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# Homogenization of a neutronic multigroup evolution model

Yves Capdeboscq

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Mailing address: Dept of Math-Hill Center  
Rutgers University  
110 Frelinghuysen Road  
Piscataway NJ 08854-8019

Tel: (1) 732-445 4877

Fax: (1) 732-445 5530

email: [ycrc@math.rutgers.edu](mailto:ycrc@math.rutgers.edu)

**Abstract.** In this paper is studied the homogenization of an evolution problem for a cooperative system of weakly coupled elliptic partial differential equations, called neutronic multigroup diffusion model, in a periodic heterogenous domain. Such a model is used for studying the evolution of the neutron flux in nuclear reactor core. In this paper, we show that under a symmetry assumption, the oscillatory behavior of the solutions is controled by the first eigenvector of a multigroup eigenvalue problem posed in the periodicity cell , whereas the global trend is asymptotically given by a homogenized evolution problem. We then turn to cases when the symmetry condition is not fulfilled. In domains without boundaries, the limit equation for the global trend is then a homogenized transport equation. Alternatively, we show that in bounded domains and with well prepared initial data, the microscopic scale does not only control the oscillatory behavior of the solutions, but also induces an exponential drift.

# 1 Introduction

We address the homogenization of a periodic model of evolution equation, corresponding to the so-called multigroup diffusion approximation, in neutron physics. In a domain  $\Omega \subset \mathbb{R}^N$ , making the hypothesis of  $K$  energy groups, this simplified version of the neutron transport equation writes as follows, see e.g. [24], for a.e.  $t \in [0, T]$ ,

$$\begin{cases} \frac{1}{v} \frac{\partial \phi}{\partial t} - \operatorname{div}(A(x)\nabla\phi) + \Sigma(x)\phi = \sigma(x)\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \\ \phi(t=0, x) = \phi_0(x) \in H_1^0(\Omega)^K \end{cases} \quad (1)$$

The neutron flux  $\phi = (\phi_\alpha)_{1 \leq \alpha \leq K}$  is a  $K$  component vector function. The velocity

$$\frac{1}{v} = \operatorname{diag}\left(\frac{1}{v_1}, \dots, \frac{1}{v_K}\right)$$

corresponds to the fixed speeds neutrons are moving at. The matrices  $\Sigma$  and  $\sigma$  are called adsorption and fission matrices. The diffusion operator  $A$  is block diagonal, and we write  $A = \operatorname{diag}(A_1, \dots, A_K)$ , in the sense that

$$A\nabla\phi = (A_1\nabla\phi_1, \dots, A_K\nabla\phi_K)^T, \quad (2)$$

where each  $(A_\alpha)_{1 \leq \alpha \leq K}$  is a symmetric  $N \times N$  matrix,  $A_\alpha \in L^\infty(\Omega)^{N \times N}$  and all are uniformly coercives. The physical issue is to know if to such a system admits a non trivial steady state. In other words, whether (1) admits a positive solution if we delete the time derivatives. Critical calculus, see e.g. [24], amounts to introduce the following abstract eigenvalue problem, called criticality problem,

$$\begin{cases} -\operatorname{div}(A(x)\nabla\phi) + \Sigma(x)\phi = \frac{1}{k_{eff}}\sigma(x)\phi & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \quad (3)$$

and to calculate its first eigenvalue, to which corresponds the only positive eigenvector, in order to know if it is equal to one. The unknown is the couple  $(k_{eff}^{-1}, \varphi)$  of the first eigenvalue and eigenvector for (3). The eigenvalue  $k_{eff}$  is a measure of the balance between production and removal of neutrons in a quasistatic limit. If  $k_{eff} < 1$ , too many neutrons are diffused or absorbed in the core compared to their production by fission : the nuclear chain reaction dies out, and the reactor is said to be sub-critical. If  $k_{eff} > 1$ , too many neutrons are created by fission, and the reactor is said to be super-critical. In such a case, absorbing media (the so-called control rods) should be added to control the reaction. Eventually, when  $k_{eff} = 1$ , the reactor is said to be critical : a perfect balance between fission and absorption-diffusion is obtained: system (1) admits a non trivial steady state.

It is worth explaining why we are concerned with the diffusion approximation of the neutron transport. Indeed, the mathematical study of the neutron transport equations was first addressed by Bensoussan, Lions and Papanicolaou [7] Larsen & al. [17, 16], and Sentis [25]. Formulas for the homogenized coefficients were then derived, and revisited in a different context by Allaire and Bal [2] for highly heterogenous media. The first reason is that diffusion

models are used in practice for industrial computations, and that the homogenization of multigroup diffusion models is actually used in neutronic computational procedures. The second reason is that the numerous theoretical results available for the diffusion models allow a study of the multigroup diffusion models with sufficiently weak assumption for then to apply to the entire range of practical configurations of interest for simulation. For example, the results of Mitidieri and Sweers [21], allows us to make only minimal assumptions on the fission operator  $\sigma$ , which makes our results applicable to every multigroup model. The results obtained for multigroup diffusion models certainly extend to transport models, but at the cost of the proof of difficult existence and regularity results, far beyond the scope of this work.

In the homogenization of the criticality problem [3], a symmetry condition (20) appeared as compulsory to be able to generalize the known results in the monogroup case [4, 20]. The case when this symmetry condition is not satisfied has been addressed in [10]. Then, asymptotic behavior of the first eigenpair  $(k_{eff}^{-1}, \varphi)$  is dramatically different. The microscopic scale does not only control the oscillatory behavior of the eigenfunctions, but also induces an exponential drift and the asymptotic limit is defined with the help of a  $\theta$ -exponential periodic eigenvalue problems. In this paper, we prove that whether this condition (20) is satisfied or not also dramatically changes the limit homogenized evolution problem. We shall obtain that under the symmetry assumption (20), the flux  $\phi$  solution of (1) can be approximated by

$$\phi_\alpha(x, t) \approx \psi_\alpha(x) \exp(\tau \cdot t) u(x, t), \quad 1 \leq \alpha \leq K, \text{ a.e. } x \in \Omega, \text{ a.e. } t \in [0, T],$$

where  $\psi$  is the first periodic eigensolution solution of an eigenvalue problem posed on the periodicity cell, and describes the local spatial oscillation of the flux,  $\tau$  is a real constant which characterizes the state of criticality of the equation, and  $u$  is solution of a homogenized evolution problem, with constant coefficients, posed on the whole domain  $\Omega$ .

When the symmetry condition (20) is not satisfied, we shall consider two different situations where the homogenized limit can be derived.

First, we will show that for well prepared initial data, the flux  $\phi$  solution of (1) can be approximated by

$$\phi_\alpha(x, t) \approx \psi_\alpha^\theta(x) \exp(\tau' \cdot t) u(x, t), \quad 1 \leq \alpha \leq K, \text{ a.e. } x \in \Omega, \text{ a.e. } t \in [0, T],$$

where at the difference of the previous case,  $\psi^\theta$  is the first eigensolution of an eigenvalue problem posed on the periodic cell, where the gradient operator has been translated by a parameter  $\theta \in \mathbb{R}^N$ . Therefore each component of  $\psi^\theta$  is the product of a periodic function and of an exponential factor, which tends to concentrate  $\phi$  on one of the boundaries of the domain  $\Omega$ .

Alternatively, without the hypothesis of well prepared initial data, but assuming that the domain  $\Omega$  does not have boundaries, i.e.  $\mathbb{R}^N$  or the torus  $\mathbb{T}^N$ , we obtain that the flux  $\phi$  solution of (1) can be approximated by

$$\phi_\alpha(x, t) \approx \psi_\alpha(x) \exp(\tau \cdot t) u(x, t), \quad 1 \leq \alpha \leq K, \text{ a.e. } x \in \Omega, \text{ a.e. } t \in [0, T],$$

where  $\psi$  is again the first periodic eigensolution solution of an eigenvalue problem posed on the periodic cell, and  $u$  is the solution of a homogenized transport equation.

This paper is organized as follows. In Section 2 are recalled the existence and regularity results for problems (1) and (3) already mentioned in [3]. In Section 3, we will adapt the

results obtained in [3] concerning critical calculus, in the drift free case. In the context of neutron transport, the absence of drift corresponds to a symmetry assumption that is currently done. Homogenization formulas in this case have already been derived [7, 17, 25]. However, it seems that it was assumed that the configuration was critical, i.e. that at cell level, the first eigenvalue was exactly equal to one. Here, no assumption is made on the criticality of the problem. The next two sections address the general case, in presence of drift. In Section 4, we show that with well prepared initial datas, the characterization of the drift phenomenon obtained in [9, 10] and with the help of  $\theta$ -exponential periodic problems can also be adapted to the evolution problem (4). In Section 5 we show that in a domain without boundary, the homogenized limit problem is a transport equation.

In truth, most of the technical results of this paper are adapted from the ones obtained in [3, 10] concerning the critical eigenvalue problem. We use in particular the results showed in these papers concerning the homogenization of associated source problems to obtain the homogenized limits. As a consequence, the actual homogenization steps are not presented here. The aim of this paper is to show that the factorization techniques, used for the spectral problem in the above mentioned papers can be adapted to the evolution model, and enables us to provide a precise description of the asymptotic behavior of the solution in the general case, i.e. without any assumption on the existence of a steady state for the evolution problem.

## 2 Existence and regularity results

We scale the space variables so that the scale of the neutron mean free path during the homogenization procedure is preserved. This scaling was first introduced by Larsen & al. [18]. Through the scope of this evolution problem, it is in fact the steady state we wish to address. Therefore we shall choose large time scales, of order 1 or 2 in  $\frac{1}{\epsilon}$ . Let us introduce the following evolution problem, where  $n$  is either 1 or 2, for a.e.  $t \in [0, T]$ ,

$$\left\{ \begin{array}{l} \epsilon^n \frac{1}{v} \frac{\partial \phi^\epsilon}{\partial t} - \epsilon^2 \operatorname{div} \left( A \left( \frac{x}{\epsilon} \right) \nabla \phi^\epsilon \right) + \Sigma \left( \frac{x}{\epsilon} \right) \phi^\epsilon = \left( \sigma \left( \frac{x}{\epsilon} \right) + \epsilon^2 \sigma' \left( x, \frac{x}{\epsilon} \right) \right) \phi^\epsilon \text{ on } \Omega \\ \phi^\epsilon = 0 \text{ on } \partial\Omega \\ \phi^\epsilon(t=0, x) = \phi_0(x) \in H_1^0(\Omega)^K \end{array} \right. \quad (4)$$

The initial condition in time is chosen positive and independent of  $\epsilon$  in order to simplify the proofs and the wording of the results.

