Eigenvalue method with symmetry and vibration analysis of cyclic structures

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Abstract. We present an application of the eigenvalue method with symmetry for solving polynomial systems arising in the vibration analysis of mechanical structure with symmetry. The search for solutions is conducted by the so called multiplication matrix method in which the symmetry of the system is taken into account by introducing a symmetry group $\mathcal{G}$ and by working with the set of invariant polynomials under the action of group $\mathcal{G}$. By using this method, we compute the periodic solutions of a simple dynamic system coming from the model of a cyclic mechanical structure subjected to nonlinearities.

1 Introduction

Many engineering problems can be modeled or approximated such that the determination of a solution goes through the resolution of a polynomial system. In this paper, we are interested in computing periodic solutions of nonlinear dynamic equations. It can be shown that the Fourier coefficients of the (approximated) periodic solutions can be obtained by solving multivariate polynomial equations resulting from the application of the Harmonic Balance Method [1, 2]. Moreover, in our applications, the dynamical system is often invariant under some transformations (cyclic permutation, change of sign, ...) due to the presence of symmetry in the mechanical structure. This implies that the polynomial system to be solved is also invariant under some transformations, and so does its solutions.

Most of the time, in mechanical engineering, polynomial systems are solved by numeric methods such as a Newton-like algorithm, which outputs only one solution of the system depending on the starting point provided. Although the Newton method is an efficient algorithm (quadratic convergence), the search for all solutions of a polynomial system cannot be conducted in a reasonable time using only this method. In the continuation methods framework [3], the study of bifurcations allows to follow new branches of solution, but does not warranty that all solutions are computed (eg disconnected solutions).

Homotopy methods [2, 4] are an alternative to the Newton algorithm when searching for all solutions of a multivariate polynomial system. Basically, homotopy methods relie on the continuation of the (known) roots of a starting polynomial $\boldsymbol{Q}$ (easy to solve) to the (unknown) roots of a target polynomial $\boldsymbol{P}$. The choice of the starting polynomial is a key point on witch depends the efficiency of the method. Indeed, if the starting polynomial has to many roots compared to $\boldsymbol{P}$, most of the continuations will lead to divergent solutions, thus wasting time and resources. Improvements such as the polyhedral homotopy aims at reducing the number of divergent path by considering a starting polynomial structurally close to the target polynomial

[5]. However, the presence of high combinatoric and/or probabilistic considerations makes the application of the method rather difficult. Moreover, it is not clear how to take into account symmetry properties in the polyhedral homotopy.

In this context, where numerical methods are not entirely satisfactory, computer algebra appears as an attractive alternative, since there exist an efficient method specially developed for solving symmetric system of polynomial equations. The method, relatively recent, is proposed by Gatermann in [6] and is called "eigenvalue method with symmetry". It is based upon the multiplication matrix method [7,8], where solutions of the polynomial system are obtained by solving an eigenvalue problem. Moreover, it takes into account the symmetry of the system by working only on a subspace of the quotient algebra. The method is very efficient since taking into account symmetry allow for reducing the size of the multiplication matrix such that only one representative of each orbit of solution can be computed.

In this paper, we propose a new application of the eigenvalue method with symmetry for computing periodic solutions of nonlinear dynamic systems solved by the harmonic balance method. It constitutes an attempt to evaluates the capabilities of computer algebra methods in the field of mechanical engineering, in which numerical methods are often the norm.

The paper is organized as follow: section 2 presents the type of system studied in this work. The motion's equations are presented along with a brief recall of the Harmonic Balance Method, and we also derive the polynomial equations solved in this study. Section 3 concentrates on polynomial systems solving. We recall some facts about the multiplication matrix method and we describe how to take into account the symmetry of the system.We also present our resolution algorithm in this section. Section 4 is dedicated to numerical examples and the paper ends with some concluding remarks.

2 Dynamic system and periodic solutions

2.1 System of interest

We aim at finding periodic solutions of (polynomial) nonlinear mechanical structures with special symmetry. For exemple, bladed disks subjected to geometric nonlinearities represent such a structure [1]. Here, only a simple cyclic system (which can be seen as a reduced order model of a bladed disk, where all blades have been reduced on their first mode of vibration) will be considered. The model consists in N duffing oscillators linearly coupled, governed by the following motion equation:

$$m\ddot{u}_i + c\dot{u}_i + (k+2k_c)u_i - k_c u_{i-1} - k_c u_{i+1} + k_{nl}u_i^3 = f_i(t),\ i = 1,\ \ldots,\ N \tag{1}$$

where $u_i(t)$ represents the temporal evolution of degree of freedom (dof) number i, and $f_i(t)$ represents the temporal evolution of the excitation force acting on dof number i. If there is no force, note that this dynamic system is invariant under the action of the dihedral group $\mathcal{D}_N$ (symmetry of a regular polygon with N vertices).

Equation (1) can be written in the following matrix form:

$$\mathbf{M}\ddot{\boldsymbol{u}} + \mathbf{C}\dot{\boldsymbol{u}} + \mathbf{K}\boldsymbol{u} + \boldsymbol{F}_{nl}(\boldsymbol{u}) = \boldsymbol{F}_{ex}(t), \tag{2}$$

were $\boldsymbol{u}(t)$ is the vector of dof of size N, $\mathbf{M} = m\mathbf{I}$ is the mass matrix, $\mathbf{C} = c\mathbf{I}$ is the damping matrix, $\mathbf{K} = (k+2k_c)\mathbf{I} - k_c\mathbf{I}^L - k_c\mathbf{I}^U$ is the stiffness matrix, and $\boldsymbol{F}_{nl}(\boldsymbol{u}) = k_{nl}\boldsymbol{u}^3$, $\boldsymbol{F}_{ex}(t) = \boldsymbol{f}(t)$

correspond to the nonlinear and excitation forces respectively. The excitation forces will be assumed to be periodic, with period $T = \frac{2\pi}{\omega}$, and we will search for periodic solutions $\boldsymbol{u}(t)$, using the harmonic balance method described hereafter.

2.2 Harmonic Balance Method

The harmonic balance method (HBM), is a widely used method in finding approximation to periodic solutions of nonlinear differential equations such as (2) [1, 9]. The solutions $\boldsymbol{u}(t)$ is approximated under the form of a truncated Fourier series, and a system of algebraic equations is derived by applying Galerkin projections. Let us recall the main steps of the method.

At first, each component $u_i(t)$ of the periodic solution $\boldsymbol{u}(t)$ is approximated by $\widehat{u}_i(t)$ under the following form:

$$\widehat{u}_i(t) = x^{(0)} + \sum_{k=1}^{H} x_i^{(k)} \cos(k\omega t) + y_i^{(k)} \sin(k\omega t),\ i = 1, \ldots, N. \tag{3}$$

We substitute (3) in (2) and we project the resulting equations on the truncated Fourier basis:

$$\begin{array}{c} \frac{2}{T}\int_0^T \boldsymbol{R}(\widehat{\boldsymbol{u}}) \times 1\ dt = \mathbf{0}, \\ \frac{2}{T}\int_0^T \boldsymbol{R}(\widehat{\boldsymbol{u}}) \times \cos(k\omega t)\ dt = \mathbf{0},\ k = 1, \ldots, H, \\ \frac{2}{T}\int_0^T \boldsymbol{R}(\widehat{\boldsymbol{u}}) \times \sin(k\omega t)\ dt = \mathbf{0},\ k = 1, \ldots, H. \end{array} \tag{4}$$

with $T = 2\pi/\omega$ and

$$\boldsymbol{R}(\widehat{\boldsymbol{u}}) = \mathbf{M}\ddot{\widehat{\boldsymbol{u}}} + \mathbf{C}\dot{\widehat{\boldsymbol{u}}} + \mathbf{K}\widehat{\boldsymbol{u}} + \boldsymbol{F}_{nl}(\widehat{\boldsymbol{u}}) - \boldsymbol{F}_{ex}(t).$$

Equations (4) corresponds to a set of $N(2H+1)$ algebraic equations with unknowns $\boldsymbol{x}$ and $\boldsymbol{y}$.

