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# On a shape derivative formula for a family of star-shaped domains

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## Abstract

In this work, we consider again the shape derivative formula for a volume cost functional which we studied in preceding papers where we used the Minkowski deformation and the support functions in the convex setting. Here, we extend it to some non convex domains, namely the star-shaped ones. The formula also happens to be an extension of a well known one in the geometric Brunn-Minkowski theory of convex bodies. Finally, we illustrate the formula by applying it to some model shape optimization problem and by giving an algorithm based on the gradient method to compute the shape derivative.

**Keywords** shape optimization, shape derivative, volume functional, convex domain, starshaped domain, support function, gauge function, Minkowski deformation, Brunn-Minkowski.

## 1 Introduction

This paper deals with a generalization of a shape derivative formula for a volume cost functional with respect to a class of convex domains, a formula that we already studied in [2, 3], and our aim is to extend it to non convex domains. To be precise, consider the shape functional  $\mathcal{J}$  defined by

$$\mathcal{J}(\Omega) = \int_{\Omega} f(x) dx,$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  and  $f$  is a fixed function defined in  $\mathbb{R}^n$ .

Using the deformation  $(1 - \varepsilon)\Omega_0 + \varepsilon\Omega$ ,  $\varepsilon \in ]0, 1[$ , of  $\Omega_0$  and a  $C^1$  function  $f$ , A. A. Niftiyev and Y. Gasimov [19] first gave the expression of the shape derivative of  $\mathcal{J}$  with respect to the class of convex domains of class  $C^2$  by means of support functions:

**Theorem** (A. Niftiyev, Y. Gasimov) *If  $\Omega_0, \Omega$  are bounded convex domains of class  $C^2$  and the function  $f$  is of class  $C^1$ , then, the limit*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{J((1 - \varepsilon)\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon}$$

*exists and is equal to*

$$\int_{\partial\Omega_0} f(x) (P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x))) d\sigma(x), \quad (1)$$

*where  $\nu_0(x)$  denotes the outward unit normal vector to  $\partial\Omega_0$  at  $x$ , and  $P_{\Omega_0}, P_\Omega$  are the support functions of the domains  $\Omega_0, \Omega$ , respectively.*

Recently, A. Boulkhemair and A. Chakib [3] extended this formula to the case where  $f$  is in the Sobolev space  $W_{loc}^{1,1}(\mathbb{R}^n)$ . Inspired by the Brunn-Minkowski theory (see, for example, R. Schneider, [22]), they also proposed a similar shape derivative formula by considering the Minkowski deformation  $\Omega_0 + \varepsilon\Omega$  of  $\Omega_0$  :

**Theorem** (A. Boulkhemair, A. Chakib) *If  $\Omega_0, \Omega$  are bounded convex domains of class  $C^2$  and the function  $f$  is in the Sobolev space  $W_{loc}^{1,1}(\mathbb{R}^n)$ , then, the limit*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{J(\Omega_0 + \varepsilon\Omega) - J(\Omega_0)}{\varepsilon}$$

*exists and is equal to*

$$\int_{\partial\Omega_0} f(x) P_\Omega(\nu_0(x)) d\sigma(x), \quad (2)$$

*where  $\nu_0(x)$  denotes the outward unit normal vector to  $\partial\Omega_0$  at  $x$ , and  $P_\Omega$  is the support function of the domain  $\Omega$ .*

In fact, this formula holds true even for bounded convex domains, see [2].

If one compares (1) and (2), one can easily remark that, unlike the first formula, the second one does not depend on the support function of  $\Omega_0$ . This suggests that (2) should hold true for non convex  $\Omega_0$ , which would be very interesting for applications in shape optimization. Unfortunately, up to now we have not been able to treat the case of general non convex domains. In this paper, we shall extend formula (2) to the case where  $\Omega_0$  is a star-shaped domain of class  $C^2$ . In fact, we were naturally led to star-shapedness because one can take again parts of the idea of proof of [2,3], mainly the use of gauge functions. Note that, by using such a method, this is the best result one can obtain since the star-shaped domains are exactly the sublevel sets of non negative continuous homogeneous functions. Thus, to be able to treat the case of more general non convex domains, one has to find new ideas. Anyhow, we shall return to this problem in a future work.

Originally, even if it is a theoretical one and like the previous ones [2,3], this work was motivated by numerical approximation in shape optimization problems, since it is indeed the most difficult aspect of this subject. We refer to [1], for example, for explanations about the issues that arise when implementing numerically the minimization of a shape integral functional, via some gradient method, by using the usual expression of the shape derivative by means of vector fields. Briefly, the reason is that, when using vector fields, at each iteration we have to extend the vector field (obtained only on the boundary) to all the domain or to re-mesh, and both approaches are expensive. On the other hand, when we use support functions, at each iteration, we get not only a set of boundary points but also a support function which, by taking its sub-differential at the origin, gives the next domain. This is why we are interested in the above formulas that is, expressions that use support functions instead of vector fields. This is illustrated in the last section, where we apply the formulas

to the computation of the shape derivative of a simple shape optimization problem by means of an algorithm based on the gradient method. Anyway, these formulas are actually applied and implemented in the forthcoming paper [6].

Another important motivation for this work came from the fact that, when  $f = 1$ , (2) is a well known formula in the Brunn-Minkowski theory of convex bodies, see [22] for example. Indeed, when  $\Omega_0$  and  $\Omega$  are bounded convex domains, we know from that theory that one can write

$$V(\overline{\Omega}_0 + t\overline{\Omega}) = \sum_{j=0}^n \binom{n}{j} t^j V_j(\overline{\Omega}_0, \overline{\Omega}), \quad t \geq 0,$$

where  $V$  denotes the volume, that is, the  $n$ -dimensional Lebesgue measure, and the coefficients  $V_j(\overline{\Omega}_0, \overline{\Omega})$  are what one calls *mixed volumes* of  $\overline{\Omega}_0$  and  $\overline{\Omega}$  and are significant in convex geometry, see [22], [17] or [23] for example. Let us first remark that the first mixed volume  $V_0(\overline{\Omega}_0, \overline{\Omega})$  is simply the volume  $V(\overline{\Omega}_0)$ . Next, it is known since a long time that

$$(V(\overline{\Omega}_0 + t\overline{\Omega}) - V(\overline{\Omega}_0))/t = \sum_{j=1}^n \binom{n}{j} t^{j-1} V_j(\overline{\Omega}_0, \overline{\Omega}) \longrightarrow n V_1(\overline{\Omega}_0, \overline{\Omega}) = \int_{\partial\Omega_0} P_{\Omega}(\nu_0(x)) \, d\sigma(x)$$

as  $t \rightarrow 0^+$ . Thus, (2) is an extension of the above formula to the case where  $f$  is not necessarily 1 and the aim of this work is to extend (2) (and the above formula, of course) to a case where  $\Omega_0$  is not necessarily convex.

Concerning the method of proof, we follow in fact the same plan as that of [3,4]. More precisely, we first assume that the deformation domain  $\Omega$  is strongly convex, which allows us to construct some parameterization of the deformed domain  $\Omega_0 + \varepsilon\Omega$  by means of some  $C^1$ -diffeomorphism defined on  $\Omega_0$ . The construction is based on some analytical and geometric properties of gauge and support functions of star-shaped domains, and reduces the problem to the usual computation of the shape derivative using vector fields. Note also that the construction of the  $C^1$ -diffeomorphism relies on a crucial result about the inclusion of the convolution of two hypersurfaces. Finally, the case of a general convex  $\Omega$  is treated by using the approximation of  $\Omega$  by a sequence of strongly convex domains and is based on some crucial analytical and geometric lemmas. We quote in particular the result on the continuity of the gauge function with respect to star-shaped domains by means of the Hausdorff distance.

The plan of the paper is as follows. In section 2, we recall some facts about star-shaped domains and give their proofs. The main results are stated in Section 3 where we also prove consequences of Formula (2) to the situation where the function  $f$  depends also on domains, which is customary in shape optimization problems. The fourth section is devoted to the proof of the main results using several lemmas. Finally, in Section 5, in order to illustrate these results, we give an application to a model shape optimization problem and an algorithm for solving this type of problem based on the gradient method.

