i,j,k=1}^{n} \langle \pi(b_{ki}, y_i), \pi(b_{kj}, y_j)\rangle - \langle \pi(b_{ki}, x_i), \pi(b_{kj}, x_j)\rangle \right)) = 0.$$

Theorem 3.4. *The class of quintuples (M, X, H, J, P), gathering a weak-* dense C^*-algebra M of X^*, a predual of a von Neumann algebra X, and a standard form (X^*, H, J, P)[4], is axiomatizable by the theory $T_{C^*,(W^*)_*,SF}$ consisting of $T_{C^*,(W^*)_*}$, the axioms of complex Hilbert spaces for H and (33)-(35).*

As in Groh theory, the morphisms for the category of axiomatization are the C^*-algebra homomorphisms, inducing normal homomorphisms of X^*, with image a von Neumann subalgebra having a conditional expectation onto it. We will see that this is enough to obtain a unique structure preserving morphism of the theory and we will in particular obtain for free a standard form homomorphism (namely a supplementary Hilbert space isometry commuting with J, Π_P and the action).

Proof : We first check that a standard form satisfies axiom (33)-(35). Most equations are obvious consequences of the definition. The bijectivity of ξ, ω on positive elements comes e.g. from [30, Th IX.1.2]. (34) comes from $Im(\Pi_P)$ self-dual cone. The last equation in (34) is the characterization Π_P as the projection on $Im(\Pi_P)$. The antepenultimate expression in (35) states that a real vector can be decomposed as a sum of positive and negative parts. It comes from [30, lemma IX.1.7]. The proof of the last equation comes from [23, lemma 3.2, 3.1] as in [1, lemma 3.21]. Indeed take $x = (x_1, ..., x_n) \in H^n, y = (y_1, ..., y_n) \in H^n$ and let

$$\epsilon = \sup_{b\in D_1(M_n(V))} \left(\sum_{i,j,k=1}^{n} \langle \pi(b_{ki}, y_i), \pi(b_{kj}, y_j)\rangle - \langle \pi(b_{ki}, x_i), \pi(b_{kj}, x_j)\rangle \right)$$

then for a positive in $M_n(X^*)$

$$\omega_y(a) \leq \omega_x(a) + \epsilon ||a||$$

The quoted lemma 3.2 then implies there is $Y \in H^n$ with $||Y - y|| \leq \sqrt{\epsilon}$ and $\omega_Y \leq \omega_x$ as state on $M_n(X^*)$. Thus by their quoted lemma 3.1, there is $c \in (M_n(X^*))' = (X^*)' I_n = JX^*JI_n$ with $c = JdJI_n$ with $||d|| \leq 1$ such that $Y_i = JdJx_i$ and thus

$$\sum_{i=1}^{n} ||JdJx_i - y_i||^2 \leq \epsilon.$$

[4] i.e. X^* acting on H, with modular conjugation J and positive self-dual cone P

By Kaplansky's density theorem giving strong-* density of the unit ball M in the unit ball of X^*, this concludes to the last equation in (35).

We already saw that a model morphism gives rise to the expected kind of morphism since preserving the structure of Groh theory is enough for that. Conversely, consider a C^*-algebra morphism extending to a normal $*$ homomorphism $i : X_1^* \to X_2^*$ with image a von Neumann subalgebra with conditional expectation iE obtained from $E : X_2^* \to X_1^*$. To build the standard form morphism giving the structure preserving map, we use the uniqueness theorem for standard forms [30, Th IX.1.14] and can assume $H_1 = L^2(i(X_1^*), \phi), H_2 = L^2(X_2^*, \phi \circ E)$ for some faithful semi-finite normal weight so that we have an isometric inclusion $u : H_1 \to H_2$. From the criteria for existence of conditional expectations [30, Th IX.4.2], the modular theory of ϕ in H_1 is computed by restriction of the one of H_2, so that $J_2 u = u J_1$ and $u(P_1) \subset P_2$ but from this and self-duality of the cones it is easy to see that $u(P_1) = P_2 \cap H_1$ and thus u commutes with the projections on the cones. Finally, since $\pi(i(x), u(y)) = u(\pi(x, y))$ and $\omega_2(u(h), u(k)) = \omega_2(h, k) \circ E$ by definition, one deduces u preserves all the other data in the structure of $T_{C^*,(W^*)_*,SF}$ which is derived : $\omega_2(h, k) = \omega_2(u(h), u(k)) \circ i$, $u\xi_\omega = \xi_{\omega \circ E}$. Moreover a standard form morphism u, as above and part of a structure preserving morphism, satisfies $u\xi_\omega = \xi_{\omega \circ E}$. Thus u is determined on the positive cone, and then by linearity on the Hilbert space. This concludes to the bijection between structure preserving morphisms and the morphisms of the considered category.

Assume given a model (M, X, H, J, Π_P) of $T_{C^*,(W^*)_*,SF}$. We already know $M \subset X^*$ is weak-* dense C^*-algebra of a von Neumann algebra and H is an Hilbert space. First, for $f \in X^*$, $f(\omega_2(\xi, \eta))$ defines a sesquilinear map on H and thus from Riesz representation theorem:

$$f(\omega_2(\xi, \eta)) = \langle A(f)\xi, \eta \rangle.$$

Moreover, if $f \in M \subset X^*$, one obtains $A(f)\xi = \pi(f, \xi)$ and the linear A is weak-* to weak operator topology continuous, thus weak-* continuous on bounded sets so that one extends the action property by weak-* density of M in X^*, so that A is a $*$-homomorphism. It is one-to-one since if $A(f) = 0$ then $A(f^* f) = 0$ and from the bijection ω, ξ between $Im(\Pi_P)$ and the positive cone of X, $f^* f$ vanishes on this cone and thus $f = 0$. Thus A is a $*$-isomorphism onto its image and is thus weak-* continuous [30, Corol III.3.10]. π has thus been extended to a normal action of X^*.

From the equations in (34), the image of Π_P is a convex cone, and since Π_P contractive, it is closed and the last equation says Π_P is the projection on this closed convex set. To check it is self-dual, first note that we know $\langle \Pi_P(x), \Pi_P(y) \rangle$ is positive from (34). Conversely, consider $h \in H$ such that $\langle h, \Pi_P(x) \rangle \geq 0$ for all $x \in H$ and also consider, from the second formula in (35), $a = h + J(h), b := \Pi_P(a) - a = \Pi_P(b)$. Then we have $\langle \Pi_P(a), \Pi_P(b) \rangle = \langle \Pi_P(a), \Pi_P(a) - a \rangle = 0$, since it is ≥ 0 from the recalled positivity and since $0 \in Im(\Pi_P)$ $\Re(\langle a - \Pi_P(a), 0 - \Pi_P(a) \rangle) = \langle a - \Pi_P(a), -\Pi_P(a) \rangle \leq 0$ from characterization of the projection. One deduces $h + J(h) = \Pi_P(a) - \Pi_P(b)$. Similarly one can get c, d with $\langle \Pi_P(c), \Pi_P(d) \rangle = 0, h - J(h) = i(\Pi_P(c) - \Pi_P(d))$ so that $2h = \Pi_P(a) - \Pi_P(b) + i(\Pi_P(c) - \Pi_P(d))$. But by assumption we have:

$$\begin{aligned} \langle 2h, \Pi_P(x) \rangle = & \langle \Pi_P(a), \Pi_P(x) \rangle - \langle \Pi_P(b), \Pi_P(x) \rangle \\ & - i\langle \Pi_P(c), \Pi_P(x) \rangle + i\langle \Pi_P(d), \Pi_P(x) \rangle \geq 0 \end{aligned}$$

Thus in taking the imaginary part $\langle \Pi_P(c), \Pi_P(x) \rangle - \langle \Pi_P(d), \Pi_P(x) \rangle = 0$, and from $x = c$ one gets $\Pi_P(c) = 0$ and from $x = d$ one gets $\Pi_P(d) = 0$. Similarly from $x = b$ on gets $-\langle \Pi_P(b), \Pi_P(b) \rangle \geq 0$ and thus $\Pi_P(b) = 0$ implying $2h = \Pi_P(a)$ so that the cone is indeed self dual.

By now we consider X^* as a von Neumann algebra on H via π. From [1, lemma 3.19], in order to check we have a standard form $(X^*, H, J, Im(\Pi_P))$, it suffices to check

1. for $x \in Im(\Pi_P)$, $J(x) = x$, which is contained in (34),
2. $aJaJ(Im(\Pi_P)) \subset Im(\Pi_P)$ which is also contained in (35) in the case $a \in M$ and extends to $a \in X^*$ by strong density of the unit ball of M in the form of Kaplansky's density theorem
3. and finally the key $JX^*J = (X^*)'$.

For that last statement, one uses the next-to-last equation in (35) to see that $JMJ \subset M' = (X^*)'$ and thus by strong-* density $J(X^*)J \subset (X^*)'$. Conversely, one uses the idea in the proof of [1, Th 3.22],

let $a \in (X^*)'$ with $||a|| \leq 1$ and show that $a \in (JMJ)''$. For take $x_1, ..., x_n \in H$. It is well-known that $M_n(X^*)$ acts on H^n with commutant $(X^*)'Id_n$. Let $y_i = ax_i \in H$. According to [23, lemma 3.1] if $x = (x_1, ..., x_n) \in H^n, y = (y_1, ..., y_n)$ and $\omega_z(.) = \langle .z, z\rangle$ the canonical state on N' with $z \in \{x, y\}$ we have $\omega_y \leq \omega_x$. This is relevant to apply the last equation in (35) since then

$$\sup_{b \in D_1(M_n(V))} \left(\sum_{i,j}^{n} \langle (b^*b)_{ji} y_i, y_j\rangle - \langle (b^*b)_{ji} x_i, x_j\rangle \right) = 0,$$

and thus by this equation

$$\inf_{a \in D_1(M)} \sum_{i=1}^{n} ||JaJx_i - y_i||^2 = 0.$$

Thus since $x_1, ..., x_n$ are arbitrary, $(X^*)'$ is in the strong operator topology closure of JMJ i.e. one gets the claimed $(X^*)' \subset J(X^*)J$. Finally, all the data of the model is determined as the expected data since J, π, Π_P are part of a standard form data, ω_2 is then determined by π in the third line of (33) and determines ω, in the second line, which determines in the same line ξ as its inverse. ∎

4. The Ando-Haagerup theory gathering Groh and Ocneanu theories

We now describe a theory for a C^*-algebra C weak-* dense in X^*, von Neumann algebra with predual X, and a σ-finite von Neumann algebra $M \simeq eX^*e$ for e the support projection of $\phi \in X$, thus inducing a faithful state φ on M. (C, X) is described by the Groh theory of the previous section with corresponding language. We will rather use for convenience the language of the Groh-Haagerup-Raynaud theory. (M, φ) is described by the Ocneanu theory of section 2. This is the kind of setting Haagerup and Ando used in [1] to relate the Ocneanu and Groh-Raynaud ultraproducts.

Let us point out that we want to axiomatize the above data even though the spectral projection e, relating X^* and M, is not axiomatizable. Indeed, it is not stable by ultrapower. If we start from $M = X^*$, then the Groh ultrapower $\prod^{\omega} M$ is usually not σ-finite, hence it cannot be equal to M^{ω}, hence a spectral projection $e^{\omega} \neq 1$ appears at this ultrapower level. The difficulty in this axiomatization consists in saying $M \simeq eX^*e$ without using e. For that purpose, following [1], we will use Raynaud's viewpoint using standard forms instead of Groh's viewpoint. However, our axiomatization differs from the proofs in [1]. In our viewpoint, the key to the characterization of the isomorphism $M \simeq eX^*e$ is to compute $\Delta^{1/2}$ in (eX^*e, ϕ) in order to state it agrees with the expected computation for (M, φ).

To have a model associated to $(C, X, H, J, Im(\Pi_P), \phi, M)$ in the setting above, we now introduce the following supplementary data in the language.

- A unary function symbol $P : D_n(V) \to D_n(U)$ for $e.e : C \to eX^*e \simeq M$
- A constant $\phi \in \mathcal{C}_1$ for a state.
- Binary Relation Symbols $\mathscr{E}_{P,\beta,r}, \mathscr{E}_{P,\beta} : (D_m(V))^2 \to \mathbb{C}$ for $\beta \in]0, 1[\cap \mathbb{Q}$, $r \in \mathbb{Q}$, meaning

$$\mathscr{E}_{P,\beta}(x, y) = \mathscr{E}_{\beta}(P(x), P(y)), \mathscr{E}_{P,\beta,r}(x, y) = \mathscr{E}_{\beta}(G_r(P(x)), P(y)).$$

- Binary function Symbols: $\mathscr{E}_{P,\beta} G^{\alpha} : (D_m(V))^2 \to D_{m^2 K_{1,1}^{\alpha,\beta}}(ILip_w(\mathbb{R}, U))$ with $\alpha, \beta \in]0, 1[\cap \mathbb{Q}, \alpha < 1/2, \alpha + \beta < 1$ for the same $K_{1,1}^{\alpha,\beta}$ as in the language of section 2.2.
- Binary Relation Symbols $\mathscr{E}_{\beta,N,\infty} : (D_m(U))^2 \to \mathbb{C}$ for $\beta \in \mathbb{Q} \cap [0, 1[, N \in \mathbb{N}$ meaning $\mathscr{E}_{\beta}(F_N^{\varphi}(.), .)$.