2.3 Equations to be solved

In our application, $\boldsymbol{F}_{nl}(\boldsymbol{u}) = k_{nl}\boldsymbol{u}^3$ is polynomial and (4) corresponds to a system of polynomial equations. In order to simplify the presentation and reduce the number of variables, we will only consider a single harmonic approximation of the periodic solution, i.e., $H = 1$ in (3). Moreover, as the nonlinearity is odd, no continuous component will be retained, i.e., $\boldsymbol{x}^{(0)} = \mathbf{0}$ in (3). Under these hypothesis, (4) corresponds to a system of $2N$ polynomial equations which can be written in the following form (dropping the harmonic index $^{(k)}$):

$$\begin{array}{l} \alpha(\omega)x_i + \delta(\omega)y_i - \beta x_{i-1} - \beta x_{i+1} + \gamma x_i(x_i^2 + y_i^2) = f_i^c,\ i = 1, \ldots, N, \\ \alpha(\omega)y_i - \delta(\omega)x_i - \beta y_{i-1} - \beta y_{i+1} + \gamma y_i(x_i^2 + y_i^2) = f_i^s,\ i = 1, \ldots, N, \end{array} \tag{5}$$

where f_i^c (resp. f_i^s) denotes the amplitude of the excitation forces relative to the $\cos(\omega t)$ (resp. $\sin(\omega t)$) term, and with the following expression for the different coefficients:

$$\alpha(\omega) = k + 2k_c - \omega^2 m, \quad \beta = k_c, \quad \gamma = \frac{3}{4}k_{nl}, \quad \delta(\omega) = \omega c.$$

In our application, we are interested in forced and free solutions.

Forced solutions. In the forced case ($\boldsymbol{f}^c \neq \mathbf{0}$ or $\boldsymbol{f}^s \neq \mathbf{0}$), the angular frequency ω is set by the excitation forces and (5) will be solved for $\boldsymbol{x}$ and $\boldsymbol{y}$. Depending on the symmetry of the excitation forces, system (5) may present some invariance properties. We will choose $f_i^c = 1$, $f_i^s = 0$ for all $i = 1, \ldots, N$ so that system (5) will be invariant under the action of the dihedral group $\mathcal{D}_N$.

Free solution. In the free case, we aims at finding solutions of an unforced, undamped version of system (2), also called Nonlinear Normal Modes (NNM) [10–12]. In order to simplify we will only search for solutions where all dof vibrate "in-phase" (monophase NNM [13]) by imposing $y_i = 0$ for all $i = 1, \ldots, N$, thus resulting in the following polynomial system with N equations:

$$\alpha(\omega)x_i - \beta x_{i-1} - \beta x_{i+1} + \gamma x_i^3 = 0,\ i = 1, \ldots, N. \tag{6}$$

The angular frequency ω will be set to an arbitrary value and system (6) will be solved for $\boldsymbol{x}$. Again (6) is invariant under the action of the dihedral group $\mathcal{D}_N$ and it is also invariant under change of sign, characterized by the group with 2 elements $\mathcal{Z}_2 = \{e,\ b \mid b^2 = e\}$ with $b(\boldsymbol{x}) = -\boldsymbol{x}$.

3 Solving multivariate polynomial systems

In this section we present the method used to solve symmetric system of polynomial equations. First, the eigenvalue method is described. Then we show how to include symmetry of the system in order to reduce the number of solution as proposed in [6], leading to the so called eigenvalue method with symmetry. Finally we propose an algorithm to summarize the process.

3.1 Gröbner Basis

We will denote by $\mathbb{C}[\boldsymbol{x}]$ the ring of multivariate polynomials with complex coefficients in the variables $\boldsymbol{x} = (x_1, \ldots, x_n)$. A polynomial in $\mathbb{C}[\boldsymbol{x}]$ has the form $f(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{S}} c(\boldsymbol{\alpha}) \boldsymbol{x}^{\boldsymbol{\alpha}}$, where $\mathcal{S} \subset \mathbb{N}^n$ is the the support of f, $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a monomial of total degree $|\boldsymbol{\alpha}| = \sum_i \alpha_i$, and $c(\boldsymbol{\alpha}) \in \mathbb{C}$ is the coefficient of monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$. We fix a monomial order on $\mathbb{C}[\boldsymbol{x}]$. In the application, we will consider the graded reverse lexicographic order $\leq_{\text{grevlex}}$ defined for $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $\mathbb{N}^n$ by:

$$\boldsymbol{\alpha} \leq_{\text{grevlex}} \boldsymbol{\beta} \equiv [|\boldsymbol{\alpha}| \leq |\boldsymbol{\beta}|] \text{ or } [\alpha_j \geq \beta_j \text{ and } \alpha_i = \beta_i \text{ for } 1 \leq j \leq i]$$

We will denote by $\text{LM}(f)$ and $\text{LC}(f)$ the leading monomial and the leading coefficient of a polynomial f, we will denote by $\text{LT}(f) = \text{LC}(f)\text{LM}(f)$ its leading term.

Consider a multivariate polynomial system given by $\boldsymbol{P}(\boldsymbol{x}) = [p_1(\boldsymbol{x}),\ \ldots,\ p_n(\boldsymbol{x})]$ with $p_j \in \mathbb{C}[\boldsymbol{x}]$ for $j = 1, \ldots, n$. We denote by $\mathcal{I} = \langle \boldsymbol{P} \rangle = \langle p_1, \ldots, p_n \rangle$ the ideal of $\mathbb{C}[\boldsymbol{x}]$ generated by the polynomial system $\boldsymbol{P}$. The reduction operation modulo $\boldsymbol{P}$ reduces a polynomial $f \in \mathbb{C}[\boldsymbol{x}]$ into a remainder of the division of f by each element of $\boldsymbol{P}$, defined by:

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} \mu_i(\boldsymbol{x}) p_i(\boldsymbol{x}) + r(\boldsymbol{x}).$$

Such a remainder is generally not unique and depends on the division order and on the monomial order. However, the reduction modulo a Gröbner basis makes the remainder unique during the reduction operation. Recall that a Gröbner basis for $\mathcal{I}$ is a finite collection of polynomials

$\boldsymbol{G} = [g_1, \ldots, g_m] \subset \mathcal{I}$ with the property that for any nonzero polynomial f in $\mathcal{I}$, $\text{LT}(f)$ is divisible by $\text{LT}(g_i)$ for some $i = 1, \ldots, m$. The remainder on division of f by a Gröbner basis is uniquely determined, thus is called normal form for f and denoted $\text{NF}(f)$. In practice, a Gröbner basis can be compute by the Buchberger algorithm [14] and its improvements, e.g. [15]. The monomial ordering choosen influences both the form of the basis $\boldsymbol{G}$ and computation time, and, in general, computation with the grevlex ordering tends to be faster than with the lexicographic ordering. We denote by $A = \mathbb{C}[\boldsymbol{x}]/\mathcal{I}$ the algebra defined as the quotient of $\mathbb{C}[\boldsymbol{x}]$ by the ideal $\mathcal{I}$. The set $\boldsymbol{G}$ being a Gröbner basis, the monomials

$$\boldsymbol{\mathcal{B}} = \{\boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{x}^{\boldsymbol{\alpha}} \notin \langle \text{LT}(\boldsymbol{G}) \rangle\}$$

form a basis of algebra A, as a vector space over $\mathbb{C}$. If the polynomial system $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{0}$ has only a finite number of solutions (say D solutions), the ideal $\mathcal{I}$ is zero-dimensional, and it can be shown [7,16] that, as a space, A is of finite dimension D.

