A uniqueness theorem for Sturm–Liouville equations with a spectral parameter linearly contained in the boundary conditions

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Abstract

A uniqueness theorem for the Sturm–Liouville equation with a spectral parameter linearly contained in both boundary conditions is proved.

Consider the Sturm–Liouville equation

$$\ell y := -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, \pi] \tag{1}$$

with the boundary conditions

$$\lambda(y'(0) - hy(0)) = h_1 y'(0) - h_2 y(0), \tag{2}$$
$$\lambda(y'(\pi) + Hy(\pi)) = H_1 y'(\pi) + H_2 y(\pi), \tag{3}$$

where $q(x) \in \mathscr{L}_2(0, \pi)$ is a real-valued function, $h, h_1, h_2, H, H_1, H_2 \in \mathbb{R}$ and

$$\delta := hh_1 - h_2 > 0, \quad \rho := HH_1 - H_2 > 0.$$

We denote this problem by $\mathcal{P}(q, h, h_1, h_2, H, H_1, H_2)$.

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of (1) satisfying the initial conditions

$$\begin{aligned} \varphi(0, \lambda) &= -\lambda + h_1, \quad \varphi'(0, \lambda) = -\lambda h + h_2, \\ \psi(\pi, \lambda) &= -\lambda + H_1, \quad \psi'(\pi, \lambda) = \lambda H - H_2. \end{aligned} \tag{4}$$

We define

$$\chi(\lambda) := \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda),$$

which is independent of $x \in [0, \pi]$. The function $\chi(\lambda)$ is entire and has zeros at the eigenvalues of the problem (1)–(3). The set of eigenvalues is countable, consists of real numbers ([1, 4]) and for each eigenvalue λ_n there exists such a number k_n that

$$\psi(x, \lambda_n) = k_n \varphi(x, \lambda_n), \quad k_n \neq 0. \tag{5}$$

In the Hilbert space $\mathcal{H} = \mathscr{L}_2(0, \pi) \oplus \mathbb{C}^2$ let an inner product be defined by

$$(F, G) := \int_0^\pi F_1(x)\overline{G_1(x)}dx + \frac{1}{\delta}F_2\overline{G_2} + \frac{1}{\rho}F_3\overline{G_3}$$

for

$$F = \begin{pmatrix} F_1(x) \\ F_2 \\ F_3 \end{pmatrix}, \quad G = \begin{pmatrix} G_1(x) \\ G_2 \\ G_3 \end{pmatrix} \in \mathcal{H}.$$

We define operator

$$A(F) := \begin{pmatrix} -F_1''(x) + q(x)F_1(x) \\ h_1F_1'(0) - h_2F_1(0) \\ H_1F_1'(\pi) + H_2F_1(\pi) \end{pmatrix}$$

with

$$D(A) = \left\{F \in \mathcal{H} | F_1(x), F_1'(x) \in \mathscr{AC}[0,\pi], \ell F_1 \in \mathscr{L}_2(0,\pi), \right. \\ \left. F_2 = F_1'(0) - hF_1(0), F_3 = F_1'(\pi) + HF_1(\pi)\right\}.$$

Then

$$\Phi_n := \begin{pmatrix} \varphi(x,\lambda_n) \\ \varphi'(0,\lambda_n) - h\varphi(0,\lambda_n) \\ \varphi'(\pi,\lambda_n) + H\varphi(\pi,\lambda_n) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda_n) \\ -\delta \\ \rho/k_n \end{pmatrix}$$

are orthogonal eigenelements of A:

$$(\Phi_m, \Phi_n) = \int_0^\pi \varphi(x,\lambda_m)\varphi(x,\lambda_n)dx + \delta + \frac{\rho}{k_m k_n} = 0, \quad m \neq n.$$

We also define *norming constants* by

$$\gamma_n := \|\Phi_n\|^2 = \int_0^\pi \varphi^2(x,\lambda_n)dx + \delta + \frac{\rho}{k_n^2}.$$

The numbers $\{\lambda_n, \gamma_n\}_{n\geq 0}$ are called the *spectral data* of the problem (1)–(3).