Of course, the last relation symbols on $D_m(U)$ could have been introduced in the Ocneanu theory from the very beginning but it will be crucial only in this section. Note that the uniform continuity in the first variable is obvious. The continuity in the second variable is obtained as the one for φ. Indeed, note that

for any $t, x = y + z$ and using the same bounds from [1, lemma 4.13] as in lemma 2.1, we have, on the one hand:

$$|\mathscr{E}_\beta(F_N^\varphi(t), x)| \leq ||\sigma^\varphi_{-i\beta}(F_N^\varphi(t))||_\varphi ||y||_\varphi + ||[(\sigma^\varphi_{-i\beta}(F_N^\varphi(t))]^* z)^*||_\varphi$$
$$\leq 3\sqrt{2} e^{(2\beta+1)N} ||t|| \sqrt{||y||_\varphi^2 + ||z^*||_\varphi^2}.$$

On the other hand, one gets:

$$|\mathscr{E}_\beta(F_N^\varphi(x), t)| \leq e^{\beta N} ||(F_N^\varphi(x))||_\varphi ||t||_\varphi \leq e^{(\beta+1/2)N} (1 + e^N)^{1/2} ||x||_\varphi^* ||t||_\varphi,$$

since from spectral theory and the proof of lemma 2.2, we can use the fact that we have:

$$||(F_N^\varphi(x))||_\varphi = ||\Delta^{-1/2}(1+\Delta)^{1/2}\Delta^{1/2}(1+\Delta)^{-1/2}(F_N^\varphi(x))\xi_\varphi|| \leq e^{N/2}(1+e^N)^{1/2} ||(F_N^\varphi(x))||_\varphi^*$$

We are now ready to introduce our supplementary axioms:

(36) For any $N, M \in \mathbb{N}^*, \beta \in \mathbb{Q} \cap [0, 1]$, $\mathscr{E}_{\beta,N,\infty}(y, F_M(x)) = \mathscr{E}_{\beta,N,M}(y, x)$,

(37) $P(1) = 1, P(x^*) = (P(x))^*$, $tr(m_1^l(a, \phi)) = \varphi(P(a))$,

$$\sup_{a \in D_n(V)} \max(0, d(P(a), 0)^2 - tr(m_1^l(aa^*, \phi))) = 0.$$

(38) For any $l, m, N \in \mathbb{N}^*$,

$$\sup_{(x,z)\in(D_l(V))^2} \sup_{y \in D_m(U)} \max\Big(0,$$
$$\mathscr{E}_{0,N,\infty}(y, P(x)) + \overline{\mathscr{E}_{0,N,\infty}(y^*, P(z^*))} - md(m_1^l(x, \phi) + m_1^r(\phi, z), 0)\Big) = 0,$$

(39) For any $m, N \in \mathbb{N}^*$,

$$\sup_{y \in D_m(U)} \inf_{z \in D_m(V)} \sup_{x \in D_1(V)} \max(|\mathscr{E}_{0,N,\infty}(y, P(x)) - tr(m_1^l(zx, \phi)|,$$
$$|\mathscr{E}_{0,N,\infty}(y^*, P(x)) - tr(m_1^l(z^*x, \phi)|) = 0,$$

$$\sup_{(y,Y)\in(D_m(U))^2} \inf_{(z,Z)\in(D_m(V))^2} \sup_{x \in D_1(V)}$$
$$\max\big(|\mathscr{E}_{0,N,\infty}(y^*, P(x)) - tr(m_1^l(z^*x, \phi)|, |\mathscr{E}_{0,M,\infty}(Y, P(x)) - tr(m_1^l(Zx, \phi)|,$$
$$|(NM+2)\mathscr{E}_{0,NM+2,\infty}(m_{N,M}(y, Y), P(x))$$
$$-(NM+1)\mathscr{E}_{0,NM+1,\infty}(m_{N,M}(y, Y), P(x)) - tr(m_1^l(Zzx, \phi)|\big) = 0,$$

(40) For any $l, m, N \in \mathbb{N}^*, \alpha, \beta \in \mathbb{Q} \cap [0, 1[, 0 < \alpha < 1/2, \alpha + \beta < 1, r \in \mathbb{Q}$

$$\sup_{x \in (D_l(V))} \sup_{y \in (D_m(V))} \max(0, |\mathscr{E}_{P,\beta,r}(x, y) - \mathscr{E}_{\beta,N,\infty}(G_r(P(x)), P(y))|$$
$$- 4me^{-r/2}(e^r + |e^r - 1|)d(F_N(P(x)), P(x))) = 0$$

$$\delta_r(\mathscr{E}_{P,\beta} G^\alpha(x, y)) = e^{\alpha r} \mathscr{E}_{P,\beta,r}(x, y), \quad \mathscr{E}_{P,\alpha+\beta}(x, y) = \frac{\cos(\alpha\pi)}{2\pi} \int_{\mathbb{R}} \mathscr{E}_{P,\beta} G^\alpha(x, y).$$

(41) For any $n, m, N, M, N_i, N_{i,j}, L, l, K \in \mathbb{N}^*, \lambda_i, \lambda_{i,j} \in \mathbb{Q} \cap [0,1], u \in \mathbb{Q}, u > 0$ with $\sum_{i=1}^n \lambda_i = 1$,:

$$\sup_{x\in(D_m(U))} \sup_{(z_1,\dots,z_n)\in(D_L(U))^n} \inf_{y\in(D_m(V))} \max_{j=1,\dots,n} |\sum_{i=1}^K \lambda_{i,j}[\mathscr{E}_{0,N_{i,j},M}(z_j,x) - \mathscr{E}_{0,N_{i,j},\infty}(z_j,P(y))]| = 0,$$

$$\sup_{x\in(D_m(V))} \max\Bigg(0, \inf_{z\in(D_l(V))} \Re(tr(\|\pi(z,\xi_\phi)\|^2 + \frac{1}{u}(\langle\pi((z-x)^*,\xi_\phi), J(\pi(z-x,\xi_\phi))\rangle$$

$$-\frac{1}{u}\mathscr{E}_{P,1/2}(x,x) - \inf_{Z\in(D_l(U))}\Bigg[\sum_{i,j=1}^n \lambda_i\lambda_j(\mathscr{E}_{0,N_i,N_j}(Z,Z) + \frac{1}{u}\mathscr{E}_{1/2,N_i,N_j}(Z,Z))$$

$$+\frac{1}{u}\sum_{i=1}^n \lambda_i(\mathscr{E}_{1/2,\infty,N_i}(P(x),Z) + \mathscr{E}_{1/2,N_i,\infty}(Z,P(x)))\Bigg]\Bigg) = 0.$$

The two most technical conditions (39),(41) will be explained in the proof below. (39) will enable us to check our expected $j : M \to eX^*e$ is a $*$-homomorphism. (41) will be related to the surjectivity of j.

Theorem 4.1. *The theory T_{AHW^*} consisting of $T_{C^*,(W^*)_*,SF}$ for (C, X, H, J, Π_P), $T_{\sigma W^*}$ for M, and (36)-(41) axiomatizes a class of septuples $(C, X, H, J, \Pi_P, \phi, M)$ exactly described as follows. C is a weak-* dense C^*-algebra of X^* and X is the predual of a von Neumann algebra. $\phi \in X$ is a state and $M \simeq eX^*e$ is a von Neumann algebra, corner of X^* for e the support projection of ϕ. $(X^*, H, J, Im(\Pi_P))$ is a standard form.*

Proof : We already know that an element in the class produces a model except for the verification of (39)-(41) which are less obvious. Seeing $F_N(y) \in eX^*e$ and $P(x) = exe$ we have

$$\mathscr{E}_{0,N,\infty}(y, P(x)) = \phi(F_N(y^*)exe) = \phi(F_N(y^*)x) = \langle F_N(y)\xi_\phi, x\xi_\phi\rangle.$$

But we know that C is strong-* dense in X^* thus there is a net $a_n \in C$ bounded in $D_m(C)$ (for $y \in D_m(M)$) with $a_n \to F_N(y^*), a_n^* \to F_N(y)$ strongly, thus $\sup_{x\in D_1(V)} |\mathscr{E}_{0,N,\infty}(y,P(x)) - \langle a_n^*\xi_\phi, x\xi_\phi\rangle| \leq ||(F_N(y) - a_n^*)\xi_\phi|| \to 0$, giving the first statement in (39) about the infimum

$$\inf_{z\in D_m(V)} \sup_{x\in D_1(V)} \max(|\mathscr{E}_{0,N,\infty}(y,P(x)) - tr(m_1^l(zx,\phi)|,$$
$$|\mathscr{E}_{0,N,\infty}(y^*,P(x)) - tr(m_1^l(z^*x,\phi)|) = 0.$$

Similarly, one takes $b_n \to F_M(Y^*)$ strongly so that

$$\langle a_n^*b_n^*\xi_\phi, x\xi_\phi\rangle \to \langle F_N(y)F_M(Y)\xi_\phi, x\xi_\phi\rangle = \phi(F_M(Y^*)F_N(y^*)x)$$
$$= (NM+2)\mathscr{E}_{0,NM+2,\infty}(m_{N,M}(y,Y), P(x))$$
$$-(NM+1)\mathscr{E}_{0,NM+1,\infty}(m_{N,M}(y,Y), P(x)),$$

and since the convergence is uniform in x we get the last equality in (39).

For the first inequality in (40), using $\Delta^\beta \leq 1 + \Delta$, the resolvent equation $G_r - G_0e^{r/2} = e^{r/2}(1 - e^r)G_0(\Delta + e^r)^{-1}$, $||(\Delta + e^r)^{-1}|| \leq e^{-r}$, $||P(y)||_\varphi^\# \leq 2||y|| \leq 2m$ and lemma 2.2, one gets:

$$|\mathscr{E}_{P,\beta,r}(x,y) - \mathscr{E}_{\beta,N,\infty}(G_r(P(x)),P(y))|$$
$$= |\mathscr{E}_\beta(G_r(P(x)) - F_N(G_r(P(x))), P(y))|$$
$$\leq ||G_r(P(x)) - F_N(G_r(P(x)))||_\varphi^\# ||P(y)||_\varphi^\#$$
$$\leq 4me^{-r/2}(e^r + |e^r - 1|)d(F_N(P(x)) - P(x))), 0).$$

The second equation in (40) is a substitute to (19) with F_L, F_K replaced by P. Note we could not have done the same with (20), this is the reason why we cannot merely define $\varphi(P(.)P(.))$ in order to express the

isomorphism $eX^*e \simeq M$ because otherwise e would have been definable while it is not stable by ultrapoduct. It is thus crucial in order to have the bounds stated in the language for $\mathscr{E}_{P,\beta}G^\alpha$ and coming from lemma 2.4 that $\alpha + \beta < 1$ in order to use the operator norm as the metric of the source space C (with sort V).

The first statement in (41) comes from the weak-* density of $P(C) = eCe$ in $M = eX^*e$. The last inequality is more technical. Recall e denotes the support projection of ϕ. First note the following computation (where we write for short $\pi(y, \xi_\phi) = y.\xi_\phi$ and use commutation of JyJ with $e \in X^*, J^2 = 1, J\xi_\phi = \xi_\phi = e\xi_\phi$):

$$\begin{aligned}\langle z.\xi_\phi, J(y.\xi_\phi)\rangle &= \langle z.\xi_\phi, JyJe\xi_\phi\rangle \\ &= \langle ez.\xi_\phi, Jy\xi_\phi\rangle \\ &= \langle y.\xi_\phi, Jez\xi_\phi\rangle \\ &= \langle ey.\xi_\phi, Jez\xi_\phi\rangle \\ &= \langle z.\xi_\phi, eJe(y.\xi_\phi)\rangle.\end{aligned} \tag{4.1}$$

We deduce that, considering the restriction of the inf on z to elements of the form eze and since the inf over $D_l(V)$ is the same as one over $D_l(X^*)$ by strong-$*$ continuity of the expression and density, we have:

$$\begin{aligned}&\inf_{z\in(D_l(V))} \Re(tr(\|\pi(ez,\xi_\phi)\|^2 + \frac{1}{u}(\langle\pi((z-x)^*,\xi_\phi), J(\pi(z-x,\xi_\phi))\rangle \\ &\geq \inf_{z\in(D_l(V))} \Re(tr(\|\pi(z,\xi_\phi)\|^2 + \frac{1}{u}(\langle\pi((z-x)^*,\xi_\phi), J(\pi(z-x,\xi_\phi))\rangle\end{aligned}$$

Now we express this formula in terms of φ on $eX^*e \simeq M$.

First note that, for e the support projection of ϕ and using commutation of e, JeJ and $J^2 = 1$, one gets:

$$J_\phi = (eJeJ)J(eJeJ) = eJeJeJ = JeJeJ = eJeJJ = eJe.$$

Here our computation of J_ϕ for the state induced by ξ_ϕ on the σ-finite von Neumann algebra eX^*e started from the computations in [21, lemmas 2.6,2.9] which characterize J_ϕ in terms of a standard form and describe qJq as the modular group of this standard from in case of qX^*q for $q = eJeJ$ as above. We also used the isomorphism of [21, corollary 2.5] in the previous computation to replace e by $q = eJeJ$ in $eX^*e \simeq qX^*q$. From this equality, we can compute:

$$\langle\pi((ze)^*,\xi_\phi), J(\pi(ez,\xi_\phi))\rangle = \langle J_\phi\Delta_\phi^{1/2}(P(z)\xi_\phi), eJe(P(z)\xi_\phi)\rangle = ||\Delta_\phi^{1/4}(P(z)\xi_\phi)||^2.$$

Combining this equality with (4.1), one gets:

$$\begin{aligned}&\Re(tr(\|\pi(ez,\xi_\phi)\|^2 + \frac{1}{u}(\langle\pi((z-x)^*,\xi_\phi), J(\pi(z-x,\xi_\phi))\rangle \\ &= \Re(tr(\|\pi(ez,\xi_\phi)\|^2 + \frac{1}{u}(\langle\pi(e(z-x)^*,\xi_\phi), J(\pi(ez-ex,\xi_\phi))\rangle \\ &= \Re\left(\varphi(P(z^*)P(z)) + \frac{1}{u}\langle\Delta_\varphi^{1/2}P(z-x)\xi_\varphi, P(z-x)\xi_\varphi\rangle\right).\end{aligned}$$

By strong-* density of $P(C)$ the infimum over $P(z), z \in D_l(V)$ is the same with $Z = P(z)$ replaced by $Z \in D_l(U)$ and one gets an upper bound by the infimum restricted to convex combinations of the form $\sum_{i=1}^n \lambda_i F_{N_i}(Z), Z \in D_l(U)$. In that form, this gives the inequality at the source of the second statement in

(41), namely for $||x|| \leq m$:

$$\inf_{z\in(D_l(V))} \Re(tr(\|\pi(z,\xi_\phi)\|^2 + \frac{1}{u}(\langle\pi((z-x)^*,\xi_\phi), J(\pi(z-x,\xi_\phi))\rangle$$
$$\leq \frac{1}{u}\mathscr{E}_{1/2,P}(x,x)+$$
$$\inf_{Z\in(D_l(U))}\left[\sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j(\mathscr{E}_{0,N_i,N_j}(Z,Z) + \frac{1}{u}\mathscr{E}_{1/2,N_i,N_j}(Z,Z))\right.$$
$$\left. +\frac{1}{u}\sum_{i=1}^n \lambda_i(\mathscr{E}_{1/2,\infty,N_i}(P(x),Z) + \mathscr{E}_{1/2,N_i,\infty}(Z,P(x)))\right].$$

One gets exactly the last formula in (41).