3.2 Multiplication Matrices Method

Given a polynomial $f \in \mathbb{C}[\boldsymbol{x}]$, we consider the map $m_f : A \to A$, defined by $m_f(h) = fh$, for any h in A. Since A is a finite-dimensional algebra the map m_f can be represented by a matrix $\mathbf{M}_f$ relative to the basis $\boldsymbol{\mathcal{B}}$. The matrix $\mathbf{M}_f$ is called *multiplication matrix* and is characterized by the following relation (modulo $\mathcal{I}$):

$$f\ \boldsymbol{\mathcal{B}} = \mathbf{M}_f\ \boldsymbol{\mathcal{B}} \mod(\mathcal{I}), \tag{7}$$

or equivalently:

$$f\ \mathcal{B}_i = \sum_{j=1}^{D} M_{i,j}^f\ \mathcal{B}_j \mod(\mathcal{I}),\ i = 1, \ldots, D.$$

The coefficients of line i of the matrix $\mathbf{M}_f$ can be obtained by computing the normal form of each product $f\ \mathcal{B}_i$ and by expressing the results as a linear combination of elements of $\boldsymbol{\mathcal{B}}$.

For particular choices of $f = x_p$, $p = 1, \ldots, n$, it can be shown that the eigenvalues of the multiplication matrices $\mathbf{M}_{x_p}$ are related to the zeros of the polynomial system. Indeed, substituting $f = x_p$ into (7), for any $\boldsymbol{x}$, we have:

$$\left(\mathbf{M}_{x_p} - x_p \mathbf{I}\right) \boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \boldsymbol{0} \mod(\mathcal{I}). \tag{8}$$

It follows that the vector $\left(\mathbf{M}_{x_p} - x_p \mathbf{I}\right) \boldsymbol{\mathcal{B}}(\boldsymbol{x})$ can therefore be expressed as a combination of the polynomials in $\boldsymbol{P}$. Now, let's suppose that $\boldsymbol{x}^*$ is a root of $\boldsymbol{P}$. Then $p_i(\boldsymbol{x}^*) = 0$ for all $i = 1, \ldots, n$, and (8) shows that x_p^* is an eigenvalue of $\boldsymbol{M}_{x_p}$ associated to the eigenvector $\boldsymbol{\mathcal{B}}(\boldsymbol{x}^*)$. Note that the eigenvector should be normalized so that its first component equals 1 (in order to mach with the associated polynomials $\mathcal{B}_1(\boldsymbol{x}) = 1$).

Going further, it can be shown [7,16] that the components of the roots are given by the eigenvalues of $\mathbf{M}_{x_p}$, $p = 1, \ldots, n$, associated with common eigenvectors $\boldsymbol{\mathcal{B}}_k$.

Here, we follow the method given in [4] (Chap.1.6.3.2), which consists in considering only one multiplication matrix associated with a linear combination of the variables $f = \sum_{i=1}^{n} c_i\ x_i$, where c_i are rational numbers chosen such that the value of $f(\boldsymbol{x}^{(k)})$ is different for each solution $\boldsymbol{x}^{(k)}$, $k = 1, \ldots, D$. Generally, random choices for coefficients c_i are sufficient to ensure this properties almost surely [4]. The search for the roots of system $\boldsymbol{P}$ is then simply conducted by solving the eigenvalue problem $(\mathbf{M}_f - f\mathbf{I})\boldsymbol{\mathcal{B}} = \boldsymbol{0}$, and by reading the solutions in the eigenvectors $\boldsymbol{\mathcal{B}}_k = \boldsymbol{\mathcal{B}}(\boldsymbol{x}^{(k)})$, $k = 1, \ldots, D$.

3.3 Introducing symmetry

Invariant polynomial systems. Due to the symmetry of the mechanical structure (change of coordinates, ...), the polynomial systems to be solved in our applications (see section 2.3) also possess a symmetric structure. Here we will consider that the polynomial system to be solved is equivariant under the action of a group $\mathcal{G}$, that is $\boldsymbol{P}(g(\boldsymbol{x})) = g(\boldsymbol{P})(\boldsymbol{x}),\ \forall g \in \mathcal{G}$, where $g \in \mathcal{G}$ is a permutation operation defined by $g(\boldsymbol{x}) = [x_{g(1)},\ \ldots,\ x_{g(n)}]$. The set of invariant polynomial under $\mathcal{G}$ is denoted $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}$ and defined by: $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}} = \{f \in \mathbb{C}[\boldsymbol{x}] \mid f(g(\boldsymbol{x})) = f(\boldsymbol{x}),\ \forall g \in \mathcal{G}\}$. We denote by $\mathcal{I}^{\mathcal{G}} = \mathcal{I} \cap \mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}$ the ideal invariant under the action of the group $\mathcal{G}$.

Quotient decomposition. It can be shown that $\mathbb{C}[\boldsymbol{x}]$ can be decomposed into a direct sum of isotypic components [6, 17], such that $\mathbb{C}[\boldsymbol{x}] = V_1 \oplus V_2 \oplus\ \ldots\ \oplus V_K$, where the V_i's are the isotypic components (related to the K irreducible representations of group $\mathcal{G}$ [6]), and where the first component is the invariant ring itself: $V_1 = \mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}$. By defining $\mathcal{I}_i = \mathcal{I} \cap V_i$, the algebra $A = \mathbb{C}[\boldsymbol{x}]/\mathcal{I}$ can be decomposed into a direct sum as follows [6]:

$$A = \mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}^{\mathcal{G}} \oplus V_2/\mathcal{I}_2 \oplus\ \ldots\ \oplus V_K/\mathcal{I}_K \tag{9}$$

The space $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}$ can be decomposed into the following direct sum (Hironaka decomposition) [6]:

$$\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}} = \oplus_i\ S_i\ \mathbb{C}[\boldsymbol{\pi}] = \mathbb{C}[\boldsymbol{\pi}] \oplus S_2\mathbb{C}[\boldsymbol{\pi}] \oplus S_3\mathbb{C}[\boldsymbol{\pi}] \oplus \cdots \oplus S_p\mathbb{C}[\boldsymbol{\pi}]$$

where $\boldsymbol{\pi} = [\pi_1, \ldots, \pi_n]$ is the set of primary polynomial invariants related to $\mathcal{G}$, and $S_2,\ \ldots,\ S_n$ correspond to the secondary polynomial invariants related to $\mathcal{G}$. The primary polynomial invariants $\boldsymbol{\pi}$ can be found by using the Reynold projection operator defined for $f \in \mathbb{C}[\boldsymbol{x}]$ by [18]:

$$\mathrm{Re}_f(\boldsymbol{x}) = \frac{1}{|G|} \sum_{g \in \mathcal{G}} f(g(\boldsymbol{x})). \tag{10}$$

Applying the Reynolds projector to any polynomial $f \in \mathbb{C}[\boldsymbol{x}]$ leads to an invariant polynomial $\mathrm{Re}_f \in \mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}$. The primary invariants can be computed by applying the Reynold projector to each monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with $|\alpha| \leq |\mathcal{G}|$. In certain cases, some monomials will lead to the same invariant, or some invariants can be obtained as a combination of the others. In those cases, we need to eliminate the redundancies by computing Gröbner basis [18]. In this work, we compute the primary invariants using the `invariant_ring` command of `Singular`. The secondary invariants corresponds to a module basis of $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}$ as a $\mathbb{C}[\pi]$-module. It can also be computed by the `invariant_ring` command.