Our major assumption is that the nuclear reactor core is periodic, i.e. all coefficients  $A(y)$ ,  $\Sigma(y)$ , and  $\sigma(y)$  are  $Y$ -periodic functions. This hypothesis is crucial for the homogenization procedure. In particular our results do not hold true any longer if the coefficients are the product of periodic functions with macroscopic modulations, as for example  $\Sigma \left( x, \frac{x}{\epsilon} \right)$  with a  $Y$ -periodic function  $\Sigma(x, y)$ . However, a positive perturbation of order  $\epsilon^2$   $\sigma'$  has been added because it does not modify the general behavior of the system in finite time. The homogenization of kinetic models involving a comparable scaling in  $\epsilon$ , and with coefficients showing macroscopic variations has been studied by Babovsky, Bardos, and Platkowski [5], Degond [11], Dumas and Golse [12], Goudon and Poupaud [13] and Kozlov and Pyatniskii [15] apart from the previously quoted authors.



**Proposition 2.1.** *For each fixed  $\epsilon > 0$  there exist a unique solution  $\phi^\epsilon$  of (4) in a distributional sense with*

$$\phi^\epsilon \in L^2(0, T; (H_0^1(\Omega))^K) \cap C([0, T]; (L^2(\Omega))^K) \quad \text{and} \quad \frac{\partial \phi^\epsilon}{\partial t} \in L^2(0, T; (H^{-1}(\Omega))^K)$$

We shall see that the critical eigenvalue problem posed in the periodicity cell  $Y$  plays a decisive part in the asymptotical behavior of system (4). This critical eigenvalue problem is the following

$$\begin{cases} -\operatorname{div}(A(y)\nabla\psi) + \Sigma(y)\psi = \lambda_0\sigma(y)\psi & 1 \leq \alpha \leq K \\ \psi \text{ Y-periodic} \end{cases} \quad (10)$$

where  $\lambda_0$  is the first eigenvalue, i.e. that of smallest modulus. We shall also need to introduce the adjoint problem of (10)

$$\begin{cases} -\operatorname{div}(A(y)\nabla\psi^*) + \Sigma^T(y)\psi^* = \lambda_0\sigma^T(y)\psi^* & 1 \leq \alpha \leq K \\ \psi^* \text{ Y-periodic} \end{cases} \quad (11)$$

To ensure that the periodic eigenvalue  $\lambda_0$  is not zero, we make the following hypothesis

$$\exists \alpha_0 \text{ s.t. } \sum_{\beta=1}^K \Sigma_{\alpha_0, \beta} > 0. \quad (12)$$

Then, the constant vector  $\mathbb{I} = (1, \dots, 1) \in \mathbb{R}^K$  satisfies

$$-\operatorname{div}(A(y)\nabla\mathbb{I}) + \Sigma(y)\mathbb{I} > 0$$

and we are then assured of the existence of a positive eigenvector for problem (10) (see e.g. [21]). The precise existence result of at least one solution for the eigenvalue problems (10) and (11) under these hypothesis is recalled by the following Theorem, see [14], [24], [21] and [3].

**Theorem 2.2.** *Under assumptions (2), (5), and (6-9) and (12),*

1. *the cell problem (10) admit at least one, and at most a countable number of possibly complex eigenvalues with associated eigenvectors in  $H_{\#}^1(Y)^K$ . Furthermore, its first eigenvalue (i.e. of smallest modulus) is real, positive and simple, and its corresponding eigenvectors can be chosen to be positive in  $Y$ .*
2. *This first eigenvalue is also the first eigenvalue of the adjoint problem (11). The corresponding eigenvector can also be chosen positive in  $Y$ .*
3. *The first eigenvectors of the cell problems (10) and (11) are Hölder continuous, i.e. belong to  $\left[H_{\#}^1(Y) \cap C_{\#}^{0,s}(Y)\right]^K$  for some  $s > 0$ .*

The cell eigenvalue problem (10) is not necessarily critical, i.e. the first eigenvalue may not be equal to one. In such situations, a qualitative argument would show that the neutron flux  $\phi^\epsilon$  solution of (4) will exponentially decay, if  $\lambda_0 > 1$ , or increase, if  $\lambda_0 < 1$ , when the time variable  $t$  tends to infinity. In order to obtain a more accurate description of the evolution

of the flux  $\phi^\epsilon$ , we will study the evolution of  $\phi^\epsilon(x, t) \exp(-\tau \frac{t}{\epsilon^n})$ , for  $\tau$  chosen such that  $\phi^\epsilon(x, t) \exp(-\tau \frac{t}{\epsilon^n})$ , admits a non trivial steady state.

This leads to the introduction of a new cell problem, shifted from (10) and (11) by a constant  $\tau \in \mathbb{R}$ .

$$\begin{cases} -\operatorname{div}(A(y)\nabla\psi_\tau) + \frac{\tau}{v}\psi_\tau + \Sigma(y)\psi_\tau = \lambda(\tau)\sigma(y)\psi_\tau & \text{in } Y \\ \psi \text{ } Y\text{-periodic} \end{cases} \quad (13)$$

Similarly, we introduce the adjoint cell problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla\psi_\tau^*) + \frac{\tau}{v}\psi_\tau^* + \Sigma^T(y)\psi_\tau = \lambda(\tau)\sigma^T(y)\psi_\tau^* & \text{in } Y \\ \psi \text{ } Y\text{-periodic} \end{cases} \quad (14)$$

We show in Theorem 2.3 that the introduction of parameter  $\tau$  enables us to tune system (13) so that it is critical.

**Theorem 2.3.** *Under the hypothesis of Theorem 2.2 there exists a unique  $\tau = \tau_\infty$  such that the first eigenvalue of the cell problems (13) and (14) is such that  $\lambda(\tau_\infty) = 1$ .*

1. *The cell problem (13) admit at least one, and at most a countable number of possibly complex eigenvalues with associated eigenvectors in  $H_{\#}^1(Y)^K$ . Furthermore, its first eigenvalue (i.e. of smallest modulus) is real and simple, and its corresponding eigenvectors can be chosen to be positive in  $Y$ .*
2. *This first eigenvalue is also the first eigenvalue of the adjoint problem (14). The corresponding eigenvector can also be chosen positive in  $Y$ .*
3. *The first eigenvectors of the cell problems (13) and (14) are Hölder continuous, i.e. belong to  $[H_{\#}^1(Y) \cap C_{\#}^{0,s}(Y)]^K$  for some  $s > 0$ .*

*Proof.* For any  $\tau \in \mathbb{R}^N$ , let us first consider the following auxiliary eigenvalue problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla\phi_\tau) + \frac{\tau}{v}\phi_\tau + \Sigma(y)\phi_\tau = \frac{\Lambda(\tau)}{v}\phi_\tau & \text{in } Y \\ \phi_\tau \text{ } Y\text{-periodic} \end{cases} \quad (15)$$

where  $\Lambda(\tau)$  is the first eigenvalue. Problem (15) corresponds to (13) in the special case when the fission matrix  $\sigma$  is diagonal and equals  $v$ . Clearly, we have

$$\Lambda(\tau) = \Lambda(0) + \tau \quad \text{and} \quad \phi_\tau = \phi_0$$

(up to a normalization of the eigenvector). Because of assumptions (8) and (12) system (15) admits a supersolution for  $\tau = 0$ , thus  $\Lambda(0) > 0$ . Consequently  $\Lambda(\tau) > 0$  if and only if  $\tau$  is in the open subset  $W \subset \mathbb{R}$  given by

$$W = ] - \Lambda(0), +\infty[. \quad (16)$$

A result of Mitidieri and Sweers [21, Theorem 5.1] show that this implies that for any  $\tau \in W$  there exists a first positive eigenvalue for the cell problem (13) and that properties 1,2 and 3 stated in Theorem 2.2 are verified by the corresponding direct and adjoint eigenvectors  $\psi_\tau$



of (13) and  $\psi_\tau^*$  of (14). It is also proved in the same article that there cannot exist a positive eigenvalue for (13) for  $\tau \notin W$ .

As a consequence, Theorem 2.3 will be proved if we obtain that there exists a unique  $\tau = \tau_\infty \in W$  such that  $\lambda(\tau_\infty) = 1$ .

Of course, for  $\tau = -\Lambda(0)$ ,  $\Lambda(\tau) = \Lambda(0) - \Lambda(0) = 0$ , and  $0 \times \frac{1}{v} = 0 \times \sigma$ , therefore eigenvalue problems (15) and (13) coincide, thus  $\psi_{-\Lambda(0)} = \phi_0$  and  $\lambda(-\Lambda(0)) = \Lambda(-\Lambda(0)) = 0$ . Consequently, the eigenvalue problems (13) and (14) are well defined for all  $\tau \in [-\Lambda(0), +\infty)$ .