Conversely, take (C, X, H, J, Π_P) model of the Groh-Haagerup-Raynaud theory and M model of the Ocneanu theory. Since we have checked $d_U(F_N(x), x) \to 0$, we indeed obtain from (36) the expected $\mathscr{E}_{\beta,M,\infty}(y,x) = \mathscr{E}_\beta(F_M^\varphi(y), x)$. Note also that the inequality in (37) implies that P is uniformly continuous from the strong operator topology on C induced from X^* to the topology of the metric d. Thus, by [25, §5.4.(4)], it extends to a uniformly continuous map we still call $P : X^* \to M$. Moreover, this extension is weak-* continuous by standard relation to the topology of d and the strong topology.

Consider the (norm) closures L, V of the spaces spanned respectively by $m_1^l(x,\phi)+m_1^r(\phi,z) \in X, x, z \in C$, and $m_1^l(x,\phi) \in X, x \in C$. It is known from the proof of [30, lemma III.3.6] that V is the smallest left C-invariant subspace containing ϕ. Hence by weak-* density (and usual applications of Hahn-Banach to identify weak-closures and norm closures), also the smallest left X^*-invariant subspace containing ϕ, which is Xe for e the support projection of ϕ. Moreover $L = V + V^* = Xe + eX$ so that the dual $L^* = eX^*e$. The inequality in (38) then states that the map:

$$i(F_N^\varphi(y^*)) : m_1^l(x,\phi) + m_1^r(\phi,z) \mapsto \varphi(F_N^\varphi(y^*)P(x)) + \varphi(P(z)F_N^\varphi(y^*))$$

extends to a continuous linear form on L giving a contractive linear map $i : A \to eX^*e$ for the operator norm (since, recall $A = Vect(F_N(x) : N \in \mathbb{N}^*, x \in M)$). Note that $i(1) = e$ and also that i is uniformly continuous on bounded sets from the strong-* topology (given by the metric $||.||_\varphi^\#$) to the weak-* topology on eX^*e. By [25, §5.4.(4)] again and Kaplansky's density theorem, it extends to a uniformly continuous map

$$j : M \to eX^*e$$

which is weak-* continuous (by standard compactness and Hahn-Banach arguments). Notice that the interpretation of terms in the two equations in (39) that we want to use:

$$(\phi.i(F_N(y^*)))(x) + (i(F_N(y^*).\phi))(z) = (x.\phi + \phi.z)(i(F_N(y^*)))$$
$$= \varphi(F_N^\varphi(y^*)P(x) + P(z)F_N^\varphi(y^*))$$

and

$$(NM+2)\mathscr{E}_{0,NM+2,\infty}(m_{N,M}(y,Y), P(x))$$
$$-(NM+1)\mathscr{E}_{0,NM+1,\infty}(m_{N,M}(y,Y), P(x))$$
$$= (\phi.i(H_{NM+1}(F_M(Y^*).F_N(y^*))))(x) = (\phi.i(F_M(Y^*).F_N(y^*)))(x).$$

Now, we can use the first equation in (39) which says exactly that for $y \in A$, there is a sequence $z_n \in C$ such that $||\phi.z_n - \phi.(i(y))||_X, ||\phi.z_n^* - \phi.(i(y^*))||_X \to 0$. Taking a subnet such that $z_n \to z$ ultraweakly in X^* which implies $\phi.z_n$ converges weakly in X to $\phi.z$, one gets using the first equation we just noticed:

$$\phi.z = \phi.(i(y)), \quad \phi.z^* = \phi.(i(y^*))$$

and thus $ez = i(y), e.z^* = i(y^*)$. Thus, extending from $y \in A$ to $x \in M$ the relation $[i(y)e]^* = [eze]^* = i(y^*)e$, we have obtained

$$[j(x^*)] = [j(x^*)]e = e[j(x)]^* = [j(x)]^*.$$

Similarly, the second equation in (39) says that for $y, Y \in A$, there are sequences $z_n, Z_n \in C$ such that $||\phi.z_n^* - \phi.(i(y^*))||_X \to 0, ||\phi.Z_n - \phi.(i(Y))||_X \to 0, ||\phi.(Z_n z_n) - \phi.(i(Y.y))||_X \to 0$. Arguing as before, one gets for limit points z, Z of z_n, Z_n in X^*,

$$ez^* = i(y^*), \quad eZ = i(Y), \quad eZz = i(Y.y)$$

and thus $i(Y.y)e = eZze = i(Y)[i(y^*)]^*$ so that extending by ultraweak continuity (separately in y, Y):

$$j(Y.y) = j(Y.y)e = j(Y)[j(y^*)]^* = j(Y)j(y)$$

and j is thus a $*$-homomorphism.

Note that making $N \to \infty$ in the relation $(x.\phi + \phi.z)(i(F_N(y^*))) = \varphi(F_N^\varphi(y^*)P(x) + P(z)F_N^\varphi(y^*))$ one gets:

$$(x.\phi + \phi.z)(j(y)) = \varphi(yP(x) + P(z)y)$$

Thus we deduce $\phi(j(y)) = \varphi(y)$ and the isometry relation implying that j is one-to-one (since φ faithful on M):

$$\phi(j(y)^* j(y)) = ||y||_\varphi^2. \tag{4.2}$$

Moreover $\phi(j(P(a))) = \varphi(P(a)) = \phi(a)$ by the last equality in (37) for $a \in C$. Note also that for $x \in C, y \in M$ $\phi(xj(y)) = \varphi(P(x)y)$ and this extends to $x \in X^*$, since P is strong to metric continuous on bounded set by (37), thus weak-* continuous. The relation thus extends to $x \in X^*$ thus

$$\varphi(P(j(P(x)))y) = \phi(j(P(x))j(y)) = \phi(j(P(x)y)) = \varphi(P(x)y),$$

and since φ is faithful on M, $P \circ j \circ P = P$ on X^* and thus $j \circ P$ is a projection on X^* to $j(M)$.

As a consequence we also deduce:

$$\varphi(P(x^*)P(x)) = \phi(x^* j(P(x))) \leq \|exe\xi_\phi\| \|j(P(x))\xi_\phi\| = \|exe\xi_\phi\| \|P(x)\xi_\varphi\| \tag{4.3}$$

Note also that this relation implies $j(1) = e$ since we know it is in eX^*e and it has the expected formula for e on the predual of eX^*e. As a consequence since from (37) $P(1) = 1$, $P(e) = P \circ j \circ P(1) = P(1) = 1$.

Let us finally check that j is onto: $j(M) = eX^*e$. The two equations in (40) determine uniquely $\mathscr{E}_{P,\beta,r}, \mathscr{E}_{P,\beta}$ so that we will be able to use $\mathscr{E}_{P,1/2}$ in (41) with its expected interpretation.

One uses the first equation (41) to note that $P(C)$ is weak-* dense in M and take $x \in D_m(U)$, $\epsilon > 0$ and $z_1, ..., z_n \in M$, and find convex combinations, from the proof of Theorem 2.7, with $\| \sum_{i=1}^K \lambda_{i,j} F_{N_{i,j}}(z_j) - z_j \|_\varphi \leq \epsilon/4m$, then find $y \in D_m(U)$ given by the first equation in (41) such that for all $j = 1, ..., n$: $|\varphi((\sum_{i=1}^K \lambda_{i,j} F_{N_{i,j}}(z_j))^*(F_M(x) - P(y)))| \leq \epsilon/2$ and thus

$$|\varphi[z_j^*(F_M(x) - P(y))]| \leq 2m\epsilon/4m + \epsilon/2 = \epsilon$$

Since M is dense in $L^1(M, \varphi)$, this concludes the weak-* density of $P(C)$ in M. Note that as a consequence $j \circ P$ is a state preserving norm 1 projection from (eXe, ϕ) onto $(j(M), \varphi \circ j^{-1})$.

We can now use the last equation of (41) and recall that we already computed the first infimum in a more explicit form and after taking an infimum over l:

$$\begin{aligned}
&\inf_{z \in C} \Re(tr(\|\pi(z, \xi_\phi)\|^2 + \frac{1}{u}(\langle \pi((z-x)^*, \xi_\phi), J(\pi(z-x, \xi_\phi))\rangle \\
&= \inf_{z \in C} \Re(tr(\|\pi(ez, \xi_\phi)\|^2 + \frac{1}{u}(\langle \pi(e(z-x)^*, \xi_\phi), J(\pi(e(z-x), \xi_\phi))\rangle \\
&= \inf_{z \in X^*} \Re\left(\|\pi(eze, \xi_\phi)\|^2 + \frac{1}{u}\langle \Delta_\phi^{1/2} e(z-x)\xi_\phi, e(z-x)\xi_\phi\rangle \right) \\
&= ||\Delta_\phi^{1/4}(u + \Delta_\phi^{1/2})^{-1/2}(exe)\xi_\phi||^2
\end{aligned}$$

with the last equation coming from a variant of lemma 2.2 for $\Delta^{1/2}$ instead of Δ.

Reasoning similarly with the right hand side and approximating Z by convex combinations of $F_{N_i}(Z)$, one gets from (41):

$$||\Delta_\phi^{1/4}(u+\Delta_\phi^{1/2})^{-1/2}(exe)\xi_\phi||^2 \leq ||\Delta_\varphi^{1/4}(u+\Delta_\varphi^{1/2})^{-1/2}(P(x)\xi_\varphi)||^2. \tag{4.4}$$

Using the variant of lemma 2.6 with again $\Delta^{1/2}$ instead of Δ, for $\alpha \in]0,1/2[$, $\Delta^{1/2-\alpha} = \frac{\sin(2\alpha\pi)}{\pi}\int_0^\infty u^{-2\alpha}\Delta^{1/2}(u+\Delta^{1/2})^{-1}$ so that integrating the previous inequality (4.4), one gets:

$$\begin{aligned}\langle\Delta_\phi^{1/2-\alpha}(exe)\xi_\phi,(exe)\xi_\phi\rangle &= \frac{\sin(2\alpha\pi)}{\pi}\int_0^\infty duu^{-2\alpha}||\Delta_\phi^{1/4}(u+\Delta_\phi^{1/2})^{-1/2}(exe)\xi_\phi||^2\\ &\leq \frac{\sin(2\alpha\pi)}{\pi}\int_0^\infty duu^{-2\alpha}||\Delta_\varphi^{1/4}(u+\Delta_\varphi^{1/2})^{-1/2}(P(x)\xi_\varphi)||^2\\ &= \langle\Delta_\varphi^{1/2-\alpha}(P(x)\xi_\varphi),P(x)\xi_\varphi\rangle.\end{aligned}$$

Finally, using the spectral theorem for Δ, the left hand side reduces to an integral

$$\langle\Delta_\phi^{1/2-\alpha}(exe)\xi_\phi,(exe)\xi_\phi\rangle = \int_0^\infty \lambda^{1/2-\alpha}\langle dE_\lambda(\Delta_\phi)(ex\xi_\phi),ex\xi_\phi\rangle,$$

with $E_\lambda(\Delta_\phi)$ the projection-valued spectral measure of this unbounded operator. Then cutting the integral and applying monotone convergence theorem for $\lambda \leq 1$ and dominated convergence theorem for $\lambda > 1$ enable us to take the limit $\alpha \to 1/2$ and obtain $\langle\Delta_\phi^{1/2-\alpha}(ex\xi_\phi),ex\xi_\phi\rangle \to ||ex\xi_\phi||^2$. Reasoning similarly on the right hand side, one concludes to the inequality:

$$||exe\xi_\phi||^2 \leq ||P(x)\xi_\varphi||^2.$$

Since the converse was already obtained in (4.3), one gets equality. Using the isometry relation (4.2) with $y = P(x)$ and the equality in (4.3), one also deduces the equality:

$$\begin{aligned}||(j(P(x))-exe)\xi_\phi||^2 &= ||j(P(x))\xi_\phi||^2 + ||exe\xi_\phi||^2 - 2\Re\phi(x^*j(P(x)))\\ &= ||exe\xi_\phi||^2 - ||P(x)\xi_\varphi||^2 = 0,\end{aligned}$$

so that $j(P(x)) = exe$ for any $x \in X^*$. Thus the expected inclusion $eX^*e \subset j(M)$ follows, implying that $j : M \to eX^*e$ is onto. Finally, via this isomorphism, $P(x) = exe$ as expected, and this defines P uniquely. This concludes the identification of the structures. ∎

As a consequence, one obtains the stability by ultraproducts from [1, Th 6.11] in the case of a general non-principal ultrafilter ω not necessarily on $\mathbb{N}$. We already used it in section 1 to obtain the existence of a first order axiomatization of III_λ factors.

Corollary 4.2. *Let $\lambda \neq 0$ fixed. Let M_n be σ-finite factors of type III_λ, and consider faithful normal states φ_n, then $(M_n,\varphi_n)^\omega$ is also a factor of type III_λ.*

Proof : $(M_n,(M_n)_*,L^2(M,\varphi_n),J,\Pi_P,\varphi_n,M_n)$ satisfies T_{AHW^*} and moreover $(M_n,(M_n)_*)$ satisfies $T_{C^*,(III_\lambda)_*}$ thus so does

$$((M_n)^\omega,((M_n)_*)^\omega,L^2(M,\varphi_n)^\omega,J^\omega,\Pi_P^\omega,\varphi_n^\omega,(M_n,\varphi_n)^\omega)$$

(with the first three spaces being the Banach space ultraproducts and the last one being the Ocneanu ultraproduct). Thus $(M_n,\varphi_n)^\omega \simeq p[((M_n)_*)^\omega]^*p$ is a corner of a III_λ factor and thus, in particular, a III_λ factor. ∎

5. Axiomatization of cross-products appearing in Connes' description of type III_0 factors

Finally, it seems interesting to axiomatize the crossed-product decomposition of III_0-factors from [7, Th 5.3.1]. We won't axiomatize in this way the class of III_0 factors, since it is not axiomatizable by the above cited result of [1], but we will describe this class as a union of uncountably many axiomatizable classes. This gives an alternative proof of [1, Prop 6.23] based on model theory. We deduce in corollary 5.2 the missing non-stability by (uncountable) ultraproducts needed in the proof of axiomatizability of type III_λ factors, for fixed $\lambda > 0$. This is the most technical work hidden in this natural result. This result could also probably have been written in operator algebraic style as in [1]. However, we believe that our explicit axiomatization will be crucial for later model theoretic investigation of III_0 factors. As for II_1, II_∞ factors, we don't expect axiomatizability without fixing the state, matrix unit and extra data in the language. Let us now describe this extra data.