Using the primary polynomial invariants. In the following, the primary invariants will be used to find the solution of an invariant system. Let's suppose that we can find the values of the primary invariant $\boldsymbol{\pi}^{(k)} = \boldsymbol{\pi}(\boldsymbol{x}^{(k)})$ for each solution $\boldsymbol{x}^{(k)}$, then by solving the following systems:

$$\boldsymbol{\pi}(\boldsymbol{x}) = \boldsymbol{\pi}^{(k)}, \quad k = 1, \ldots, D_{\mathcal{G}},$$

for $\boldsymbol{x}$ by a Newton-like method, one can compute an unique occurrence of solution $\boldsymbol{x}^{(k)}$ and the other can be generated by applying the group's actions on $\boldsymbol{x}^{(k)}$, i.e., $g(\boldsymbol{x}^{(k)}), \forall g \in \mathcal{G}$.

We will compute the values of the primary invariants $\boldsymbol{\pi}^{(k)}$ for each solution $\boldsymbol{x}^{(k)}$ with the multiplication matrix method. However, as shown in [6], the multiplication matrices related to

the primary invariants are redundant as they contain the same eigenvalues several times. In a suited basis of A, it is even shown that the multiplication matrices associated to the primary invariants are block diagonal [6, Thm. 3], with each block containing the same eigenvalues [6, Prop. 8]. Thus, only the first diagonal block (related to the subspace $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}^{\mathcal{G}}$) is of interest to compute the values of the primary invariants.

All that is left to do here, is to find a basis $\boldsymbol{\mathcal{B}'}$ of A that makes the multiplication matrices block diagonal. More precisely, it is sufficient to find a basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}} = [\mathcal{B}'_1,\ \ldots, \mathcal{B}'_{D_{\mathcal{G}}}]$ of $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}^{\mathcal{G}}$ in agreement with the direct sum decomposition in (9).

Construction of an adapted basis. The goal is to find a basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ of $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}^{\mathcal{G}}$ (with $\#\boldsymbol{\mathcal{B}}^{\mathcal{G}} = D_{\mathcal{G}}$) in agreement with the direct sum decomposition in (9), in order to construct the first block of a multiplication matrix. As in the previous section, the multiplication matrix will be related to a polynomial $f = \sum_{i=1}^{n} c_i\ \pi_i$, where c_i are rational coefficients chosen randomly.

The basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ should only contains invariant polynomials, and their normal forms should be sufficient to express all remainders r in the division of $f\ \mathcal{B}_i^{\mathcal{G}}$ by $\mathcal{I}$ (i.e., $r = \sum_{j=1}^{D_{\mathcal{G}}}\ M_{i,j}^{\mathcal{G}}\ \textsc{nf}(\mathcal{B}_j^{\mathcal{G}})$).

We suppose that a Gröbner basis $\boldsymbol{G}$ of $\mathcal{I}$ is known. Let $\textsc{nf}$ the normal form operator for $\boldsymbol{G}$. At start, we set $\mathcal{B}_1^{\mathcal{G}} = 1$.

The construction of the basis then goes as follows. For $\mathcal{B}_i^{\mathcal{G}}$ in $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ we compute the normal form $r = \textsc{nf}(f\ \mathcal{B}_i^{\mathcal{G}})$. Then, until the remainder r equals zero, we search if there exists $\mathcal{B}_j^{\mathcal{G}}$ in $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ such that $\textsc{lm}(\textsc{nf}(\mathcal{B}_j^{\mathcal{G}})) = \textsc{lm}(r)$, that is $\textsc{lt}(r) = q\ \textsc{lt}(\textsc{nf}(\mathcal{B}_j^{\mathcal{G}}))$, with $q \in \mathbb{C}$

- if such a $\mathcal{B}_j^{\mathcal{G}}$ exists, then we divide r by $\textsc{nf}(\mathcal{B}_j^{\mathcal{G}})$: $r = M_{i,j}^{\mathcal{G}}\ \textsc{nf}(\mathcal{B}_j^{\mathcal{G}}) + h$ and we save the (numeric) matrix coefficient $M_{i,j}^{\mathcal{G}}$. Finally, we affect $r = h$, and search for a new divisor of $\textsc{lt}(r)$.
- if not, we will create a new basis term $\mathcal{B}_k^{\mathcal{G}}$ whose leading monomial equals $\textsc{lm}(r)$ by considering the Reynold projection of $\textsc{lm}(r)$, ie: $\mathcal{B}_k^{\mathcal{G}} = \mathrm{Re}_{\textsc{lm}(r)}$. However, it may happen that $\textsc{lm}(\textsc{nf}(\mathrm{Re}_{\textsc{lm}(r)})) \neq \textsc{lm}(r)$. In that case, we modify the Reynold projection by subtracting the high order term until $\textsc{lm}(\textsc{nf}(\mathrm{Re}_{\textsc{lm}(r)})) = \textsc{lm}(r)$. This is done by searching into the basis an element $\mathcal{B}_{j_0}^{\mathcal{G}}$ such that $\textsc{lm}(\textsc{nf}(\mathcal{B}_{j_0}^{\mathcal{G}})) = \textsc{lt}(\textsc{nf}(\mathrm{Re}_{\textsc{lm}(r)}))$ and by modifying the Reynold projection : $\mathrm{Re}_{\textsc{lm}(r)} = \mathrm{Re}_{\textsc{lm}(r)} - c_{j_0}\mathcal{B}_{j_0}^{\mathcal{G}}$. Once the invariant is computed, we divide r by the new element : $r = M_{i,k}^{\mathcal{G}}\ \mathcal{B}_k^{\mathcal{G}} + h$, and we can save the (numeric) matrix coefficient. Finally, we affect $r = h$, and search for a new divisor of $\textsc{lt}(r)$.

This process is repeated until all products $f\ \mathcal{B}_i,\ i = 1, \ldots, D_{\mathcal{G}}$, have been computed. The basis construction is summarized in Algorithm 1.

Algorithm 1. Computation of a basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ of the invariant space $\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}$, and construction of the multiplication matrix of the invariant variable $f = \sum c_j \pi_j$

#Preliminaries
compute a Gröbner basis $\boldsymbol{G}$ of $\boldsymbol{P}$ with the grevlex order
initialize $f = \sum_j c_j \pi_j$, $\mathcal{B}_1^{\mathcal{G}} = 1$, $n = 1$
#Basis Computation
$j = 0$
while $j < n$ **do**
  $j = j + 1$

compute the normal form $r = \text{NF}(f\mathcal{B}_j^{\mathcal{G}})$
while $r \neq 0$ **do**
for $k = 1, \ldots, n$ **do**
if $\text{LM}(\text{NF}(b_k)) = \text{LM}(r)$ **then**
reduce $r : r = q\mathcal{B}_k^{\mathcal{G}} + h$
save $M_{j,k} = q$ and update : $r = h$
end if
end for
if $\text{LM}(r) \notin \boldsymbol{\mathcal{B}}^{\mathcal{G}}$ **then**
compute the Reynold projection $Re(\boldsymbol{x}) = Re_{\text{LM}(r)}(\boldsymbol{x})$
if $\text{LM}(\text{NF}(\text{Re}))=\text{LM}(\text{r})$ **then**
affect $\mathcal{B}_{n+1}^{\mathcal{G}} = Re$
else
while $\text{LM}(\text{NF}(Re)) \neq \text{LM}(r)$, reduce the Reynold projection: $Re = Re - c_k\mathcal{B}_k^{\mathcal{G}}$
affect $\mathcal{B}_{n+1}^{\mathcal{G}} = Re$
end if
reduce the normal form $r : r = qRe + h$
save $M_{j,n+1} = q$ and update : $n = n + 1$, $r = h$
end if
end while
end while
return the multiplication matrix $\boldsymbol{M}_f$ and the basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$

4 Numerical applications

In this section, we apply the eigenvalue method with symmetry to the system given in Section 2.3. The numerical application will be conducted for system with $N = 2, 4$ degrees of freedom. In the two cases, free and forced analysis are conducted. Solutions for a particular frequency are computed with the multiplication matrice method, and we give an overview of the system dynamics by applying continuation methods [3]. Finally, an NNM analysis is carried for $2 \leq N \leq 6$ in order to show the decrease in the number of solutions.