Let  $\tau$  and  $\tau'$  be in  $[-\Lambda(0), +\infty)$ . From the variational formulation of  $\psi_{\alpha,\tau'}$  (13) tested against  $\psi_{\alpha,\tau}^*$ , and that of  $\psi_{\alpha,\tau}^*$  (14) tested against  $\psi_{\alpha,\tau'}$  we obtain

$$(\tau - \tau') \sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\alpha,\tau'} \psi_{\alpha,\tau}^* dy = (\lambda(\tau) - \lambda(\tau')) \sum_{\alpha,\beta=1}^K \int_Y \sigma_{\alpha,\beta}(y) \psi_{\beta,\tau'} \psi_{\alpha,\tau}^* dy \quad (17)$$

Thus  $\tau \rightarrow \lambda(\tau)$  is strictly increasing, because of the positivity of  $\sigma$ ,  $v_\alpha$ ,  $\psi_{\tau'}$  and  $\psi_\tau^*$ . Because of assumption (6) and (7) we have

$$\sum_{\alpha,\beta=1}^K \int_Y \sigma_{\alpha,\beta}(y) \psi_{\beta,\tau'} \psi_{\alpha,\tau}^* dy \geq \sum_{\alpha=1}^K \int_Y \sigma_{\alpha,\alpha}(y) \psi_{\alpha,\tau'} \psi_{\alpha,\tau}^* dy \geq C \sum_{\alpha=1}^K \int_Y \frac{1}{v} \psi_{\alpha,\tau'} \psi_{\alpha,\tau}^* dy$$

therefore

$$|\lambda(\tau) - \lambda(\tau')| \leq C|\tau - \tau'|$$

and  $\tau \rightarrow \lambda(\tau)$  is a Lipschitz continuous function. To conclude, we shall use the fact that  $\lambda(\tau)$  is bounded below by the first eigenvalue of the symmetrical part of the system (13). Remark that

$$\lambda(\tau) = \frac{\sum_{\alpha=1}^K \int_Y A_\alpha(y) \nabla \psi_{\alpha,\tau} \cdot \nabla \psi_{\alpha,\tau} + \sum_{\alpha,\beta=1}^K \int_Y \Sigma_{\alpha,\beta} \psi_{\beta,\tau} \psi_{\alpha,\tau} + \sum_{\alpha=1}^K \int_Y \frac{\tau}{v_\alpha} \psi_{\alpha,\tau} \cdot \psi_{\alpha,\tau}}{\sum_{\alpha,\beta=1}^K \int_Y \sigma_{\alpha,\beta}(y) \psi_{\beta,\tau} \psi_{\alpha,\tau} dy}$$

If we define  $\lambda_s$  the first eigenvalue of the symmetrical part of system (13), it is characterized by

$$\lambda_s = \min_{\varphi \in H_{\#}^1(Y)} \frac{\sum_{\alpha=1}^K \int_Y A_\alpha(y) \nabla \varphi_\alpha \cdot \nabla \varphi_\alpha + \sum_{\alpha,\beta=1}^K \int_Y \Sigma_{\alpha,\beta} \varphi_\beta \varphi_\alpha}{\sum_{\alpha,\beta=1}^K \int_Y \sigma_{\alpha,\beta}(y) \varphi_\beta \varphi_\alpha dy}$$

Therefore  $\lambda(\tau)$  satisfies

$$\lambda(\tau) \geq \lambda_s + \tau \frac{\sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\alpha,\tau}(y) \psi_{\alpha,\tau}(y) dy}{\sum_{\alpha,\beta=1}^K \int_Y \sigma_{\alpha,\beta}(y) \psi_{\beta,\tau}(y) \psi_{\alpha,\tau}(y) dy}. \quad (18)$$

if we note that

$$\sum_{\alpha,\beta=1}^K \int_Y \sigma_{\alpha,\beta}(y) \psi_{\beta,\tau} \psi_{\alpha,\tau} dy \leq C \int_Y \left( \sum_{\alpha=1}^K \psi_{\alpha,\tau} \right)^2 dy \leq 2C \sum_{\alpha=1}^K \int_Y \psi_{\alpha,\tau}(y) \psi_{\alpha,\tau}(y) dy$$

we obtain

$$\lambda(\tau) \geq \lambda_s + \tau C$$

with  $C > 0$ . Thus  $\tau \rightarrow \lambda(\tau)$  is a continuous monotone function mapping  $[-\Lambda(0), +\infty)$  into  $[0, +\infty)$ , and  $\lambda(\tau) = 1$  for a unique  $\tau = \tau_\infty \in W$ .  $\square$

**Remark 2.4.** In what follows, we shall denote  $\psi_\infty$  and  $\psi_\infty^*$  the first eigenvectors of the cell problems (13) and (14) for  $\tau = \tau_\infty$ , positive and normalized in the following way

$$\sum_{\alpha=1}^K \int_Y \psi_{\infty,\alpha}^2(y) dy = 1 \text{ and } \sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\infty,\alpha}(y) \psi_{\infty,\alpha}^*(y) dy = 1. \quad (19)$$

### 3 The drift free case

Our first result concerns the homogenization of the evolution problem (4) when it is well behaved, i.e. when an asymptotic homogenized evolution problem can be found. In this section we shall assume that a symmetry condition (20) stands. For the neutron transport model, and with  $\tau_\infty = 0$ , (in the notations that we use in Theorem 3.1) this result is similar to that of Larsen [17]. It is an adaptation of Theorem 3.2 in [3] to the evolution problem (4), with an appropriate time scale, i.e.  $n = 2$ .

**Theorem 3.1.** Let  $\tau = \tau_\infty$  be defined as in Theorem 2.3. Suppose that the corresponding eigenvectors  $\psi_\infty$  of the cell problem (13) and  $\psi_\infty^*$  of the adjoint cell problem (14) satisfy the following symmetry condition

$$\sum_{\alpha=1}^K \int_Y A_\alpha(y) (\psi_{\infty,\alpha} \nabla \psi_{\infty,\alpha}^* - \psi_{\infty,\alpha}^* \nabla \psi_{\infty,\alpha}) dy = 0. \quad (20)$$

Then, each component of the solution of problem (4) with  $n = 2$  can be factorized in the following way

$$\phi_\alpha^\epsilon(t, x) = u_\alpha^\epsilon(t, x) \psi_{\infty,\alpha} \left( \frac{x}{\epsilon} \right) \exp \left( \tau_\infty \frac{t}{\epsilon^2} \right) \quad \forall \alpha \in \{1, \dots, K\} \quad \forall x \in \Omega \quad \text{et } \forall t \in [0, T]$$

where each  $u_\alpha^\epsilon$  converges weakly in  $L^2(0, T; H_0^1(\Omega))$  towards the same limit  $u$  which is the unique solution of the following scalar evolution equation

$$\left\{ \begin{array}{l} \frac{1}{\bar{v}} \frac{\partial u}{\partial t} - \text{div}(\bar{D} \nabla u(t, x)) = \bar{\sigma}^l(x) u(t, x) \text{ in } \Omega, \text{ a.e. } t \in [0, T] \\ u = 0 \text{ on } \partial\Omega, \text{ a.e. } t \in [0, T] \\ u(t = 0, x) = \sum_{\alpha=1}^K \bar{v} \left( \int_Y \psi_{\infty,\alpha}^*(y) \frac{1}{v_\alpha} dy \right) \phi_{0,\alpha}(x) \text{ in } \Omega \end{array} \right. \quad (21)$$

The homogenized coefficients are

$$\bar{v} = \left( \sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\infty,\alpha}(y) \psi_{\infty,\alpha}^*(y) dy \right)^{-1} \quad (22)$$

$$\bar{\sigma}'(x) = \sum_{\alpha,\beta=1}^K \int_Y \sigma'_{\alpha,\beta}(x,y) \psi_{\infty,\beta}(y) \psi_{\infty,\alpha}^*(y) dy, \quad (23)$$

And  $\bar{D}$  is a  $N \times N$  positive definite matrix defined by its entries

$$\begin{aligned} \bar{D}_{i,j} = & \sum_{\alpha=1}^K \int_Y A_\alpha \psi_{\infty,\alpha} \psi_{\infty,\alpha}^* \nabla(y_i + \xi_{i,\alpha}) \nabla(y_j + \xi_{j,\alpha}) dy \\ & + \sum_{\alpha,\beta=1}^K \int_Y \frac{1}{2} \psi_{\infty,\alpha}^* \psi_{\infty,\beta} (\sigma_{\alpha,\beta} - \Sigma_{\alpha,\beta}) (\xi_{i,\alpha} - \xi_{i,\beta}) (\xi_{j,\alpha} - \xi_{j,\beta}) dy \end{aligned} \quad (24)$$

where, for each  $1 \leq i \leq N$ , the components  $(\theta_{i,\alpha})_{1 \leq \alpha \leq K}$  are defined by

$$\xi_{i,\alpha} = \frac{\zeta_{i,\alpha}}{\psi_\alpha},$$

and  $\zeta_i = (\zeta_{i,\alpha})_{1 \leq \alpha \leq K}$  is the solution of

$$\begin{cases} -\operatorname{div}(A \nabla(y) \nabla \zeta_i) + \frac{\tau_\infty}{v} \zeta_i + \Sigma(y) \zeta_i = \sigma(y) \zeta_i + Z_i & 1 \leq \alpha \leq K \text{ in } Y, \\ y \rightarrow \zeta_i(y) \text{ } Y\text{-periodic.} \end{cases} \quad (25)$$

where the right hand side  $Z_i$  has components  $Z_{i,\alpha} = \frac{1}{\psi_{\infty,\alpha}(y)} \operatorname{div}(A_\alpha(y) \psi_{\infty,\alpha}^2(y) \nabla y_i)$  for  $1 \leq \alpha \leq K$ .

**Remark 3.2.** In the case when the symmetry condition (20) is not satisfied, we obtain that at such a time scale, the factorized limit  $u^\epsilon$  converges strongly to zero. This gives only little information of the behavior of neutron flux  $\phi_\epsilon$ , because of the exponential term  $\exp(\tau_\infty \frac{t}{\epsilon^2})$ . This issue will be partially addressed in Sections 4 and 5.

The symmetry hypothesis (20) is the existence condition for the correctors  $\zeta_i$  defined by system (25), by a Fredholm alternative.

The proof of Theorem 3.1 heavily relies on results exposed in [3]. Let us describe here its main steps. At first, we shall prove that without loss of generality, it suffices to focus on the fundamental cell problem (10), i.e.  $\tau_\infty = 0$ . This will result of a factorization by a space-independent function  $t \rightarrow \exp(\tau_\infty \frac{t}{\epsilon^2})$ . Then in Proposition 3.3 we shall briefly recall why the factorization by the direct and adjoint eigenvectors of the cell problems (10)  $\psi$  and (11)  $\psi^*$  is licit. Because of this factorization, we will turn to a new evolution equation (26), that we shall write has a source problem for a compact operator. Then, Proposition 3.4 and a priori estimates proved in Lemma 3.5 will allow us to conclude.

