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ADDITIVE COMPLEMENTS FOR A GIVEN ASYMPTOTIC DENSITY

ALAIN FAISANT, GEORGES GREKOS, RAM KRISHNA PANDEY,
AND SAI TEJA SOMU

ABSTRACT. We investigate the existence of subsets A and B of $\mathbb{N} := \{0, 1, 2, \dots\}$ such that the sumset $A + B := \{a + b ; a \in A, b \in B\}$ has given asymptotic density. We solve the particular case in which B is a given finite subset of \mathbb{N} and also the case when $B = A$; in the last situation, we generalize our result to $kA := \{x_1 + \dots + x_k : x_i \in A, i = 1, \dots, k\}$.

1. INTRODUCTION

The purpose of this paper is to introduce a new, up to our knowledge, subject of research, resolving a few particular cases.

Two subsets, not necessarily distinct, A and B of $\mathbb{N} := \{0, 1, 2, \dots\}$ are called *additive complements* if $A + B$ contains all, except finitely many, positive integers; that is, if the set $\mathbb{N} \setminus (A + B)$ is finite. From the huge literature on this topic, we just mention the papers [Lo], [Nar] and [D] among the initial ones, and the papers [FC] and [R] among the last ones. The reader may find there information and more references. A set A being additive complement of itself, that is a set A such that $\mathbb{N} \setminus (2A)$ be finite (where we put $2A := A + A$), is called an *asymptotic basis of order 2*.

In this paper we are interested in what happens when one asks that the density of the sumset $A + B$ is equal to a given value α , $0 \leq \alpha \leq 1$. As density concept we use the asymptotic density, defined below.

Definition 1.1. Let X be a subset of \mathbb{N} and x a real number. For $x \geq 1$, we put $X(x) := |X \cap [1, x]|$. For $x < 1$, we put $X(x) = 0$. We define the *asymptotic* (also called *natural*) *density* of X as

$$dX := \lim_{x \rightarrow +\infty} \frac{X(x)}{x}$$

provided that the above limit exists. The *lower* and the *upper asymptotic densities*, denoted by $\underline{d}X$ and $\overline{d}X$, respectively, are defined by taking in the

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above formula the lower and the upper limits, respectively, which always exist.

In the classical situation the existence of additive complements is obvious and the main problem is to find “thin” subsets of \mathbb{N} being additive complements or asymptotic bases. In our study, a first question is the existence of such sets A and B . Precisely, in this paper, we establish the existence of A given a *finite* set B and we consider also the case $B = A$ (see below in the introduction). Once the existence has been established, two questions arise naturally:

- to find “thin” sets verifying the required conditions;
- to find “thick” sets verifying the required conditions.

Notations. $|S|$ denotes, according to the context, either the cardinality of the finite set S or the length of the interval S . All small letters, except f, g, c, d, x, α and θ , represent nonnegative integers. Let x be a real number. We denote by $\lfloor x \rfloor$ the “integer part” of x and by $\{x\}$ the “fractional part” of x . Thus

$$x = \lfloor x \rfloor + \{x\}$$

where $\lfloor x \rfloor$ is an integer and $0 \leq \{x\} < 1$, this writing being *unique*.

In Section 2, we indicate or prove some properties of asymptotic density and in Section 3, we prove the next theorem.

Theorem 1.2. *Let α be a real number, $0 \leq \alpha \leq 1$, and B a finite subset of \mathbb{N} . Then there exists a set $A \subset \mathbb{N}$ such that $d(A + B) = \alpha$.*

Let us now suppose that α is given and $A = B$. Our goal was to prove the existence of a set A such that $d(2A) = \alpha$. The proof(s) can be generalized to sumsets of more summands A ; precisely, we prove the following more general result, showing that, for the constructed set A , the density of the sumsets jA , $1 \leq j \leq k$, increases regularly with j . Proofs are of different form, according to the nature of α : rational or irrational. Here is the formulation of the theorem proved in Section 4.

Theorem 1.3. *Let α be a real number, $0 \leq \alpha \leq 1$, and k an integer, $k \geq 2$. Then there exists a subset A of \mathbb{N} such that for every j , $1 \leq j \leq k$, one has*

$$d(jA) = \frac{j\alpha}{k}.$$

In particular, $d(kA) = \alpha$.