## 2 Preliminaries on star-shaped domains

There are several definitions of what is called a star-shaped set in the literature. Here, we shall use the following one:

**Definition 1.** *An open subset (or a domain)  $\Omega$  of  $\mathbb{R}^n$  is said to be star-shaped with respect to some  $x_0 \in \Omega$ , if for all  $x \in \overline{\Omega}$ ,  $\Omega$  contains the segment  $[x_0, x] = \{(1-t)x_0 + tx; 0 \leq t < 1\}$ .*

It follows from this definition that the domain  $\Omega$  is convex if and only if it is star-shaped with respect to each  $x_0 \in \Omega$ .

In what follows, we shall often work with bounded domains which are star-shaped with respect to 0. The reason for this is that such domains are naturally associated to *gauge functions* like the convex domains. So, let  $\Omega$  be a bounded domain which is star-shaped with respect to 0. For each  $x \in \mathbb{R}^n$ , consider the following set of positive real numbers

$$\{\lambda; \lambda > 0, x \in \lambda\Omega\},$$

which is always non empty since  $\Omega$  is a neighborhood of 0. By definition, the gauge function associated to  $\Omega$  is the real function  $J_\Omega : \mathbb{R}^n \rightarrow \mathbb{R}_+$  given by

$$J_\Omega(x) = \inf\{\lambda; \lambda > 0, x \in \lambda\Omega\}.$$

Like for the convex bodies, the gauge functions characterize the star-shaped domains they are associated to. In the following proposition, we summarize their main properties.

**Proposition 1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain which is star-shaped with respect to 0. Then, the gauge function  $J_\Omega$  is a non negative continuous positively homogeneous function of degree 1. More precisely, we have the following properties:*

- (i)  $J_\Omega(0) = 0, J_\Omega(x) > 0, \forall x \neq 0.$
- (ii)  $J_\Omega(tx) = tJ_\Omega(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}_+.$
- (iii)  $\Omega = \{x \in \mathbb{R}^n; J_\Omega(x) < 1\}.$
- (iv)  $\partial\Omega = \{x \in \mathbb{R}^n; J_\Omega(x) = 1\}.$
- (v)  $J_\Omega : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous.
- (vi) If  $\Omega'$  is another domain which is star-shaped with respect to 0 and  $\Omega' \subset \Omega$ , then,  $J_\Omega \leq J_{\Omega'}.$

**Proof.** (i), (ii) and (vi) are easy consequences of the definition of  $J_\Omega$  and the fact that  $\Omega$  is a bounded neighborhood of 0.

(iii) : As it follows from the definition, if  $J_\Omega(x) < 1$ , we have  $x \in \lambda\Omega$  for all  $\lambda > J_\Omega(x)$ , and in particular for  $\lambda = 1$ . Conversely, if  $x \in \Omega$ , we have, by definition, only  $J_\Omega(x) \leq 1$ . But the fact that  $\Omega$  is open and star-shaped with respect to 0 implies that  $x \in \lambda\Omega$  for some  $\lambda < 1$ ; hence,  $J_\Omega(x) < 1$ .

(iv) : It follows from (iii) that if  $x \in \partial\Omega$ , then  $J_\Omega(x) \geq 1$ . Now, by star-shapedness, we have  $tx \in \Omega, \forall t \in [0, 1[$  which implies  $J_\Omega(tx) = tJ_\Omega(x) < 1, \forall t \in [0, 1[$ ; hence,  $J_\Omega(x) < \lambda, \forall \lambda > 1$  which implies  $J_\Omega(x) \leq 1$ , and so  $J_\Omega(x) = 1$ . Conversely, if  $J_\Omega(x) = 1$ , we have  $x \notin \Omega$  and  $x \in \lambda\Omega$  for all  $\lambda > 1$  which implies that  $tx \in \Omega, \forall t \in ]0, 1[$ . Since  $tx \rightarrow x$  when  $t \rightarrow 1^-$ , we obtain that  $x \in \partial\Omega$ .

(v) : Since  $J_\Omega$  is a positively homogeneous function, to prove its continuity it suffices to show that  $\{J_\Omega < 1\}$  is an open set and that  $\{J_\Omega \leq 1\}$  is a closed set. But this follows from (iii) and (iv) since  $\{J_\Omega < 1\} = \Omega$  is open and  $\{J_\Omega \leq 1\} = \overline{\Omega}$  is closed. ■

It is well known in convex analysis that the gauge function of any convex domain is Lipschitz continuous. This is no longer true for star-shaped domains. Since such a Lipschitz regularity will be needed in the sequel, in fact, we shall work exactly with the star-shaped domains whose gauge functions are Lipschitz continuous. In order to be able to describe geometrically this subfamily of domains, let us give the following definition.

**Definition 2.** *An open set  $\Omega \subset \mathbb{R}^n$  is said to be star-shaped with respect to a subset  $G \subset \Omega$ , if it is star-shaped with respect to any point of  $G$ .*

This definition allows us to characterize in a simple manner the star-shaped domains whose gauge functions are Lipschitz continuous. This is done in the following result for which we provide a new and simple proof (see also [7, 11]).

**Proposition 2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain which is star-shaped with respect to 0. Then, its gauge function  $J_\Omega$  is Lipschitz continuous if and only if  $\Omega$  is star-shaped with respect to some ball  $B(0, r) \subset \Omega$  centered at 0 and with radius  $r > 0$ . Moreover, when this condition is satisfied, one can take  $1/r$  as a Lipschitz constant for  $J_\Omega$ .*

**Proof.** Assume first that  $J_\Omega$  satisfies the inequality  $|J_\Omega(y) - J_\Omega(x)| \leq \frac{1}{r}|y - x|$  for all  $x, y \in \mathbb{R}^n$ , and let us show that  $\Omega$  is star-shaped with respect to the ball  $B(0, r)$ . For all  $y \in B(0, r)$ ,  $x \in \overline{\Omega}$  and  $t \in [0, 1[$ , it follows from the assumption that

$$J_\Omega((1-t)y + tx) \leq J_\Omega(tx) + \frac{1}{r}|(1-t)y| < t.1 + \frac{1}{r}(1-t)r = 1,$$

which says exactly that  $(1-t)y + tx \in \Omega$  for all  $y \in B(0, r)$ ,  $x \in \overline{\Omega}$  and  $t \in [0, 1[$ , that is,  $\Omega$  is star-shaped with respect to the ball  $B(0, r)$ .

Conversely, assume that  $\Omega$  is star-shaped with respect to the ball  $B(0, r)$ ,  $r > 0$ . For each  $x \in \partial\Omega$ , consider the convex hull of the set  $\overline{B}(0, r) \cup \{x\}$  and denote by  $\Omega_x$  its interior. Clearly,  $\Omega_x$  is a convex domain and a subset of  $\Omega$ . Thus, it follows from Proposition 1 (vi) that

$$J_\Omega \leq J_{\Omega_x} \leq J_{B(0, r)}.$$

Hence, we can write, for all  $y \in \mathbb{R}^n$ ,

$$J_\Omega(y) \leq J_{\Omega_x}(y) \leq J_{\Omega_x}(x) + J_{\Omega_x}(y - x) \leq J_\Omega(x) + J_{B(0, r)}(y - x) \leq J_\Omega(x) + \frac{1}{r}|y - x|,$$

since  $J_{\Omega_x}$  is a convex function,  $J_\Omega(x) = 1 = J_{\Omega_x}(x)$  and  $J_{B(0, r)}(z) = |z|/r$ . So,  $J_\Omega(y) - J_\Omega(x) \leq \frac{1}{r}|y - x|$  under the assumption  $x \in \partial\Omega$ . This is also true if  $x = 0$  and when  $x \neq 0$ , it follows from this inequality, since  $x/J_\Omega(x)$  is on  $\partial\Omega$ , that

$$J_\Omega\left(\frac{y}{J_\Omega(x)}\right) - J_\Omega\left(\frac{x}{J_\Omega(x)}\right) \leq \frac{1}{r}\left|\frac{y}{J_\Omega(x)} - \frac{x}{J_\Omega(x)}\right|,$$

which implies by homogeneity that  $J_\Omega(y) - J_\Omega(x) \leq \frac{1}{r}|y - x|$  for all  $x, y \in \mathbb{R}^n$ , and, by symmetry, the Lipschitz continuity of  $J_\Omega$ . ■

We shall also need the following technical result. Note here that the scalar product in  $\mathbb{R}^n$  of  $x$  by  $y$  is denoted in what follows by  $\langle x, y \rangle$  or sometimes by  $x.y$ .