We want to axiomatize cross products by $\mathbb{Z}$ (with implementing unitary U) of a II_∞ von Neumann algebra N. This algebra N will be the centralizer for a lacunary normal faithful semifinite weight ψ (thus with spectrum included in $(\mathbb{R}-]\log(\lambda_0), -\log(\lambda_0)[) \cup \{0\}$ for some $\lambda_0 \in]0,1[$). We will use a matrix unit $(w_{i,j})_{i,j\geq 0}$ for $B(H)$ with some geometric state φ playing the role of our basic state in our previous language. We will model the semifinite weight by Connes cocycle derivative with respect to our faithful state (see e.g. [30]) and various compressions by well chosen projections related to the matrix unit. We will also need some of the modular theory for ψ.

Unfortunately, saying that U is a unitary is not obvious in our non-tracial setting. Hence, we will use what we called before Ando-Haagerup theory, to have the C^*-algebra in Groh's theory to require u in it (or rather $u = eUe + 1 - e$ or any other unitary projecting to $eue = U$ and the theory will depend on the choice of this unitary in the C^*-algebra). This is the key part restricting the choice of the language and which is crucial to get some stability results by ultraproducts and not only ultrapowers as in [1].

This will however enable to easily express the unitarity condition on eue since more maps can be defined on the C^*-algebra. However, this will not be completely easy since $\varphi(P(.)P(.))$ has not been defined in section 3 (and could not have been defined in a first order theory). Our solution won't be completely satisfactory since it will axiomatize only a class of cross-products depending on parameters such as λ_0 above and we won't axiomatize every cross product with the same theory which would require to axiomatize the union over all parameters. We don't know if this is possible but we conjecture it is not. This will be enough to obtain in corollary 5.2 the stability by ultrapower we are aiming at, thus extending [1].

We will fix $\Lambda_j, j \geq 0$ increasing in j satisfying: $\lim_{j\to\infty} \Lambda_j = 1$ and such that for $i \geq 0$:

$$\langle \Delta(e^{-j} + \Delta)^{-1}(u\xi_\varphi), (u\xi_\varphi)\rangle \geq \Lambda_j; \quad \sum_{l=0}^{j} \varphi(uw_{l,l}u^*) \geq \Lambda_j. \tag{5.1}$$

Of course, in any cross-product (for instance in any III_0-factor), there is a choice of parameters satisfying this if we take Λ_j the minimum of the two values it has to bound, which is increasing in j and converges as expected to 1.

We are now ready to state what we need to add in the language of the Ando-Haagerup theory.

- The constant u in $D_1(V)$ for a preimage of the implementing unitary.
- Constant symbols $W_{i,j} \in D_1(V), i, j \in \mathbb{N}$ with $w_{i,j} = P(W_{i,j}) \in D_1(U), i, j \in \mathbb{N}$ for a matrix unit. Unary function symbols $P_{i,j} : D_1(U) \to D_1(U), i, j \in \mathbb{N}$ meaning $P_{i,j}(x) = w_{i,i}xw_{j,j}$.
- Constant symbols $u_t, t \in \mathbb{Q}$ in $D_1(U)$ for Connes' cocycle derivatives $(D\varphi : D\psi)_t$ and unary function symbols Σ_t (for σ_t^ψ).
- Unary function symbols for $(m, l) \in \mathbb{Q}^*, m > 0$ $\Psi_{m,l} : D_n(U) \to D_n(U), \Psi_{N,0} = \Psi_N$ (for Fejer's map $F^\psi_{m,l}$)
- Binary function symbols for $N \in \mathbb{N}$ $m_{N,\infty,P} : D_n(U) \times D_m(V) \to D_{nm}(U)$, $m_{\infty,N,P} : D_n(V) \times D_m(U) \to D_{nm}(U)$ (meaning $F^\varphi_N(.)P(.), P(.)F^\varphi_N(.)$)

- A Ternary Relation Symbol $\mathscr{E}_{0,N,\infty,M} : (D_m(U))^3 \to \mathbb{C}$ for meaning $\varphi(F_N^{\varphi}(.)^*.F_M^{\varphi}(.)^*)$, $\mathscr{E}_{0,\infty,M}(.,.) = \mathscr{E}_{0,1,\infty,M}(1,.,.)$.
- Unary function symbols for $r \in \mathbb{Q}$ $\Gamma_r : D_n(U) \to D_n(U)$, (meaning G_r^{ψ})
- Binary relation symbols $\psi_{r,n,m,P}$ for $(n,m) \in \mathbb{N}^2, r \in \mathbb{Q}$ meaning

$$\psi_{r,n,m,P}(x,y) = \langle G_r^{\psi}(w_{m,m}P(x)w_{n,n})\xi_{\psi}, P(y)w_{n,n}\xi_{\psi}\rangle,$$

  Binary relation symbols $\mathscr{E}_{\beta,P}^{\psi,n,m}$for $(n,m) \in \mathbb{N}^2, \beta \in \mathbb{Q}\cap]0,1/2[$,meaning

$$\mathscr{E}_{\beta,P}^{\psi,n,m}(x,y) = \langle \Delta_{\psi}^{\beta}(w_{m,m}P(x)w_{n,n}\xi_{\psi}), P(y)w_{n,n}\xi_{\psi}\rangle,$$

- Curves valued function symbols

  $g_s\Sigma : D_n(U) \to D_{nK'_s}(ILip_w(\mathbb{R},U)), s \in \mathbb{Q}, \quad f_{m,l}\Sigma : D_n(U) \to D_{nK'_{m,l}}(ILip_w(\mathbb{R},U)), (m,l) \in \mathbb{Q}^2, m > 0$

  and

$$\psi_{N,M,P}^{\alpha} : D_n(U) \times D_{\nu}(U) \to D_{n\nu K_{N,M}^{\alpha,0}}(ILip_w(\mathbb{R},U)), \alpha \in \mathbb{Q} \cap [0,1/2[, N, M, n, \nu \in \mathbb{N}^*$$

  with same constant as in section 2.2 and: $K'_s = \left\lceil (\pi + s + 8) \right\rceil$, $K'_{m,l} = \left\lceil \max((\frac{2}{\pi m} + \frac{m}{2\pi}), (\frac{m^2}{\pi} + \frac{ml}{2\pi}) + 8\frac{m}{\pi}) \right\rceil$. The modulus of uniform continuity are also those determined in lemmas 2.4 and 2.6 with ψ replacing φ using $\psi(w_{n,n}) = 1$.
- Unary function symbols for $N, k \in \mathbb{N}^*$

$$\Pi_{N,k}, E_{N,k}, EU_{N,k}, \theta EP, \overline{\theta} EP, uEP, u^*EP : D_m(V) \to D_m(U)$$

  for

$$\Pi_{N,k}(x) = P(u^k)\Psi_N(P(x)), E_{N,k}(x) = E_{M_{\psi}}(P(u^k)\Psi_N(P(x))),$$
$$EU_{N,k}(x) = P((u^k)^*)E_{M_{\psi}}(P(u^k)\Psi_N(P(x))), uEP(x) = P(u)E_{M_{\psi}}(P(x)),$$
$$\theta EP(x) = P(u)E_{M_{\psi}}(P(x))P(u^*), u^*EP(x) = P(u^*)E_{M_{\psi}}(P(x)),$$
$$\overline{\theta} EP(x) = P(u^*)E_{M_{\psi}}(P(x))P(u).$$

The uniform continuity constants are similar to previous sections. For instance, we bound $||\Sigma_t(x)||_{\varphi}^* = ||\sigma_t^{\varphi}(u_t x u_t^*)||_{\varphi}^* \leq 9||x||_{\varphi}^*$ from lemma 2.1 since u_t is in the centralizer. From the uniform continuity bounds in section 3, one gets: $||F_N^{\varphi}(y)P(x)||_{\varphi}^* \leq ||(F_N^{\varphi}(y)P(x))^*||_{\varphi} \leq ||x||\ ||(F_N^{\varphi}(y))^*||_{\varphi} \leq e^{N/2}\sqrt{1+e^N}||x||\ ||y||_{\varphi}^*$.

We define, for $N = (N_1, ..., N_n), \lambda = (\lambda_1, ..., \lambda_n)$:

$$\begin{aligned}
&\epsilon_N(\lambda)^2 \\
&= 2 - \sum_{i=1}^{n} \overline{\lambda_i}\mathscr{E}_{0,N_i,\infty,N_1}(P(u),P(u),1) - \sum_{j=1}^{n} \lambda_j \mathscr{E}_{0,N_1,\infty,N_j}(1,P(u^*),P(u^*)) \\
&+ \sum_{i,j=1}^{n} \lambda_i\overline{\lambda_j}\mathscr{E}_{0,N_i,\infty,N_j}(P(u),1,P(u^*)) + \sum_{i,j=1}^{n} \lambda_j\overline{\lambda_i}\mathscr{E}_{0,N_i,\infty,N_j}(P(u^*),1,P(u)) \\
&- \sum_{i=1}^{n} \lambda_i\mathscr{E}_{0,N_i,\infty,N_1}(P(u^*),P(u^*),1) - \sum_{j=1}^{n} \overline{\lambda_j}\mathscr{E}_{0,N_1,\infty,N_j}(1,P(u),P(u))
\end{aligned} \tag{5.2}$$

meaning

$$\epsilon_N(\lambda) = ||P(u) - \sum_{i=1}^{n} \lambda_i F^{\varphi}_{N_i}(P(u))||^{\#}_{\varphi} \tag{5.3}$$

if we would know that $\varphi(P(u)P(u^*)) = 1 = \varphi(P(u^*)P(u))$. Our axiom (42) based on the control of the spectrum of u will insure these equalities via new maps we will introduce in axioms (43) and (49).

We finally consider the following supplementary axioms beyond Ando-Haagerup theory and (24), depending on parameters $\lambda_0 < 1, \Lambda_j$:

(42) For $N \in \mathbb{Q}\cap]0,\infty[, l \in \mathbb{Q}^*, m \in \mathbb{N}^*$ with $]l-N, l+N[\subset]\ln(\lambda_0), -\ln(\lambda_0)[-\{0\}$

$$\sup_{x\in D_m} d_U(\Psi_{N,l}(x)), 0) = 0,$$

and if $]l-N, l+N[\subset]\ln(\lambda_0), \infty[$

$$\sup_{x\in D_m} d_U(\Psi_{N,l}(P(u))), 0) = 0,$$

and (25) with x, y replaced by $2\Psi_{2n}(x) - \Psi_n(x), 2\Psi_{2n}(y) - \Psi_n(y)$ with $2n \in \mathbb{Q}\cap]0, |\log(\lambda_0)|[$,

$$2\Psi_{2n}(w_{i,j}) - \Psi_n(w_{i,j}) = w_{i,j}.$$

For all $k, l \in \mathbb{N}$,

$$tr(m^l_1(\phi, xW_{k,l})) = 2\mathscr{E}_{0,\infty,2}(P(x), w_{k,l}) - \mathscr{E}_{0,\infty,1}(P(x), w_{k,l})$$

(43) With the notation in (25)

$$\sup_{x\in D_m(U)} \max(0, d(P_{k,j}(x), w_{k,k}F_N(x)w_{j,j}) - 9d(x, F_N(x))) = 0$$

For $t \in \mathbb{Q}$, $P_{k,k}(u_t) = 2^{(k+1)it}w_{k,k}$,$P_{k,j}(u_t) = 0, k \neq j$ and $2N \in \mathbb{Q}\cap]0, |\log(\lambda_0)|[$,

$$2F_{2N}(u_t) - F_N(u_t) = u_t, 2\Psi_{2N}(u_t) - \Psi_N(u_t) = u_t,$$

$$2\Psi_{2N}(\theta EP(x)) - \Psi_N(\theta EP(x)) = \theta EP(x),$$
$$2\Psi_{2N}(\overline{\theta} EP(x)) - \Psi_N(\overline{\theta} EP(x)) = \theta EP(x).$$

(44) For $N \in \mathbb{Q}\cap]0,\infty[, l \in \mathbb{Q}^*, M, m \in \mathbb{N}^*, K, L \in \mathbb{N}, L \geq 2K$

$$\Sigma_l(F_M(x)) = \sigma_l(M_{(M+4,0)}(M_{(0,M+1)}(u_l, F_M(x)), u^*_l), \quad \delta_0(f_{N,l}\Sigma(x)) = \frac{N}{2\pi}x$$

$$\delta_t(f_{N,l}\Sigma(x)) = e^{ilt}\frac{1-\cos(Nt)}{\pi Nt^2}\Sigma_t(x), \qquad \Psi_{N,l}(x) = \int_{\mathbb{R}} f_{N,l}\Sigma(x).$$

(45) For any $K, m, n \in \mathbb{N}^*, \beta \in \mathbb{Q}\cap]0, 1/2[$

$$2^{-1}e^{K/2}\mathscr{E}_{P,1/2,-K}(u, u) \geq \Lambda_K$$

$$\sum_{l=0}^{m}\sum_{k=0}^{n} 2^{-k-1}\mathscr{E}^{\psi,k,l}_{\beta,P}(u^*, u^*) \geq \lambda_0^{-\beta}[\Lambda_m - 2^{-n-1}(m+1)].$$

(46) For any $N, M, L \in \mathbb{N}^*$ $m_{\infty,N,P}(y,x) = m_{N,\infty,P}(x^*, y^*)^*$

$$\mathscr{E}_{0,N,\infty,L}(x, P(y), z) = \mathscr{E}_{0,\infty,L}(m_{N,\infty,P}(x^*, y), z)$$

For $r \in \mathbb{Q}, M \in \mathbb{N}, x \in D_M(U)$:

$$\delta_r(g_s\Sigma(x)) = \frac{2e^{-isr}}{e^{\pi r} + e^{-\pi r}}\Sigma_r(x), \qquad G_s(x) = \int_{\mathbb{R}} g_s\Sigma(x).$$

$$\begin{aligned}\sup_{(x,y)\in D_M(V)^2} \max(0, -\frac{8}{\pi\sqrt{N}}M^2 - \frac{2(n+m+2+\cosh(r))}{\sqrt{N}}M^2 + |\psi_{r,n,m,P}(x,y)\\ -2^{n+1}\mathscr{E}_{0,N,\infty}\left[\Gamma_r(2m_{2,\infty,P}\left(w_{m,m}, 2m_{\infty,2,P}(x, w_{n,n}) - m_{\infty,1,P}(x, w_{n,n})\right)\right)\\ -m_{1,\infty,P}\left(w_{m,m}, 2m_{\infty,2,P}(x, w_{n,n}) - m_{\infty,1,P}(x, w_{n,n})\right)\right),\\ (2m_{\infty,2,P}(y, w_{n,n}) - m_{\infty,1,P}(y, w_{n,n}))]|) = 0\end{aligned}$$