We give a list of open questions and prospects for further research in Section 5.

2. AUXILIARY RESULTS

The next lemma is very useful. We omit the proof.

Lemma 2.1. *Let the set X in Definition 1.1 be infinite, $X =: \{a_1 < a_2 < \dots\}$. Then*

$$\underline{d}X := \liminf_{x \rightarrow +\infty} \frac{X(x)}{x} = \liminf_{k \rightarrow +\infty} \frac{k}{a_k},$$

and

$$\overline{d}X := \limsup_{x \rightarrow +\infty} \frac{X(x)}{x} = \limsup_{k \rightarrow +\infty} \frac{k}{a_k}.$$

As a consequence, we get the next property.

Property (i) Let X be a subset of \mathbb{N} and θ a real number, $\theta \geq 1$. Put $\theta.X := \{\lfloor \theta a \rfloor; a \in X\}$. Then $\underline{d}(\theta.X) = \theta^{-1} \underline{d}X$ and $\overline{d}(\theta.X) = \theta^{-1} \overline{d}X$.

Put $\mathbb{N}_+ := \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$. The uniform distribution of the sequence $(\{n\theta\})_{n \in \mathbb{N}_+}$ when θ is irrational ([KN], p. 8; [SP], p. 2 - 72) amounts to:

Property (ii) Let I be a subinterval of $[0, 1[$ and θ an irrational real number. Then

$$d\{n \in \mathbb{N}_+; \{n\theta\} \in I\} = |I|.$$

As a corollary of the previous properties, we get:

Corollary 2.2. *Let I be a subinterval of $[0, 1[$ and θ an irrational real number, $\theta > 1$. Then*

$$d\{\lfloor n\theta \rfloor; n \in \mathbb{N}_+, \{n\theta\} \in I\} = |I|/\theta.$$

3. PROOF OF THEOREM 1.2

If α is 0 or 1, the answer is easy: take, respectively, A to be the set of powers of 2 or $A = \mathbb{N}$.

We shall suppose in the sequel that $0 < \alpha < 1$.

Lemma 3.1. *Without loss of generality, we can suppose that $\min B = 0$.*

Proof. Suppose that the theorem is proved when $b_1 := \min B = 0$. Now let $b_1 > 0$. Put $B' := B - b_1$. By the case $\min B = 0$, there is $A' \subset \mathbb{N}$ such that $d(A' + B') = \alpha$. Put $A_1 := A' - b_1 \subset \{-b_1, -b_1 + 1, \dots, -1\} \cup \mathbb{N}$ and $A := \{a \in A_1; a \geq 0\}$. Notice that

$$A + B \subset A_1 + B = A' + B' \subset \mathbb{N}$$

so $A_1 + B$ has asymptotic density α . Then notice that

$$(A_1 + B) \setminus (A + B) \subset \cup_{k=-1}^{-b_1} \{k + B\}.$$

The set in the right hand member is finite. It yields that $d(A + B) = d(A_1 + B) = \alpha$ which proves the Lemma. \square

Continuation of the proof of Theorem 1.2. We suppose from now on that $\min B = 0$. Let also $b := \max B$, $k := |B|$. Of course the theorem is trivial if $k = 1$. We suppose in the sequel that $k \geq 2$; consequently $b \geq 1$. The set $A = \{a_1, a_2, \dots\}$, $a_1 < a_2 < \dots$, will be defined by induction. We define $a_1 := \min A$ in the following manner:

$$(1) \quad a_1 := \min\{a \geq 0; \max_{n \geq 1} \frac{(a+B)(n)}{n} \leq \alpha\}.$$

Remark on (1). Notice that $a_1 = 0$ if and only if, for all $n \geq 1$, $B(n) \leq \alpha n$. For given a and all $n \geq 1$, let $f(n) := \frac{(a+B)(n)}{n}$. The function f takes nonnegative values and is decreasing when $n \geq a + b$; so, for fixed a , the maximum exists. This maximum is attained for (at least) an $n = n_1$, $a \leq n_1 \leq a + b$, and verifies, when $a \neq 0$,

$$\max_{n \geq 1} f(n) = f(n_1) \leq \frac{k}{n_1} \leq \frac{k}{a}.$$

So this maximum is less than or equal to α provided that $a \geq \frac{k}{\alpha}$. It follows that the minimum in formula (1) exists and hence a_1 is well defined and $a_1 \leq \lceil \frac{k}{\alpha} \rceil$.