**Lemma 1.** *If  $\Omega \subset \mathbb{R}^n$  is a bounded domain which is star-shaped with respect to a ball  $B(0, r)$ , then, we have*

$$\langle \nu(x), x \rangle \geq r,$$

for almost every  $x \in \partial\Omega$ , where  $\nu(x)$  is the outward unit normal vector at  $x$ .

**Proof.** It follows from Proposition 1 that

$$\Omega = \{x \in \mathbb{R}^n; J_\Omega(x) < 1\}, \quad \partial\Omega = \{x \in \mathbb{R}^n; J_{\Omega_0}(x) = 1\}, \quad (3)$$

and from Proposition 2 that  $J_\Omega$  is Lipschitz continuous with a Lipschitz constant equal to  $\frac{1}{r}$ , that is, for all  $x, y \in \mathbb{R}^n$ ,

$$|J_\Omega(x) - J_\Omega(y)| \leq \frac{1}{r}|x - y|,$$

which implies that  $|\nabla J_\Omega| \leq \frac{1}{r}$  a.e. according to Rademacher's theorem (see [14], for example), and that the outward unit normal vector is given by

$$\nu(x) = \frac{\nabla J_\Omega(x)}{|\nabla J_\Omega(x)|}$$

for almost every  $x \in \partial\Omega$ . Moreover, since  $J_\Omega$  is positively homogeneous, we have, for every  $x \in \partial\Omega$ ,

$$\langle \nabla J_\Omega(x), x \rangle = \lim_{t \rightarrow 0} \frac{J_\Omega(x + tx) - J_\Omega(x)}{t} = \lim_{t \rightarrow 0} \frac{(1+t)J_\Omega(x) - J_\Omega(x)}{t} = J_\Omega(x) = 1,$$

which implies that

$$\langle \nu(x), x \rangle = \left\langle \frac{\nabla J_\Omega(x)}{|\nabla J_\Omega(x)|}, x \right\rangle = \frac{1}{|\nabla J_\Omega(x)|} \geq r,$$

for almost every  $x \in \partial\Omega$ . ■

Like for the convex domains, the regularity of a star-shaped domain with respect to a ball is that of its gauge function:

**Lemma 2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain which is star-shaped with respect to a ball  $B(0, r)$ ,  $r > 0$ . Then,  $\Omega$  is of class  $C^k$ ,  $k \geq 1$ , if and only if its gauge function  $J_\Omega$  is of class  $C^k$  in  $\mathbb{R}^n \setminus \{0\}$ .*

The proof of this last result is the same as that given in [3] in the case of convex domains and makes use of the fact that  $\langle \nu(x), x \rangle$  does not vanish which is insured by Lemma 1 in our case. So, we refer to it.

Finally, the following result will also be used:

**Proposition 3.** *Let  $(\Phi_\varepsilon)_{0 \leq \varepsilon \leq \varepsilon_0}$  be a family of  $C^1$  diffeomorphisms from  $\mathbb{R}^n$  onto  $\mathbb{R}^n$  such that  $\Phi_0(x) = x$  and  $(\varepsilon, x) \mapsto \Phi_\varepsilon(x)$  and  $(\varepsilon, y) \mapsto \Phi_\varepsilon^{-1}(y)$  are of class  $C^1$  in  $[0, \varepsilon_0] \times \mathbb{R}^n$ . Then, for all  $f \in W_{loc}^{1,1}(\mathbb{R}^n)$ , the limit  $\lim_{\varepsilon \rightarrow 0^+} (f(\Phi_\varepsilon(x)) - f(x))/\varepsilon$  exists in  $L_{loc}^1(\mathbb{R}^n)$  and is equal to  $\nabla f(x) \cdot \frac{d}{d\varepsilon} \Phi_\varepsilon(x)|_{\varepsilon=0}$ .*

For a proof of this lemma, see [16], Chapter 5.

### 3 Main results

Let us first define the set of admissible domains  $\mathcal{U}$  to be the set of bounded open subset of  $\mathbb{R}^n$  which are of class  $C^2$  and star-shaped with respect to some ball.

Recall that the support function  $P_\Omega$  of a bounded convex domain  $\Omega$  is given by

$$P_\Omega(x) = \sup_{y \in \Omega} x \cdot y = \sup_{y \in \bar{\Omega}} x \cdot y,$$

where  $x \cdot y$  denotes the standard scalar product of  $x$  and  $y$  in  $\mathbb{R}^n$ , a product that we shall also denote sometimes by  $\langle x, y \rangle$ .

We can now state the first result of this paper which concerns the shape derivative of the volume functional

$$\Omega \mapsto \mathcal{J}(\Omega) = \int_{\Omega} f(x) dx.$$

**Theorem 1.** *Let  $\Omega_0 \in \mathcal{U}$ ,  $\Omega$  be a bounded convex domain and  $f \in W^{1,1}(D)$  where  $D$  is a large smooth bounded domain which contains all the sets  $\Omega_0 + \varepsilon\Omega$ ,  $\varepsilon \in [0, 1]$ . Then, we have*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x) P_{\Omega_0}(\nu_0(x)) d\sigma(x). \quad (4)$$

where  $\nu_0$  denotes the outward unit normal vector on  $\partial\Omega_0$ .

The proof of this theorem will be given in the following section. Here, we state and prove a corollary of this result which treats a case that occurs frequently in the applications, that is, the case where the function  $f$  itself depends on the parameter  $\varepsilon$ .

**Corollary 1.** Let  $\Omega_0$ ,  $\Omega$  and  $D$  be as in Theorem 1, let  $(f_\varepsilon)$ ,  $0 \leq \varepsilon \leq 1$ , be a family of functions in  $L^1(D)$  such that  $f_0 \in W^{1,1}(D)$  and let  $h$  be a function such that  $(f_\varepsilon - f_0)/\varepsilon \rightarrow h$  in  $L^1(D)$  as  $\varepsilon \rightarrow 0^+$ . Let us set  $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$  and

$$I(\varepsilon) = \int_{\Omega_\varepsilon} f_\varepsilon(x) dx.$$

Then, we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{I(\varepsilon) - I(0)}{\varepsilon} = \int_{\Omega_0} h(x) dx + \int_{\partial\Omega_0} f_0(x) P_\Omega(\nu_0(x)) d\sigma(x). \quad (5)$$

**Proof.** We write

$$\frac{I(\varepsilon) - I(0)}{\varepsilon} = \int_{\Omega_\varepsilon} \left( \frac{1}{\varepsilon} (f_\varepsilon - f_0)(x) - h(x) \right) dx + \int_{\Omega_\varepsilon} h(x) dx + \frac{1}{\varepsilon} \left( \int_{\Omega_\varepsilon} f_0(x) dx - \int_{\Omega_0} f_0(x) dx \right),$$

and then we study each of the three terms of the right hand side of this equality. It follows from the assumption that

$$\left| \int_{\Omega_\varepsilon} \left( \frac{1}{\varepsilon} (f_\varepsilon - f_0)(x) - h(x) \right) dx \right| \leq \int_D \left| \frac{1}{\varepsilon} (f_\varepsilon - f_0)(x) - h(x) \right| dx \xrightarrow{\varepsilon \rightarrow 0^+} 0.$$

On the other hand, since the characteristic functions of  $\Omega_\varepsilon$  converge almost everywhere to the characteristic function of  $\Omega_0$  when  $\varepsilon \rightarrow 0^+$ , it follows from the Lebesgue convergence theorem and from (4) that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{I(\varepsilon) - I(0)}{\varepsilon} = \int_{\Omega_0} h(x) dx + \int_{\partial\Omega_0} f_0(x) P_\Omega(\nu_0(x)) d\sigma(x).$$

■

## 4 Proof of Theorem 1

Note first that one can assume that  $f \in W_{loc}^{1,1}(\mathbb{R}^n)$  or even  $f \in W^{1,1}(\mathbb{R}^n)$ . Indeed, one can reduce to this case just by extending the function  $f$  to  $\mathbb{R}^n$  by means of the usual results on Sobolev spaces.

