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COLEMAN MAP IN COLEMAN FAMILIES

by

Filippo A. E. Nuccio Mortarino Majno di Capriglio & Tadashi Ochiai

Abstract. — In this paper, we construct a two-variable Coleman map for a given p-adic family of eigen cuspforms with a fixed non-zero slope (Coleman family). A Coleman map is a machinery which transforms a hypothetical p-adic family of zeta elements to a p-adic L-function. The result obtained in this paper is a non-ordinary generalization of a two-variable Coleman map for a given Hida deformation obtained by the second-named author in the paper [Och03].

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1. Introduction

In the celebrated papers [Col96] and [Col97b], Coleman constructed p-adic families of cuspforms

$$\mathcal{F} = \sum_{n=1}^{\infty} a_n(\mathcal{F}) q^n \in \mathscr{A}_{\mathscr{X}}[\![q]\!]$$

on the ring of rigid functions $\mathscr{A}_{\mathscr{X}}$ over an affinoid $\mathscr{X}$, the so-called Coleman families. In fact, there is a Zariski-dense subset $\mathscr{Z}$ of $\mathscr{X}$ such that the specialization f_x of $\mathcal{F}$ at x is a classical cuspform for every point $x \in \mathscr{Z}$. It is known that the p-adic valuation of $a_p(f_x)$ is constant when $x \in \mathscr{Z}$ varies. We call this valuation the *slope* of the Coleman family $\mathscr{V}$.

Associated to $\mathcal{F}$, there is a family $\mathscr{V}$ of p-adic Galois representations of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. The family of Galois representations $\mathscr{V}$ is free of rank two over $\mathscr{A}_{\mathscr{X}}$. Moreover, at each $x \in \mathscr{Z}$ of the above-mentioned set $\mathscr{Z} \subset \mathscr{X}$, the specialization $x(\mathscr{V})$ is isomorphic to the Galois representation V_x for f_x.

Along the spirit of Iwasawa theory, we believe that it is interesting to study p-adic variations of some arithmetic invariants on such families of Galois representations. One of the most interesting invariants among such p-adic variations, is a p-adic L-function on $\mathscr{X}$. A p-adic L-function should interpolate the critical special values $L(f_x, j)$ divided by complex periods when $x \in \mathscr{Z}$ and integers j satisfying $1 \leq j \leq k_x - 1$ vary, where $k_x \in \mathscr{Z}$ is the weight of the cuspform f_x.

Before stating our main result, let us give a very rough sketch of a nearly ordinary situation which motivates this work. In the setting of Hida theory, namely in the ordinary prototype of Coleman families, we have a local ring $\mathbb{I}$ finite flat over $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$ and we have analogous families of $\mathbb{I}$-linear Galois representations $\mathbb{T}$ of $G_{\mathbb{Q}}$, isomorphic to $\mathbb{I} \oplus \mathbb{I}^*$ where $\mathbb{I}^*$ is the $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$-linear dual of $\mathbb{I}$. It is known that there is a Zariski-dense subset $S_{\mathbb{T}}$ of $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$ such that for each $x \in S_{\mathbb{T}}$, the specialization V_x of $\mathbb{T}$ via x is isomorphic to the p-adic Galois representation associated to a certain ordinary cuspform f_x.

Let G_{cyc} be the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ and let $\Lambda(G_{\mathrm{cyc}})$ be the complete group ring $\mathbb{Z}_p[\![G_{\mathrm{cyc}}]\!]$. Let us denote by $\chi_{\mathrm{cyc}} : G_{\mathrm{cyc}} \xrightarrow{\sim} \mathbb{Z}_p^\times$ the p-adic cyclotomic character. Among all continuous characters $\eta : G_{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p^\times$, characters of the form $\eta = \chi_{\mathrm{cyc}}^j \phi$ where j is an integer and ϕ is of finite order play an important role and we call them *arithmetic characters* of G_{cyc}. Note that each continuous character $\eta : G_{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p^\times$ induces a continuous ring homomorphism $\eta : \Lambda(G_{\mathrm{cyc}}) \longrightarrow \overline{\mathbb{Q}}_p$.