Recursion. Suppose that a_1, a_2, \dots, a_m have been defined, and let $A_m := \{a_1, a_2, \dots, a_m\}$. We suppose that, for all $n \geq 1$,

$$\frac{(A_m + B)(n)}{n} \leq \alpha.$$

Then we define a_{m+1} in the following manner:

$$(2) \quad a_{m+1} := \min\{a > a_m; \max_{n \geq a} \frac{((A_m \cup \{a\}) + B)(n)}{n} \leq \alpha\}.$$

Remark on (2). Notice that $a_{m+1} = a_m + 1$ if and only if, for all $n \geq a_m + 1$, $((A_m \cup \{a_m + 1\}) + B)(n) \leq \alpha n$. Otherwise, $a_{m+1} > a_m + 1$. For fixed $a > a_m$, let $g(n) := \frac{((A_m \cup \{a\}) + B)(n)}{n}$, for all $n \geq a$. The function g takes nonnegative values and is decreasing when $n \geq a + b$; so, for fixed a , the maximum exists. This maximum is attained for (at least) an $n = n_{m+1}$, $a \leq n_{m+1} \leq a + b$, and verifies

$$\max_{n \geq a} g(n) = g(n_{m+1}) \leq \frac{k(m+1)}{n_{m+1}} \leq \frac{k(m+1)}{a}.$$

So this maximum is less than or equal to α provided that $a \geq \frac{k(m+1)}{\alpha}$. It follows that the minimum in formula (2) exists and hence a_{m+1} is well defined

and $a_{m+1} \leq \lceil \frac{k(m+1)}{\alpha} \rceil$.

Let us now observe that, for all $n \in \mathbb{N}$, we have

$$(A + B)(n) \leq \alpha n.$$

This is clear when $n = 0$ or $n < a_1$. Otherwise, there is $m \geq 1$ such that $a_m \leq n < a_{m+1}$. Then we have $(A + B)(n) = (A_m + B)(n) \leq \alpha n$.

This implies that $\bar{d}(A + B) \leq \alpha$.

It remains to prove that $\underline{d}(A + B) \geq \alpha$. To do that, it is sufficient to prove that, for any $\varepsilon > 0$, we have $\underline{d}(A + B) \geq \alpha - \varepsilon$.

In what follows, we shall need the next property of the counting function $n \mapsto (A + B)(n)$ of the set $A + B$. In its (short) proof, we shall need the fact that $0 \in B$.

Property (iii) If $a \in A$, $a > 1$, then

$$\frac{(A + B)(a)}{a} \geq \frac{(A + B)(a - 1)}{a - 1}.$$

Proof. Let $y := (A + B)(a - 1)$. Since $a \in A$ and $0 \in B$, we get that $a = a + 0 \in A + B$ and so $(A + B)(a) = y + 1$. We have to show that $(y + 1)/a \geq y/(a - 1)$. The verification is straightforward and this proves Property (iii). \square

Continuation of the proof of Theorem 1.2. Let $0 < \varepsilon < \alpha$. We shall prove that $\underline{d}(A + B) \geq \alpha - \varepsilon$ by contradiction: suppose that $\underline{d}(A + B) < \alpha - \varepsilon$. Then the set

$$S := \{n \in \mathbb{N}; (A + B)(n) < (\alpha - \varepsilon)n\}$$

is infinite.

We observe that, for $0 < \alpha < 1$, the constructed set A is neither finite nor cofinite. A is a collection of finite “blocks”, each block consisting of one or of a finite number of consecutive integers, and two consecutive blocks are separated by a “hole” of length at least 2.

The preceding Property (iii) implies that if an element a of A , $a > 1$, belongs to S , then $a - 1$ belongs also to S . And since a belongs to a block of A , the last (the biggest) element of the hole just before the block to which a belongs, is also an element of S . We conclude that the set $S' := S \setminus A$ is infinite.

From S' we can extract an infinite, strictly increasing, sequence of positive integers $(N_t)_{t \geq 1}$ such that to each $t \geq 1$ corresponds an index m_t , verifying :

- (i) $a_{m_t} < N_t < a_{m_t+1}$, and
- (ii) $1 \leq m_1 < m_2 < \dots$.