We follow the same idea as [3], that is, we treat first the case where the deformation domain  $\Omega$  is strongly convex, the general case being obtained by means of an appropriate approximation.

To be able to use gauge functions, we have to assume that  $\Omega_0$  and  $\Omega$  are neighborhoods of 0. However, this is not a restriction of generality. Indeed, assume that Theorem 1 (and hence also Corollary 1) is proved in this case, then, if  $\Omega_0$  and  $\Omega$  are neighborhoods of 0 and  $c_0, c \in \mathbb{R}^n$ , we have, by obvious changes of variables,

$$\begin{aligned} (\mathcal{J}(c_0 + \Omega_0 + \varepsilon(c + \Omega)) - \mathcal{J}(c_0 + \Omega_0))/\varepsilon &= (\mathcal{J}(c_0 + \varepsilon c + \Omega_\varepsilon) - \mathcal{J}(c_0 + \Omega_0))/\varepsilon \\ &= \frac{1}{\varepsilon} \left( \int_{\Omega_\varepsilon} f(c_0 + \varepsilon c + x) dx - \int_{\Omega_0} f(c_0 + x) dx \right). \end{aligned}$$

It follows then from Proposition 3 that

$$\frac{f(x + c_0 + \varepsilon c) - f(x + c_0)}{\varepsilon} \longrightarrow \nabla f(x + c_0) \cdot c = \operatorname{div}(f(x + c_0)c) \quad \text{in } L_{loc}^1(\mathbb{R}^n)$$

as  $\varepsilon \rightarrow 0^+$ , and from Corollary 1 that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(c_0 + \Omega_0 + \varepsilon(c + \Omega)) - \mathcal{J}(c_0 + \Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x + c_0) P_\Omega(\nu_0(x)) d\sigma(x) + \int_{\Omega_0} \operatorname{div}(f(x + c_0)c) dx.$$



Now, it remains to apply the divergence formula to get

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(c_0 + \Omega_0 + \varepsilon(c + \Omega)) - \mathcal{J}(c_0 + \Omega_0)}{\varepsilon} &= \int_{\partial\Omega_0} f(x + c_0) P_\Omega(\nu_0(x)) d\sigma + \int_{\partial\Omega_0} f(x + c_0) c \cdot \nu_0(x) d\sigma \\
&= \int_{\partial\Omega_0} f(x + c_0) P_{c+\Omega}(\nu_0(x)) d\sigma \\
&= \int_{\partial(c_0 + \Omega_0)} f(x) P_{c+\Omega}(\nu_{c_0 + \Omega_0}(x)) d\sigma,
\end{aligned}$$

where  $\nu_{c_0 + \Omega_0}$  is the exterior unit normal vector to  $\partial(c_0 + \Omega_0)$  at  $x$ , which establishes the formula in the case where the domains are not necessarily neighbourhoods of 0.

In what follows  $\Omega_0$  is thus assumed to be star-shaped with respect to the ball  $B(0, r)$  and  $\Omega$  is a neighborhood of 0.

## 4.1 Case where the deformation domain is strongly convex

Assume that  $\Omega$  is strongly convex, that is, near each point of its boundary, the open set  $\Omega$  is defined by  $\{\varphi < 0\}$  and its boundary  $\partial\Omega$  by  $\{\varphi = 0\}$ , with some  $C^2$  function  $\varphi$  whose Hessian matrix is strictly positive. Such an assumption allows us to do some geometrical construction to show that the domain  $\Omega_0 + \varepsilon\Omega$  is the deformation of  $\Omega_0$  via some diffeomorphism. This reduces the problem to a well known situation of deformations with vector fields, see [16] for example. The construction relies on several lemmas and starts with the following:

**Lemma 3.** *Let  $\Omega_0$  and  $\Omega$  be bounded open subsets of  $\mathbb{R}^n$  of class  $C^2$  and assume that  $\Omega$  is strongly convex. Then, there exists a map  $a_0 : \partial\Omega_0 \rightarrow \partial\Omega$ , such that*

- (i) *For all  $x \in \partial\Omega_0$ ,  $P_\Omega(\nu_0(x)) = \nu_0(x) \cdot a_0(x)$ .*
- (ii) *For all  $x \in \partial\Omega_0$ ,  $\nu(a_0(x)) = \nu_0(x)$  where  $\nu(y)$  denotes the exterior unit normal vector to  $\partial\Omega$  at  $y$ .*
- (iii) *The map  $a_0 : \partial\Omega_0 \rightarrow \partial\Omega$  is of class  $C^1$ .*

The proof of this lemma indeed do not assume a particular geometry for  $\Omega_0$  and is the same as that of Lemma 1 of [3], so we refer to it.

Now, we would like to extend  $a_0$  to a map from  $\Omega_0$  to  $\Omega$  and even from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . This is done by using homogeneity.

**Lemma 4.**  *$\Omega_0$  and  $\Omega$  being as in the preceding lemma, assume moreover that  $\Omega_0$  is star-shaped with respect to a ball centered at 0. Then, there exists a map  $a$  defined from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , satisfying the following properties:*

- (i)  *$a = a_0$  on  $\partial\Omega_0$ .*
- (ii)  *$a(\Omega_0) \subset \Omega$  and  $a(\mathbb{R}^n \setminus \overline{\Omega_0}) \subset \mathbb{R}^n \setminus \overline{\Omega}$ .*
- (iii)  *$a$  is positively homogeneous of degree 1, Lipschitz continuous on  $\mathbb{R}^n$  and of class  $C^1$  in  $\mathbb{R}^n \setminus \{0\}$ .*

**Proof.** We define  $a$  on  $\mathbb{R}^n$  by

$$a(x) = \begin{cases} 0 & \text{if } x = 0 \\ J_{\Omega_0}(x) a_0(x/J_{\Omega_0}(x)) & \text{if } x \neq 0. \end{cases}$$

Using Proposition 1, Proposition 2, Lemma 2 and Lemma 3, it is easy to check that  $a$  satisfies (i), (ii) and (iii). ■

Using the vector field  $a$ , let us now consider the map

$$\Phi_\varepsilon(x) = x + \varepsilon a(x), \quad x \in \mathbb{R}^n, \quad \varepsilon > 0.$$

Since  $a$  is lipschitz continuous on  $\mathbb{R}^n$ , it is a classical fact (and easy to check) that, if  $\varepsilon$  is sufficiently small,  $\Phi_\varepsilon$  is a Lipschitz homeomorphism from  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ . Moreover, it follows from the inverse function theorem that  $\Phi_\varepsilon$  is a  $C^1$ -diffeomorphism from  $\mathbb{R}^n \setminus 0$  onto  $\mathbb{R}^n \setminus 0$ . We shall use  $\Phi_\varepsilon$  to parameterize the set  $\Omega_0 + \varepsilon\Omega$ . In order to be able to do that, we need the following result which estimates the boundary of the Minkowski sum of two subsets of  $\mathbb{R}^n$  using the convolution of hypersurfaces.