In the paper [Och03] of the second author, we proved the following theorem:

Theorem. — [Och03, Theorem 3.13] *There exists a free $\mathbb{I}$-module $\mathbb{D}$ of rank one whose specialization at each $x \in S_{\mathbb{T}}$ is canonically isomorphic to $D_{\mathrm{dR}}(V_x)/\mathrm{Fil}^1 D_{\mathrm{dR}}(V_x)$. We have a unique $\mathbb{I}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$-linear big exponental map*

$$\mathrm{EXP}_{\mathbb{T}} : \mathbb{D}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}}) \longrightarrow \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbb{T})$$

such that, for each $x \in S_{\mathbb{T}}$ and for each arithmetic character $\chi_{\mathrm{cyc}}^j\phi : G_{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p^\times$ with $j \in \mathbb{Z}_{\geq 1}$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{D}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}}) \arrow[r, "\mathrm{EXP}_{\mathbb{T}}"] \arrow[d, "{(x,\chi_{\mathrm{cyc}}^j\phi)}"'] & \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbb{T}) \arrow[d, "{(x,\chi_{\mathrm{cyc}}^j\phi)}"] \\
D_{\mathrm{dR}}(V_x \otimes \chi_{\mathrm{cyc}}^j\phi)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V_x \otimes \chi_{\mathrm{cyc}}^j\phi) \arrow[r] & H^1(\mathbb{Q}_p, V_x \otimes \chi_{\mathrm{cyc}}^j\phi)
\end{tikzcd}$$

where the bottom horizontal map is equal to $(-1)^j(j-1)!E_p(f_x, j, \phi)\exp_{V_x\otimes\chi_{\mathrm{cyc}}^j\phi}$ with

$$E_p(f_x, j, \phi) = \begin{cases} 1 - \dfrac{p^{j-1}}{a_p(f_x)} & \phi = 1, \\ \left(\dfrac{p^{j-1}}{a_p(f_x)}\right)^{\mathrm{ord}(\phi)} & \phi \neq 1. \end{cases} \tag{1.1}$$

In the above, $\exp_{V_x\otimes\chi_{\mathrm{cyc}}^j\phi}$ denotes the Bloch–Kato's p-adic exponential map for the Galois representation $V_x \otimes \chi_{\mathrm{cyc}}^j\phi$.

As a corollary of the above result, we obtain:

1. another construction of the two-variable p-adic L-function of Mazur, Kitagawa and Greenberg–Stevens via Beilinson–Kato Euler system (cf. [Och06, Theorem 5.10]).
2. one of divisibilities of the two-variable Iwasawa Main Conjecture for Hida families combined with the result of [Och05] (under some technical conditions).

In this paper, we want to discuss a non-ordinary generalization of this theory. That is, we want to construct an interpolation of Bloch–Kato exponential maps for certain non-ordinary Galois deformations.

For this purpose, it is important to introduce a formal structure of a given Coleman family. Starting with a classical p-stabilized cuspform f of weight k_0 and a finite slope $\alpha < k_0 - 1$, we will construct a Coleman family $\mathcal{F}$ of slope α passing through f over $\mathscr{X}_{(k_0,i)}[r_0]$ with certain radius r_0 in Theorem 2.5 (see also Definition 2.7). Let $\mathcal{K}$ be the field of coefficients associated to the Coleman family $\mathcal{F}$, which is a finite extension of $\mathbb{Q}_p$ such that Fourier coefficients of f_x are contained in $\mathcal{K}$ for all $x \in \mathscr{Z}$. In Proposition 2.10, we will show that this Coleman family has a formal structure over $\mathbf{A}^0_{/\mathcal{K}}$, which is the ring of formal power series over the ring of integers $\mathcal{O}_{\mathcal{K}}$ of $\mathcal{K}$ and is contained in $\mathscr{A}_{\mathscr{X}}$.