Let us fix now an index $t \geq 1$. Recall that

$$(3) \quad (A + B)(N_t) = (A_{m_t} + B)(N_t) < (\alpha - \varepsilon)N_t .$$

Let $A' := A_{m_t} \cup \{N_t\}$. By the formula (2), we get that

$$\max_{n \geq N_t} \frac{1}{n} (A' + B)(n) > \alpha .$$

So there is an integer n' , $N_t \leq n' \leq N_t + b$, such that

$$(4) \quad (A' + B)(n') > \alpha n' .$$

By the construction,

$$(5) \quad (A_{m_t} + B)(n') \leq \alpha n' .$$

We observe that

$$(A_{m_t} + B)(n') - (A_{m_t} + B)(N_t) \leq n' - N_t \leq b ,$$

which implies, using also (3), that

$$(6) \quad (A_{m_t} + B)(n') \leq (A_{m_t} + B)(N_t) + b < (\alpha - \varepsilon)N_t + b .$$

We also observe that

$$(A' + B)(n') - (A_{m_t} + B)(n') \leq k ,$$

which implies, using now (4), that

$$(7) \quad (A_{m_t} + B)(n') \geq (A' + B)(n') - k > \alpha n' - k \geq \alpha N_t - k .$$

The left member of (6) and (7) is the same. Comparing their right members, we get that $\varepsilon N_t < k + b$.

This is not true for any t , since N_t tends to infinity. This implies that the hypothesis $\underline{d}(A + B) < \alpha - \varepsilon$ is false and completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.3

If α is 0 or 1, the answer is easy: take, respectively, A to be the set of powers of 2 or $A = \mathbb{N}$.

We shall suppose in the sequel that $0 < \alpha < 1$.

We distinguish two cases

♣ **Case A:** α rational; say, $\alpha = \frac{m}{n}$, where m, n are integers, $1 \leq m \leq n - 1$. It is not necessary that $\gcd(m, n) = 1$. Our construction of such a set A is **simpler** (see also the remark at the end of the proof of Case A) when

$m \geq 3$. So multiplying, if necessary, the terms of the fraction for α by 2 or by 3, we will suppose in the sequel that $3 \leq m \leq n - 1$.

Let $H := \{0, 1, \dots, m - 2, m\}$. We shall prove that the set

$$A := \cup_{h \in H}(nk \cdot \mathbb{N} + h)$$

verifies $d(jA) = j\alpha/k$, for all j , $1 \leq j \leq k$. To prove that, let us first observe that

$$jA = nk \cdot \mathbb{N} + jH;$$

hereon it is easy to verify that each element of the left member belongs to the right member and vice versa.

Notice that $k \max H = km < nk$. The set jA being a finite union of mutually disjoint arithmetic progressions of difference nk , we have

$$d(jA) = \sum_{t \in jH} d(nk \cdot \mathbb{N} + t) = |jH| \frac{1}{nk}.$$

But $jH = \{0, 1, \dots, jm - 2, jm\}$ because $jm = j \max H \in jH$, $jm - 1 \notin jH$, and every nonnegative integer less than $jm - 1$ belongs to jH . For example, $jm - 2 = (j - 1)m + (m - 2) \in jH$; or $jm - 3 = (j - 1)m + (m - 3) \in jH$. So

$$d(jA) = |jH| \frac{1}{nk} = \frac{jm}{nk} = \frac{j\alpha}{k}, \quad 1 \leq j \leq k.$$

Remark 4.1. It is possible to invent specific constructions for $m = 1$ and for $m = 2$.

♣ **Case B:** α irrational, $0 < \alpha < 1$.

We put $\theta := 1/\alpha$. We recall our notation $\mathbb{N}_+ := \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$. We shall prove that the set

$$A := \{[n\theta]; n \in \mathbb{N}_+, \{n\theta\} < \frac{1}{k}\}$$

verifies $d(jA) = j\alpha/k$, for all j , $1 \leq j \leq k$.

For $j = 1$, this follows from Corollary 2.2.

We suppose in the sequel that $2 \leq j \leq k$.

♠ We firstly prove that $\bar{d}(jA) \leq j\alpha/k$.