**Lemma 5.** *Let  $A, B \subset \mathbb{R}^n$  be open, bounded and of class  $C^1$ . Consider the following set*

$$\partial A \star \partial B := \{x + y : x \in \partial A, y \in \partial B \text{ and } \nu_A(x) = \nu_B(y)\} \quad (6)$$

where  $\nu_A$  and  $\nu_B$  are the outward unit normal vectors to  $\partial A$  and  $\partial B$  respectively. Then, we have

$$\partial(A + B) \subset \partial A \star \partial B.$$

**Proof.** Recall that the Minkowski sum of two subsets  $A, B$  of  $\mathbb{R}^n$  can also be written as

$$A + B = \{x \in \mathbb{R}^n ; (-A + x) \cap B \neq \emptyset\}. \quad (7)$$

Let  $x \in \partial(A + B)$ . It follows from (7) that  $(-A + x) \cap B = \emptyset$  and that  $\overline{(-A + x)} \cap \overline{B} \neq \emptyset$ ; hence,  $\partial(-A + x) \cap \partial B \neq \emptyset$ . Let  $y \in \partial(-A + x) \cap \partial B$ . Since  $-A + x \subseteq \mathbb{R}^n \setminus B$ , the hypersurfaces  $\partial(-A + x)$  and  $\partial B$  are tangent at  $y$  and we have  $T_y \partial(-A + x) = T_y \partial B$  and  $\nu_{(-A+x)}(y) = -\nu_B(y)$ . Now,  $y \in \partial(-A + x) = -\partial A + x$ , so that  $x \in \partial A + y$  and there exists  $a \in \partial A$  such that  $x = y + a$ . Moreover, since  $-A + x$  is the image of  $A$  by the diffeomorphism  $z \mapsto -z + x$ , we also have

$$-\nu_A(a) = \nu_{(-A+x)}(y) = -\nu_B(y),$$

which achieves the proof of the lemma. ■

We will also need the following result:

**Lemma 6.** *Let  $\Omega$  be a bounded and strongly convex domain of class  $C^2$  and let  $\nu$  denote the outward unit vector field normal to  $\partial\Omega$ . Then,  $\nu : \partial\Omega \mapsto S^{n-1}$  is injective.*

**Proof.** There exists a strongly convex  $C^2$  function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\Omega = \{x \in \mathbb{R}^n / \varphi(x) < 0\}$ ,  $\partial\Omega = \{x \in \mathbb{R}^n / \varphi(x) = 0\}$ ,  $\nabla\varphi \neq 0$  on  $\partial\Omega$  and  $\nu = \nabla\varphi/|\nabla\varphi|$ . Let  $x, y \in \partial\Omega$  be such that  $\nu(x) = \nu(y)$  and let us show that  $x = y$ . Assume that  $x \neq y$ . Since  $\varphi$  is strongly convex, it follows from Taylor's formula that

$$\langle \nabla\varphi(x), y - x \rangle < \varphi(y) - \varphi(x) \quad \text{and} \quad \langle \nabla\varphi(y), x - y \rangle < \varphi(x) - \varphi(y).$$

Since  $\varphi(x) = \varphi(y) = 0$ , multiplying respectively by  $\frac{1}{|\nabla\varphi(x)|}$  and  $\frac{1}{|\nabla\varphi(y)|}$  yields

$$\langle \nu(x), y - x \rangle < 0 \quad \text{and} \quad \langle \nu(y), x - y \rangle < 0,$$

which gives a contradiction since  $\nu(x) = \nu(y)$ . So,  $x = y$  and the lemma is proved. ■

The above lemmas allow us to prove the following crucial one which concerns the parameterization of the perturbed domain  $\Omega_\varepsilon$  by means of  $\Omega_0$  and  $\Phi_\varepsilon$ .

**Lemma 7.** *Let  $\Omega_0 \in \mathcal{U}$  and  $\Omega$  be a bounded and strongly convex domain of class  $C^2$  in  $\mathbb{R}^n$ . Consider the set  $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$  and the map  $\Phi_\varepsilon : x \mapsto x + \varepsilon a(x)$ ,  $\varepsilon > 0$ , where  $a$  is as in Lemma 4. Then, if  $\varepsilon$  is sufficiently small, we have the following:*

- (i)  $\Phi_\varepsilon(\partial\Omega_0) = \partial\Omega_0 \star \varepsilon\partial\Omega$  and  $\partial\Omega_\varepsilon \subseteq \partial(\Phi_\varepsilon(\Omega_0))$ .
- (ii)  $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$ .

Here,  $\nu_0$  and  $\nu$  denote the outward unit normal vector to  $\Omega_0$  and  $\Omega$  respectively.

**Proof.** Let  $x \in \partial\Omega_0$ . According to Lemma 4 and Lemma 3,  $a(x) = a_0(x) \in \partial\Omega$  and  $\nu_0(x) = \nu(a(x))$ ; hence,  $\Phi_\varepsilon(x) = x + \varepsilon a(x) \in \partial\Omega_0 \star \varepsilon\partial\Omega$ . Conversely, if  $z \in \partial\Omega_0 \star \varepsilon\partial\Omega$ , there exists  $(x, y) \in \partial\Omega_0 \times \partial\Omega$  such that  $z = x + \varepsilon y$  and  $\nu_0(x) = \nu_\varepsilon(\varepsilon y) = \nu(y)$ . Applying once again Lemma 4, we have  $a(x) \in \partial\Omega$  and  $\nu_0(x) = \nu(a(x)) = \nu(y)$ . Next, applying Lemma 6 yields  $a(x) = y$ . Therefore,  $z = x + \varepsilon a(x) = \Phi_\varepsilon(x) \in \Phi_\varepsilon(\partial\Omega_0)$ . Thus, we have proved that  $\Phi_\varepsilon(\partial\Omega_0) = \partial\Omega_0 \star \varepsilon\partial\Omega$ . Now, according to Lemma 5, we have

$$\partial\Omega_\varepsilon = \partial(\Omega_0 + \varepsilon\Omega) \subseteq \partial\Omega_0 \star \partial(\varepsilon\Omega) = \partial\Omega_0 \star \varepsilon\partial\Omega = \Phi_\varepsilon(\partial\Omega_0) = \partial\Phi_\varepsilon(\Omega_0),$$

which achieves the proof of (i).

To show (ii), note first that  $\Phi_\varepsilon(\Omega_0) \subset \Omega_\varepsilon$  is an obvious consequence of Lemma 4. To prove the other inclusion, let us first remark that it follows from the homogeneity of  $\Phi_\varepsilon$  that  $\Phi_\varepsilon(\Omega_0)$  is also a star-shaped domain with respect to 0 as it can be checked easily. Next, assume that there exists  $x \in \Omega_\varepsilon$  such that  $x \in \mathbb{R}^n \setminus \Phi_\varepsilon(\Omega_0)$ . Then, it follows from Proposition 1 that  $0 < J_{\Omega_\varepsilon}(x) < 1$  and  $J_{\Phi_\varepsilon(\Omega_0)}(x) \geq 1$ . Now, consider  $x_* = x/J_{\Omega_\varepsilon}(x) \in \partial\Omega_\varepsilon$ . Clearly,  $J_{\Phi_\varepsilon(\Omega_0)}(x_*) = J_{\Phi_\varepsilon(\Omega_0)}(x)/J_{\Omega_\varepsilon}(x) > 1$ , that is,  $x_* \notin \partial(\Phi_\varepsilon(\Omega_0))$ , which contradicts (i). Thus,  $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$ . ■

Lemma 7 provides the main tool in the proof of Theorem 1 in the case where  $\Omega$  is strongly convex and of class  $C^2$ . Indeed, according to this lemma,  $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)$  and the problem is reduced to the case of a deformation of  $\Omega_0$  by a diffeomorphism or, more precisely, a Lipschitz homeomorphism. According to [21] for example, we have the following shape derivative formula

$$\left. \frac{d}{d\varepsilon} \mathcal{J}(\Omega_0 + \varepsilon\Omega) \right|_{\varepsilon=0^+} = \int_{\partial\Omega_0} f(x) a(x) \cdot \nu_0(x) d\sigma,$$

and according to Lemma 3 and Lemma 4, we have

$$a(x) \cdot \nu_0(x) = a_0(x) \cdot \nu_0(x) = P_\Omega(\nu_0(x)).$$

Hence the formula

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x) P_\Omega(\nu_0(x)) d\sigma(x),$$

and this achieves the proof of Theorem 1 in the case where  $\Omega$  is strongly convex. ■

## 4.2 The general case

The domain  $\Omega$  is now assumed to be bounded and (only) convex. We shall approximate it by a sequence of strongly convex ones. To do that, let us recall the following approximation result used in [3].