A formal model provides us a p-adic families of cuspforms

$$\mathbb{F} = \sum_{n=1}^{\infty} a_n(\mathcal{F})q^n \in \mathbf{A}^0_{/\mathcal{K}}[\![q]\!]$$

with which we recover $\mathcal{F}$ as the image of $\mathbb{F}$ via $\mathbf{A}^0_{/\mathcal{K}}[\![q]\!] \hookrightarrow \mathscr{A}_{\mathscr{X}}[\![q]\!]$. There is a Zariski-dense subset $\mathscr{Z}$ of the spectrum of $\mathbf{A}_{/\mathcal{K}} := \mathbf{A}^0_{/\mathcal{K}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that the specialization f_x of $\mathbb{F}$ at x is a classical cuspform for every point $x \in \mathscr{Z}$.

Thanks to the formal structure over $\mathbf{A}^0_{/\mathcal{K}}$, we associate a Galois representation $\boldsymbol{\rho}_{\mathbb{F}} \colon G_{\mathbb{Q},S} \longrightarrow \mathrm{GL}_2\left(\mathbf{A}^0_{/\mathcal{K}}\right)$ to our fixed Coleman family which interpolates Galois representations for classical forms at weights $k \in \mathscr{Z}$. We thus also obtain $\boldsymbol{\rho}_{\mathcal{F}} \colon G_{\mathbb{Q},S} \longrightarrow \mathrm{GL}_2\left(\mathscr{A}^0_{\mathscr{X}}\right)$ in Corollary 2.13. Let $\mathbf{T} \cong (\mathbf{A}^0_{/\mathcal{K}})^{\oplus 2}$ be the Galois module associated to $\boldsymbol{\rho}_{\mathbb{F}}$. We denote by $\mathbf{V}$ the Galois module $\mathbf{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let us consider the following conditions.

(NonInt) There is no $x \in \mathscr{Z}$ such that $a_p(f_x)$ for the p-stabilized form f_x is a natural number which is a power of p.

(DoubRt) There is no $x \in \mathscr{Z}$ such that $D^+_{\mathrm{cris}}(V_x)$ is crystalline and the characteristic polynomial for φ has a double root $a_p(x)$).

We remark that the condition (NonInt) is automatically satisfied if the slope of the given Coleman family is not an integer. We remark that by [CE98, Theorem 2.1] it is known that $a_p(x)$ is not a double root when $k_x = 2$ and that by [CE98, Corollary 3.2] it is not a double root for general k_x provided φ_D acts semisimply on $D^+_{\mathrm{cris}}(V_x)$. Milne has proven that semisimplicity of this action follows from Tate conjecture on the dimension of Chow groups for smooth, projective varieties over finite fields, as quoted in [CE98, Introduction].

***Main Theorem* (Theorem 5.6).** — *Let us assume the conditions* (NonInt) *and* (DoubRt) *for our Coleman family and let $h \in \mathbb{Z}_{\geq 0}$ be equal to or greater than the slope of our Coleman family $\mathcal{F}$. Assume that the residual representation $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ associated to forms f_x is irreducible when restricted to $G_{\mathbb{Q}_p(\mu_p)}$.*

Then, we have a unique $\mathbf{A}_{/\mathcal{K}}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$-linear big exponential map

$$\mathrm{EXP}_{\mathbf{V}} : \mathbf{D}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}}) \longrightarrow \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}\big)\right) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})}\mathcal{H}_h(G_{\mathrm{cyc}})$$

such that, for each $x \in \mathscr{Z}$ and for each arithmetic character $\chi^j_{\mathrm{cyc}}\phi \colon G_{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p^{\times}$ with $j \in \mathbb{Z}_{\geq 1}$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{D}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}}) & \xrightarrow{\mathrm{EXP}_{\mathbf{V}}} & \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}\big)\right) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})}\mathcal{H}_h(G_{\mathrm{cyc}}) \\
{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)}\big\downarrow & & \big\downarrow{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)} \\
D_x \otimes D_{\mathrm{dR}}(\chi^j_{\mathrm{cyc}}\phi) & \longrightarrow & H^1(\mathbb{Q}_p, V_x \otimes \chi^j_{\mathrm{cyc}}\phi)
\end{array}$$

where $\mathcal{H}_h(G_{\mathrm{cyc}})$ is the module of distribution on G_{cyc} of logarithmic order h (see §4 for an explanation of $\mathcal{H}_h(G_{\mathrm{cyc}})$) and the bottom horizontal map is equal to $(-1)^j(j-1)!E_p(f_x, j, \phi)\exp_{V_x \otimes \chi^j_{\mathrm{cyc}}\phi}$ with $E_p(f_x, j, \phi)$ as defined in (1.1).