Lemma 1. *The following equality holds:*

$$\dot{\chi}(\lambda_n) = k_n\gamma_n, \quad n \geq 0,$$

where $\dot{\chi}(\lambda) = \dfrac{d}{d\lambda}\chi(\lambda)$.

Analogously to [3, Theorem 1(i)] the following theorem can be proved.

Theorem 1. *For* $F \in \mathcal{H}$

$$\|F\|^2 = \sum_{n=0}^{\infty} \frac{1}{\gamma_n}|(F, \Phi_n)|^2.$$

By taking $F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in this theorem we obtain

$$\frac{1}{\delta} = \sum_{n=0}^{\infty} \frac{1}{\gamma_n}, \qquad \frac{1}{\rho} = \sum_{n=0}^{\infty} \frac{1}{k_n^2\gamma_n}. \tag{6}$$

Theorem 2. *Following asymptotics hold:*

$$\sqrt{\lambda_n} = n - 2 + \frac{\omega}{n\pi} + \frac{\zeta_n}{n}, \quad \{\zeta_n\} \in l_2, \tag{7}$$

$$\gamma_n = (n-2)^4 \left(\frac{\pi}{2} + \frac{\zeta_n'}{n}\right), \quad \{\zeta_n'\} \in l_2, \tag{8}$$

where

$$\omega = h + H + \frac{1}{2}\int_0^\pi q(x)dx.$$

Proof. We denote $s := \sqrt{\lambda}$. Then from

$$\varphi(x,\lambda) = -s^2\cos sx - \left(h + \frac{1}{2}\int_0^x q(t)dt\right) s\sin sx - \\ - \frac{1}{2}\int_0^x q(t)s\sin s(x-2t)dt + O\left(e^{|Imsx|}\right),$$

$$\varphi'(x,\lambda) = s^3\sin sx - \left(h + \frac{1}{2}\int_0^x q(t)dt\right) s^2\cos sx - \\ - \frac{1}{2}\int_0^x q(t)s^2\cos s(x-2t)dt + O\left(|s|\mathrm{e}^{|\mathrm{Imsx}|}\right)$$

using (4) we obtain

$$\chi(\lambda) = s^5\sin s\pi - \left(h + H + \frac{1}{2}\int_0^\pi q(x)dx\right) s^4\cos s\pi + I(s)s^4, \tag{9}$$

where

$$I(s) = -\frac{1}{2}\int_0^\pi q(t)\cos s(\pi - 2t)dt + O\left(\frac{e^{|Ims\pi|}}{|s|}\right).$$

Now using Bessel's inequality it's easy to obtain (7) and (8).

Lemma 2. *Let $v(t,\lambda)$ be any function entire on λ with*

$$v(t,\lambda) = O\left(e^{|Im\sqrt{\lambda}|t}\right), \quad t \in [0,\pi].$$

Then

$$\sum_{n=0}^{\infty} \frac{\varphi(x,\lambda_n)v(t,\lambda_n)}{\gamma_n} = 0, \tag{10}$$

uniformly on $0 \le t \le x \le \pi$.

Proof. Using (5), Lemma 1. and the residue theorem we get

$$\sum_{n=0}^{N} \frac{\varphi(x,\lambda_n)v(t,\lambda_n)}{\gamma_n} = \sum_{n=0}^{N} \frac{\psi(x,\lambda_n)v(t,\lambda_n)}{k_n\gamma_n} = \sum_{n=0}^{N} \frac{\psi(x,\lambda_n)v(t,\lambda_n)}{\dot{\chi}(\lambda_n)} = \\ = \sum_{n=0}^{N} \operatorname*{Res}_{\lambda=\lambda_n} \frac{\psi(x,\lambda)v(t,\lambda)}{\chi(\lambda)} = \frac{1}{2\pi i}\int_{C_N} \frac{\psi(x,\lambda)v(t,\lambda)}{\chi(\lambda)}d\lambda,$$

where $C_N = \{\lambda : |\lambda| = (N-3/2)^2\}$. Equality (9) implies that

$$\chi(\lambda) = \lambda^2\sqrt{\lambda}\sin\sqrt{\lambda}\pi + O\left(|\lambda|^2 e^{|Im\sqrt{\lambda}|\pi}\right).$$