**Lemma 8.** *Let  $\Omega$  be a bounded convex domain in  $\mathbb{R}^n$ . Then, there exists a sequence  $(\Omega^k)_{k \in \mathbb{N}}$  of strongly convex smooth open subsets of  $\Omega$  such that*

$$d^H(\overline{\Omega^k}, \overline{\Omega}) \xrightarrow[k \rightarrow \infty]{} 0,$$

where  $d^H$  denotes the Hausdorff distance.

Such an approximation is used to prove the following lemma which is an important step in the proof of our theorem.

**Lemma 9.** Let  $\Omega_0 \in \mathcal{U}$ ,  $\Omega$  be a bounded convex domain in  $\mathbb{R}^n$  and  $(\Omega^k)_{k \in \mathbb{N}}$  the sequence given by Lemma 8 which approximates  $\Omega$ . Then, for all  $\varepsilon \in [0, 1]$  and for all  $k \in \mathbb{N}$ , we have

$$d^H(\overline{\Omega}_\varepsilon^k, \overline{\Omega}_\varepsilon) \leq \varepsilon d^H(\overline{\Omega}^k, \overline{\Omega}).$$

where  $\Omega_\varepsilon^k = \Omega_0 + \varepsilon \Omega^k$  et  $\Omega_\varepsilon = \Omega_0 + \varepsilon \Omega$ .

**Proof.** We have  $\overline{\Omega}_\varepsilon^k = \overline{\Omega}_0 + \varepsilon \overline{\Omega}^k$  and  $\overline{\Omega}_\varepsilon = \overline{\Omega}_0 + \varepsilon \overline{\Omega}$ , thus according to [22], page 64, we have

$$d^H(\overline{\Omega}_\varepsilon^k, \overline{\Omega}_\varepsilon) = d^H(\overline{\Omega}_0 + \varepsilon \overline{\Omega}^k, \overline{\Omega}_0 + \varepsilon \overline{\Omega}) \leq d^H(\overline{\Omega}_0, \overline{\Omega}_0) + d^H(\varepsilon \overline{\Omega}^k, \varepsilon \overline{\Omega}).$$

Since  $\Omega_k \subseteq \Omega$ , then  $d^H(\varepsilon \overline{\Omega}^k, \varepsilon \overline{\Omega}) = \sup_{x \in \varepsilon \overline{\Omega}} d(x, \varepsilon \overline{\Omega}^k) = \varepsilon d^H(\overline{\Omega}^k, \overline{\Omega})$ . Thus,  $d^H(\overline{\Omega}_\varepsilon^k, \overline{\Omega}_\varepsilon) \leq \varepsilon d^H(\overline{\Omega}^k, \overline{\Omega})$ . ■

We need also the following result.

**Proposition 4.** Let  $A, B \subset \mathbb{R}^n$  be two bounded domains which are star-shaped with respect to the ball  $B(0, r)$ ,  $r > 0$ , and such that  $A \subseteq B$ . Then, we have

$$\sup_{S^{n-1}} |J_A - J_B| \leq \frac{1}{r^2} d^H(\overline{A}, \overline{B}) \quad (8)$$

**Proof.** Let  $x \in \partial B$ . Since  $\overline{A} \subset \overline{B}$ , there exists  $y_x \in \partial A$  such that  $d(x, \overline{A}) = |x - y_x|$ . According to Proposition 2, the gauge functions  $J_A, J_B$  are Lipschitz functions with Lipschitz constant  $\frac{1}{r}$ . Therefore, since  $J_A(y_x) = 1 = J_B(x)$  and  $\overline{A} \subset \overline{B}$ , we can write

$$\begin{aligned} |J_A(x) - J_B(x)| &\leq |J_A(x) - J_A(y_x)| + |J_A(y_x) - J_B(x)| = |J_A(x) - J_A(y_x)| \\ &\leq r^{-1} |x - y_x| = r^{-1} d(x, \overline{A}) \\ &\leq r^{-1} \sup_{z \in \overline{B}} d(z, \overline{A}) = r^{-1} d^H(\overline{A}, \overline{B}), \end{aligned}$$

an inequality that holds for  $x \in \partial B$ . Now, if  $x \in S^{n-1}$ , we have  $\frac{x}{J_B(x)} \in \partial B$ , and by using the homogeneity of the gauge functions we obtain

$$|J_A(x) - J_B(x)| = J_B(x) |J_A(\frac{x}{J_B(x)}) - J_B(\frac{x}{J_B(x)})| \leq J_B(x) r^{-1} d^H(\overline{A}, \overline{B}).$$

Since  $B(0, r) \subset B$ , we have  $J_B(x) \leq J_{B(0, r)}(x) = |x|/r = 1/r$ , which implies the desired inequality. ■

The last lemma is crucial for our proof.

**Lemma 10.** Let  $\Omega_0 \in \mathcal{U}$ ,  $\Omega$  be a bounded convex domain and  $f \in W_{loc}^{1,1}(\mathbb{R}^n)$ , and let  $(\Omega^k)_{k \in \mathbb{N}}$  be as in Lemma 8. Then, there exists a constant  $C > 0$  such that, for all  $k \in \mathbb{N}$  and all  $\varepsilon \in [0, 1]$ , we have

$$\left| \int_{\Omega_\varepsilon^k} f(x) dx - \int_{\Omega_\varepsilon} f(x) dx \right| \leq C \varepsilon d^H(\overline{\Omega}^k, \overline{\Omega}),$$

where  $\Omega_\varepsilon^k = \Omega_0 + \varepsilon \Omega^k$  et  $\Omega_\varepsilon = \Omega_0 + \varepsilon \Omega$ .

**Proof.** We follow the idea of [4]. Let  $r > 0$  be such that  $\Omega_0$  is star-shaped with respect to  $B(0, r)$  and let  $B(0, R)$  be a large ball which contains all the sets  $\Omega_0 + \varepsilon \Omega$ ,  $\varepsilon \in [0, 1]$ . As one can easily check,  $\Omega_\varepsilon$  and  $\Omega_\varepsilon^k$  are star-shaped with respect to  $B(0, r)$ , and we shall denote by  $J_\varepsilon$  and  $J_\varepsilon^k$  respectively their gauge functions. Let us denote by  $I_\varepsilon^k(f)$  the difference

$$\int_{\Omega_\varepsilon} f(x) dx - \int_{\Omega_\varepsilon^k} f(x) dx.$$

Since  $\Omega_\varepsilon = \{J_\varepsilon < 1\}$  and  $\Omega_\varepsilon^k = \{J_\varepsilon^k < 1\}$ , by using polar coordinates, we can write

$$I_\varepsilon^k(f) = \int_{S^{n-1}} \int_0^{\frac{1}{J_\varepsilon(\omega)}} f(\rho\omega) \rho^{n-1} d\rho d\omega - \int_{S^{n-1}} \int_0^{\frac{1}{J_\varepsilon^k(\omega)}} f(\rho\omega) \rho^{n-1} d\rho d\omega = \int_{S^{n-1}} \int_{\frac{1}{J_\varepsilon^k(\omega)}}^{\frac{1}{J_\varepsilon(\omega)}} f(\rho\omega) \rho^{n-1} d\rho d\omega.$$

This allows us to estimate  $I_\varepsilon^k(f)$  as follows :

$$|I_\varepsilon^k(f)| \leq \int_{S^{n-1}} \left| \frac{J_\varepsilon(\omega) - J_\varepsilon^k(\omega)}{J_\varepsilon(\omega) J_\varepsilon^k(\omega)} \right| \sup_{\rho \in [\frac{1}{J_\varepsilon^k(\omega)}, \frac{1}{J_\varepsilon(\omega)}]} |f(\rho\omega) \rho^{n-1}| d\omega.$$