As corollary, we have also a Coleman map over affinoid space as follows:

Corollary. — *Let $\mathscr{X}$, $\mathscr{V} \cong \mathscr{A}_{\mathscr{X}}^{\oplus 2}$ and $\mathscr{Z}$ be as above. There exists a free $\mathscr{A}_{\mathscr{X}}$-module $\mathcal{D}$ of rank one whose specialization at each $x \in \mathscr{Z}$ is isomorphic to $D_x := D_{\mathrm{cris}}^{\varphi=a_p(f_x)}(V_x)$. Let us assume the conditions* (NonInt) *and* (DoubRt) *for our Coleman family and let $h \in \mathbb{Q}_{\geq 0}$ be equal to or greater than the slope of the Coleman family $\mathscr{V}$. Assume that the residual representation $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ associated to forms f_x is irreducible when restricted to $G_{\mathbb{Q}_p(\mu_p)}$. Let $\mathscr{A}_{\mathscr{X}}^0$ be the subring of $\mathscr{A}_{\mathscr{X}}$ which consist of power bounded functions and let us choose a $G_{\mathbb{Q}}$-stable free $\mathscr{A}_{\mathscr{X}}^0$-module $\mathscr{T}$ of rank two giving an integral structure of $\mathscr{V}$, which recovers $\mathscr{V}$ by $\mathscr{V} = \mathscr{T} \otimes_{\mathscr{A}_{\mathscr{X}}^0} \mathscr{A}_{\mathscr{X}}$.*

Then, we have an $\mathscr{A}_{\mathscr{X}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}})$-linear big exponental map

$$\mathrm{EXP}_{\mathscr{V}} : \mathcal{D} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) \longrightarrow \varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathscr{T}\big) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})} \mathcal{H}_h(G_{\mathrm{cyc}})$$

such that, for each $x \in \mathscr{Z}$ and for each arithmetic character $\chi_{\mathrm{cyc}}^j \phi : G_{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p^{\times}$ with $j \in \mathbb{Z}_{\geq 1}$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) & \xrightarrow{\mathrm{EXP}_{\mathscr{V}}} & \varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathscr{T}\big) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})} \mathcal{H}_h(G_{\mathrm{cyc}}) \\ {\scriptstyle (x, \chi_{\mathrm{cyc}}^j \phi)} \big\downarrow & & \big\downarrow {\scriptstyle (x, \chi_{\mathrm{cyc}}^j \phi)} \\ D_x \otimes D_{\mathrm{dR}}(\chi_{\mathrm{cyc}}^j \phi) & \longrightarrow & H^1(\mathbb{Q}_p, V_x \otimes \chi_{\mathrm{cyc}}^j \phi) \end{array}$$

where the bottom horizontal map is equal to

$$(-1)^j (j-1)! E_p(f_x, j, \phi) \exp_{V_x \otimes \chi_{\mathrm{cyc}}^j \phi}$$

with $E_p(f_x, j, \phi)$ as in (1.1). We remark that $\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathscr{T}\big) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})} \mathcal{H}_h(G_{\mathrm{cyc}})$ and the map $\mathrm{EXP}_{\mathscr{V}}$ do not depend on the choice of $\mathscr{T}$.

By definition, we recover the $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module $\mathcal{D}$ by taking the base extension $-\otimes_{\mathbf{A}/\mathcal{K}} \mathscr{A}_{\mathscr{X}/\mathcal{K}}$ of $\mathbf{D}$. To obtain Corollary from Main Theorem is straightforward by taking the base extension $- \otimes_{\mathbf{A}/\mathcal{K}} \mathscr{A}_{\mathscr{X}/\mathcal{K}}$.

Remark 1.1. — The condition (NonInt) excludes the case where our Coleman family contains a classical specialization f_x whose cyclotomic p-adic L-function has a trivial zero. In fact, throughout the proof of Main Theorem, the condition (NonInt) is required only when we apply Theorem 4.25 and Corollary 4.26 under the assumption of the condition (a) stated there (the precise place where we use this is §5.4 after we reduce the Main Theorem to Theorem 5.8).