4.1 Simple example with 2 degrees of freedom

As a first application, we study a system with $N = 2$ degree of freedom. In this case, (2) reduces to the following dynamic system:

$$\begin{aligned} m\ddot{u}_1 + c\dot{u}_1 + (k + k_c)u_1 - k_c u_2 + k_{nl}u_1^3 &= f_1(t), \\ m\ddot{u}_2 + c\dot{u}_2 + (k + k_c)u_2 - k_c u_1 + k_{nl}u_2^3 &= f_2(t). \end{aligned} \tag{11}$$

The application of the HBM with only one harmonic ($u_i = x_i \cos(\omega t) + y_i \sin(\omega t)$) leads to the following system of polynomial equations:

$$\begin{aligned} \alpha x_1 - \beta x_2 + \delta y_1 + \gamma x_1(x_1^2 + y_1^2) &= f_c, \\ \alpha y_1 - \beta y_2 - \delta x_1 + \gamma y_1(x_1^2 + y_1^2) &= f_s, \\ \alpha x_2 - \beta x_1 + \delta y_2 + \gamma x_2(x_2^2 + y_2^2) &= f_c, \\ \alpha y_2 - \beta y_1 - \delta x_2 + \gamma y_2(x_2^2 + y_2^2) &= f_s, \end{aligned} \tag{12}$$

with $\alpha = k + k_c - \omega^2 m$, $\beta = k_c$, $\gamma = \frac{3}{4}k_{nl}$ and $\delta = \omega c$. The frequency parameter will be set to $\omega = \frac{25}{10}$ (however the search for multiple solution can be conducted for any value of ω), leading to the following numerical values:

$$\alpha = \frac{-17}{4},\ \beta = 1, \gamma = \frac{3}{4},\ \delta = \frac{1}{10},\ f_c = 1,\ f_s = 0. \tag{13}$$

Monophase NNM analysis. We search for monophase NNM solutions of (12) (undamped, unforced). In this case, the system (6) reduces to the following:

$$\begin{aligned} \alpha x_1 - \beta x_2 + \gamma x_1^3 &= 0, \\ \alpha x_2 - \beta x_1 + \gamma x_2^3 &= 0. \end{aligned} \tag{14}$$

We consider the order grevlex with $x_1 > x_2$. Since the leading term of each equation are coprime, the polynomial system $\boldsymbol{P}$ is already in a Gröbner basis form. We computed a normal set and we show the algebra $A = \mathbb{C}[\boldsymbol{x}]/\langle \boldsymbol{P} \rangle$ is of dimension 9 (i.e., the system has 9 solutions).

The system (14) is invariant under permutation of variable and under change of sign. This invariance property corresponds to the group $\mathcal{G} = \mathbb{C}_2 \times \mathbb{Z}_2$, where $\mathcal{C}_2 = \{e,\ a \mid a^2 = e\ \}$, where $a[(x_1, x_2)] = (x_2, x_1)$ and $\mathcal{Z}_2 = \{e,\ b \mid b^2 = e\ \}$, where $b[(x_1, x_2)] = (-x_1, -x_2)$. All element $g \in \mathcal{G}$ can be represented by a matrix $\mathbf{M}_g = \mathbf{A}^{i_g}\mathbf{B}^{i_g}$ where $\mathbf{A}$ and $\mathbf{B}$ are given by the following:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using `Singular`, we know that the primary invariant of $\mathcal{G}$ are $\pi_1 = x_1 x_2$ and $\pi_2 = \frac{1}{2}(x_1^2 + x_2^2)$. We set $f = \pi_1 + \frac{2}{3}\pi_2$, and we construct the multiplication matrix of f in an symmetry adapted basis of $A^{\mathcal{G}}$ using Algorithm 1. The basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ of $A^{\mathcal{G}}$ and the multiplication matrix $\mathbf{M}_f$ are given by

$$\boldsymbol{\mathcal{B}}^{\mathcal{G}} = [1,\ \frac{1}{2}(x_1^2 + x_2^2),\ x_1 x_1,\ x_1^2 x_2^2],\ \mathbf{M}_f = \begin{bmatrix} 0 & \frac{4}{3} & 1 & 0 \\ 0 & \frac{46}{39} & \frac{59}{9} & \frac{2}{3} \\ 0 & \frac{19}{9} & \frac{68}{9} & 1 \\ 0 & \frac{471}{27} & \frac{1187}{27} & \frac{68}{9} \end{bmatrix}.$$

The computation of eigenvalues $\boldsymbol{\lambda} = \boldsymbol{f}(\boldsymbol{x}^*)$ and eigenvectors $\boldsymbol{\mathcal{B}}^{\mathcal{G}}(\boldsymbol{x}^*)$ of $\mathbf{M}_f$ gives (after normalization of the first component):

$$\boldsymbol{\lambda} = \begin{bmatrix} 0 \\ 16.3333 \\ 2.4444 \\ 1.4444 \end{bmatrix}, \qquad \boldsymbol{\mathcal{B}}^{\mathcal{G}}(\boldsymbol{x}^*) = \begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 \\ 0 & 7.00 & 2.83 & 4.33 \\ 0 & 7.00 & -1.33 & -4.33 \\ 0 & 49.00 & 1.77 & 18.77 \end{bmatrix}.$$

Here π_1 and π_2 belong to the invariant basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ ($\pi_1 = \mathcal{B}_3^{\mathcal{G}}$ and $\pi_2 = \mathcal{B}_2^{\mathcal{G}}$), so that their values $\boldsymbol{\pi}(\boldsymbol{x}^*)$ can directly be read into the eigenvectors $\boldsymbol{\mathcal{B}}^{\mathcal{G}}(\boldsymbol{x}^*)$ (at line 3 and line 2), leading to the 4 following systems of equations:

$$(\pi_1(\boldsymbol{x}),\ \pi_2(\boldsymbol{x})) \in \{\ (0,0),\ (7,7),\ (-1.33, 2.83),\ (-4.33, 4.33)\ \} \tag{15}$$

The nonlinear system in (15) are solved by a Newton Raphson method. Four different solutions are obtained, see (16), and they are depicted on Fig.1. We verified that those solutions are

actually solutions of $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{0}$ by computing the values of $\|\boldsymbol{P}(\boldsymbol{x}^*)\|$ in Table 4.1. To assess the quality of the real solutions, we compare them with refined solutions obtained with a Newton algorithm applied on $\boldsymbol{P}$ with starting points $\boldsymbol{x}_0 = \boldsymbol{x}^*$, see Table 4.1. It is seen that solutions from the eigenvalue method are indeed very close to the actual roots of $\boldsymbol{P}$, as their relative differences lie below 0.5%. In any cases, a few Newton iterations should be applied to overcome the numerical error due to numerical rounding of rational numbers in the multiplication matrix.