For $\alpha \in \mathbb{Q}\cap]0, 1/2[, \epsilon = 1/2 - \alpha, \delta = \min(\epsilon, \alpha), m, K, L, n \in \mathbb{N}^*, r \in \mathbb{Q}, (x,y) \in D_m^2(V)$ we have

$$\delta_r(\psi^\alpha_{n,m,P}(x,y)) = e^{\alpha r}\psi_{r,n,m,P}(x,y), \qquad \mathscr{E}^{\psi,n,m}_{\alpha,P}(x,y) = \int_{\mathbb{R}} \psi^\alpha_{n,m,P}(x,y).$$

(47) For $N = (N_1, ..., N_n), N_i, M, k, m \in \mathbb{N}^*, \lambda = (\lambda_1, ..., \lambda_n), \lambda_i \in \mathbb{C}$

$$\sup_{x\in D_m(U)} \max(0, |\mathscr{E}_{0,M,\infty,N_1}(x,1,1) - \sum_{i=1}^n \lambda_i \mathscr{E}_{0,M,\infty,N_i}(x, P(u), P(u))| - m\epsilon_N(\lambda)) = 0,$$

$$\sup_{x\in D_m(U)} \max(0, |\mathscr{E}_{0,N_1,\infty,M}(1,1,x) - \sum_{i=1}^n \lambda_i \mathscr{E}_{0,N_i,\infty,M}(P(u), P(u), x)| - m\epsilon_N(\lambda)) = 0,$$

(48) For $M = (M_1, ..., M_n), M_i, m, p, n, L, N \in \mathbb{N}, \lambda = (\lambda_1, ..., \lambda_n), 2Q \in \mathbb{Q}\cap]0, |\log(\lambda_0)|[$, we have:

$$\sup_{\substack{y\in D_p(V)\\ x\in D_m(U)}} \max(0,$$
$$|\sum_{i=1}^n \lambda_i \mathscr{E}_{0,M_i,\infty,L}(P(u^*), (2\Psi_{2Q} - \Psi_Q)(P(y)), x) - \mathscr{E}_{0,\infty,L}(uEP(y), x)| - mp\epsilon_M(\lambda))= 0,$$

$$\sup_{\substack{y\in D_p(V)\\ x\in D_m(U)}} \max(0, |\mathscr{E}_{0,L,\infty}(x, \theta EP(y)) - \sum_{i=1}^n \lambda_i \mathscr{E}_{0,L,\infty,M_i}(x, uEP(y), P(u))| - mp\epsilon_M(\lambda))= 0,$$

$$\sup_{\substack{y\in D_p(V)\\ x\in D_m(U)}} \max(0,$$
$$|\sum_{i=1}^n \lambda_i \mathscr{E}_{0,M_i,\infty,L}(P(u), (2\Psi_{2Q} - \Psi_Q)(P(y)), x) - \mathscr{E}_{0,\infty,L}(u^*EP(y), x)| - mp\epsilon_L(\lambda))= 0$$

$$\sup_{\substack{y\in D_p(V)\\ x\in D_m(U)}} \max(0, |\mathscr{E}_{0,L,\infty}(x, \overline{\theta} EP(y)) - \sum_{i=1}^n \lambda_i \mathscr{E}_{0,L,\infty,M_i}(x, u^*EP(y), P(u))| - mp\epsilon_M(\lambda))= 0.$$

(49) $2\Psi_{2M}(\Pi_{N,k}(y)) - \Psi_M(\Pi_{N,k}(y)) = E_{N,k}(y)$ for $M < |\log(\lambda_0)|/2$

$$\sup_{x\in D_m(U), y\in D_l(V)} \max(0,$$
$$|\sum_{i=1}^{n} \lambda_i \mathscr{E}_{0,M_i,\infty,L}((P(u^k))^*, \Psi_N(P(y)), x) - \mathscr{E}_{0,M_i,\infty,L}(1, \Pi_{N,k}(y), x)|^2$$
$$- \sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} \varphi(m_{M_j,M_i}(P(u^k), P(u^k)^*) m^2 l^2 + \sum_{j=1}^{n} \lambda_j \mathscr{E}_{0,M_j,\infty,L}(P(u^k)^*, P((u^k)^*), 1) m^2 l^2$$
$$+ \sum_{j=1}^{n} \overline{\lambda_j} \mathscr{E}_{0,L,\infty,M_j}(1, P(u^k), P((u^k))) m^2 l^2 - m^2 l^2) = 0$$

$$\sup_{x\in D_m(U), y\in D_l(V)} \max(0, |\sum_{i=1}^{n} \lambda_i \mathscr{E}_{0,M_i,\infty,L}((u^k), E_{N,k}(y), x) - \mathscr{E}_{0,M_i,\infty,L}(1, EU_{N,k}(y), x)|^2$$
$$- \sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} \varphi(m_{M_j,M_i}(P(u^k)^*, P(u^k)) m^2 l^2$$
$$+ \sum_{j=1}^{n} \lambda_i \mathscr{E}_{0,M_j,\infty,L}(P(u^k), P((u^k)), 1) m^2 l^2$$
$$+ \sum_{j=1}^{n} \overline{\lambda_j} \mathscr{E}_{0,L,\infty,M_j}(1, P(u^k)^*, P((u^k)^*)) m^2 l^2 - m^2 l^2) = 0$$

For $K = \lfloor \frac{N}{|\log(\lambda_0)|} \rfloor + 1$

$$\Psi_N(P(x)) = E_{N,0}(P(x)) + \sum_{k=1}^{K} EU_{N,k}(x) + [EU_{N,k}(x^*)]^*$$

(50) For $2Q \in \mathbb{Q} \cap]0, |\log(\lambda_0)|[$ and writing $P_{\leq n}(x) = \sum_{i,j=0}^{n} W_{i,i} x W_{j,j}$, we have:

$$\sup_{x\in D_1(V)} \max(0,$$
$$\sum_{j=0}^{n} 2^j \varphi(P_{j,j}(\theta E P(P_{\leq m}(xx^*)))) - \lambda_0 \sum_{j=0}^{m} 2^j \varphi(P_{j,j}[2\Psi_{2Q}(P(xx^*)) - \Psi_Q(P(xx^*))]) = 0$$

(51) For any $K, N \in \mathbb{N}$,

$$\varphi(E_{N,0}(P(x)) + \sum_{k=1}^{K} EU_{N,k}(x) + [EU_{N,k}(x^*)]^*) = \varphi(E_{N,0}(P(x))).$$

Stated in words, (42) expresses the various spectral properties of constants, spectral gap and traciality of φ on the compression of the centralizer of ψ by $w_{0,0}$. (43) defines u_t and expresses preservation of the centralizer by the actions. (44) , (46) define various spectral theory for Δ_ψ. (47) expresses that u is unitary, based on the inequalities obtained in (45) using (5.1). (49) expresses the cross-product decomposition based on maps defined in (48). (50) states the relation $\psi(u.u^*) \leq \lambda_0 \psi(.)$. (51) identifies φ on the cross product as the dual state of a geometric state. This geometric state defined on the centralizer of ψ was already identified in (24) and (42).

For a geometric state φ on $N = N_0 \otimes B(H)$ for N_0 finite, we call associated trace of φ the trace: $\tau = \varphi|_{N_0} \otimes Tr$.

Theorem 5.1. *We will fix $\lambda_0 \in]0,1[$ and $\Lambda = (\Lambda_j)_{j\geq 0}$, increasing in j and such that $\lim_{j\to\infty}\Lambda_j = 1$.*

The theory $T_{III_0}(\lambda_0,\Lambda)$ consisting of T_{AHW^*}, (24) and (42)-(51) axiomatizes a class of nonuples $(C,X,H,J,\Pi_P,\phi,M,u,W)$ exactly described as follows. C is a weak- dense C^*-algebra of X^*, a von Neumann algebra in standard form $(X^*,H,J,Im(\Pi_P))$. X is the predual of X^* containing a state $\phi \in X$. $M \simeq eX^*e$ is a von Neumann algebra for e the support projection of ϕ, having a crossed-product decomposition $M = N \rtimes_\theta \mathbb{Z}$ for a von Neumann algebra N of type II_∞. ϕ induces a faithful normal state φ on M. ϕ is the dual weight of a geometric state on N for the matrix unit $w = (w_{i,j} = P(W_{i,j})) \in N, W_{i,j} \in C$ satisfying (5.1). θ is an automorphism of N implemented by the image $U := eue \in M$ of $u \in C$ decreasing the trace τ associated to $\phi|_N$ by a factor $\lambda_0 < 1$ fixed (i.e. $\tau(\theta(x)) \leq \lambda_0\tau(x)$) and such that the image eue belongs to $M(\sigma^{\hat\tau},]-\infty,\log(\lambda_0)])$ in the modular theory of the dual weight $\psi = \hat\tau$.*

By Connes' result [7, Th 5.3.1], every type III_0-factor is of this type with θ ergodic on the center of N which is diffuse and conversely by his [7, Prop 5.1.1]. We leave to the reader the identification of the category of axiomatization from structure preserving morphisms.

PROOF : Let us start with such a nonuple and check this gives a model of $T_{III_0}(\lambda_0,\Lambda)$ with the interpretation suggested in the language description with $\psi = \hat\tau$. We thus define all the data in this way.

For further use, we notice the following inequalities:

$$||\sigma_t^\psi(x) - x||_\varphi^* \leq ||\sigma_t^\phi(x) - x||_\varphi^* + 2||x||||u_t - 1||_\varphi^\# \leq c|t|\ ||x||, \tag{5.4}$$

with $c = 6\ln(2) + 2 \leq 8$ since for $C = 6\ln(2)^2$:

$$(||u_t - 1||_\varphi^\#)^2 = \sum_{k=0}^{\infty} |2^{(k+1)it} - 1|^2 2^{-k-1} \leq \sum_{k=0}^{\infty} \ln(2)^2(k+1)^2|t|^2 2^{-k-1} = C|t|^2. \tag{5.5}$$

Similarly, since $\sigma_t^\phi = \sigma_t^\psi(u_t^*.u_t)$, $\Gamma_r(w_{m,m}xw_{n,n}) = w_{m,m}\Gamma_r(x)w_{n,n}$ and $w_{n,n}u_t = w_{n,n}2^{(n+1)it}$,

$$\begin{aligned}
&|\langle[\sigma_t^\phi(\Gamma_r(w_{m,m}xw_{n,n})) - \Gamma_r(w_{m,m}xw_{n,n})].\xi_\psi, yw_{n,n}.\xi_\psi\rangle| \\
&\leq |\langle[\sigma_t^\psi(\Gamma_r(w_{m,m}xw_{n,n})) - \Gamma_r(w_{m,m}xw_{n,n})].\xi_\psi, yw_{n,n}.\xi_\psi\rangle| \\
&+ |\langle[u_t^*\Gamma_r(w_{m,m}xw_{n,n})u_t - \Gamma_r(w_{m,m}xw_{n,n})].\xi_\psi, \sigma_{-t}^\psi(yw_{n,n}.\xi_\psi)\rangle| \\
&\leq |2^{(m+1)it} - 1|||x||\ ||y|| + |2^{(n+1)it} - 1|||x||\ ||y|| + \|yw_{n,n}.\xi_\psi\| \\
&\times \sqrt{e^r\langle\frac{|\Delta_\psi^{it} - 1|^2}{\Delta_\psi + \Delta_\psi^{-1}}(\Delta_\psi^2 + 1)(e^r + \Delta_\psi)^{-2}(w_{m,m}xw_{n,n})\xi_\psi, w_{m,m}xw_{n,n}\xi_\psi\rangle} \\
&\leq \|yw_{n,n}\xi_\psi\|\sqrt{(e^r + e^{-r})(2t)^2e^{-2}\|xw_{n,n}\xi_\psi\|^2} + (n+m+2)t\ln(2)\|x\|\|y\| \\
&\leq 2(n+m+2+\cosh(r))t\|x\|\|y\|
\end{aligned}$$

The second inequality in (46) is a straightforward consequence, once one cuts the integral defining F_N^φ from the modular group at $t = \pm 1/\sqrt{N}$.

We already have a model of T_{AHW^*} and (24) that expresses we have a matrix unit. The first part of (42) means that $M = M(\sigma^\psi,]-\infty,\log(\lambda_0)] \cup \{0\} \cup [-\log(\lambda_0),\infty[)$. This is indeed the case since, by definition, N is in the centralizer of ψ and by assumption $U := eue \in M(\sigma^{\hat\tau},]-\infty,\log(\lambda_0)])$ so that $NU^k \in M(\sigma^{\hat\tau},]-\infty,k\log(\lambda_0)])$ thus we see that any finite sum of terms in N, NU^k and $(U^*)^kN$ is in $M(\sigma^\psi,]-\infty,\log(\lambda_0)] \cup \{0\} \cup [-\log(\lambda_0),\infty[)$ and since those finite sums are dense, one obtains the equality with M (we also see from the crossed-product decomposition that N is exactly the centralizer for ψ). The modified (25) in (42) means that (25) is true for the centralizer $N = M_\psi = N_0 \otimes B(H)$ (in particular, N_0 is the centralizer for φ). The end of (42) means that the matrix unit is in this centralizer. All those relations are satisfied since N is of type II_∞.

The second part of (43) then means that u_t is in the centralizer for both φ, ψ. The first part identifies its components on the matrix unit. The first part uses the notation of (25) and defines $P_{k,j}$ as expected. Thus

the first equation for u_t means that it is diagonal in $B(H)$ with the expected expression. In order to see that this is indeed the expected cocycle between the trace and the geometric state, we refer to [30, lemma VIII.2.10]. The two last equations in (43) express the invariance of the centralizer M_ψ by $u.u^*$ and $u^*.u$. Then (44) means that Σ_t, the modular group for ψ, is indeed related as it should to σ_t using the cocycle, and $\Psi_{N,l}$ is indeed the corresponding modular map.