Let us introduce  $\tilde{\phi}_\alpha^\epsilon = \phi_\alpha^\epsilon \exp(-\tau_\infty \frac{t}{\epsilon^2})$ . This new function then satisfies

$$\epsilon^2 \frac{1}{v} \frac{\partial \tilde{\phi}^\epsilon}{\partial t} - \epsilon^2 \operatorname{div} \left( A \left( \frac{x}{\epsilon} \right) \nabla \tilde{\phi}^\epsilon \right) + \tilde{\Sigma} \left( \frac{x}{\epsilon} \right) \tilde{\phi}^\epsilon = \left( \sigma \left( \frac{x}{\epsilon} \right) + \epsilon^2 \sigma' \left( x, \frac{x}{\epsilon} \right) \right) \tilde{\phi}^\epsilon$$

where  $\tilde{\Sigma}_{\alpha,\beta} = \Sigma_{\alpha,\beta} + \frac{\tau_\infty}{v_\alpha} \delta_{\alpha,\beta}$ . Velocity  $\tau_\infty$  has been chosen such that the cell problem (13) is critical, i.e. of first eigenvalue equal to 1. Thus, problem (4) stated in terms of this new function  $\tilde{\phi}^\epsilon$  is such that the corresponding eigenvalue cell problem is critical. Therefore, we shall suppose without loss of generality that equation (10) satisfies  $\lambda_0 = 1$ .

**Proposition 3.3.** *For  $1 \leq \alpha \leq K$ , let  $T_\alpha$  et  $T_\alpha^*$  be the following operators*

$$\begin{aligned} T_\alpha : H_0^1(\Omega) &\rightarrow H_0^1(\Omega) \quad \text{et} \quad T_\alpha^* : H_0^1(\Omega) &\rightarrow H_0^1(\Omega) \\ \phi(x) &\rightarrow \frac{\phi(x)}{\psi_\alpha \left( \frac{x}{\epsilon} \right)} &\phi(x) &\rightarrow \frac{\phi(x)}{\psi_\alpha^* \left( \frac{x}{\epsilon} \right)} \end{aligned}$$

Then  $T_\alpha^*$  and  $T_\alpha$  are bicontinuous linear operators.

This proposition is proved in [3]. We can therefore introduce the following factorization

$$\phi^\epsilon(x) = \psi \left( \frac{x}{\epsilon} \right) u^\epsilon(x) \text{ a.e. in } \Omega$$

Expressed in terms of  $u^\epsilon$ , The evolution problem (4) rewrites as follows

$$\left\{ \begin{array}{l} c \left( \frac{x}{\epsilon} \right) \frac{\partial u^\epsilon}{\partial t} - \operatorname{div} \left( D \left( \frac{x}{\epsilon} \right) \nabla u^\epsilon \right) + \frac{1}{\epsilon^2} Q^\epsilon(u^\epsilon) = B' \left( x, \frac{x}{\epsilon} \right) u^\epsilon \text{ in } \Omega, \text{ a.e. } t \in [0, T], \\ u^\epsilon = 0 \text{ on } \partial\Omega, \text{ a.e. } t \in [0, T], \\ u_\alpha^\epsilon(t=0, x) = \frac{\phi_{0,\alpha}(x)}{\psi_\alpha \left( \frac{x}{\epsilon} \right)} \in H_0^1(\Omega), \forall \alpha \in \{1, \dots, K\} \end{array} \right. \quad (26)$$

here, each of the  $K$  component of  $c$  represents a  $Y$ -periodic group velocity. It is defined, for all  $1 \leq \alpha \leq K$  by

$$c_\alpha(y) = \psi_\alpha(y) \psi_\alpha^*(y) \frac{1}{v_\alpha}. \quad (27)$$

The diffusion operator is block-diagonal, and each diffusion matrix  $D_\alpha$  is  $Y$ -periodic, given by

$$D_\alpha(y) = A_\alpha(y) \psi_\alpha^*(y) \psi_\alpha(y) \quad (28)$$

and  $Q^\epsilon$  is the following positive bounded bilinear form, with  $Y$  periodic coefficients

$$Q^\epsilon(u) = \sum_{\alpha=1}^K \epsilon J_\alpha \left( \frac{x}{\epsilon} \right) \nabla u_\alpha + \tilde{Q} \left( \frac{x}{\epsilon} \right) u \quad (29)$$

with

$$J_\alpha(y) = A_\alpha(y) (\psi_\alpha(y) \nabla_y \psi_\alpha^*(y) - \psi_\alpha^*(y) \nabla_y \psi_\alpha(y)), \quad (30)$$

and where  $\tilde{Q}$  is a  $K \times K$  matrix, with  $Y$  periodic entries given by

$$\begin{aligned}\tilde{Q}_{\alpha,\beta}(y) &= (\Sigma_{\alpha,\beta}(y) - \sigma_{\alpha,\beta}(y)) \psi_\beta(y) \psi_\alpha^*(y) \leq 0 \text{ if } \alpha \neq \beta, \\ \tilde{Q}_{\alpha,\alpha}(y) &= - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^K \tilde{Q}_{\alpha,\beta}(y) \geq 0.\end{aligned}\tag{31}$$

Finally, the  $K \times K$  matrix  $B'$  is the factorized form of the perturbation  $\sigma'$

$$B'_{\alpha,\beta}(x, y) = \psi_\alpha^*(y) \psi_\beta(y) \sigma'_{\alpha,\beta}(x, y).\tag{32}$$

Let us now introduce the linear operator  $S_\epsilon$  defined by

$$\begin{aligned}S_\epsilon : L^2(\Omega)^K &\rightarrow L^2(\Omega)^K \\ f = (f_\alpha)_{1 \leq \alpha \leq K} &\rightarrow u = (u_\alpha)_{1 \leq \alpha \leq K} \text{ unique solution in } H_1^0(\Omega) \text{ of}\end{aligned}\tag{33}$$

$$\begin{cases} -\operatorname{div} \left( D \left( \frac{x}{\epsilon} \right) \nabla u \right) + \frac{1}{\epsilon^2} Q^\epsilon(u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Remark that problem (26) can be written in terms of  $S_\epsilon$  in the following way

$$\begin{cases} (S_\epsilon)^{-1} u^\epsilon = B' \left( x, \frac{x}{\epsilon} \right) u^\epsilon - c \left( \frac{x}{\epsilon} \right) \frac{\partial u^\epsilon}{\partial t}, \\ u_\alpha^\epsilon(t=0) = \frac{\phi_{0,\alpha}(x)}{\psi_\alpha \left( \frac{x}{\epsilon} \right)} \quad 1 \leq \alpha \leq K \end{cases}$$

We chose to write it under this particular form because the homogenized limit operator for  $S_\epsilon$  has been already obtained in a previous work [3]. The following proposition describes this limit operator  $S$ .

**Proposition 3.4.** *Let  $f^\epsilon$  be a sequence which converges weakly in  $L^2(\Omega)^K$  to  $f = (f_\alpha)_{1 \leq \alpha \leq K}$ . Then, the sequence  $u^\epsilon = S_\epsilon(f^\epsilon)$  converges weakly in  $H_0^1(\Omega)^K$  to  $(u^0, \dots, u^0)$  which is defined by  $u^0 = S \left( \sum_{\alpha=1}^K f_\alpha \right)$ .*

*If the symmetry condition (20) is not satisfied, then  $S = 0$ . If the symmetry condition (20) is satisfied,  $S$  is the following compact operator*

$$\begin{aligned}S : L^2(\Omega) &\rightarrow L^2(\Omega) \\ f &\rightarrow u \text{ unique solution of} \\ &\begin{cases} -\operatorname{div}(\bar{D} \nabla u(x)) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}\end{aligned}$$

where  $\bar{D}$  is the constant positive definite matrix defined by (24).

The last necessary ingredients for the proof of Theorem 3.1 are a priori estimates, in order to assert the existence of a converging subsequence, at least weakly in  $L^2(\Omega)^K$ . The uniqueness of the solution to the limit problem will allow us to deduce the convergence of all the sequence. These estimates are given by the Lemma below.

**Lemma 3.5.** *The solution of (26) in  $L^2(0, T; (H_0^1(\Omega))^K)$  satisfies to the following estimate*

$$\sum_{\alpha=1}^K \|u_\alpha^\epsilon\|_{L^2(0, T; H_0^1(\Omega))} + \frac{1}{\epsilon} \sum_{\alpha, \beta=1}^K \|u_\alpha^\epsilon - u_\beta^\epsilon\|_{L^2(0, T; L^2(\Omega))} \leq C \sum_{\alpha=1}^K \|\phi_{0, \alpha}\|_{L^2(\Omega)} \quad (34)$$

where  $C > 0$  is a constant independent of  $\epsilon$ .

*Proof.* Testing the variational formulation of equation (26) against  $u^\epsilon$ , we obtain after an integration by parts

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \left( \sum_{\alpha=1}^K \int_{\Omega} c_\alpha \left( \frac{x}{\epsilon} \right) u_\alpha^\epsilon \cdot u_\alpha^\epsilon \right) + \sum_{\alpha=1}^K \int_{\Omega} D_\alpha \left( \frac{x}{\epsilon} \right) \nabla u_\alpha^\epsilon \cdot \nabla u_\alpha^\epsilon \\ + \frac{1}{\epsilon^2} \int_{\Omega} Q^\epsilon(u^\epsilon) \cdot u^\epsilon = \int_{\Omega} B' \left( x, \frac{x}{\epsilon} \right) u^\epsilon \cdot u^\epsilon \end{aligned}$$

The estimates satisfied by  $Q^\epsilon$  show that there exist a constant  $c > 0$  such that

$$c \sum_{\alpha, \beta=1}^K \|u_\alpha - u_\beta\|_{L^2(\Omega)}^2 \leq \int_{\Omega} Q^\epsilon(u) \cdot u dx \quad (35)$$

The coercivity of the diffusion matrices  $(D_\alpha)_{1 \leq \alpha \leq K}$  allows us to deduce from (35) the following inequality

$$\sum_{\alpha=1}^K \frac{\partial}{\partial t} \int_{\Omega} (u_\alpha^\epsilon)^2 + \sum_{\alpha=1}^K \int_{\Omega} \nabla u_\alpha^\epsilon \cdot \nabla u_\alpha^\epsilon + \frac{1}{\epsilon^2} \sum_{\alpha, \beta=1}^K \int_{\Omega} (u_\alpha^\epsilon - u_\beta^\epsilon)^2 \leq C \sum_{\alpha, \beta=1}^K \int_{\Omega} |u_\alpha^\epsilon u_\beta^\epsilon| \quad (36)$$