To do that, we begin by proving that $jA \subset T_j$ where

$$(8) \quad T_j := \{[m\theta]; m \in \mathbb{N}_+, \{m\theta\} < j/k\}.$$

An element of jA is of the form $a_1 + \dots + a_j$ where $a_i = [n_i\theta] \in A$, $1 \leq i \leq j$.

This yields

$$a_1 + \dots + a_j = [n_1\theta] + \dots + [n_j\theta] = n_1\theta - \{n_1\theta\} + \dots + n_j\theta - \{n_j\theta\}$$

and consequently

$$(n_1 + \cdots + n_j)\theta = a_1 + \cdots + a_j + \{n_1\theta\} + \cdots + \{n_j\theta\}.$$

We have

$$0 < \{n_1\theta\} + \cdots + \{n_j\theta\} < j\frac{1}{k} \leq 1,$$

and so $\{n_1\theta\} + \cdots + \{n_j\theta\}$ is the fractional part and $a_1 + \cdots + a_j$ is the integer part of $(n_1 + \cdots + n_j)\theta$. In other terms and according to definition (8), $a_1 + \cdots + a_j$ belongs to T_j . Since, by Corollary 2.2, T_j has asymptotic density $j\alpha/k$, the desired inequality follows.

♠ We shall now prove that $\underline{d}(jA) \geq j\alpha/k$.

This will be done in the following way. We fix a real number ε , $0 < \varepsilon < \frac{1}{4k}$, and we shall prove that

$$(9) \quad \underline{d}(jA) \geq \alpha\left(\frac{j}{k} - \varepsilon\right).$$

Taking the limit for ε tending to zero in (9) gives the desired inequality for $\underline{d}(jA)$.

To prove (9), we introduce the set

$$(10) \quad B := \{[N\theta]; N \in \mathbb{N}_+, \frac{\varepsilon}{2} \leq \{N\theta\} < \frac{j}{k} - \frac{\varepsilon}{2}\}$$

which, by Corollary 2.2, has asymptotic density $(\frac{j}{k} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2})/\theta = \alpha(\frac{j}{k} - \varepsilon)$ and we will verify that **almost all** (that is, all except a finite number) **elements of B belong to jA** ; this implies (9).

Let $\ell := \frac{j}{k} - \varepsilon$, which is a positive real number ($\ell > \frac{2}{k} - \frac{1}{4k} > 0$) less than 1 ($\ell \leq \frac{k}{k} - \varepsilon < 1$). We split the interval $[\frac{\varepsilon}{2}, \frac{j}{k} - \frac{\varepsilon}{2})$ into j intervals of equal length ℓ/j :

$$[\frac{\varepsilon}{2}, \frac{j}{k} - \frac{\varepsilon}{2}) = \cup_{i=0}^{j-1} I_i$$

where

$$I_i := [\frac{\varepsilon}{2} + \frac{i\ell}{j}, \frac{\varepsilon}{2} + \frac{(i+1)\ell}{j}), \quad 0 \leq i \leq j-1.$$

The set B splits into j sets $B = \cup_{i=0}^{j-1} B_i$, where $B_i := \{[N\theta]; N \in \mathbb{N}_+, \{N\theta\} \in I_i\}$, and it will be sufficient (and necessary!) to prove that, for each i , all except finitely many elements of B_i lie in jA . Here is the procedure.

First, one can easily verify that

$$0 \leq \frac{\varepsilon}{2} + \frac{(i+1)\ell}{j} - \frac{1}{k} < \frac{\varepsilon}{2} + \frac{i\ell}{j}.$$

By the uniform distribution modulo 1 of the sequence $(\{n\theta\})_n$ (Property (ii)), there is a positive integer m_i such that

$$(11) \quad \frac{\varepsilon}{2} + \frac{(i+1)\ell}{j} - \frac{1}{k} < (j-1)\{m_i\theta\} < \frac{\varepsilon}{2} + \frac{i\ell}{j}.$$

We have that $\lfloor m_i\theta \rfloor$ belongs to A since

$$\begin{aligned} \{m_i\theta\} &< \frac{1}{j-1}\left(\frac{\varepsilon}{2} + \frac{i\ell}{j}\right) \leq \frac{1}{j-1}\left(\frac{\varepsilon}{2} + \frac{(j-1)\ell}{j}\right) = \frac{\varepsilon}{2(j-1)} + \frac{1}{j}\ell = \\ &= \frac{\varepsilon}{2(j-1)} + \frac{1}{j}\left(\frac{j}{k} - \varepsilon\right) = \frac{1}{k} - \varepsilon\left(\frac{1}{j} - \frac{1}{2(j-1)}\right) = \frac{1}{k} - \varepsilon\frac{j-2}{2j(j-1)} \leq \frac{1}{k}. \end{aligned}$$

We shall prove that, for every $N > (j-1)m_i$ such that $\lfloor N\theta \rfloor$ belongs to B_i , $\lfloor N\theta \rfloor$ belongs also to jA .