Even if we do not assume the condition (NonInt), we can still apply Theorem 4.25 and Corollary 4.26 assuming the condition (b). In this case, only a weaker variant of the Main Theorem and Theorem 5.6 in which $\mathrm{EXP}_{\mathbf{V}}$ is defined only on a smaller subgroup of $\mathbf{D} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}})$. However, for the application to the trivial zero conjecture, only such a weaker variant will be sufficient.

We have several other remarks.

Remark 1.2. —

1. The construction of the above rank-one module $\mathbf{D}$ which will be done in Theorem 3.5 is basically based on the theory of Sen and the theory of

Kisin [Kis03]. We have a free $\mathbf{A}^0_{/\mathcal{K}}$-lattice $\mathbf{D}^0$. In Theorem 5.6 which is a finer version of the above Main Theorem, we take care of the denominators of the image of the map $\mathrm{EXP}_{\mathbf{V}}$ with respect to an open ball $\mathcal{H}_h^+(G_{\mathrm{cyc}}) \subset \mathcal{H}_h(G_{\mathrm{cyc}})$ for the Banach module structure on $\mathcal{H}_h(G_{\mathrm{cyc}})$ which we define and study carefully in §4.

2. The technical background of the map $\mathrm{EXP}_{\mathbf{V}}$ is based on the work [PR94] of Perrin-Riou, but we need a more careful treatment of the integral structure associated to the integral part $\mathcal{H}_h^+(G_{\mathrm{cyc}})$ of the Banach module $\mathcal{H}_h(G_{\mathrm{cyc}})$ since we treat a family of p-adic representations, not the cyclotomic tower of a single p-adic representation. The integral structure $\mathcal{H}_h^+(G_{\mathrm{cyc}})$ of the Banach module $\mathcal{H}_h(G_{\mathrm{cyc}})$ was not necessary in the work [PR94] of Perrin-Riou, hence the integral structure $\mathcal{H}_h^+(G_{\mathrm{cyc}})$ which we define later does not appear in [PR94]. On the way of working with integral structures, we also needed to make some of constants in [PR94] explicit, which were left nonexplicit in [PR94] (see constants c_1 to c_4, $c_{h,\lambda}$ in §4). We also filled some technical details of the previous work [PR94] which we could not follow.
3. Recall that the work by Cherbonnier and Colmez [CC99] opened another way to approach the result in [PR94] by using (φ, Γ)-modules. We could have worked with (φ, Γ)-modules for the construction of the map in the main theorem. But, in fact, period rings of Robba type in the theory of (φ, Γ)-modules do not match well with $\mathcal{H}_h(G_{\mathrm{cyc}})$ and it seems that we can not recover the statement as good as our Main theorem formulated with $\mathcal{H}_h(G_{\mathrm{cyc}})$ if we worked purely through the method of (φ, Γ)-modules. Hence, we have opted for the Perrin-Riou's original approach to keep delicate estimate of denominators and of integral structures when the weights of the modular forms f_x vary.
4. On a very quick glance, it might seem that the formulation and the statement of the main result is a bit similar and parallel to that of Perrin-Riou [PR94] except that we consider a family of Galois representations in place of a single Galois representation. However, it is a crucial idea to restrict ourselves to the rank-one subspace $D_{\mathrm{cris}}(V_x)^{\varphi = a_p(f_x)}$ of $D_{\mathrm{cris}}(V_x)$. In fact, when the weight k_x of f_x tends to infinity, we have unbounded denominators if we consider the whole of $D_{\mathrm{cris}}(V_x)$ as in the formulation of the usual Coleman map of Perrin-Riou. We finally have to keep track of all denominators in the construction and we succeeded in showing that such denominators are bounded with respect to $x \in \mathscr{Z}$ when we restrict the exponential map to the rank-one subspace $D_{\mathrm{cris}}(V_x)^{\varphi = a_p(f_x)}$. As far as we know, before this work, there was no such construction of a Coleman map using fixed slope subspace of crystals. Also, if we look into the detail of our construction, there are improvements and simplifications compared to [PR94] (see the strategy in Section 5.4).
5. The residual representation of ordinary Hida deformation is always reducible as a $G_{\mathbb{Q}_p}$-module and thus our actual technical assumption of the residual representation as a $G_{\mathbb{Q}_p}(\mu_p)$-module excludes the ordinary case. This assumption might be rather technical, but it is used in a lot of steps of the proof. Especially

it is used to assure freeness of Galois cohomology, which implies the uniqueness statement of the Coleman map.