We denote $G_\delta = \{\lambda : |\sqrt{\lambda} - n| \geq \delta, n = 0, \pm1, \pm2, \dots\}$ for some small fixed $\delta > 0$ and recall that (see e.g. [6, p. 15])

$$|\sin\sqrt{\lambda}\pi| \geq 2C_\delta e^{|Im\sqrt{\lambda}|\pi}, \quad \lambda \in G_\delta,$$

where C_δ does not depend on λ. Therefore we obtain

$$|\chi(\lambda)| \geq C_\delta|\lambda^2\sqrt{\lambda}|e^{|Im\sqrt{\lambda}|\pi}, \quad \lambda \in G_\delta, |\lambda| \geq \lambda_\delta$$

for sufficiently large λ_δ. Since

$$|\psi(x,\lambda)| = O\left(|\lambda|e^{|Im\sqrt{\lambda}|(\pi-x)}\right), \quad x \in [0,\pi],$$

we obtain

$$|\psi(x,\lambda)v(t,\lambda)| = O\left(|\lambda|e^{|Im\sqrt{\lambda}|(\pi-x+t)}\right) = O\left(|\lambda|e^{|Im\sqrt{\lambda}|\pi}\right), \quad 0 \leq t \leq x \leq \pi.$$

Thus the equality (10) holds. Uniformity of this convergence can be obtained from (7) and (8) by applying Weierstraß' theorem.

Putting $v(t,\lambda) \equiv 1$ and $x = 0$ in (10) we obtain

$$\sum_{n=0}^{\infty} \frac{h_1 - \lambda_n}{\gamma_n} = 0.$$

Similarly, we have

$$\sum_{n=0}^{\infty} \frac{H_1 - \lambda_n}{k_n^2\gamma_n} = 0.$$

Using (6) in these equalities we get the representation of h_1 and H_1 by the spectral data:

$$h_1 = \delta\sum_{n=0}^{\infty}\frac{\lambda_n}{\gamma_n}, \qquad H_1 = \rho\sum_{n=0}^{\infty}\frac{\lambda_n}{k_n^2\gamma_n}. \tag{11}$$

Using the transformation operators (see [5]), we can write

$$\varphi(x,\lambda) = \mathring{\varphi}(x,\lambda) + \int_{-x}^{x} G(x,\tau)\mathring{\varphi}(\tau,\lambda)d\tau, \tag{12}$$

where

$$\mathring{\varphi}(x,\lambda) := (h_1 - \lambda)\cos\sqrt{\lambda}x + (h_2 - \lambda h)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$$

is the solution of the equation $-y''(x) = \lambda y(x)$ satisfying the initial conditions

$$\mathring{\varphi}(0,\lambda) = -\lambda + h_1, \quad \mathring{\varphi}'(0,\lambda) = -\lambda h + h_2,$$

$G(x,\tau)$ is a real-valued continuous function and

$$G(x,x) = \frac{1}{2}\int_0^x q(t)dt.$$

Theorem 3. *For each fixed $x \in (0,\pi]$ the kernel $G(x,\tau)$ satisfies the following equation:*

$$A(x,t)+\int_{-x}^{x} G(x,\tau)A(\tau,t)d\tau=0, \quad 0\le t\le x\le \pi, \tag{13}$$

where

$$A(x,t):=\sum_{n=0}^{\infty}\frac{\mathring{\varphi}(x,\lambda_n)\cos\sqrt{\lambda_n}t}{\gamma_n}.$$

Proof. Using equality (12) we obtain

$$\sum_{n=0}^{N}\frac{\varphi(x,\lambda_n)\cos\sqrt{\lambda_n}t}{\gamma_n}=\sum_{n=0}^{N}\frac{\mathring{\varphi}(x,\lambda_n)\cos\sqrt{\lambda_n}t}{\gamma_n}+$$
$$+\sum_{n=0}^{N}\frac{\cos\sqrt{\lambda_n}t}{\gamma_n}\int_{-x}^{x}G(x,\tau)\mathring{\varphi}(\tau,\lambda_n)d\tau.$$

Applying Lemma 2. to $v(t,\lambda)=\cos\sqrt{\lambda}t$ we get the statement of the theorem.