$$(x_1, x_2) \in \{\ (0,0),\ (-2.65,-2.65),\ (2.31,-0.58),\ (2.08,-2.08)\ \} \tag{16}$$

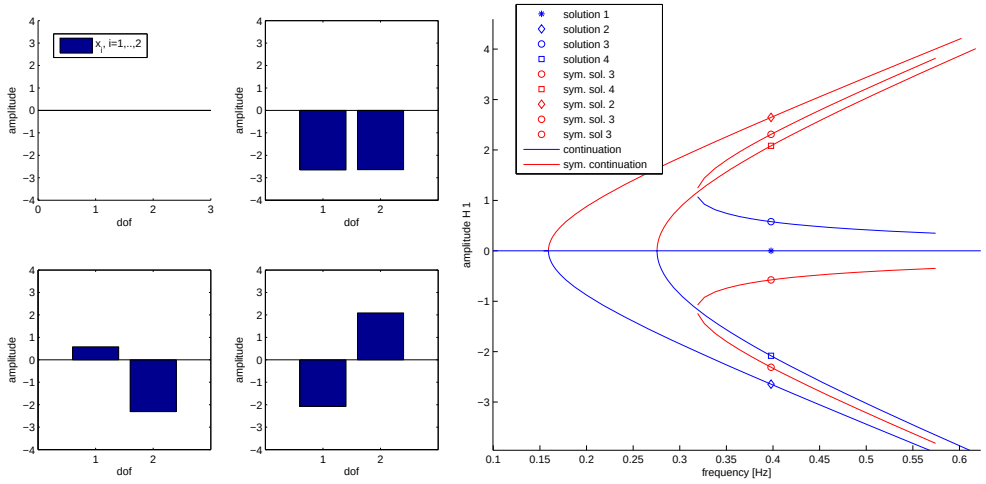

Fig. 1. Left:Form of the real solutions of system (14) found by the invariant multiplication matrix method. Right: Frequency continuation of the solution obtained at $f = \frac{1}{2\pi}\frac{25}{10}$ and their symmetric relative to the group operation

solution	1	2	3	4
value $\|\boldsymbol{P}(\boldsymbol{x}^*)\|$	0.04	0.11	0.00	0.04
relative diff. from NR sol. (%)	x	0.23	0.00	0.32

Table 1. assessment of the solution quality of (14) at $\omega = \frac{25}{10}$

The application of the group actions generates 5 other solutions. At the end the total set of solutions contains 9 elements as indicated by the dimension of the quotient space. However, the use of symmetry decreased the size of the eigenvalue problem from 9 to 4, leading to only 4 solutions (one for each orbit of solutions).

In order to give an overview of the system dynamics, we use the four solutions in (16) as starting points for a continuation procedure on the parameter ω. The results are depicted on Fig. 1 and correspond to the monophase nonlinear normal modes of the systems. Three types of solution can be identified, an in-phase solution (sol. 1), an out-of-phase solution (sol. 4) and a localized solution (sol. 3) which corresponds to a bifurcation of the out-of-phase solution.

Forced analysis. We now turn to the forced analysis of system (12). We compute a Gröbner basis $\boldsymbol{G}$ with 12 elements relatively to the grevlex order with $y_2 < y_1 < x_2 < x_1$. We compute $\dim(A) = 11$, thus the system has 11 solutions. The system is invariant under the action of

$\mathcal{G} = \mathcal{C}_2 = \{e,\ a \mid a^2 = e\}$ with $a(x_1, y_1, x_2, y_2) = (x_2, y_2, x_1, y_1)$. The representation of $\mathcal{G}$ is chosen such that a is represented by $\mathbf{M}_a = \begin{bmatrix} \mathbf{0} & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0} \end{bmatrix}$.

The primary invariant of $\mathcal{G}$ are given by $\pi_1 = \frac{1}{2}(x_1 + x_2)$, $\pi_2 = \frac{1}{2}(y_1 + y_2)$, $\pi_3 = x_1 x_2$ and $\pi_4 = y_1 y_2$; and the multiplication matrix is computed for $f = \pi_1 + \pi_2 + \pi_3 + \pi_4$.

By using Algorithm 1 we compute a basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ of $A^{\mathcal{G}}$ with 7 elements.

All primary invariants are in $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ except for π_3. Thus, the normal form of π_3 is computed and the result is expressed in terms of elements of $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$: $\pi_3 = \boldsymbol{c}^T \boldsymbol{\mathcal{B}}^{\mathcal{G}}$. After solving the eigenvalue problem, the values of π_3 at the solutions point are given by $\pi_3(\boldsymbol{x}^*) = \boldsymbol{c}^T \boldsymbol{\mathcal{B}}^{\mathcal{G}}(\boldsymbol{x}^*)$.

The solution of $\boldsymbol{P}(\boldsymbol{x}) = \mathbf{0}$ are then evaluated by solving the 7 nonlinear systems $\boldsymbol{\pi} = \boldsymbol{\mathcal{B}}^{\mathcal{G}}(\boldsymbol{x}^*)$ corresponding to each eigenvector: 7 solutions (5 real and 2 complex) are found by a Newton algorithm, and the form of the real solutions are depicted in Fig.2.

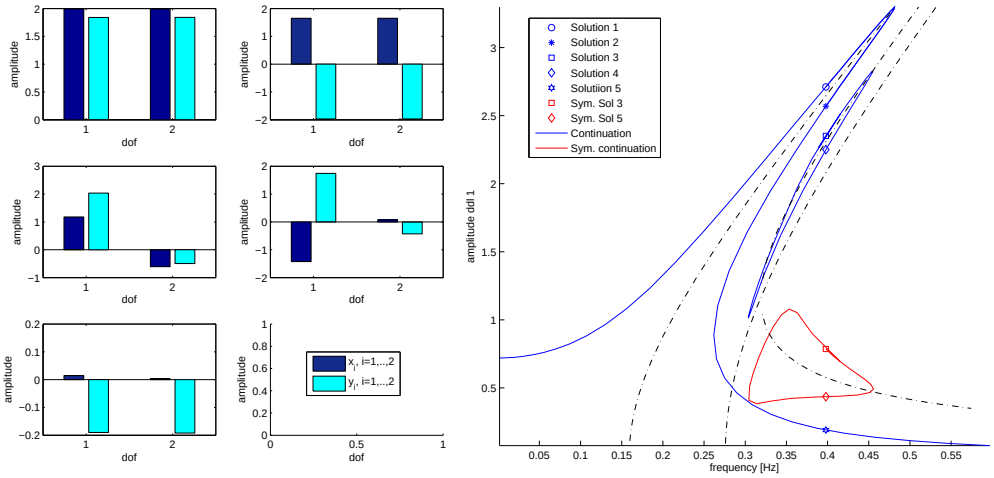

Fig. 2. left: Form of the real solutions of system (12) found by the invariant multiplication matrix method. Right: Frequency continuation of the solution obtained at $f = \frac{1}{2\pi}\frac{25}{10}$ and their symmetric relative to the group operation

Assessment of the solution's quality is given in Table 2. Note that solutions from the eigenvalue method are close to the actual roots of $\boldsymbol{P}$, as their relative differences lie below 3%.

solution	1	2	3	4	5
value $\|\boldsymbol{P}(\boldsymbol{x}^*)\|$	0.00	0.00	0.00	0.00	0.02
relative diff. from NR sol. (%)	0.03	0.02	0.00	0.00	2.80

Table 2. Assessment of the solutions quality for (12) at $\omega = \frac{25}{10}$

To obtain the full set of solution, we apply the group actions and generate 4 more solutions, leading to a total of 11 solutions (7 real and 4 complex) as indicated by the dimension of the quotient space.