Note that conversely, (24) implies that $w_{i,j}$ is a matrix unit so that $M = M_0 \otimes B(H)$ and then (43) implies u_t is diagonal in $B(H)$. We can thus compute from φ a (not necessarily unique) weight ψ [30, Th VIII.3.8] with corresponding cocycle derivative and (43) implies we have the corresponding modular theory and Fejer map. (42) then describes the centralizer as some $N_0 \otimes B(H)$ and ψ as lacunary, and the spectral condition on $P(u)$.

Let us come back to checking that our algebra gives a model. (45) is satisfied since $P(u) = eue$ is a unitary, based on (5.1), (46) defines Γ_s, $\psi_{r,n,m,P}, \mathscr{E}_{\beta,P}^{\psi,n,m}$ as explained at the beginning of the proof or in a way similar to section 2 (for instance we use once more a variant of (19)).

Note that for Q small enough, since the weight ψ is lacunary, we have:

$$E_{M_\psi} = 2\Psi_{2Q} - \Psi_Q. \tag{5.6}$$

(47)-(48) are easy to check from the definitions of the forms $\mathscr{E}_{0,M,\infty,N}$ and ϵ_N (in the form (5.3)), combined, for (47), with the fact that $P(u)$ is unitary, and, for (48), with the definitions of $uEP, \theta EP, u^*EP, \overline{\theta}EP$ and the starting point (5.6). For instance, by Cauchy-Schwarz and $P(u)P(u)^* = 1$:

$$\begin{aligned}
|\varphi(F_N^\varphi(x)^* 1 F_M^\varphi(1)^*) - \sum_{i=1}^n \lambda_i \varphi(F_N^\varphi(x)^* P(u) F_{N_i}^\varphi(P(u))^*)| \\
&= |\varphi(F_N^\varphi(x)^* P(u)(P(u^*) - \sum_{i=1}^n \lambda_i F_{N_i}^\varphi(P(u))^*))| \\
&\leq \sqrt{\varphi(F_N^\varphi(x)^* P(u) P(u^*) F_N^\varphi(x))}||P(u) - \sum_{i=1}^n \lambda_i F_{N_i}^\varphi(P(u))||_\varphi^\#.
\end{aligned}$$

The three first formulas in (49) are straightforward from (5.6), the definitions of $\Pi_{N,k}, E_{N,k}, EU_{N,k}$ and Cauchy-Schwarz inequality. The last formula is the crucial one and follows from the proof of [1, Proposition 6.23]. Let us explain it for the reader's convenience. First since

$$p_n := \bigvee_{k=0}^\infty (u^k(w_{0,0} + ... + w_{n,n})u^{-k}) \in M_\psi$$

we can use σ^ψ and thus Ψ_N are M_ψ bimodular: $\Psi_N(p_n P(x) p_n) = p_n \Psi_N(P(x)) p_n$. Now if $U = P(u)$, $Up_nUp_n = Up_n\theta(p_n)U = U\theta(p_n)U = U^2 p_n$ since by definition $\theta(p_n) \leq p_n$. Similarly we have $[Up_n]^k = U^k p_n \in M(\sigma^{\hat{\tau}},]-\infty, k\log(\lambda_0)])$ and $[p_n U^*]^k = p_n (U^*)^k \in M(\sigma^{\hat{\tau}}, [-k\log(\lambda_0), \infty[)$. Finally, one gets by (2.2) again

$$U^k \Psi_N(p_n P(x) p_n) \in M(\sigma^{\hat{\tau}},]-\infty, k\log(\lambda_0) + N]).$$

Thus for $k \geq K$, $k\log(\lambda_0) + N < 0$ so that

$$E_{N,k}(p_n x p_n) = E_\psi(U^k \Psi_N(P(p_n x p_n))) = 0$$

and similarly $E_{N,k}(p_n x^* p_n) = 0$. The crossed product expansion for $M_\psi \rtimes_\theta \mathbb{Z}$ thus gives

$$\Psi_N(p_n x p_n) = E_{N,0}(p_n x p_n) + \sum_{k=1}^\infty U E_{N,k}(p_n x p_n) + [U E_{N,k}(p_n x^* p_n)]^*.$$

This is actually the expected finite sum stopping at index K. Since σ_t^ψ is strongly continuous, so are the maps in the sum and $p_n \to 1$ so that taking the limit, we have thus finished checking (49).

Let us write $p_{n,0} = w_{0,0} + ... + w_{n,n}$. (50) is equivalent to

$$\tau(p_{n,0}\theta(p_{m,0}E_{M_\psi}(P(xx^*))p_{m,0})) \leq \lambda_0\tau(p_{m,0}E_{M_\psi}(P(xx^*)))$$

which comes from $p_{n,0} \leq 1$ and $\tau\circ\theta \leq \lambda_0\tau$. (51) is then standard for a dual weight (see e.g. after polarization [30, Th X.1.17 (i)]).

Conversely, assume given a model. We have already noticed that we have a lacunary weight ψ with II_∞ centralizer. We have to check that M is indeed a cross-product. We use [7, Th 5.3.1] in the variant of [1, Lemma 6.25]. We know that $E_{M_\psi} = 2\Psi_{2N} - \Psi_N$ is a conditional expectation. (45) implies that $\varphi(P(u)P(u)^*) = 1 = \varphi(P(u)^*P(u))$. (Note that the first uses the limits $\beta \to 0$, $n \to \infty$ and finally $m \to \infty$.) This uses that the spectral theory maps have the right interpretation from (44), (46).

As explained before, equation (45) guaranties

$$\epsilon_N(\lambda) = ||P(u) - \sum_{i=1}^{n} \lambda_i F_{N_i}^\varphi(P(u))||_\varphi^\#$$

and then from Hahn-Banach and the proof of theorem 2.7, there is a net u_n of convex combinations of the form $\sum_{i=1}^{m_n} \lambda_i F_{N_i}^\varphi(P(u))$ such that $||P(u) - u_n||_\varphi^\# \to 0$. This is the starting point to use (44) which then gives by taking a limit

$$\varphi(F_M(x^*)(1 - P(u)P(u^*))) = 0, \qquad \varphi((1 - P(u^*)P(u))F_M(x^*)) = 0.$$

Since x, M are arbitrary, using density, we get $1 = P(u)P(u^*) = P(u^*)P(u)$ and thus $P(u)$ is unitary as expected.

The last relations in (43) also implies that $P(u).P(u)^*, P(u)^*.P(u)$ leave stable M_ψ by using the weak-* density of the image of $E_{M_\psi}P$. This gives the automorphism $\theta(x) = P(u)xP(u)^*$ of M_ψ. From the equivalent version of (50) above, and letting $n \to \infty, m \to \infty$, one gets $\tau(\theta(E_{M_\psi}(P(xx^*)) \leq \lambda_0\tau(E_{M_\psi}(P(xx^*)))$ and we can replace by density[5] $E_{M_\psi}(P(xx^*))$ by any positive y in M_ψ. Then from [7, Prop 5.1.1], one deduces $p(\theta^k) = 0$ for $k \neq 0$ and of course $U = P(u)$ so that $\theta(x) = UxU^*$. The last equation of (49) implies $\Psi_N(P(x))$ is in the algebra generated by N, U so that by density of image of P, one deduces that $\Psi_N(M)$ is in the von Neumann algebra generated by N, U. Note that using $||\sigma_t^\psi(x) - x||_\varphi^* \leq c|t|\ ||x||$, from (5.4), one gets $||\Psi_N(x) - x||_\varphi^* \to_{N\to\infty} 0$ for any $x \in M$. Taking $N \to \infty$ on elements of $\Psi_N(M)$, one concludes that M is generated by N, U. Hence, it only remains to check that $M'_\psi \cap M \subset M_\psi$. But, for $x \in M'_\psi \cap M$, we have $\Psi_N(x) \in M'_\psi \cap M$ (since Ψ_N is M_ψ-bimodular). Moreover, it has a finite decomposition $\Psi_N(x) = x_0 + \sum_{k=1}^K x(k)U^k + x(-k)(U^*)^k$ from the uniqueness of the decomposition (coming from the explicit formulas for $x(k)$), one deduces that for any $a \in M_\psi$, $x(k)\theta^k(a) = ax(k), \theta^k(a)x(-k) = x(-k)a$ for $k > 0$. By [7, Rmq 1.5.3 (a)], for $k \neq 0$, one deduces from $p(\theta^k) = 0$ that $x(k) = 0$. Thus $\Psi_N(x) \in M_\psi$ and taking $N \to \infty$, one deduces $x \in M_\psi$ as expected. This satisfies the assumptions in [1, Lemma 6.25] and thus M is indeed a cross product as expected.

It remains to check the data is uniquely determined as it should. (46) is similar to previous sections.

Similarly, the 4 equations in (48) give the definitions of $\theta EP, \overline{\theta}EP$, uEP, u^*EP. Reasoning as before, the three first relations in (49) define $\Pi_{N,k}, E_{N,k}, EU_{N,k}$.

Equation (51) finally characterizes the state φ as determined from its restriction on M_ψ as it should be for a dual weight and all the data is determined as expected. At this stage we can also choose $\psi = \hat{\tau}$ the dual weight of the determined trace and it has the expected modular theory. ∎

We can thus finish with the analogue for uncountable ultraproducts of [1, Th 6.16].

Corollary 5.2. *Let M be a σ-finite factor of type III_0 with faithful normal state φ and assume ω is a countably incomplete ultrafilter, then $(M, \varphi)^\omega$ is not a factor.*

[5] using Kaplansky's density theorem e.g. [30, Th I.4.24]

PROOF : Using the crossed product decomposition in [7], one gets $M \simeq (N \otimes B(H)) \rtimes \mathbb{Z}$ with N finite and there is λ_0, Λ so that $(M, M_*, \varphi, M, w, u)$ satisfies $T_{III}(\lambda_0, \Lambda)$ and thus so does $(M^\omega, (M_*)^\omega, \varphi^\omega, (M_n, \varphi)^\omega, w, u)$.

Thus $(M, \varphi)^\omega \simeq (N^\omega \otimes B(H)) \rtimes \mathbb{Z}$ and the center $(\mathcal{Z}(N))^\omega \simeq \mathcal{Z}(N^\omega)$ by [12, Corol 4.3] and it suffices to find a non trivial element in the fixed point algebra of the action of $\mathbb{Z}$, since such an element will be a non trivial element in the center of $(M, \varphi)^\omega$. This follows from [1, lemmas 6.19, 6.22]. Indeed, since ω is countably incomplete on I, there is a sequence $J_n \in \omega$, with $\cap_{n \in \mathbb{N}} J_n = \emptyset, J_0 = I$ and we can assume J_n decreasing. Define the net k_i by $k_i = n$ if $i \in J_{n-1} - J_n$ so that $\{i \in I, k_i \geq n\} = J_{n-1} \in \omega$ and thus $\lim_{n \to \omega} \frac{1}{k_n} = 0$. Take $p = (1_{B_{k_n}})^\omega$ with B_n built in their lemma 6.22. This gives as in the proof of their lemma 6.19, a central element in N with $\varphi^\omega(p) = 1/2$ since $\varphi(B_n) = 1/2$. Moreover $||upu^* - p|| \to 0$ since $\lim_{n \to \omega} \mu(TB_{k_n} \Delta B_{k_n}) \leq \lim_{n \to \omega} 2/k_n = 0$ for $\mathcal{Z}(N) \simeq L^\infty(\mu)$ with $u.u^*$ acting via T on the measure space. ■

6. Proof of theorem 1.1

We apply [13, Th 5.6]. We can do this since our Ocneanu theory (theorem 2.7) and Groh theory (theorem 3.3) are axiomatized in a separable language (in subsection 2.4 we even wrote an explicit countable language). A separable von Neumann algebra is exactly a separable structure[6] in those theories. Combining (1) and (2) in their theorem, whether the theory is stable or not, if we assume (CH) the model theoretic ultrapowers are isomorphic and thus, by identification with Ocneanu and preduals of Groh ultraproducts in our quoted theorems, we deduce the first point of our theorem.

Assume now that the continuum hypothesis fails. Since we assumed M is a factor not of type III_0, we treat each remaining type of factors for M separately. Types I_n, II_1 are known from [12, Th 4.7]. Type I_∞ is a consequence of the canonical isomorphism $(B(H))^\omega = B(H)$ (see [1, section 6.1] or the proof of our corollary 2.13).