We denote

$$F(x,t):=\frac{(h_1-\lambda_0)\mathring{\varphi}(x,\lambda_0)\cos\sqrt{\lambda_0}t}{\gamma_0}+\frac{(h_1-\lambda_1)\mathring{\varphi}(x,\lambda_1)\cos\sqrt{\lambda_1}t}{\gamma_1}+$$
$$+\sum_{n=0}^{\infty}\left(\frac{(h_1-\lambda_{n+2})\mathring{\varphi}(x,\lambda_{n+2})\cos\sqrt{\lambda_{n+2}}t}{\gamma_{n+2}}-\frac{\left(\cos nx+h\frac{\sin nx}{n}\right)\cos nt}{\alpha_n^0}\right),$$

where

$$\alpha_n^0=\begin{cases}\dfrac{\pi}{2}, & n\ge 1\\ \pi, & n=0.\end{cases}$$

By differentiating the equation (13) twice with respect to t we find

$$F(x,t)+h+G(x,t)+G(x,-t)+$$
$$+h\int_t^x\left(G(x,\tau)-G(x,-\tau)\right)d\tau+\int_{-x}^{x}G(x,\tau)F(\tau,t)d\tau=0.$$

Putting $x=t=0$ we obtain

$$h=-\frac{(h_1-\lambda_0)^2}{\gamma_0}-\frac{(h_1-\lambda_1)^2}{\gamma_1}-\sum_{n=2}^{\infty}\left(\frac{(h_1-\lambda_n)^2}{\gamma_n}-\frac{1}{\alpha_{n-2}^0}\right). \tag{14}$$

Analogously, we have

$$H=-\frac{(H_1-\lambda_0)^2}{k_0^2\gamma_0}-\frac{(H_1-\lambda_1)^2}{k_1^2\gamma_1}-\sum_{n=2}^{\infty}\left(\frac{(H_1-\lambda_n)^2}{k_n^2\gamma_n}-\frac{1}{\alpha_{n-2}^0}\right). \tag{15}$$

Theorem 4. *Suppose that $\{\lambda_n,\gamma_n\}_{n\ge0}$ and $\{\widetilde{\lambda}_n,\widetilde{\gamma}_n\}_{n\ge0}$ are the spectral data of the problems $\mathcal{P}(q,h,h_1,h_2,H,H_1,H_2)$ and $\mathcal{P}(\widetilde{q},\widetilde{h},\widetilde{h_1},\widetilde{h_2},\widetilde{H},\widetilde{H}_1,\widetilde{H}_2)$, respectively. If*

$$\lambda_n=\widetilde{\lambda}_n,\quad \gamma_n=\widetilde{\gamma}_n,\qquad\qquad n\ge 0,$$

then

$$q(x) = \widetilde{q}(x) \ \textit{a.e. on} \ (0,\pi),$$

$$h = \widetilde{h}, \quad h_1 = \widetilde{h_1}, \quad h_2 = \widetilde{h_2}, \qquad H = \widetilde{H}, \quad H_1 = \widetilde{H}_1, \quad H_2 = \widetilde{H}_2.$$

Proof. Using (6), (11), (14) and (15) we obtain

$$h = \widetilde{h}, \quad h_1 = \widetilde{h_1}, \quad h_2 = \widetilde{h_2}, \quad H = \widetilde{H}, \quad H_1 = \widetilde{H}_1, \quad H_2 = \widetilde{H}_2.$$

The result now follows from [2, Theorem 5.2].

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