As a hypothetical application of our Main Theorem, we construct a two-variable p-adic L-function from a hypothetical family of Beilinson–Kato elements. In fact, by taking the Kummer dual of the main theorem above, we have a unique $\mathbf{A}_{/\mathcal{K}}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$-linear big dual exponental map

$$\mathrm{EXP}^*_{\mathbf{V}} : \varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}^*(1)\big) \longrightarrow \mathbf{D}^*\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{H}_h(G_{\mathrm{cyc}})$$

which interpolates dual exponential maps $\exp^* : H^1(\mathbb{Q}_p, V_x^*(1)\otimes\chi_{\mathrm{cyc}}^{1-j}\phi) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(, V_x^*(1)\otimes\chi_{\mathrm{cyc}}^{1-j}\phi)$ for any $x\in\mathscr{Z}$, any critical integer j for f_x and any finite character ϕ of G_{cyc}. If we take a hypothetical Beilinson–Kato element $\mathcal{Z}\in\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}^*(1)\big)$ and if we decompose the image $\mathrm{EXP}^*_{\mathbf{V}}(\mathcal{Z})\in\mathbf{D}^*\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{H}_h(G_{\mathrm{cyc}})$ as $\mathbb{F}\otimes_{\mathbf{A}_{/\mathcal{K}}} L_p(\mathcal{F})$ with $\mathbb{F}\in\mathbf{D}^*$ a canonical element associated to a Coleman family, the element $L_p(\mathcal{F})\in\mathbf{A}_{/\mathcal{K}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{H}_h(G_{\mathrm{cyc}})$ satisfies the desired interpolation properties of a two-variable p-adic L-function for our given Coleman family. This should be a non-ordinary generalization of the construction of two-variable p-adic L-functions associated to a given Hida family obtained in the paper [Och03] (see [Och06] for an optimization of [Och03]). In the case of [Och03] (and [Och06]), the existence of the Beilinson–Kato element $\mathcal{Z}$ was not hypothetical and we were able to construct Beilinson–Kato Euler system on Hida family. However, in the non-ordinary case of Coleman family, it seems not as straightforward as in the ordinary case of Hida deformation, mainly because the construction of Coleman families in the non-ordinary case is much more "analytic" than the construction of Hida families. Under certain strong assumption on the Galois image of the global residual representation, we construct Beilinson–Kato element with help of the result of this paper (see [Och18]). However, in general, a veritable proof of the construction of Beilinson–Kato Euler systems on Coleman family requires a serious understanding of the complex and p-adic periods on Coleman family.

Although we started announcing the result of this paper more than three years ago, it took much time to finalize the full detail. Meanwhile, we have learned that David Hansen [Han15] and Shangwen Wang [Wan14] have results which might partially overlap with the statement of our Main Theorem. However, we could not follow their argument in [Han15] and [Wan14]. Also, their statements look different from ours and their techniques relying on (φ,Γ)-modules seem completely different from ours. In a certain sense, we are rather going in a complementary direction.

Notation

Throughout the paper, we fix an odd prime p. We fix embeddings of an algebraic closure $\overline{\mathbb{Q}}$ into an algebraic closure $\overline{\mathbb{Q}_p}$ and into $\mathbb{C}$. We normalize the p-adic absolute value in $\overline{\mathbb{Q}_p}$ by $|p|=\frac{1}{p}$. We denote by $\mu_{p^k}(R)$ the group of roots of unity of order p^k (for $k\leq\infty$) inside a ring R and we write $\mathbb{C}_p$ for a completion of $\overline{\mathbb{Q}_p}$ with respect to the absolute value $\|\ \|$.