Let M of type II_∞. Consider on M a geometric state φ whose theory is described in theorem 2.10 in the language with a matrix unit added. This state is lacunary thus there exists $N \in \mathbb{Q}$ such that the De la Vallée Poussin map (obtained from Fejer's map F_N^φ) $2F_{2N}^\varphi - F_N^\varphi$ is the conditional expectation on the centralizer as explained in the use of axiom (22) and is in the theory with language of σ-finite W^*-probability spaces (without matrix unit added). Consider the formula of this theory

$$f(x_1, y_1, x_2, y_2) = \varphi(M_{(3N,3N)}([\tilde{m}_{(N,N)}(x_1, y_2) - \tilde{m}_{(N,N)}(y_2, x_1)], \\ [\tilde{m}_{(N,N)}(x_1, y_2) - \tilde{m}_{(N,N)}(y_2, x_1)]^*)),$$

with the notations of subsection 2.2 and recall the definition (2.12) in subsection 2.4 Recall also that $m_{(N,M)}(x, y) = F_N^\varphi(x).F_M^\varphi(y)$ is one of the smeared product we have in our theory. In the lacunary case it means

$$f(x_1, y_1, x_2, y_2) = |||[E_{M_\varphi}(x_1), E_{M_\varphi}(y_2)]|||_2^2$$

computed with the $||.||_2$ norm of the centralizer M_φ. This formula is thus the formula witnessing the order property in [12] for the centralizer. We thus obtain that the theory of M with this geometric state in the language of σ-finite algebras is unstable. By [13, Th 5.5], it suffices to check that it has the order property as witnessed by the formula above and this is the case since the centralizer (which is of the form $N \otimes Z$, N II_1 factor and Z a commutative algebra) contains unitally $M_{2^n}(\mathbb{C})$ so that one can use [12, lemma 3.2]. Thus, by [13, Th 5.6] (we use the implication mostly coming from [15]), there exists $2^\mathfrak{c}$ ultrafilters with the model theoretic ultraproducts not isomorphic. Assume that, for two such ultrafilters $\mathcal{U}, \mathcal{V}$ we have $(M, \varphi)^\mathcal{U} \simeq (M, \varphi)^\mathcal{V}$ as von Neumann algebras (recall we can compute Ocneanu ultrapower with any state and this gives the same result). We want to count how many ultrafilters of that type there can be. Since the models are non-isomorphic if and only if the states are non-isomorphic, we want to count how many

[6] separable unit ball for $||.||_\varphi^*$, which is the same as separable for the strong topology or the weak-* topology by a standard application of Banach-Saks theorem, or separable predual

non-isomorphic geometric states there can be on the same ultraproduct von Neumann algebra of a geometric state. Since we took φ to be a geometric state described in the language with a matrix unit added, one obtains $\varphi^{\mathcal{U}}, \varphi^{\mathcal{V}}$ are geometric states with the same matrix unit coming from the one in M. By a standard result, if $e = w_{00}$ is the first projection in the matrix unit, we have $(M,\varphi)^{\mathcal{U}} \simeq e(M,\varphi)^{\mathcal{U}} e \otimes B(H)$. Note that if the isomorphism class of $e(M,\varphi)^{\mathcal{U}}e$ is determined, there is only one isomorphism class of geometric state on $(M,\varphi)^{\mathcal{U}}$ and thus the model as σ-finite W^*-probability space is determined (all the remaining part of the theory, (smeared) product, modular theory is determined by the state), and thus there is at most one $\mathcal{U}$ within the family fixed before. Fix a trace Tr_N on $N \simeq (M,\varphi)^{\mathcal{U}} \simeq (M,\varphi)^{\mathcal{V}}$. It is well-known that the equivalence classes of finite projections e is characterized by $Tr(e) \in [0,\infty)$ and thus if $e_{\mathcal{U}}, e_{\mathcal{V}}$ are the images in the common algebra N we thus have $e_{\mathcal{U}} N e_{\mathcal{U}} \simeq e_{\mathcal{V}} N e_{\mathcal{V}}$ if $Tr(e_{\mathcal{U}}) = Tr(e_{\mathcal{V}})$.

As a consequence the isomorphism invariance classes as von Neumann algebras among the family of $2^{\mathfrak{c}}$ non-isomorphic models have at most $\mathfrak{c}$ members, so that there are again $2^{\mathfrak{c}}$ non-isomorphic ultrapowers as von Neumann algebras.

Now assume M is of type III_1, let us show that $Th(M) = Th(M^{\omega})$ has the order property as witnessed by the formula

$$f(x_1,y_1,x_2,y_2) =$$
$$\varphi(M_{(2,2)}([M_{(0,0)}(x_1,y_2) - M_{(0,0)}(y_2,x_1)],[M_{(0,0)}(x_1,y_2) - M_{(0,0)}(y_2,x_1)]^*)).$$

For, one uses from [1, Th 4.20,Prop 4.24] according to which the centralizer $(M^{\omega})_{\varphi^{\omega}}$ is a type II_1 factor. If x_i, y_i are in the centralizer, the formula is interpreted by:

$$f(x_1,y_1,x_2,y_2) = ||x_1y_2 - y_2x_1||_2^2.$$

This is again the formula that witnesses the order property in the type II_1 case by [12, lemma 3.2] since such an algebra contains unitally $M_{2^n}(\mathbb{C})$. Thus one can take the sequence for the order property for M in the centralizer of M^{ω}. Again, from [13, Th 5.5,5.6], one gets two ultrafilters with $(M,\varphi)^{\mathcal{U}} \not\simeq (M,\varphi)^{\mathcal{V}}$ as models in the language of σ-finite von Neumann algebras. But if we had an isomorphism as III_1 factors, since those ultraproducts are strictly homogeneous by [1, Th 4.20] again, the two states $\varphi^{\mathcal{U}}, \varphi^{\mathcal{V}}$ would be unitarily conjugated, establishing the isomorphism as models in the language of σ-finite W^* probability spaces, a contradiction.

Consider finally the case where M is of type III_{λ}, $0 < \lambda < 1$. One uses [7, Th 4.3.2] and its proof. One can fix φ a periodic state (a faithful normal state of period $T_0 = 2\pi/\log(\lambda)$ from the computation of the invariant T and its alternative definition in his remark 1.3.3 in the σ-finite case). Then $M \simeq M \otimes B(H)$ and $\varphi \otimes Tr$ is Connes' construction of a generalized trace on M. Since φ is lacunary and M_{φ} is a II_1 factor (by [7, Th 4.2.6]) it is a finite factor and its tensor product with $B(H)$ is a II_{∞} factor from his corollary 4.3.3). We will use that the same formula as for the II_{∞} case gives the order property for the theory of (M,φ) as a σ-finite W^* probability space. But we won't only use [13, Th 5.6] to get $2^{\mathfrak{c}}$ non-isomorphic models $(M,\varphi)^{\mathcal{U}}$, we will rather deduce a huge non-isomorphic class from the II_{∞} case.

Consider first two ultrafilters with $(M,\varphi)^{\mathcal{U}} \simeq (M,\varphi)^{\mathcal{V}}$ as von Neumann algebras. We know from the axiomatization result theorem 2.10 that they are III_{λ} factors with a periodic state. Thus as above, on $(M,\varphi)^{\mathcal{U}} \otimes B(H)$, $\varphi^{\mathcal{U}} \otimes Tr$ is a generalized trace, thus from [7, Th 4.3.2], it would be proportional to a unitary conjugate of $\varphi^{\mathcal{V}} \otimes Tr$. In particular, both would have unitarily conjugated centralizers $((M,\varphi)^{\mathcal{U}})_{\varphi^{\mathcal{U}}} \otimes B(H) \simeq ((M,\varphi)^{\mathcal{V}})_{\varphi^{\mathcal{V}}} \otimes B(H)$. By the lacunarity again, the ultraproduct of centralizers is nothing but the centralizer of the ultraproduct [1, section 4.3] and thus

$$(M_{\varphi} \otimes B(H))^{\mathcal{U}} \simeq (M_{\varphi})^{\mathcal{U}} \otimes B(H) \simeq (M_{\varphi})^{\mathcal{V}} \otimes B(H) \simeq (M_{\varphi} \otimes B(H))^{\mathcal{V}}.$$

From the II_{∞} factor case above, we deduce we have $2^{\mathfrak{c}}$ non-isomorphic von Neumann algebras $(M_{\varphi} \otimes B(H))^{\mathcal{U}}$ and thus $2^{\mathfrak{c}}$ non-isomorphic von Neumann algebras $(M,\varphi)^{\mathcal{U}}$.

This completes the proof of Theorem 1.1.

We conclude by a consequence in the spirit of [1, Th 6.11]

Corollary 6.1. *Let M be a III_λ factor with separable predual and $0 < \lambda \leq 1$ and assume the continuum hypothesis fails. Then there are two ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ such that the Groh ultrapowers are not isomorphic $\prod^{\mathcal{U}} M \not\simeq \prod^{\mathcal{V}} M$ as von Neumann algebras.*

PROOF : It suffices to take the same ultrafilters as in theorem 1.1. If the isomorphism $\prod^{\mathcal{U}} M \simeq \prod^{\mathcal{V}} M$ were true, since any σ-finite projection of a type III-factor are equivalent (see e.g. [30, Prop V.1.39]), one would deduce from [1, Corol 3.28] that for some σ-finite (support) projections q, p:

$$(M, \varphi)^{\mathcal{U}} \simeq q(\prod^{\mathcal{U}} M)q \simeq p(\prod^{\mathcal{V}} M)p \simeq (M, \varphi)^{\mathcal{V}}.$$

This would contradict theorem 1.1. ∎

7. Appendix: Hölder paths sorts $Lip^{\alpha}_{w}(I, U)$

In order to axiomatize the modular group, we used various equations from spectral theory involving Riemann integrals of uniformly continuous maps on intervals. To improve the readability of those axioms, it was convenient to add sorts involving spaces of Hölder continuous maps on products of intervals.

This appendix explains how it is always possible to add a sort for certain Lipschitz curves since those curves have an explicit universal axiomatization. On this sort, we have integral maps and evaluations at points. This setting is not necessary, and a previous preprint version contained repetitive Riemann sum formulas instead of this appendix. We think that a more detailed axiomatization of Besov type spaces, beyond the elementary Hölder curves we consider here, could be important for applications of continuous model theory to mainstream analysis. This is the reason why we think this appendix is of independent interest.

Typically, following [13], we had a sort U, say for the von Neumann algebra, and domains of quantifications $D_n(U)$ given by balls of radius n for some norm, for instance for the operator norm unit balls. In particular, we always have vector space operations enabling convex combinations and appropriate dilations between those sorts. We recall we have a bounded distance d on each $D_n(U)$ making them complete. We can write $||.||$ for the norm giving the ball $D_n(U)$. This norm may be different from the one giving the distance d. Moreover, we also assume that d comes from a norm. In this case, the norm $||.||$ corresponding to balls is not part of the language and should not appear in formulas.

Fix $\alpha \in]0, 1]$ and $I = [a, b]$ or $I =]a, b[$, or $I =]a, b]$ or $I = [a, b[$ an interval (with $a = -\infty$, $b = +\infty$ allowed in the open side case). Let also w a continuous positive function on I converging to 0 at open endpoints among a, b for the interval I. It will serve as weight and it is also assumed integrable in case of $ILip$. We will add a sort $Lip^{\alpha}_{w}(I, U)$ (resp. $ILip^{\alpha}_{w}(I, U)$) for bounded and dominated by a multiple of w α-Hölder continuous maps from I to U. The space is thus the same for $ILip$, only the condition on the weight w differ and this will allow to define extra structures. It will have domains of quantification $D_n(Lip^{\alpha}_{w}(I, U))$ (resp. $D_n(ILip^{\alpha}_{w}(I, U))$), interpreted as α-Hölder continuous maps with value in the domain of quantification $D_n(U)$ and with

$$d(f(s), f(t)) \leq n|s - t|^{\alpha}$$

and also satisfying $\sup_{s \in I} \frac{d(0, f(s))}{w(s)} \leq n, \sup_{s \in I} \frac{||f(s)||}{w(s)} \leq n$. In other words, $D_n(Lip^{\alpha}_{w}(I, U))$ and $D_n(ILip^{\alpha}_{w}(I, U))$ are the balls of radius n and center 0 for the norm

$$\max\Big(\sup_{s \in I} ||f(s)||, \sup_{s \neq t} \frac{d(f(s), f(t))}{|s - t|^{\alpha}}, \sup_{s \in I} \frac{d(0, f(s))}{w(s)}, \sup_{s \in I} \frac{||f(s)||}{w(s)}\Big). \tag{7.1}$$

In this way we can apply iteratively the definition and obtain for instance $Lip^{\alpha}_{w}(I, Lip^{\alpha}_{w}(I, U))$.

In the main text, when $\alpha = 1$, we wrote for short $(I)Lip_w(I, U)$ instead of $(I)Lip^1_w(I, U)$.

We always define the following distance of uniform convergence for $f, g \in D_n((I)Lip^{\alpha}_{w}(I, U))$:

$$d(f, g) = \sup_{s \in I} d(f(s), g(s)).$$

We need the following

Lemma 7.1. *Interpreted as above, $D_N(Lip^\alpha_w(I,U)) = D_N(ILip^\alpha_w(I,U))$ is complete for d.*

PROOF : Taking a Cauchy sequence f_n it converges uniformly to f in $C^0(I, \overline{\cup_m D_m(U)}^d)$ with value in the completion for d. The Hölder property of the limit is obvious and so is the first weight condition $d(0, f(s)) \leq Nw(s)$. For any s, $f_n(s)$ is in $D_N(U)$ and d-Cauchy there, thus converges there so that its limit for d, $f(s)$ is in $D_N(U)$ by the assumed completeness of this ball. Similarly, $f(s)/w(s)$ is also in $D_N(U)$ so that the limit satisfies the second weight condition. ∎

We then call $\mathcal{L}(Lip^\alpha_w(I,U))$ (resp. $\mathcal{L}(ILip^\alpha_w(I,U))$) the following language that will enable an easy axiomatization for the new sorts (and an integral map).

- The maps $\delta_s : D_n((I)Lip^\alpha_w(I,U)) \to D_n(U), w\delta_s : D_n((I)Lip^\alpha_w(I,U)) \to D_n(U)$, $s \in \mathbb{Q} \cap I$ for $\delta_s(f) = f(s), w\delta_s(f) = w(s)f(s)$.
- For $t_0 < t_1 < ... < t_n < t_{n+1} \in Q \cap I$ The maps $\iota^m_{t_0,t_1,...,t_n,t_{n+1}} : D_{m_1}(U) \times ... \times D_{m_n}(U) \to D_m((I)Lip^\alpha_w(I,U))$ with $m_i = \min(m, \lfloor m \inf_{t\in]t_{i-1},t_{i+1}[} w(t) \rfloor)$. They correspond to some canonical parametrization giving a linear interpolation. It satisfies the following properties: It starts at 0 before t_0, with value the i-th argument at t_i and arriving at zero after t_{n+1}, before dividing by an appropriate constant not smaller than 1 to meet the weight and Hölder continuity constraint (see fourth axiom below for a formula).
- For $\mathcal{L}(ILip^\alpha_w(I,U))$ we add an integral map $\int_I : D_n(ILip^\alpha_w(I,U)) \to D_N(U)$ for $\int_I f = \int_I dsf(s)$ with $N = \lceil n \int_I dsw(s) \rceil$

The operators $\iota_{t_0,t_1,...,t_n,t_{n+1}}$ will produce a dense family of Lipschitz curves enabling to identify exactly the models of the theory we now introduce. Explicit uniform modulus of continuity must be associated to those maps. We leave to the reader the explicit computation of a modulus for ι, we only make explicit one for the integral based on the following inequality for $[c,d] \subset I$:

$$d(\int_I f, \int_I g) \leq 2 \int_{I-[c,d]} w(s) + (d-c)d(f,g)$$

so that for c_ϵ, d_ϵ such that $\int_{I-[c_\epsilon,d_\epsilon]} w(s) \leq \epsilon/4$, the modulus is $\eta(\epsilon) = \epsilon/2(d_\epsilon - c_\epsilon)$ enabling to get $d(\int_I f, \int_I g) \leq \epsilon$ for $d(f,g) \leq \eta(\epsilon)$.