The application of the continuation procedure for the 5 real solutions from the invariant system (Fig.2) shows that 3 solutions belong to the principale resonance curve, and that 2 solutions belong to closed curves corresponding to a localized motion. The application of the group action generates another closed curve solution corresponding to the change of coordinates $(u_1, u_2) \to (u_2, u_1)$ in the dynamic system (11). All forced solutions are positioned around the backbone curves coming from the monophase NNM analysis.

4.2 Simple example with 4 degrees of freedom

For $N = 4$, the application of the HBM with one harmonic on (2) leads to the following system:

$$\begin{array}{l} \alpha x_i - \beta x_{i+1} - \beta x_{i-1} + \delta y_i + \gamma x_i(x_i^2 + y_i^2) = f_i^c, \quad i = 1, \ldots, 4, \\ \alpha y_i - \beta y_{i+1} - \beta y_{i-1} - \delta x_i + \gamma y_i(x_i^2 + y_i^2) = f_i^s, \quad i = 1, \ldots, 4, \end{array} \tag{17}$$

with $\alpha = k + 2k_c - \omega^2 m$, $\beta = k_c$, $\gamma = \frac{3}{4}k_{nl}$ and $\delta = \omega c$. In the NNM analysis, the frequency parameter will be set to $\omega = \frac{31}{10}$, leading to the following numerical values:

$$\alpha = \frac{-661}{100},\ \beta = 1, \gamma = \frac{3}{4},\ \delta = \frac{1}{10},\ f_c = 1,\ f_s = 0.$$

In the forced analysis, the angular frequency will be set by $\omega = \frac{25}{10}$, leading to the numerical values in (13) several values of the frequency parameter will be considered.

Monophase NNM analysis. For the monophase analysis the system is the following:

$$\alpha x_i - \beta x_{i+1} - \beta x_{i-1}\gamma x_i^3 = 0, \quad i = 1, \ldots, 4. \tag{18}$$

As in the previous example, the system is already in a gröbner basis form for the grevlex order, and the dimension of the quotient pace is given by $\dim(A) = 81$ (the system has 81 solutions).

The invariance group is taken as $\mathcal{G} = \mathcal{C}_4 \times \mathcal{Z}_2$, where $\mathcal{C}_4$ correspond to the cyclic group with 4 elements (cyclic symmetry), and $\mathcal{Z}_2$ is the group relative to the change of sign as in the previous section. The primary invariant of $\mathcal{G}$ are given by:

$$\pi_1 = x_1x_3 + x_2x_4,\ \pi_2 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1,\ \pi_3 = x_1^2 + x_2^2 + x_3^2 + x_4^2,\ \pi_4 = x_1x_2x_3x_4.$$

The application of Algorithm 1 leads to the construction of a basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ with 14 elements. Following method exposed in the previous section, 14 real solutions are obtained by solving the invariant systems, and their forms are depicted in Fig.3. The assessment of the solutions quality is given in Table 3, showing that all solutions of the invariant systems are indeed solutions of the polynomial system $\boldsymbol{P}$.

solution	1	2	3	4	5	6	7	8	9	10	11	12	13	14
residual $\|\boldsymbol{P}(\boldsymbol{x}^*)\|$	0.58	0.68	0.00	0.23	0.00	0.97	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
rel. diff. from NR sol. (%)	x	0.74	0.00	0.90	0.01	2.11	0.01	0.01	0.01	0.01	0.01	0.00	0.00	0.00

Table 3. Assessment of the solutions quality for (18) at $\omega = \frac{31}{10}$

solution	1	2	3	4	5	6	7	8	9	10	11	12	13	14	total
occurence	1	2	8	2	8	8	4	8	8	8	4	4	8	8	81

Table 4. Application of the group action to the solution of (18): number of generated solutions

The total set of solution is generated by applying the group action (see Table 4) leading to 81 solutions.

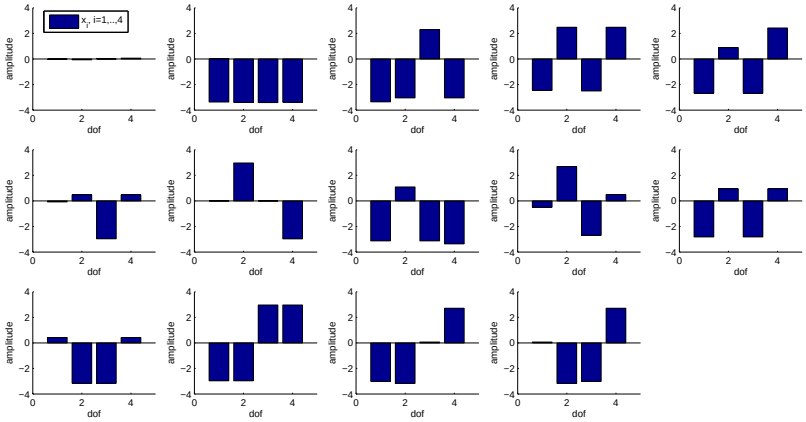

Fig. 3. Form of the real solutions of system (18) found by the invariant multiplication matrix method

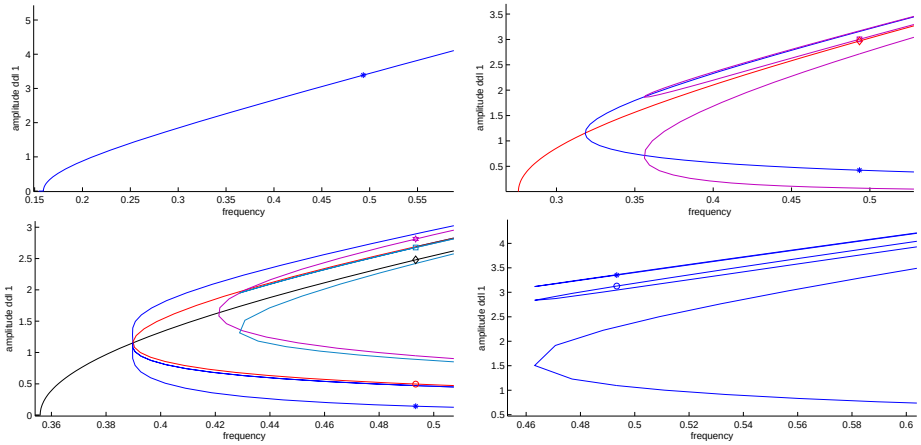

Fig. 4. Frequency continuation of the solution obtained at $f = \frac{1}{2\pi}\frac{31}{10}$ and their symmetric relative to the group operation (only positive amplitudes of the first dof are depicted). From top left to botom right: Mode 1 (solution 2); Mode 2 (solutions 7, 11, 12, 13, 14); Mode 3 (solutions 4, 5 ,6 ,9, 10); Disconnected solutions (solutions 3, 8)

Forced analysis. We now turn to the forced analysis of system (17). First, the angular frequency parameter is set to $\omega = \frac{25}{10}$. In this case the computation of a Gröbner basis and a normal set for the grevlex order tells us that the quotient space A is of dimension 147. The invariant group $\mathcal{G}$ is the dihedral group $\mathcal{D}_4$ of order 4 represented in $\mathbb{R}^8$ by the following matrices:

$$\mathbf{M}_r = \begin{bmatrix} 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \\ I_2 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}_s = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \\ 0 & I_2 & 0 & 0 \end{bmatrix}.$$