Note that, since  $B(0, r) \subset \Omega_\varepsilon^k \subset \Omega_\varepsilon \subset B(0, R)$ , we have  $J_{B(0, R)} \leq J_\varepsilon \leq J_\varepsilon^k \leq J_{B(0, r)}$ . Recall that  $J_{B(0, r)}(x) = |x|/r$ ,  $J_{B(0, R)}(x) = |x|/R$ . Hence,

$$|I_\varepsilon^k(f)| \leq R^2 \int_{S^{n-1}} |J_\varepsilon(\omega) - J_\varepsilon^k(\omega)| \sup_{\rho \in [r, R]} |f(\rho\omega) \rho^{n-1}| d\omega.$$

Since  $\Omega_\varepsilon$  and  $\Omega_\varepsilon^k$  are star-shaped with respect to  $B(0, r)$  and  $\Omega_\varepsilon^k \subseteq \Omega_\varepsilon$ , it follows from Lemma 4 and Lemma 9 that

$$\sup_{S^{n-1}} |J_\varepsilon - J_\varepsilon^k| \leq \frac{1}{r^2} d^H(\overline{\Omega}_\varepsilon^k, \overline{\Omega}_\varepsilon) \leq \frac{\varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}).$$

Therefore,

$$|I_\varepsilon^k(f)| \leq \frac{R^2 \varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{S^{n-1}} \sup_{\rho \in [r, R]} |f(\rho\omega) \rho^{n-1}| d\omega.$$

It remains to apply the following classical inequality for functions of one real variable :

$$\|\varphi\|_{L^\infty(I)} \leq \frac{1}{|I|} \int_I |\varphi(t)| dt + \int_I |\varphi'(t)| dt, \quad \varphi \in W^{1,1}(I), \quad (9)$$

where  $I$  is a bounded interval and  $|I|$  is its length. In fact, this is just a precise version of Sobolev's inequality. Its proof is easy when  $\varphi$  is of class  $C^1$  on  $\bar{I}$  and the general case is obtained by a density argument and is left to the reader. Applying (9) to the function  $\rho \mapsto f(\rho\omega) \rho^{n-1}$  yields

$$\begin{aligned} |I_{\varepsilon, k}(f)| &\leq \frac{R^2 \varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{S^{n-1}} \int_r^R \left( |f(\rho\omega)| \left( \frac{\rho^{n-1}}{R-r} + (n-1)\rho^{n-2} \right) + |\nabla f(\rho\omega)| \rho^{n-1} \right) d\rho d\omega \\ &\leq \frac{R^2 \varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{r \leq |x| \leq R} \left( \frac{|f(x)|}{R-r} + \frac{n-1}{|x|} |f(x)| + |\nabla f(x)| \right) dx \\ &\leq \frac{R^2 \varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \left( \frac{1}{R-r} + \frac{n-1}{r} + 1 \right) \int_{r \leq |x| \leq R} (|f(x)| + |\nabla f(x)|) dx \\ &\leq \frac{R^2 \varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \left( \frac{1}{R-r} + \frac{n-1}{r} + 1 \right) \|f\|_{W^{1,1}(D)}, \end{aligned}$$

which achieves the proof of the lemma. ■

Using the above lemmas, we can now finish the proof of Theorem 1. Let  $\delta > 0$  be arbitrary. We can write:

$$\frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} - \int_{\partial\Omega_0} f(x) P_\Omega(\nu_0(x)) d\sigma(x) = \frac{1}{\varepsilon} \left( \int_{\Omega_\varepsilon} f(x) dx - \int_{\Omega_\varepsilon^k} f(x) dx \right)$$

$$\begin{aligned}
& + \frac{1}{\varepsilon} \left( \int_{\Omega_\varepsilon^k} f(x) dx - \int_{\Omega_0} f(x) dx \right) - \int_{\partial\Omega_0} f(x) P_{\Omega^k}(\nu_0)(x) d\sigma(x) \\
& + \int_{\partial\Omega_0} f(x) \left( P_{\Omega^k}(\nu_0(x)) - P_{\Omega}(\nu_0(x)) \right) d\sigma(x).
\end{aligned} \tag{10}$$

For the first term in the righthand side of (10), according to Lemma 8 and Lemma 10, there exists  $k_0 \in \mathbb{N}$ , such for all  $k \geq k_0$  and for all  $\varepsilon \in ]0, 1]$ , we have

$$\frac{1}{\varepsilon} \left| \left( \int_{\Omega_\varepsilon} f(x) dx - \int_{\Omega_\varepsilon^k} f(x) dx \right) \right| \leq \delta. \tag{11}$$

Using the formula  $\|P_A - P_B\|_{L^\infty(S^{n-1})} = d^H(A, B)$  for compact convex sets, (see for example [22], page 66), we can estimate the last term in the righthand side of (10) as follows :

$$\left| \int_{\partial\Omega_0} f \left( P_{\Omega^k}(\nu_0) - P_{\Omega}(\nu_0) \right) d\sigma \right| \leq \|P_{\Omega^k} - P_{\Omega}\|_{L^\infty(S^{n-1})} \int_{\partial\Omega_0} |f| d\sigma = d^H(\bar{\Omega}^k, \bar{\Omega}) \int_{\partial\Omega_0} |f| d\sigma,$$

which implies that it tends to 0 when  $k \rightarrow \infty$  by virtue of Lemma 8. Hence, there exists  $k_1 \in \mathbb{N}$ , such that, for all  $k \geq k_1$ , we have

$$\left| \int_{\partial\Omega_0} f \left( P_{\Omega^k}(\nu_0) - P_{\Omega}(\nu_0) \right) d\sigma \right| \leq \delta. \tag{12}$$

Now, if  $k_2 = \max\{k_0, k_1\}$ , since  $\Omega^{k_2}$  is strongly convex, it follows from the first part of the proof that there exists  $\varepsilon_\delta$  such that for all  $\varepsilon \leq \varepsilon_\delta$ , we have

$$\left| \frac{1}{\varepsilon} \left( \int_{\Omega_\varepsilon^{k_2}} f(x) dx - \int_{\Omega_0} f(x) dx \right) - \int_{\partial\Omega_0} f(x) P_{\Omega^{k_2}}(\nu_0(x)) d\sigma(x) \right| \leq \delta. \tag{13}$$

By taking  $k = k_2$  in (10) and using (11), (12) and (13), we obtain that, for all  $\varepsilon \leq \varepsilon_\delta$ ,

$$\left| \frac{1}{\varepsilon} \left( \int_{\Omega_\varepsilon} f(x) dx - \int_{\Omega_0} f(x) dx \right) - \int_{\partial\Omega_0} f(x) P_{\Omega}(\nu_0(x)) d\sigma(x) \right| \leq 3\delta,$$

which achieves the proof of Theorem 1. ■

## 5 Application

To illustrate our work, we give an algorithm based on the gradient method to indicate how our formula could be applied to a shape optimization problem and we compute the shape derivative of some functional related to the solution of a partial differential equation. This is done without implementing to keep our paper in a reasonable length. We mention, however, that we are successfully implementing such an algorithm to another shape optimization problem (of Bernoulli type) in [6].

Let us define the set  $\mathcal{U}_{ad}$  of admissible domains by

$\Omega \in \mathcal{U}_{ad} \iff \Omega$  is a  $C^3$  open subset of  $\mathbb{R}^n$  which is star-shaped with respect to some ball of radius  $r$ .