We call $Th(Lip^\alpha_w(I,U))$ the following statements in the above language:

- For any $s,t \in I \cap Q$, $w\delta_s(f) = w(s)\delta_s(f)$ and $\sup_{f\in D_n(Lip^\alpha_w(I,U))} \max(0, d(\delta_s(f), \delta_t(f)) - n|s-t|^\alpha) = 0$, and
$$\sup_{f\in D_n(Lip^\alpha_w(I,U))} \max(0, d(\delta_s(f), 0) - nw(s)) = 0.$$
- $\max(0, d(\delta_s(f), \delta_s(g)) - d(f,g)) = 0$,
- For any $c < d \in I$, $n, m \in \mathbb{N}$
$$\sup_{(f,g)\in D_m(Lip^\alpha_w(I,U))^2} \max\Big(0, d(f,g) - \max_{i=0,...,n} d(\delta_{c+(d-c)i/n}(f), \delta_{c+(d-c)/n}(g)) - 2\frac{|c-d|^\alpha}{n^\alpha}m - 2m \sup_{s\in]a,c]\cup[d,b[} w(s)\Big) = 0,$$
- For $t_0 < t_1 < ... < t_n < t_{n+1} \in Q \cap I$, then $\delta_s(\iota^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n)) = 0$ for $s \leq t_0$, or $s \geq t_{n+1}$ and for $0 \leq i \leq n, t \in [t_i, t_{i+1}]$
$$\delta_t(\iota^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n)) = \frac{1}{\lambda^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n)}\Big(\frac{t_{i+1}-t}{t_{i+1}-t_i}x_i + \frac{t-t_i}{t_{i+1}-t_i}x_{i+1}\Big).$$
with the following abbreviation ($x_0 = x_{n+1} = 0$):
$$\lambda^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n) = \max(1, \max_{0\leq i<j\leq n+1} \frac{d(x_i,x_j)}{m|t_i-t_j|^\alpha}, \max_i \frac{d(x_i,0)}{m_i}).$$

Note that in the third point the *sup* of w is not a logical formula, merely a number depending on the parameters a, b, c, d, w. The anti-derivative W of w below is also merely considered a bunch of numbers $W(x)$.

For $b = +\infty$ let $b_n = n^\alpha$, and otherwise $b_n = \min(b - 1/n, n^\alpha)$, for $a = -\infty$, let $a_n = -n^\alpha$ and otherwise $a_n = \max(-n^\alpha, a + 1/n), c_n = \lfloor (b_n - a_n)/2 \rfloor \geq 0$ if $n \geq 2/(b-a)$. We call $Th(ILip^\alpha_w(I,U))$ the following statement added to $Th(Lip^\alpha_w(I,U))$:

- Let $W(x) = \int_a^x w(s)ds$. For any integer $n \geq 2/(b-a)$

$$\sup_{f \in D_m(ILip^\alpha_w(I,U))} \max\Big(0, d\Big(\frac{1}{n^2}\sum_{k=-n^2c_n}^{n^2c_n-1} \delta_{\frac{a_n+b_n}{2}+\frac{k}{n^2}}(f), \int_I f\Big) - \frac{2m}{n^\alpha} - m(W(a_n) + W(b) - W(b_n))\Big) = 0.$$

Note that both theories are universally axiomatizable in the above languages.

Lemma 7.2. *Given a model M for the type U. The theory $Th(Lip^\alpha_w(I,U))$ axiomatizes $||.||$-bounded d-α-Hölder continuous maps from I to M dominated by the weight w in both d and $||.||$. The theory $Th(ILip^\alpha_w(I,U))$, defined when the weight w is integrable, axiomatizes the subclass of maps which are also integrable with integral on I valued in some $D_n(M)$.*

Proof : First, we have to check that for M a model of U, $Lip^\alpha_w(I,M)$ is a model of both theories. The first two axioms follow from definitions.

If $s \in]c,d]$ and f, g are Hölder-continuous of constant at most m, then for any n, there is $i \leq n$ such that $s \in]c_i, c + (d-c)(i+1)/n], c_i = c + (d-c)i/n$ so that

$$d(\delta_s(f), \delta_s(g)) \leq d(\delta_{c_i}(f), \delta_{c_i}(g)) + 2m|s - c_i|^\alpha. \tag{7.2}$$

This explains the bound :

$$\sup_{s \in]c,d]} d(\delta_s(f), \delta_s(g)) \leq \max_{i=0,\ldots,n} d(\delta_{c+(d-c)i/n}(f), \delta_{c+(d-c)/n}(g)) + 2\frac{|c-d|^\alpha}{n^\alpha}m. \tag{7.3}$$

For $s \in I -]c,d]$, one can use that f, g are bounded by the weight mw to get

$$\sup_{s \in I -]c,d]} d(\delta_s(f), \delta_s(g)) \leq 2m \sup_{s \in]a,c] \cup [d,b[} w(s).$$

The right hand side is considered as a number we can compute from w. Combining (7.2) and (7.3) gives the third axiom.

Note right now that the second axiom implies $\sup_{s \in I} d(\delta_s(f), \delta_s(g)) \leq d(f,g)$ and that conversely, the third axiom implies

$$\begin{aligned} d(f,g) &\leq \sup_{s \in I} d(\delta_s(f), \delta_s(g)) + 2\frac{|c-d|^\alpha}{n^\alpha}m + 2m \sup_{s \in]a,c] \cup [d,b[} w(s)) \\ &\to_{n \to \infty} \sup_{s \in I} d(\delta_s(f), \delta_s(g)) + 2m \sup_{s \in]a,c] \cup [d,b[} w(s) \to_{d \to b, c \to a} \sup_{s \in I} d(\delta_s(f), \delta_s(g)). \end{aligned}$$

We used the assumed limit 0 of the weight if the endpoint is open, only case where $c = a$, $d = b$ cannot be taken in the inequality. In the closed side case, the sup over $]a,c], [d,b[$ can be taken empty and don't require a limit. This gives in any case the equality:

$$\sup_{s \in I} d(\delta_s(f), \delta_s(g)) = d(f,g). \tag{7.4}$$

The fourth point in $Th(Lip^\alpha_w(I,U))$ is only a definition of $\iota^m_{t_0,t_1,\ldots,t_n,t_{n+1}}(x_1,\ldots,x_n)$ and has only to be checked for consistency with the space of value of this map $\iota^m_{t_0,t_1,\ldots,t_n,t_{n+1}} : D_{m_1}(U) \times \ldots \times D_{m_n}(U) \to D_m(Lip^\alpha(I,U))$.

We need to estimate the Lipschitz constant of $f = \iota^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n), x_i \in D_{m_i}(U)$. We call $\lambda = \lambda^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n) \geq 1$.

First for $t,s \in [t_i,t_{i+1}]$, $f(t)-f(s) = \frac{(t-s)}{\lambda(t_{i+1}-t_i)}(x_i - x_{i+1})$ so that

$$d(f(t),f(s)) \leq \frac{|t-s|}{\lambda(t_{i+1}-t_i)} d(x_i,x_{i+1}) \leq \frac{|t-s|^\alpha}{\lambda(t_{i+1}-t_i)^\alpha} d(x_i,x_{i+1}) \leq m|t-s|^\alpha.$$

where the last inequality comes from $\lambda(t_{i+1}-t_i)^\alpha \geq d(x_i,x_{i+1})/m$. More generally for $i<j$, let $t \in [t_i,t_{i+1}], s \in [t_j,t_{j+1}]$ and write for $\mu_1,\mu_2 \in [0,1]$, $t = \mu_1 t_i + (1-\mu_1)t_{i+1}$, $s = \mu_2 t_j + (1-\mu_2)t_{j+1}$ so that

$$(s-t) = \mu_1\mu_2(t_j - t_i) + (1-\mu_1)(1-\mu_2)(t_{j+1}-t_{i+1}) + \mu_1(1-\mu_2)(t_{j+1}-t_i) + (1-\mu_1)\mu_2(t_j - t_{i+1}).$$

We decompose similarly since d comes from a norm and f piecewise linear and $d(f(t_i),f(t_j)) \leq m|t_i - t_j|^\alpha$ from the definition of λ:

$$\begin{aligned}
d(f(t),f(s)) &\leq \mu_1\mu_2 d(f(t_j),f(t_i)) \\
&\quad + (1-\mu_1)(1-\mu_2)d(f(t_{j+1}),f(t_{i+1})) + \mu_1(1-\mu_2)d(f(t_{j+1}),f(t_i)) + (1-\mu_1)\mu_2 d(f(t_j),f(t_{i+1})) \\
&\leq m\big(\mu_1\mu_2(t_j-t_i)^\alpha + (1-\mu_1)(1-\mu_2)(t_{j+1}-t_{i+1})^\alpha + \mu_1(1-\mu_2)(t_{j+1}-t_i)^\alpha + (1-\mu_1)\mu_2(t_j-t_{i+1})^\alpha\big) \\
&\leq m|t-s|^\alpha
\end{aligned}$$

where the last inequality comes from concavity of x^α and the formula above writing $(s-t)$ as convex combination. Therefore, f satisfies the right Hölder continuity condition. Let us check the weight conditions. For $t \in [t_i,t_{i+1}]$,

$$||f(t)|| \leq \max(||x_i||,||x_{i+1}||)/\lambda \leq \max(m \inf_{s\in]t_{i-1},t_{i+1}[} w(s), m \inf_{s\in]t_i,t_{i+2}[} w(s)) \leq m w(t)$$

Similarly, we have using $\lambda m_i \geq d(x_i,0)$:

$$d(0,f(t)) \leq \max(d(0,x_i),d(0,x_{i+1}))/\lambda \leq \max(m_i,m_{i+1}) \leq \max(m \inf_{s\in]t_{i-1},t_{i+1}[} w(s), m \inf_{s\in]t_i,t_{i+2}[} w(s)) \leq m w(t).$$

As a conclusion, $f \in D_m(Lip^\alpha_w(I,U))$ as expected.

Finally let's consider the last statement for $Th(ILip^\alpha_w(I,U))$. Let $I_n = [\frac{a_n+b_n}{2} - c_n, \frac{a_n+b_n}{2} + c_n] \subset [a_n,b_n]$ and $f \in D_m(ILip^\alpha_w(I,U))$ for the norm associated to d, one gets:

$$\begin{aligned}
&\Big\|\frac{1}{n^2}\sum_{k=-n^2c_n}^{n^2c_n-1} f(\frac{a_n+b_n}{2} + \frac{k}{n^2}) - \int_I f\Big\|_d \leq \\
&\qquad \Big\|\sum_{k=-n^2c_n}^{n^2c_n-1} \int_{\frac{a_n+b_n}{2}+\frac{k}{n^2}}^{\frac{a_n+b_n}{2}+\frac{k+1}{n^2}} \Big(f(\frac{a_n+b_n}{2}+\frac{k}{n^2}) - f(s)\Big)ds\Big\|_d + \Big\|\int_{I-I_n} f\Big\|_d \\
&\qquad \leq m\frac{1}{n^{2\alpha}}2c_n + m\Big|\int_a^{a_n} w + \int_{b_n}^b w\Big| \leq \frac{2m}{n^\alpha} + m\Big|\int_a^{a_n} w + \int_{b_n}^b w\Big|.
\end{aligned}$$

This is the expected inequality once introduced W. Moreover the weight condition implies $||\int_I f(s)|| \leq m\int_I w(s)$ which gives the space of value of the integral.

Conversely, consider a model M for the type U and N forming jointly with M a model of $Th(Lip^\alpha_w(I,U))$ and let us see that $N = Lip^\alpha_w(I,M)$. Indeed, using δ_s, $J : f \mapsto \delta.(f)$ send N to $Lip^\alpha_w(I,M)$ since the first axiom imply the Hölder-continuity and weight conditions. As we have just seen in proving (7.4), the second and third axiom insure that the distance d on N has the right meaning, hence J is isometric. Any compactly supported piecewise linear curve with value in a ball of M can be written in the form $\iota^m_{t_0,t_1,...,t_n,t_{n+1}}(x_1,...,x_n)$, for μ large enough. Hence, $J(N)$ contains all those maps. We are ready to use the

completeness of the balls of N for d. To prove surjectivity of J, it suffices to see that any $f \in D_\mu(Lip^\alpha_w(I,U))$ can be approximated uniformly by such curves. Using that the weight goes to 0 at boundaries and regularity, it is easy to find n large enough such that with $t_i = c + i(d-c)/n$, f is uniformly close of the piecewise linear function g with value 0 outside $[c,d]$ and value $g(t_i) = f(t_i)$. Fix $\epsilon = 1/q, q \in \mathbb{N}^*$ large, $m = \mu q$. From the condition on f, one can assume n large enough such that $(1-\epsilon)qf(t_i) \in D_{m_i}(U)$. Recall $m_i = \min(m, \lfloor m \inf_{s\in]t_{i-1},t_{i+1}[} w(s) \rfloor$ for n large enough is involved in the domain of ι^m. Also note that $1 \leq \lambda^m_{t_0,t_1,...,t_n,t_{n+1}}((1-\epsilon)qf(t_1),...,(1-\epsilon)qf(t_n)) \leq max(1, \max_i \frac{(q-1)\mu w(t_i)}{m_i}) = C_n(w)$. This constant depends only on the weight and n (and also $[c,d],q$) and using a modulus of uniform continuity of w on $[c,d]$, one gets $C_n(w) \sim_{n\to\infty} max(1, \frac{(q-1)\mu}{m}) = 1$ so that once $[c,d], \epsilon$ fixed, one can choose n large enough to get some g 2ϵ-close of some $\frac{1}{q}\iota^m_{t_0,t_1,...,t_n,t_{n+1}}((1-\epsilon)qf(t_1),...,(1-\epsilon)qf(t_n))$. This gives a Cauchy-sequence of such curves whose limit must exit in $D_m(N)$ and therefore $J : D_m(N) \to D_m(Lip^\alpha_w(I,U))$ is indeed surjective (and thus bijective). This concludes the identification $N = Lip^\alpha_w(I,M)$ as models of $Th(Lip^\alpha_w(I,U))$. To conclude the same for $Th(ILip^\alpha_w(I,U))$, it suffices to notice that $\int_I f$ is uniquely determined by the extra axiom (since $W(a_n) \to 0, W(b_n) \to W(b)$) and with only possible value the expected integral. ∎

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Note added in proof

While this paper was in review, an alternative approach to our section 2 was obtained in [17], among other axiomatization results. It provides an axiomatizability result in a different language rather than an explicit axiomatization.

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