In particular this imply that

$$\sum_{\alpha=1}^K \frac{\partial}{\partial t} \int_{\Omega} (u_\alpha^\epsilon)^2 \leq C \sum_{\alpha=1}^K \int_{\Omega} (u_\alpha^\epsilon)^2$$

The other right-hand-side terms being positives. We deduce that for all  $t \in [0, T]$

$$\sum_{\alpha=1}^K \int_{\Omega} (u_\alpha^\epsilon)^2 \leq \sum_{\alpha=1}^K e^{C \cdot T} \|\phi_{0, \alpha}\|_{L^2(\Omega)}^2.$$

Then, inserting this last inequality in (36), and integrating the time variable on  $[0, T]$  we obtain the announced estimates.  $\square$

**Remark 3.6.** *Both factorizations, by a time exponential and by the eigensolutions of the cell problem, are needed to obtain these a priori estimates. If we had tried to directly obtain estimates on the solution of the starting equation (4), we would have had non vanishing terms of order  $\epsilon^{-2}$  of various signs, and thus no information on the boundedness of the sequence  $\phi^\epsilon$ . If we had omitted the factorization by a time exponential, we could only have derived a qualitative result on the time behavior of the solution.*

*Proof of Theorem 3.1.* From inequality (34) we deduce that there exists a subsequence, still indexed by  $\epsilon$ , such that  $u^\epsilon(t, x)$  weakly converges in  $L^2((0, T); H_0^1(\Omega)^K)$  towards a limit which has  $K$  identical components,  $(u_0(t, x), \dots, u_0(t, x))$ .

Let  $\varphi(t)$  be a smooth function of  $C^\infty([0, T])$ , with  $\varphi(T) = 0$ . Let us now introduce the new sequence  $f^\epsilon$ , with  $K$  components, defined by

$$f_\alpha^\epsilon = \int_0^T \varphi(t) \left( \sum_{\beta=1}^K B'_{\alpha,\beta} \left( x, \frac{x}{\epsilon} \right) u_\beta^\epsilon - \frac{\partial u_\alpha^\epsilon}{\partial t} c_\alpha \left( \frac{x}{\epsilon} \right) \right) dt \quad \forall \alpha \in \{1, \dots, K\}$$

Recall that  $y \rightarrow B'(x, y)$  is a periodical matrix with bounded coefficients in  $L^\infty(Y)$ , thus for all  $\alpha, \beta \in \{1, \dots, K\}$  we have

$$\int_0^T \varphi(t) B'_{\alpha,\beta} \left( x, \frac{x}{\epsilon} \right) u_\beta^\epsilon(t, x) dt \rightharpoonup \int_0^T \varphi(t) \int_Y B'_{\alpha,\beta}(x, y) u_0(t, x) dy \quad (37)$$

in weakly in  $L^2(\Omega)$ , since  $\int_0^T \varphi(t) u_\epsilon(x, t) dt \rightarrow \int_0^T \varphi(t) u_0(x, t) dt$  in  $H^1(\Omega)$ -weak, thus  $L^2(\Omega)$ -strong. On the other hand, after an integration by parts, we obtain, for all  $\alpha$ ,  $1 \leq \alpha \leq K$ ,

$$\int_0^T \varphi(t) \frac{\partial u_\alpha^\epsilon}{\partial t} c_\alpha \left( \frac{x}{\epsilon} \right) dt \rightharpoonup \int_0^T \frac{\partial \varphi(t)}{\partial t} \int_Y c_\alpha(y) u_0(t, x) dy - \phi_{0,\alpha}(x) \int_Y \psi_\alpha^*(y) \frac{1}{v_\alpha(y)} \varphi(0) dy$$

in the sense of the weak convergence in  $L^2(\Omega)$ . Therefore, with the help of Proposition 3.4 we deduce that  $\int_0^T \varphi(t) u^\epsilon(t, x) dt$  converges weakly in  $H_0^1(\Omega)^K$  towards  $(f_0(x), \dots, f_0(x))$  with  $f_0(x) = \int_0^T \varphi(t) u^0(t, x) dt$ , and that this limit is the unique solution of

$$\begin{cases} -\operatorname{div}(\overline{D} \nabla f_0(x)) = \overline{\sigma'} f_0(x) - \frac{1}{\overline{v}} \int_0^T \frac{\partial \varphi(t)}{\partial t} u_0(t, x) + \sum_{\alpha=1}^K \int_Y \frac{\psi_\alpha^*(y)}{v_\alpha(y)} dy \phi_{0,\alpha}(x) \varphi(0) & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega, \end{cases} \quad (38)$$

with  $v$  defined in (22),  $\overline{\sigma'}$  defined in (23) and  $\overline{D}$  defined in (24).

Theorem 3.1 is then proved once remarked that system (38) is a weak formulation of (21).  $\square$

## 4 The drift case with well-prepared initial data

In a previous work [10] it was shown that for the eigenvalue problem, the case when the symmetry condition (20) was not satisfied could be solved by factorizing the neutron flux by the eigensolution of  $\theta$ -exponential cell problems. In this section, we will show that this technique can also be used to homogenize the evolution problem (4), provided that the initial data  $\phi_0$  is also  $\theta$ -exponential. For the rest of this section, we shall make an additional regularity assumption of the diffusion matrix, that is

$$\forall \epsilon \in \{1, \dots, K\} \forall i, j \in \{1, \dots, N\}^2 y \rightarrow A_{\alpha,i,j}(y) \in C^1(Y). \quad (39)$$

It is a technical assumption used in Theorem 4.2 for the existence of a positive eigenvector for the eigenvalue problems (40) and (41). It is probably not necessary. Consider the following  $\theta$ -exponential cell eigenvalue problem, shifted by the constant

$$\text{vector } \tau \quad \begin{cases} -\operatorname{div}(A(y)\nabla\psi_{\tau,\theta}) + \frac{\tau}{v}\psi_{\tau,\theta} + \Sigma(y)\psi_{\tau,\theta} = \mu(\tau,\theta)\sigma(y)\psi_{\tau,\theta} \\ y \rightarrow \psi_{\tau,\theta}(y)e^{-\theta\cdot y} \quad Y\text{-periodic} \end{cases} \quad (40)$$

and the corresponding adjoint problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla\psi_{\tau,\theta}) + \frac{\tau}{v}\psi_{\tau,\theta} + \Sigma^T(y)\psi_{\tau,\theta} = \mu(\tau,\theta)\sigma^T(y)\psi_{\tau,\theta} \\ y \rightarrow \psi_{\tau,\theta}(y)e^{\theta\cdot y} \quad Y\text{-periodic} \end{cases} \quad (41)$$

Where  $(\tau, \theta)$  are in  $U$  defined by

$$U = \{\tau \in \mathbb{R}, \theta \in \mathbb{R}^N \text{ s.t. } \mu(\tau, \theta) > 0\}. \quad (42)$$

Such a family of  $\theta$ -exponential cell problems, that is where the gradient operator has been shifted by a constant vector  $\theta$  with real components, have been introduced in [6, Ch. 4, p.690] for the heat equation.

The existence of such a set is given by the following Proposition.

**Proposition 4.1.** *The set  $U$  defined by (42) is an open subset of  $\mathbb{R} \times \mathbb{R}^N$  with non empty interior. Furthermore, if we define*

$$T = \{\tau \in \mathbb{R} \text{ s.t. } \exists \theta \in \mathbb{R}^N \text{ s.t. } (\tau, \theta) \in U\}$$

*then  $T$  is of non empty interior, and for each  $\tau \in T$ , the set  $U(\tau) = \{\theta \in \mathbb{R}^N \text{ s.t. } (\tau, \theta) \in U\}$  is a bounded, connected open subset of  $U$ .*

*Proof.* It is shown by Mitidieri and Sweers in [21] that the set  $U$  is the set where the first eigenvalue  $M(\tau, \theta)$  of the following problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla\phi_{\tau,\theta}) + \frac{\tau}{v}\phi_{\tau,\theta} + \Sigma(y)\phi_{\tau,\theta} = \frac{M(\tau,\theta)}{v}\phi_{\tau,\theta} & \text{in } Y \\ \phi_{\tau,\theta} e^{-\theta\cdot y} & Y\text{-periodic} \end{cases}$$

is positive. In [10, Section 4, Step 1], it is proved that  $\theta \rightarrow M(0, \theta)$  belongs to  $C^\infty(\mathbb{R}^N, \mathbb{R})$ , and is uniformly concave, i.e.

$$\forall \theta \in \mathbb{R}^N, \forall \zeta \in \mathbb{R}^N - \sum_{i,j=1}^N \frac{\partial M}{\partial \theta_i \partial \theta_j}(0, \theta) \zeta_i \cdot \zeta_j \geq C \sum_{i=1}^N \zeta_i \cdot \zeta_i$$

where  $C$  is a constant independent of  $\theta$ . Thus  $U(\tau) = M(0, \cdot)^{-1}(]-\tau, +\infty))$  is a bounded, open and connected subset of  $U$ . Since  $M(0, 0) = \Lambda(0) > 0$ , the set  $M(0, \cdot)^{-1}(]0, +\infty))$  is non empty, and included in  $U(0)$ , thus  $U$  is non empty. We have also

$$T = ]\tau_{inf}, +\infty)$$

with  $\tau_{inf} = -\max\{M(0, \theta), \theta \in U(0)\}$ . The set  $T$  is therefore non empty. We consequently have obtained that  $U = \bigcup_{\tau \in T} M(0, \cdot)^{-1}(]-\tau, +\infty))$  is a non empty open subset of  $\mathbb{R} \times \mathbb{R}^N$ .  $\square$



The following result is a consequence of Proposition 2.6, (due to Mitidieri and Sweers [21]) and of Theorem 2.7 in [10].