The primary invariant of $\mathcal{G}$ are given by:

$$\pi_1 = y_1 + y_2 + y_3 + y_4, \quad \pi_2 = x_1 + x_2 + x_3 + x_4, \quad \pi_3 = y_1y_3 + y_2y_4,$$
$$\pi_4 = y_1x_3 + y_3x_1 + y_2x_4 + y_4x_2, \quad \pi_5 = x_1x_3 + x_2x_4, \quad \pi_6 = y_1y_2 + y_2y_3 + y_3y_4 + y_4y_1,$$
$$\pi_7 = x_1x_2x_3x_4, \quad \pi_8 = x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_1^2 + y_1y_2y_3y_4.$$

With Algorithm 1 we compute a basis $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ with 33 elements, and the multiplication matrix associated to the polynomial $f = \sum_i c_i \pi_i$ is also of size 33. In this case all primary invariant are in the basis except for π_7, for which we compute its normal form and express it in term of elements of $\boldsymbol{\mathcal{B}}^{\mathcal{G}}$ as $\pi_7 = \boldsymbol{c}^T \boldsymbol{\mathcal{B}}^{\mathcal{G}}$. The solution of the eigenvalue problem then leads to 33 possible values (5 real and 28 complex) for the primary invariants. Finally the solution of the 5 real invariant systems lead to 5 real solutions of the polynomial system $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{0}$ depicted on Fig. 5.

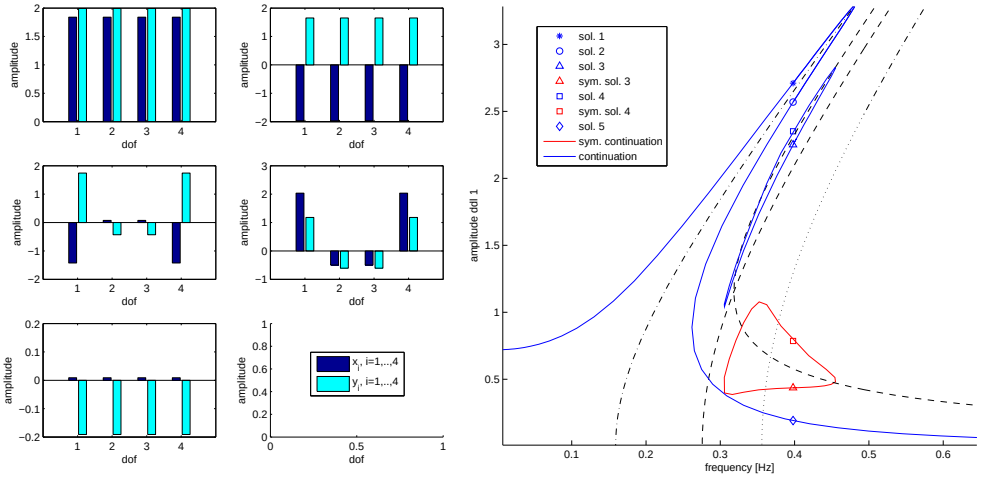

Fig. 5. Left: Form of the real solutions of system (17) found by the invariant multiplication matrix method at $\omega = \frac{25}{10}$. Right: Frequency continuation of the solution obtained at $f = \frac{1}{2\pi}\frac{25}{10}$ and their symmetric relative to the group operation. The backbone curve of NNM 1, NNM 2, NNM3 and a bifurcation of NNM 2 are also depicted

The application of the group's actions on the real solutions generates only two other solutions (i.e., the symmetric of solution 3 and 4). The frequency continuation of the solutions is depicted on Fig. 5. Again, three solutions belong to the principal resonance curve (corresponding to a motion shape on the first NNM), and two solutions belong to a closed curve solution corresponding to a motion shape on a bifurcation of the second NNM (i.e., a localized motion on only two dof corresponding to the monophase NNM solution 11 in Fig. 3).

4.3 NNM analysis for $3 \leq N \leq 6$

In this last application, we consider the monophase NNM analysis of system (2). The application of the harmonic balance method, leads to the polynomial system (6). In order to illustrate the reduction in the number of solution, Algorithm 1 is applied for N from 3 to 6. The invariance groupe is taken as $\mathcal{G} = \mathcal{C}_N \times \mathcal{Z}_2$, where $\mathcal{Z}_2$ is related to the transformation $\boldsymbol{x} \to -\boldsymbol{x}$. Results are summarized in Table 5.

N	$\dim(\mathbb{C}[\boldsymbol{x}]/\mathcal{I})$	$\dim(\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}^{\mathcal{G}})$	reduction ratio
3	27	6	22.22%
4	81	14	17.2%
5	243	26	10.70%
6	729	68	9.33%

Table 5. Application of Algorithm 1 on (6) for $3 \leq N \leq 6$

It can be seen that taking into account symmetry decrease the number of solution down to 10% of the total number of solution. This number should be even smaller if taking into account invariance by reflection (i.e., $\mathcal{G} = \mathcal{C}_N \times \mathcal{Z}_2 \times \mathcal{Z}_2$). In all cases, the resolution of the invariant problems leads to a maximum number of real solutions for the polynomial system (6) (i.e., the system has $\dim(\mathbb{C}[\boldsymbol{x}]^{\mathcal{G}}/\mathcal{I}^{\mathcal{G}})$ real solutions).

This application also shows the limitation of the proposed method. Indeed, the computation of primary invariants for the dihedral group $\mathcal{D}_N$ is very time consuming when $N > 6$. However, further investigations should be carried to see if there exist a way to directly compute the primary invariant of the dihedral group for large N.

5 Discussion, Conclusion

This paper present the application of the so called eigenvalue method with symmetry for solving polynomial systems arising in the vibrations study of nonlinear mechanical structures by the harmonic balance method. The system under consideration correspond to N duffing oscillators, linearly coupled. The application of the harmonic balance method with one harmonic on this system generates polynomial equations, which are invariant under some transformations (cyclic permutation, change of sign, ...).

The application of the eigenvalue method with symmetry for solving the invariant polynomial system shows that this method is well adapted for this kind of problem. Indeed, taking into account symmetry can greatly decrease the size of the multiplication matrix. Each obtained solution is different and corresponds to a unique orbit of solutions that can be generated by applying the group's actions. Moreover, the obtained solutions are very close to the actual solutions of the polynomial system, even in the presence of rounding-off errors.

The best results are obtained when searching for free solutions (NNM) of the dynamic system. In the forced case, the method is only interesting when the spacial distribution of the excitation also presents symmetry properties. In the worst case scenario (symmetry breaking excitation) the system is not longer invariant, and the method no longer applicable.

Further applications to larger systems seems limited by several factors. The first drawback is related to Gröbner basis computation. For large number of variables, it can take a great amount of time even with the grevlex ordering. Second, it is not clear how to efficiently find primary invariants of large groups such a $\mathcal{D}_N$ or $\mathcal{D}_N \times \mathcal{Z}_2$ for large N. However, the computation of the invariants is needed only once per invariance group as they can be reused for any subsequent computation on system having the same invariance properties.

Although this method has limitations, we have to recall that numerical methods, such has homotopie, are also subjected to limitations that restrict the size of the polynomial system to be solved. In this context, the fact that the eigenvalue method with symmetry automatically sorts the solutions (i.e., computes only one representative of each orbits) is an improvement as it simplifies the analysis of the system.

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