Throughout the article, we fix a positive integer N which is not divisible by p. The number N will be the tame conductor of our Coleman family.

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2. Families of Galois representations

2.1. Some basics on affinoids and weight spaces. — We refer the reader to [BGR84] for our conventions and basic results about rigid analytic spaces in the sense of Tate. Let K be a complete subfield of $\mathbb{C}_p$. The field K can be either a finite extension of $\mathbb{Q}_p$ or an infinite extension of $\mathbb{Q}_p$. Let $\mathscr{X}$ be an affinoid space defined over K. We write $\mathscr{A}_{\mathscr{X}}$ for the ring of analytic functions on $\mathscr{X}$ and $\mathscr{A}^0_{\mathscr{X}}$ for the subring of power-bounded elements (see [BGR84, 1.2.5]). They will always be endowed with their Gauß semi-norm (which is a norm and coincides with the sup-norm if $\mathscr{X}$ is reduced). When K is a discrete valuation field, the ring of power-bounded elements $\mathscr{A}^0_{\mathscr{X}}$ is noetherian because it is a quotient of $\mathcal{O}_K\langle T_1,\dots,T_n\rangle$ and $\mathcal{O}_K\langle T_1,\dots,T_n\rangle$ is the p-adic completion of a polynomial algebra over $\mathcal{O}_K$. The ring $\mathscr{A}_{\mathscr{X}} = \mathscr{A}^0_{\mathscr{X}}[\frac{1}{p}]$ is noetherian whether K is a discrete valuation field or not. For every maximal ideal $\mathfrak{m} \subseteq \mathscr{A}_{\mathscr{X}}$, $\mathscr{A}_{\mathscr{X}}/\mathfrak{m}$ is a finite extension of K. Whereas, for every non-zero prime ideal $\mathfrak{p} \subseteq \mathscr{A}^0_{\mathscr{X}}$, the quotient $\mathscr{A}^0_{\mathscr{X}}/\mathfrak{p}$ is finite over $\mathcal{O}_K$ (for these facts, see [BGR84, Cor. 3, 6.1.2/3 and Theorem 5.2.7/7]).

We consider the following definition:

Definition 2.1. — Let K be a complete subfield of $\mathbb{C}_p$ and let $\mathscr{X}$ be an affinoid space over K. A subset $\mathscr{Z}$ of the set of K-valued points $\mathscr{X}(K)$ is a Zariski-dense subset in $\mathscr{X}$ if we have $U(K) \cap \mathscr{Z} \neq \emptyset$ for every non-empty Zariski-open subspace $U \subseteq \mathscr{X}$.

Given $x_0 \in K$ and $r \in p^{\mathbb{Q}}$, we denote by $\mathscr{B}[x_0, r]_K$ and $\mathscr{B}(x_0, r)_K$, respectively, the closed and open ball of radius r and center x_0, seen as K-rigid analytic spaces (see [deJ95, Sect. 7] for a description of the second space). We note that we normalize the p-adic absolute value $|\ \ |$ so that $|p| = \frac{1}{p}$. For example, in case $r = 1$ and $x_0 \in K$, $\mathscr{A}^0_{\mathscr{B}[x_0,1]_K}$ is isomorphic to the following ring of restricted power series with coefficients in $\mathcal{O}_K$:

$$\mathcal{O}_K\langle T - x_0\rangle = \Big\{ \sum_{i=0}^{\infty} c_i (T - x_0)^i \in \mathcal{O}_K[\![T - x_0]\!] \ \Big|\ \lim_{i\to\infty} |c_i| = 0 \Big\}.$$

Finally, given any complete subfield $L \subseteq \mathbb{C}_p$, we also need the notation $B[a, r]_L$ and $B(a, r)_L$ for the set of all $x \in L$ such that $|x - a| \leq r$ (respectively, such that $|x - a| < r$). When $K = \mathbb{Q}_p$, we denote $B[x_0, r]_{\mathbb{Q}_p}$ (resp. $B(x_0, r)_{\mathbb{Q}_p}$) by $B[x_0, r]$ (resp. $B(x_0, r)$) dropping the subscript.