**Theorem 4.2.** *Under the hypothesis of Theorem 2.2 and assuming (39), for each  $\tau \in T$ ,*

1. *for each  $\theta \in U(\tau)$ , there is a unique normalized strictly positive eigenfunction  $\psi_{\tau,\theta} \in \left(W_{loc}^{2,N}(Y) \cap C(\bar{Y})\right)^K$  to (40) and  $\psi_{\tau,\theta}^* \in \left(W_{loc}^{2,N}(Y) \cap C(\bar{Y})\right)^K$  to (41). The first eigenvalue  $\mu(\tau, \theta)$  of (40) and (41) is the same. It is positive and of geometric and algebraic multiplicity equal to one.*
2. *The application  $\theta \rightarrow \mu(\tau, \theta)$  is in  $C^\infty(U(\tau), \mathbb{R}_+^*)$  and admits a maximum  $\mu_\infty(\tau)$  which is obtained for a unique  $\theta = \theta_\infty(\tau)$ . It is characterized by the following relation*

$$\sum_{\alpha=1}^K \int_Y a(y) (\psi_{\tau,\theta_\infty,\alpha}(y) \nabla \psi_{\tau,\theta_\infty,\alpha}^*(y) - \psi_{\tau,\theta_\infty,\alpha}^*(y) \nabla \psi_{\tau,\theta_\infty,\alpha}(y)) = 0 \quad (43)$$

In a similar manner to the drift free case, the introduction of the parameter  $\tau$  enables us to tune system (40) so that it is critical, as is shown by the following Lemma

**Lemma 4.3.** *There exists a unique couple  $(\tau_\infty, \theta_\infty) \in U$  such that the first eigencouples  $(\psi_{\tau_\infty, \theta_\infty}, \mu_\infty)$  of (40) and  $(\psi_{\tau_\infty, \theta_\infty}^*, \mu_\infty)$  of (41) satisfy*

$$\mu_\infty = 1 \text{ and } \sum_{\alpha=1}^K \int_Y a(y) (\psi_{\tau_\infty, \theta_\infty, \alpha}(y) \nabla \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) - \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) \nabla \psi_{\tau_\infty, \theta_\infty, \alpha}(y)) = 0.$$

*Proof.* Let  $(\tau, \theta)$  and  $(\tau', \theta')$  be two elements of  $U$ . Suppose  $\tau' < \tau$ : then  $(\tau, \theta')$  is also in  $U$ . Like in the proof of Theorem 2.3, we obtain the following identity similar to (17)

$$(\tau - \tau') \sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\alpha, \theta, \tau'} \psi_{\alpha, \theta, \tau}^* dy = (\mu(\tau, \theta') - \mu(\tau', \theta')) \sum_{\alpha, \beta=1}^K \int_Y \sigma_{\alpha, \beta}(y) \psi_{\beta, \theta, \tau'} \psi_{\alpha, \theta, \tau}^* dy \quad (44)$$

which yields, with the help of assumption (7)

$$|\mu(\tau, \theta') - \mu(\tau', \theta')| \leq C|\tau - \tau'|$$

where  $C$  is a constant independent of  $\tau$  and  $\theta$ . Therefore

$$|\mu(\tau, \theta) - \mu(\tau', \theta')| \leq C|\tau - \tau'| + |\mu(\tau, \theta') - \mu(\tau, \theta)|. \quad (45)$$

With the help of Theorem 4.2 and Proposition 4.1,  $\theta \rightarrow \mu(\tau, \theta)$  is uniformly continuous on  $U(\tau)$ , thus (45) yields that  $(\tau, \theta) \rightarrow \mu(\tau, \theta)$  is continuous on  $U$ .

As a consequence,  $\tau \rightarrow \mu_\infty(\tau) = \max\{\mu(\theta, \tau), \theta \in U(\tau)\}$  is continuous on  $T$ . From identity (44) we see that  $\tau \rightarrow \mu(\tau, \theta)$  is strictly increasing for each  $\theta \in \Theta$ , and so is  $\tau \rightarrow \mu_\infty(\tau)$ . Therefore, if we obtain that  $\mu_\infty(T) = ]0, +\infty)$ , Lemma 4.3 is proved.

We have the following generalization of inequality (18)

$$\mu(\tau, \theta) \geq \mu_s(\theta) + C\tau \quad (46)$$

where  $C$  is another constant depending only on the coefficients, therefore

$$\lim_{\tau \rightarrow +\infty} \mu_\infty(\tau) = +\infty$$

On the other hand, note that because of the positivity of  $\psi_{\tau,\theta}$  and of assumption (7), there exists a constant  $C$  independent of  $\tau$  and  $\theta$  such that

$$\mu(\tau, \theta) \sigma(y) \psi_{\tau,\theta} \geq C \frac{\mu(\tau, \theta)}{v} \psi_{\tau,\theta}$$

Therefore,  $\mu(\tau, \theta) \leq C(M(0, \theta) + \tau) = C(\tau - \tau_{inf})$  and

$$\lim_{\tau \rightarrow \tau_{inf}} \mu_\infty(\tau) = 0.$$

□

We are now able to state our result, concerning the homogenization of equation (4).

**Theorem 4.4.** *Let  $\tau = \tau_\infty$  and  $\theta = \theta_\infty$  be defined as in Lemma 4.3. Suppose that the initial condition  $\phi_0$  of problem (4) with  $n = 2$  is of the following form*

$$\phi_{0,\alpha}^\epsilon(x) = \exp\left(\theta_\infty \cdot \frac{x}{\epsilon}\right) \tilde{\phi}_{0,\alpha}(x), \quad 1 \leq \alpha \leq K \text{ a.e. } x \in \Omega \quad (47)$$

with  $\tilde{\phi}_0 \in L^2(\Omega)^K$ . Then, each component of the solution of problem (4) with  $n = 2$  can be factorized in the following way

$$\phi_\alpha^\epsilon(t, x) = u_\alpha^\epsilon(t, x) \psi_{\tau_\infty, \theta_\infty, \alpha}\left(\frac{x}{\epsilon}\right) \exp\left(\tau_\infty \frac{t}{\epsilon^2}\right) \quad \forall \alpha \in \{1, \dots, K\} \quad \forall x \in \Omega \text{ et } \forall t \in [0, T]$$

where each  $u_\alpha^\epsilon$  converges weakly in  $L^2(0, T; H_0^1(\Omega))$  towards the same limit  $u^0$  which is the unique solution of the following scalar evolution equation

$$\left\{ \begin{array}{l} \frac{1}{\bar{v}} \frac{\partial u}{\partial t} - \operatorname{div}(\bar{D} \nabla u(t, x)) = \bar{\sigma}'(x) u(t, x) \text{ in } \Omega, \text{ a.e. } t \in [0, T] \\ u = 0 \text{ on } \partial\Omega, \text{ a.e. } t \in [0, T] \\ u(t = 0, x) = \sum_{\alpha=1}^K \bar{v} \left( \int_Y \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) \exp(\theta_\infty \cdot y) \frac{1}{v_\alpha} dy \right) \tilde{\phi}_{0,\alpha}(x) \text{ in } \Omega \end{array} \right. \quad (48)$$

The homogenized coefficients are

$$\bar{v} = \left( \sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\tau_\infty, \theta_\infty, \alpha}(y) \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) dy \right)^{-1} \quad (49)$$

$$\bar{\sigma}'(x) = \sum_{\alpha, \beta=1}^K \int_Y \sigma'_{\alpha, \beta}(x, y) \psi_{\tau_\infty, \theta_\infty, \beta}(y) \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) dy, \quad (50)$$

And  $\bar{D}$  is a  $N \times N$  positive definite matrix defined by its entries

$$\begin{aligned} \bar{D}_{i,j} = & \sum_{\alpha=1}^K \int_Y A_\alpha \psi_{\tau_\infty, \theta_\infty, \alpha} \psi_{\tau_\infty, \theta_\infty, \alpha}^* \nabla(y_i + \xi_{i,\alpha}) \nabla(y_j + \xi_{j,\alpha}) dy \\ & + \sum_{\alpha, \beta=1}^K \int_Y \frac{1}{2} \psi_{\tau_\infty, \theta_\infty, \alpha}^* \psi_{\tau_\infty, \theta_\infty, \beta} (\sigma_{\alpha, \beta} - \Sigma_{\alpha, \beta}) (\xi_{i,\alpha} - \xi_{i,\beta}) (\xi_{j,\alpha} - \xi_{j,\beta}) dy \end{aligned} \quad (51)$$

where, for each  $1 \leq i \leq N$ , the components  $(\xi_{i,\alpha})_{1 \leq \alpha \leq K}$  are solutions of

$$\begin{cases} -\operatorname{div}(D_\infty(y) \nabla(\xi_i(y) + y_i \mathbb{I})) + Q_\infty(\xi_i(y) + y_i \mathbb{I}) = 0 \\ y \rightarrow \xi_i(y) \quad Y\text{-periodic.} \end{cases} \quad (52)$$

with  $D_\infty$  defined in (55) and  $Q_\infty$  defined in (56).

**Remark 4.5.** For  $\theta_\infty = 0$ , the drift free case, Theorem 4.4 is simply Theorem 3.1. Because of assumption (47), this new result only addresses the homogenization of problem (4) in the special case of well conditioned initial datas. In particular, it does not provides any information on the behavior of the solution of problem (4) as  $\epsilon$  goes to zero, for a fixed initial data  $\phi_0 \in L^2(\Omega)^K$  when a drift phenomenon appears, i.e. when  $\theta_\infty \neq 0$ . We shall return to that problem in Section 5.