Since $\lfloor N\theta \rfloor$ belongs to B_i , we have that

$$(12) \quad \frac{\varepsilon}{2} + \frac{i\ell}{j} \leq \{N\theta\} < \frac{\varepsilon}{2} + \frac{(i+1)\ell}{j}.$$

Putting (11) and (12) together, gives

$$(13) \quad 0 < \{N\theta\} - (j-1)\{m_i\theta\} < \frac{1}{k}.$$

From the equality $N\theta = (N - (j-1)m_i)\theta + (j-1)m_i\theta$, taking the integer part and the fractional part of each multiple of θ , we get

$$(14) \quad \lfloor N\theta \rfloor + \{N\theta\} - (j-1)\{m_i\theta\} = \lfloor (N - (j-1)m_i)\theta \rfloor + \{(N - (j-1)m_i)\theta\} + (j-1)\lfloor m_i\theta \rfloor.$$

By the uniqueness of decomposition of a real number into its integer part and its fractional part, the inequality (13) implies that the fractional parts appearing in (14) verify

$$(15) \quad \{N\theta\} - (j-1)\{m_i\theta\} = \{(N - (j-1)m_i)\theta\}$$

and this, combined with (14), gives that

$$(16) \quad \lfloor N\theta \rfloor = \lfloor (N - (j-1)m_i)\theta \rfloor + (j-1)\lfloor m_i\theta \rfloor.$$

As observed before, $\lfloor m_i\theta \rfloor \in A$. Because of (15) and (13), $\lfloor (N - (j-1)m_i)\theta \rfloor$ belongs to A and (16) gives us that $\lfloor N\theta \rfloor \in jA$. This completes the proof of (9) and of the whole Theorem 1.3.

Added in proof.- In [V] the author resolves in a more general context (\mathbb{Z}^t instead of \mathbb{N}) a problem which, in some sense, contains as special case the problem solved in the above theorem. When $k = 2$, the meaning of our sentence “in some sense” is as follows: Given two positive real numbers α_1 and α_2 such that $\alpha_1 + \alpha_2 \leq 1$ and a third real number γ , $\alpha_1 + \alpha_2 \leq \gamma \leq 1$, Bodo Volkmann [V] constructs sets A_1, A_2 of natural numbers satisfying $d(A_1) = \alpha_1$, $d(A_2) = \alpha_2$ and $d(A_1 + A_2) = \gamma$. The construction uses ideas similar to the ours. The principle is the same: to use uniform distribution of fractional parts in order to obtain sets of integers with prescribed density.

In the case when $\alpha_1 = \alpha_2$, the sets A_1, A_2 are not equal. The set A_1 is constructed in a way similar to the one used in our proof, but the set A_2 is constructed in a specific way in order to obtain $d(A_1 + A_2) = \gamma$. Even in the case when $\gamma = 2\alpha_1 = 2\alpha_2$, the set A_2 is different from A_1 . But we observed that in this particular case Volkmann's construction can be slightly modified to give $A_2 = A_1$ thus providing another proof of our theorem. A similar remark is valid when $k \geq 3$.

5. FUTURE PROSPECTS

We separate this section into questions: *Q1* to *Q7*.

Q1 - Thin sets. Concerning the case $B = A$, it would be interesting to study the existence of thin sets A verifying $d(kA) = \alpha$. For *additive bases* A (that is, for sets A verifying $kA = \mathbb{N}$ for some k , called "order" of the basis A) this was done by Cassels [Ca] (see also [HR], p. 35-43) where, for every $k \geq 2$, a "thin" basis of order k was found. As in the case of additive bases, in our situation, with $\alpha > 0$, the condition $A(x) \geq cx^{1/k}$ is necessary. In Cassels' construction, the basis corresponding to the order k verifies $A(x) \leq c'x^{1/k}$. The question here is to find sets A verifying $d(kA) = \alpha > 0$ and $A(x) \leq c''x^{1/k}$. Here is a related question: is it possible to extract from Cassels' thin basis A (such that $kA = \mathbb{N}$) a set $A' \subset A$ such that $d(kA') = \alpha$? If this is possible, the condition $A(x) \leq c''x^{1/k}$ is automatically verified.