If  $D$  is an open bounded (convex) and non empty subset of  $\mathbb{R}^n$ , let us consider the problem

$$(PO) \quad \begin{cases} \text{Find } \Omega_* \in \mathcal{U}_{ad}(D) \text{ such that } \mathcal{J}(\Omega_*) = \inf_{\Omega \in \mathcal{U}_{ad}(D)} \mathcal{J}(\Omega) \\ \text{where } \mathcal{U}_{ad}(D) = \{\Omega \in \mathcal{U}_{ad} ; \Omega \subset D\}, \\ \mathcal{J}(\Omega) = \int_{\Omega} (u_{\Omega} - u_d)^2 dx \text{ and } u_{\Omega} \text{ is the solution of} \\ (PE) \quad \begin{cases} -\Delta v + v = f & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = g & \text{on } \partial\Omega \end{cases} \end{cases}$$

where  $u_d \in H^1(D)$ ,  $f \in L^2(D)$  and  $g \in H^2(D)$ . Let  $u_0$  be the solution of  $(PE)$  on  $\Omega_0 \in \mathcal{U}_{ad}(D)$  and  $u_\varepsilon$  be the solution of  $(PE)$  on  $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$ ,  $\varepsilon \in [0, 1]$ . Assuming that  $\Omega$  is a strongly convex domain, we know from Lemma 7 that  $\Omega_\varepsilon$  can be considered as a deformation of the domain  $\Omega_0$  by the vector field  $a$ , that is  $\Omega_\varepsilon = (\text{Id}_{\mathbb{R}^n} + \varepsilon a)(\Omega_0)$  for small enough  $\varepsilon$ . Therefore, at least when  $f \in H^1(D)$ , according to [1], we can write

$$\tilde{u}_\varepsilon = \tilde{u}_0 + \varepsilon u'_0 + \varepsilon v_\varepsilon$$

where  $u'_0 \in H^2(\Omega_0)$  is the shape derivative of  $\tilde{u}_0$  with respect to the vector field  $a$  and  $v_\varepsilon \rightarrow 0$  in  $L^2(D)$  as  $\varepsilon \rightarrow 0^+$ . It follows from that result that

$$\frac{1}{\varepsilon}[(\tilde{u}_\varepsilon - u_d)^2 - (\tilde{u}_0 - u_d)^2] - 2u'_0(\tilde{u}_0 - u_d) \xrightarrow{\varepsilon \rightarrow 0^+} 0 \quad \text{in } L^1(D),$$

which allows one to apply Corollary 1 to obtain

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\Omega_0} 2u'_0(\tilde{u}_0 - u_d)dx + \int_{\partial\Omega_0} (\tilde{u}_0 - u_d)^2 P_\Omega(\nu_0) d\sigma(x).$$

Now, in this expression of the shape derivative of  $\mathcal{J}$ , even the domain integral can be written as a boundary integral. Indeed, by following, for example, the same method as [16], one can show that  $u'_0$  satisfies the boundary value problem

$$\begin{cases} -\Delta u'_0 + u'_0 = 0 & \text{in } \Omega_0 \\ \frac{\partial}{\partial \nu_0} u'_0 = (\frac{\partial g}{\partial \nu_0} - \frac{\partial^2 u_0}{\partial \nu_0^2}) \langle a, \nu_0 \rangle + \nabla u_0 \nabla_{\partial\Omega_0} \langle a, \nu_0 \rangle & \text{on } \partial\Omega_0 \end{cases} \quad (14)$$

where

$$\frac{\partial^2 u_0}{\partial \nu_0^2} = \sum_{i,j=1}^n \frac{\partial^2 u_0}{\partial x_i \partial x_j} n_{i,0} n_{j,0}$$

and  $\nabla_{\partial\Omega_0}$  is the tangential gradient (see [16]). Note here that, since  $\Omega$  is of class  $\mathcal{C}^3$ ,  $u_0$  is in fact in  $H^3(\Omega_0)$ ,  $u'_0 \in H^2(\Omega_0)$  (see [16]), so that the second derivative  $\frac{\partial^2 u_0}{\partial \nu_0^2}$  is well defined on  $\partial\Omega_0$ . Now, using the solution of the following adjoint state boundary value problem

$$\begin{cases} -\Delta \psi + \psi = -2(u_0 - u_d) & \text{in } \Omega_0 \\ \frac{\partial \psi}{\partial \nu_0} = 0 & \text{on } \partial\Omega_0, \end{cases} \quad (15)$$

we obtain the following expression of the shape derivative of the functional  $\mathcal{J}$  (see [1])

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} d_{\Omega_0} \langle a, \nu_0 \rangle d\sigma,$$

where

$$d_{\Omega_0} = (u_0 - u_d)^2 + \langle \nabla u_0, \nabla \psi \rangle + \psi(u_0 - f) - \frac{\partial(g\psi)}{\partial \nu_0} - \mathcal{H}g\psi.$$

Now, since  $\Omega$  is strongly convex, it follows from Lemma 3 and Lemma 4 that

$$\langle a(x), \nu_0(x) \rangle = P_\Omega(\nu_0(x)) \quad \text{on } \partial\Omega_0, \quad (16)$$

so that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} d_{\Omega_0} P_\Omega \circ \nu_0 d\sigma.$$

The last thing we propose is an algorithm to solve the shape optimization problem  $(PO)$ .

**Algorithm.**

1. Choose  $\Omega_0 \in \mathcal{U}(D)$ ,  $\rho \in ]0, 1[$  and a precision  $\varepsilon$ .
2. Solve the state equation :

$$(PE) \begin{cases} -\Delta u_0 + u_0 = f & \text{in } \Omega_0, \\ \frac{\partial u_0}{\partial \nu_0} = 0 & \text{on } \partial\Omega_0; \end{cases} \quad (17)$$

3. Solve the adjoint state problem

$$(PEA) \begin{cases} -\Delta \psi + \psi = -2(u_0 - u_d) & \text{in } \Omega_0, \\ \frac{\partial \psi}{\partial \nu_0} = 0 & \text{on } \partial\Omega_0; \end{cases} \quad (18)$$

4. Calculate  $\hat{p}_0$  solution of

$$\arg \min_{p \in \mathcal{E}} J_0(p)$$

where

$$\mathcal{E} = \{\varphi \in C(\mathbb{R}^2) ; \varphi \text{ is convex and homogeneous of degree 1 and } \varphi \leq P_D\},$$

and

$$J_0(p) = \int_{\partial\Omega_0} \left( (u_0 - u_d)^2 + \langle \nabla u_0, \nabla \psi \rangle + \psi(u_0 - f) - \frac{\partial(g\psi)}{\partial \nu_0} - \mathcal{H}g\psi \right) p \circ \nu_0 d\sigma.$$

5. At step  $k$ ,

$$\text{if } \|u_k - u_d\|_{L^2(\Omega_k)} < \varepsilon, \quad \mathbf{Go to 7.}$$

where  $u_k$  is the solution of the state equation in  $\Omega_k$  (the domain at step  $k$ ).

6. Compute

$$\Omega_{k+1} = \Omega_k + \rho \hat{\Omega}_k \quad \text{where} \quad \hat{\Omega}_k = \partial \hat{p}_k(0)$$

and **Go to 2.**

7. **End.**

**Remark 1.** In the above algorithm, the sequence of domains  $(\Omega_k)_{k \in \mathbb{N}}$  is constructed in such a way that  $(\mathcal{J}(\Omega_k))_{k \in \mathbb{N}}$  is decreasing. Indeed, let  $k \in \mathbb{N}^*$ , then, for a small  $\rho \in ]0, 1[$ , we have

$$\begin{aligned} \mathcal{J}(\Omega_{k+1}) - \mathcal{J}(\Omega_k) &= \mathcal{J}(\Omega_k + \rho \hat{\Omega}_k) - \mathcal{J}(\Omega_k) \\ &= \rho \left( \int_{\partial\Omega_k} d_{\Omega_k} P_{\hat{\Omega}_k} \circ \nu_k d\sigma \right) + O(\rho^2). \end{aligned}$$

Now, since  $\hat{p}_k = P_{\hat{\Omega}_k}$  is a solution of  $\arg \min_{p \in \mathcal{E}} J_k(p)$ , then

$$J_k(\hat{p}_k) = \int_{\partial\Omega_k} d_{\Omega_k} P_{\hat{\Omega}_k} \circ \nu_k d\sigma \leq J_k(0) = 0,$$

which guarantees the decrease of the objective function  $\mathcal{J}$ . Consequently,  $\hat{\Omega}_k$  defines a descent direction for  $\mathcal{J}$ .



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