*Proof.* We introduce the following factorization (which is valid, by a variant of Proposition 3.3)

$$\phi^\epsilon(x) = \exp\left(\tau_\infty \frac{t}{\epsilon^2}\right) \psi_{\tau_\infty, \theta_\infty}\left(\frac{x}{\epsilon}\right) u^\epsilon(x) \text{ a.e. in } \Omega.$$

Expressed in terms of  $u^\epsilon$  the evolution problem (4) with an initial condition  $\phi_0^\epsilon$  of the form (47) rewrites as follows

$$\begin{cases} c_\infty\left(\frac{x}{\epsilon}\right) \frac{\partial u^\epsilon}{\partial t} - \operatorname{div}\left(D_\infty\left(\frac{x}{\epsilon}\right) \nabla u^\epsilon\right) + \frac{1}{\epsilon^2} Q_\infty^\epsilon(u^\epsilon) = B'_\infty\left(x, \frac{x}{\epsilon}\right) u^\epsilon \text{ in } \Omega, \text{ a.e. } t \in [0, T], \\ u^\epsilon = 0 \text{ on } \partial\Omega, \text{ a.e. } t \in [0, T], \\ u_\alpha^\epsilon(t=0, x) = \frac{\exp\left(\theta_\infty \cdot \frac{x}{\epsilon}\right) \tilde{\phi}_{0,\alpha}(x)}{\psi_{\tau_\infty, \theta_\infty, \alpha}\left(\frac{x}{\epsilon}\right)} \in H_0^1(\Omega), \forall \alpha \in \{1, \dots, K\}. \end{cases} \quad (53)$$

Note that because of the particular form of  $\phi_0^\epsilon$ ,  $u^\epsilon(t=0, \cdot)$  is bounded in  $L^2(\Omega)^K$  independently of  $\epsilon$ . Here, each of the  $K$  component of  $c_\infty$  represents a  $Y$ -periodic group velocity. It is defined, for all  $1 \leq \alpha \leq K$  by

$$c_{\infty, \alpha}(y) = \psi_{\tau_\infty, \theta_\infty, \alpha}(y) \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) \frac{1}{v_\alpha}. \quad (54)$$

The diffusion operator is block-diagonal, and each diffusion matrix  $D_\alpha$  is  $Y$ -periodic, given by

$$D_{\infty, \alpha}(y) = A_\alpha(y) \psi_{\tau_\infty, \theta_\infty, \alpha}^*(y) \psi_{\tau_\infty, \theta_\infty, \alpha}(y) \quad (55)$$

and  $Q^\epsilon$  is the following positive bounded bilinear form, with  $Y$  periodic coefficients

$$Q_\infty^\epsilon(u) = \sum_{\alpha=1}^K \epsilon J_{\infty,\alpha} \left( \frac{x}{\epsilon} \right) \nabla u_\alpha + \tilde{Q}_\infty \left( \frac{x}{\epsilon} \right) u \quad (56)$$

with

$$J_{\infty,\alpha}(y) = A_\alpha(y) (\psi_{\tau_\infty,\theta_\infty,\alpha}(y) \nabla_y \psi_{\tau_\infty,\theta_\infty,\alpha}^*(y) - \psi_{\tau_\infty,\theta_\infty,\alpha}^*(y) \nabla_y \psi_{\tau_\infty,\theta_\infty,\alpha}(y)), \quad (57)$$

and where  $\tilde{Q}_\infty$  is a  $K \times K$  matrix, with  $Y$  periodic entries given by

$$\begin{aligned} \tilde{Q}_{\infty,\alpha,\beta}(y) &= (\Sigma_{\alpha,\beta}(y) - \mu^\infty \sigma_{\alpha,\beta}(y)) \psi_{\tau_\infty,\theta_\infty,\beta}(y) \psi_{\tau_\infty,\theta_\infty,\alpha}^*(y) \leq 0 \text{ if } \alpha \neq \beta, \\ \tilde{Q}_{\infty,\alpha,\alpha}(y) &= - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^K \tilde{Q}_{\infty,\alpha,\beta}(y) \geq 0. \end{aligned} \quad (58)$$

Finally, the  $K \times K$  matrix  $B'_\infty$  is the factorized form of the perturbation  $\sigma'$

$$B'_{\infty,\alpha,\beta}(x, y) = \psi_{\tau_\infty,\theta_\infty,\alpha}^*(y) \psi_{\tau_\infty,\theta_\infty,\beta}(y) \sigma'_{\alpha,\beta}(x, y). \quad (59)$$

From Lemma 4.3 we obtain  $\sum_{\alpha=1}^K \int_Y J_{\infty,\alpha}(y) dy = 0$ , and the rest of the proof is similar to that of Theorem 3.1.  $\square$

## 5 The drift case in a domain without boundary

This section is devoted to the study of (4) when the symmetry condition (20) is not satisfied, in the case when the domain  $\Omega$  does not have boundaries, i.e.

$$\Omega = \mathbb{R}^N \text{ or the unit torus } \mathbb{T}^N. \quad (60)$$

As we mentioned in Remark 3.2, in such a case Theorem 3.1 only shows that the neutron flux  $\phi^\epsilon$  is of the following form

$$\phi^\epsilon = \exp\left(\tau_\infty \frac{t}{\epsilon^2}\right) r_\epsilon, \text{ with } r_\epsilon \rightarrow 0 \text{ in } L^2(\Omega)^K$$

To obtain more information on the asymptotic behavior of the solution of (4) in such a case, without assuming that the initial data are well prepared, we shall choose a smaller time scale, i.e.  $n = 1$ . Namely, we will consider the following variant of problem (4)

$$\begin{cases} \epsilon \frac{1}{v} \frac{\partial \phi^\epsilon}{\partial t} - \epsilon^2 \operatorname{div} \left( A \left( \frac{x}{\epsilon} \right) \nabla \phi^\epsilon \right) + \Sigma \left( \frac{x}{\epsilon} \right) \phi^\epsilon = \left( \sigma \left( \frac{x}{\epsilon} \right) + \epsilon^2 \sigma' \left( x, \frac{x}{\epsilon} \right) \right) \phi^\epsilon \text{ on } \Omega \\ \phi^\epsilon(t = 0, x) = \phi_0(x) \in H^1(\Omega)^K \end{cases} \quad (61)$$

We then have the following result

**Theorem 5.1.** Let  $\tau = \tau_\infty$  be defined as in Theorem 2.3. Suppose that the corresponding eigenvectors  $\psi_\infty$  of the cell problem (13) and  $\psi_\infty^*$  of the adjoint cell problem (14) are such that

$$\sum_{\alpha=1}^K \int_Y A_\alpha(y) (\psi_{\infty,\alpha} \nabla \psi_{\infty,\alpha}^* - \psi_{\infty,\alpha}^* \nabla \psi_{\infty,\alpha}) dy \neq 0, \quad (62)$$

Then each component of the solution  $\phi^\epsilon$  of (61) with  $n = 1$  writes as the product of three terms

$$\phi_\alpha^\epsilon(t, x) = u_\alpha^\epsilon(t, x) \psi_{\infty,\alpha} \left( \frac{x}{\epsilon} \right) \exp \left( \tau_\infty \frac{t}{\epsilon} \right) \quad \forall \alpha \in \{1, \dots, K\} \quad \forall x \in \Omega \quad \text{et} \quad \forall t \in [0, T]$$

where each  $u_\alpha^\epsilon$  converges weakly in  $L^2(0, T; H^1(\Omega))$  towards the same limit  $u^0$  which is the unique solution of the following scalar transport equation

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t, x) - b \cdot \nabla u(t, x) = 0 \text{ in } \Omega, \text{ a.e. } t \in [0, T] \\ u(t = 0, x) = \frac{\sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\infty,\alpha}^*(y) dy}{\sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\infty,\alpha}(y) \psi_{\infty,\alpha}^*(y) dy} \phi_{0,\alpha}(x) \text{ in } \Omega \end{array} \right. \quad (63)$$

where the homogenized constant velocity is defined by

$$b = \frac{\sum_{\alpha=1}^K \int_Y A_\alpha(y) (\psi_{\infty,\alpha} \nabla \psi_{\infty,\alpha}^* - \psi_{\infty,\alpha}^* \nabla \psi_{\infty,\alpha}) dy}{\sum_{\alpha=1}^K \int_Y \frac{1}{v_\alpha} \psi_{\infty,\alpha}(y) \psi_{\infty,\alpha}^*(y) dy} \quad (64)$$

For the same reason that in Section 3, it is sufficient to prove this result in the case when the fundamental cell problem (10) is critical, i.e.  $\lambda_0 = 1$ . We shall perform the same factorization as before, component by component, by the first eigenvectors of the cell problems (10) and (11). This first step was detailed in Section 3. Expressed in terms of the factorized flux  $u^\epsilon$

$$\phi^\epsilon = \psi \left( \frac{\cdot}{\epsilon} \right) u^\epsilon$$

the evolution problem (4) then becomes

$$\left\{ \begin{array}{l} \frac{1}{\epsilon} c \left( \frac{x}{\epsilon} \right) \frac{\partial u^\epsilon}{\partial t} + \operatorname{div} \left( D \left( \frac{x}{\epsilon} \right) \nabla u^\epsilon \right) + \frac{1}{\epsilon^2} Q^\epsilon(u^\epsilon) = B' \left( x, \frac{x}{\epsilon} \right) u^\epsilon \text{ on } \Omega \text{ and a.e. } t \in [0, T], \\ u_\alpha^\epsilon(t = 0, x) = \frac{\phi_{0,\alpha}(x)}{\psi_\alpha \left( \frac{x}{\epsilon} \right)} \in H^1(\Omega), \forall \alpha \in \{1, \dots, K\} \end{array} \right. \quad (65)$$

the velocity  $c$  is given by (27), the block diagonal diffusion operator  $D$  is defined in (28), the collision kernel  $Q^\epsilon$  is defined in (29) and the matrix  $B'$  is defined by its entries in (32). The

improvement this new formulation brings is that the left-hand-side, apart from the first order terms in times corresponds to a bilinear positive operator, and that the right hand side is a bounded operator.

Starting from the variational formulation of problem (65), we shall first prove that it allows us to derive estimates on the sequence  $u^\epsilon$ , independently of  $\epsilon$ . Then we will be entitled to introduce the weak limits of  $u^\epsilon$  and  $\nabla u^\epsilon$  in the sense of the two-scale convergence, for a converging subsequence. The ultimate step corresponds to the derivation of the actual homogenized problem.

**Lemma 5.2.** *The solution of (65) in  $L^2((0, T); H_0^1(\Omega)^K)$  satisfies to the following estimates*

$$\|u^\epsilon\|_{L^2((0, T); L^2(\Omega)^K)} \leq C \|\phi_0\|_{L^2(\Omega)^K} \quad (66)$$

$$\sum_{\alpha=1}^K \|\nabla u_\alpha^\epsilon\|_{L^2((0, T); L^2(\Omega)^N)} + \frac{1}{\epsilon} \sum_{\alpha, \beta=1}^K \|u_\alpha^\epsilon - u_\beta^\epsilon\|_{L^2((0, T); L^2(\Omega))} \leq \frac{1}{\sqrt{\epsilon}} C \sum_{\alpha=1}^K \|\phi_{0, \alpha}\|_{L^2(\Omega)} \quad (67)$$

where  $C > 0$  is a constant independent of  $\epsilon$ .