Remark.- The above mentioned Cassels' thin bases allow to answer the above question when $\alpha = 1/n$. Take the Cassels asymptotic basis $C = \{c_1 < c_2 < \dots\}$ [Ca] (see also [HR], p. 37) of order n . It verifies $c_m = \beta m^n + O(m^{n-1})$. Now a solution to the above asked question is to take $A := \{nx; x \in C\}$. For other values of α the question remains open.

Concerning the case of a given finite set B considered in this paper, we see two questions:

- (1) To determine the more thin set A that verifies $d(A + B) = \alpha$: A necessary condition is that $\underline{d}A \geq \alpha/|B|$.
- (2) To evaluate the density (or the upper and the lower densities) of the greedily constructed set A . The constructed set seems to be "the thickest".

Q2 - Thick sets. In the case of bases, the thickest set A verifying $kA = \mathbb{N}$ is $A = \mathbb{N}$. What are thick sets A satisfying $d(kA) = \alpha$ when $\alpha < 1$? In the case $\alpha = \frac{1}{r}$, where r is an integer, $r \geq 2$, the answer is trivial: take $A = \{0, r, 2r, 3r, \dots\}$. But in general the answer is not obvious. It may depend on the nature of α : rational or irrational.

Q3 - B infinite. What happens for given infinite B ? Find necessary or sufficient (or both!) conditions on B (on the upper and lower densities of B) such that A exists. Our greedy method of Section 2, with some supplementary considerations on formulas (1) and (2), allows to construct a set A such that $\bar{d}(A + B) \leq \alpha$ when $dB = 0$. But we are not able to prove that $d(A + B) = \alpha$; nor to disprove it for a specific set B .

Q4 - Other densities. Replace asymptotic density by other densities. See [G] for a list of definitions. For instance, the *exponential density*, defined as [compare with Definition 1.1 of asymptotic density in Section 1]

$$\varepsilon X := \lim_{x \rightarrow +\infty} \frac{\log X(x)}{\log x},$$

could be of interest. That is, given B and α , is there A such that the exponential density of $A + B$ is equal to α ? Instead of using specific (concrete) definitions of density, one could use axiomatically defined densities which generalize some of the usual concepts of density; see [FS], [LT1] and [LT2]. The cases of *positive* Schniremann density and of *positive lower* asymptotic density were already considered in [Le], [Ch] and [Nat], where best possible results to Mann's and Kneser's theorems are given.

Q5 - Couple of densities. It is possible to consider the initial problem and to ask all the above questions replacing α by two real numbers α' and α'' , $0 \leq \alpha' \leq \alpha'' \leq 1$, that will be, respectively, the lower and the upper densities. For instance, given α' and α'' as above and $k \geq 2$, find a set A such that $\underline{d}(kA) = \alpha'$ and $\bar{d}(kA) = \alpha''$.

Q6 - Generalization. Given a subset B of \mathbb{N} , finite or of zero asymptotic density, a real number α , $0 \leq \alpha \leq 1$, and an integer $k \geq 2$, is there a set $A \subset \mathbb{N}$ such that $d(B + kA) = \alpha$? Search for "thin" and for "thick" such sets A .

Q7 - Last but not least: A and B . The more studied question on classical additive complements is to compare the functions $(A+B)(x)$ and $A(x)B(x)$. Obviously $(A+B)(x) \leq A(x)B(x)$. So to have "thin" sets A and B means that $A(x)B(x)$ is not much bigger than $(A+B)(x)$. In the classical case, $(A+B)(x)$ is equal, up to a constant, to x . In our case, with $\alpha > 0$, $(A+B)(x)$ is "equivalent" to $\alpha x : \lim_{x \rightarrow +\infty} (A+B)(x)/\alpha x = 1$. So questions studied in [Nar], [D] or [FC] and finally in [R] can be formulated with αx in place of x .

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