*Proof.* Testing the variational formulation of equation (26) against  $u^\epsilon$ , and after an integration by parts we obtain

$$\begin{aligned} \frac{1}{2\epsilon} \frac{\partial}{\partial t} \left( \sum_{\alpha=1}^K \int_{\Omega} c_\alpha \left( \frac{x}{\epsilon} \right) u_\alpha^\epsilon \cdot u_\alpha^\epsilon \right) + \sum_{\alpha=1}^K \int_{\Omega} D_\alpha \left( \frac{x}{\epsilon} \right) \nabla u_\alpha^\epsilon \cdot \nabla u_\alpha^\epsilon \\ + \frac{1}{\epsilon^2} \int_{\Omega} Q^\epsilon(u^\epsilon) \cdot u^\epsilon = \int_{\Omega} B' \left( x, \frac{x}{\epsilon} \right) u^\epsilon \cdot u^\epsilon \end{aligned}$$

Estimate (35) on the collision kernel  $Q^\epsilon$  and the coercivity of the matrices  $(D_\alpha)_{1 \leq \alpha \leq K}$  allows us to deduce from the above expression the following inequality

$$\sum_{\alpha=1}^K \frac{1}{\epsilon} \frac{\partial}{\partial t} \int_{\Omega} (u_\alpha^\epsilon)^2 + \sum_{\alpha=1}^K \int_{\Omega} \nabla u_\alpha^\epsilon \cdot \nabla u_\alpha^\epsilon + \frac{1}{\epsilon^2} \sum_{\alpha, \beta=1}^K \int_{\Omega} (u_\alpha^\epsilon - u_\beta^\epsilon)^2 \leq C \sum_{\alpha, \beta=1}^K \int_{\Omega} |u_\alpha^\epsilon u_\beta^\epsilon| \quad (68)$$

In particular, we have

$$\sum_{\alpha=1}^K \frac{\partial}{\partial t} \int_{\Omega} (u_\alpha^\epsilon)^2 \leq C \epsilon \sum_{\alpha=1}^K \int_{\Omega} (u_\alpha^\epsilon)^2$$

The other right hand side terms being positives. Therefore we have for all  $t \in [0, T]$

$$\sum_{\alpha=1}^K \int_{\Omega} (u_\alpha^\epsilon)^2 dx \leq \sum_{\alpha=1}^K e^{C\epsilon T} \|\phi_{0, \alpha}\|_{L^2(\Omega)}^2$$

And we obtained the first estimate (66) of the Lemma. inserting this last inequality in (68), and integrating in the time variable over  $[0, T]$ , we obtain

$$\sum_{\alpha=1}^K \int_{\Omega} \nabla u_\alpha^\epsilon \cdot \nabla u_\alpha^\epsilon + \frac{1}{\epsilon^2} \sum_{\alpha, \beta=1}^K \int_{\Omega} (u_\alpha^\epsilon - u_\beta^\epsilon)^2 \leq (1 + \frac{1}{\epsilon}) C \sum_{\alpha=1}^K \|\phi_{0, \alpha}\|_{L^2(\Omega)}^2$$

and this imply the second estimate (67) of the Lemma.  $\square$

We shall now recall classical results from the theory of two scale convergence (see [1], [22]), which we shall use to prove the existence of weak limits for  $u^\epsilon$  and its gradient.

**Proposition 5.3.** *1. Let  $u_\epsilon$  be bounded sequence in  $L^2(\Omega)$ . There exists a subsequence, still indexed by  $\epsilon$ , and a limit  $u_0(x, y) \in L^2(\Omega; L^2_\#(Y))$  such that  $u_\epsilon$  two scale converges towards  $u_0$  in the sense that*

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} u_\epsilon(x) \phi\left(x, \frac{x}{\epsilon}\right) dx = \int_{\Omega} \int_Y u_0(x, y) \phi(x, y) dx dy$$

*For every function  $\phi(x, y) \in L^2(\Omega; C_\#(Y))$ .*

*2. Let  $u_\epsilon$  and  $\epsilon \nabla u_\epsilon$  be two bounded sequences in  $L^2(\Omega)$ . Then, there exists a function  $u_0(x, y)$  in  $L^2(\Omega, H^1_\#(Y))$  such that  $u_\epsilon$  and  $\epsilon \nabla u_\epsilon$  two scale converge to  $u_0(x, y)$  and  $\nabla_y u_0(x, y)$  respectively.*

*Proof of Theorem 5.1.* From estimate (66) we deduce that there exist a subsequence, still indexed by  $\epsilon$ , such  $u^\epsilon(t, x)$  weakly converges in  $L^2(O, T; (L^2(\Omega))^K)$  to a limit  $(u_{0,1}(t, x), \dots, u_{0,K}(t, x))$ . From the second estimate (67) we obtain

$$\begin{aligned} u_{0,\alpha} &= u_0 \quad \forall \alpha, \quad 1 \leq \alpha \leq K \\ \epsilon \nabla u_\alpha^\epsilon &\rightarrow 0 \text{ in } L^2(0, T; (L^2(\Omega))^N) \quad \forall \alpha, \quad 1 \leq \alpha \leq K \end{aligned}$$

and this imply, taking into Proposition 5.3,§2, that the two scale limit of  $u^\epsilon$  is its weak limit in  $L^2(O, T; (L^2(\Omega))^K)$ .

Let  $\varphi(t, x)$  be a smooth function in  $C^\infty([O, T] \times \Omega)$  with compact support in  $\Omega$  and such that  $\varphi(T, \cdot) = 0$ . The variational formulation of (65) against the  $K$ -component function  $(\epsilon\varphi, \dots, \epsilon\varphi)$  writes

$$\begin{aligned} & \sum_{\alpha=1}^K \int_{\Omega} \int_0^T c_\alpha\left(\frac{x}{\epsilon}\right) \frac{\partial u_\alpha^\epsilon}{\partial t}(t, x) \cdot \varphi(t, x) dt dx \\ & + \sum_{\alpha=1}^K \epsilon \int_{\Omega} \int_0^T D_\alpha\left(\frac{x}{\epsilon}\right) \nabla u_\alpha^\epsilon(t, x) \cdot \nabla \varphi(t, x) dt dx \\ & + \frac{1}{\epsilon} \int_{\Omega} \int_0^T u^\epsilon(t, x) \cdot (Q^\epsilon)^*(\varphi(t, x)) dt dx = \epsilon \int_{\Omega} \int_0^T B'\left(\frac{x}{\epsilon}\right) u^\epsilon(t, x) \cdot \varphi(t, x) dt dx \end{aligned} \tag{69}$$

where  $(Q^\epsilon)^*$  is the adjoint operator of  $Q^\epsilon$ , defined by

$$Q^*(u) = -\epsilon J\left(\frac{x}{\epsilon}\right) \nabla u + \tilde{Q}^a\left(\frac{x}{\epsilon}\right) u.$$

The  $K \times N$  tensor noted  $J$  is defined by (30), and  $\tilde{Q}^a$  is such that its off-diagonal entries are that of the transpose of  $\tilde{Q}$  (31) and its diagonal is such that, for all  $1 \leq \alpha \leq K$ ,

$$\sum_{\beta=1}^K \tilde{Q}_{\alpha,\beta}^a = 0.$$

The kernel of  $\tilde{Q}^a$  is spanned by the  $K$ -component vector  $(1, \dots, 1)$ . Taking into account estimates (66) and (67) we have that  $u^\epsilon$  and  $\sqrt{\epsilon}\nabla u^\epsilon$  are bounded,. Thus we deduce that system (69) simplifies in

$$\sum_{\alpha=1}^K \int_{\Omega} \int_0^T c_{\alpha} \left( \frac{x}{\epsilon} \right) \frac{\partial u_{\alpha}^{\epsilon}}{\partial t}(t, x) \cdot \varphi(t, x) - \sum_{\alpha=1}^K \int_{\Omega} \int_0^T u_{\alpha}^{\epsilon} \cdot J_{\alpha} \left( \frac{x}{\epsilon} \right) \nabla \varphi = r(\epsilon) \quad (70)$$

where  $r(\epsilon)$  is a bounded term going to zero with  $\epsilon$ . Tensor  $J(y)$  is  $Y$ -periodic by its definition (30) thus we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \sum_{\alpha=1}^K \int_0^T \int_{\Omega} u_{\alpha}^{\epsilon}(t, x) \cdot J_{\alpha} \left( \frac{x}{\epsilon} \right) \nabla \varphi(t, x) dx dt \\ &= \int_0^T \int_{\Omega} u_0(t, x) \cdot \left( \sum_{\alpha=1}^K \int_Y J_{\alpha}(y) dy \right) \nabla \varphi(t, x) dx dt \end{aligned} \quad (71)$$

The first term writes, after an integration by parts on the time variable,

$$- \sum_{\alpha=1}^K \int_{\Omega} \int_0^T c_{\alpha} \left( \frac{x}{\epsilon} \right) u_{\alpha}^{\epsilon}(t, x) \cdot \frac{\partial \varphi}{\partial t} dt dx + \sum_{\alpha=1}^K \int_{\Omega} \psi_{\alpha}^* \left( \frac{x}{\epsilon} \right) \phi_{0,\alpha}(x) \varphi(0, x) dx$$

And under this form, we see that the limit as  $\epsilon$  goes to zero is

$$\begin{aligned} & - \left( \sum_{\alpha=1}^K \int_Y c_{\alpha}(y) dy \right) \int_0^T \int_{\Omega} u_0(t, x) \cdot \frac{\partial \varphi(t, x)}{\partial t} dx dt \\ & + \int_{\Omega} \sum_{\alpha=1}^K \left( \int_Y \psi_{\alpha}^*(y) \frac{1}{v_{\alpha}(y)} dy \phi_{0,\alpha}(x) \right) \cdot \varphi(0, x) dx \end{aligned} \quad (72)$$

We then obtain a system (63) under a weak form by adding the two limits expression (71) et (72).  $\square$

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