Lemma 2.2. — *Let K be a complete subfield of $\mathbb{C}_p$ which is a discrete valuation field.*

1. *When $\mathscr{X}$ is a reduced affinoid defined over K and $f \in \mathscr{A}_{\mathscr{X}}$ vanishes on every point of a Zariski-dense subset Z, we have $f = 0$.*
2. *Let $x_0 \in K$. Every infinite set inside $\mathscr{B}[x_0, 1]_K(K)$ is Zariski-dense.*

Proof. — For the first assertion, suppose $f \neq 0$ and consider the Zariski-open subset $U_f = \{x \in \mathscr{X} \text{ such that } f(x) \neq 0\}$ of $\mathscr{X}$. Since $\mathscr{X}$ is reduced and $f \neq 0$, we have $U_f \neq \emptyset$. By the assumption that $Z \subset \mathscr{X}$ is a Zariski-dense subset, we have $Z \cap U_f(K) \neq \emptyset$. For any point $z \in Z \cap U_f(K)$, we have $f(z) = 0$, contradicting the definition of U_f.

We pass to the second assertion. By Weierstraß preparation Theorem ([BGR84, Theorem 5.2.2/1]), every function $f \in \mathscr{A}_{\mathscr{B}[0,1]_K}$ can be factored as $f = P \cdot U$ where $P \in K[T - x_0]$ is a polynomial and $U \in \mathscr{A}^{\times}_{\mathscr{B}[x_0,1]_K}$ is an invertible power series which does not vanish on $\mathscr{B}[x_0, 1]_K$. It follows that every such f has only finitely many zeroes and that $\mathscr{A}_{\mathscr{B}[x_0,1]_K}$ is a PID, showing that non-trivial Zariski-closed sets in $\mathscr{B}[x_0, 1]_K$ consist of finitely many points. □

One of the main rigid spaces of interest for us is the weight space $\mathscr{W}_N$, which is isomorphic to $\varphi(Np)$ copies of $\mathscr{B}(1, 1)_K$ indexed by

$$\mathfrak{D} = \mathrm{Hom}\big((\mathbb{Z}/Np\mathbb{Z})^{\times}, \mathbb{C}_p^{\times}\big).$$

For generalities about $\mathscr{W}_N$, we refer to [Col97b, Sect. B1], [CM98, Sect. 1.4]. For more detailed accounts, we refer to [Gou88, Chap. I.3, §4 and Appendix] and [Buz07, page 103]. By definition, the weight space satisfies

$$\mathscr{W}_N(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\varprojlim_n (\mathbb{Z}/Np^n\mathbb{Z})^{\times}, \mathbb{C}_p^{\times}).$$

Following Coleman and Mazur, we give the following definition:

Definition 2.3. — We denote by $\omega : \mu_p(\mathbb{Z}_p) \to \mathbb{Z}_p^{\times}$ the Teichmüller character and by $\langle\langle\ \rangle\rangle : \mathbb{Z}_p^{\times} \to 1 + p\mathbb{Z}_p$ the projection $x \mapsto x/\omega(x)$. For every integer k and $\chi \in \mathfrak{D}$ of finite order, the point $\chi\langle\langle\ \rangle\rangle^k \in \mathscr{W}_N(\mathbb{Q}_p)$ is called an accessible weight-character with coordinates (χ, k).

As detailed in [CM98, Definition, Section 1.4] the accessible weight-characters are parametrized by the rigid analytic subspace $\mathscr{W}_N^* = \mathfrak{D} \times \mathscr{B}^* \subseteq \mathscr{W}_N$ where

$$\mathscr{B}^* \cong \mathscr{B}(0, p^{\frac{p-2}{p-1}}) \tag{2.1}$$

is the subdisk of $\mathscr{B}(1, 1)$ which is the image of $\mathscr{B}(0, p^{\frac{p-2}{p-1}})$ via the map $s \mapsto (1 + p)^s$. In the notation introduced in Definition 2.3, the character $\chi\langle\langle\ \rangle\rangle^k$ is represented by the point $(\chi, (1 + p)^k) \in \mathscr{W}_N(\mathbb{Q}_p)$ which gets mapped to (χ, k) by the identification in (2.1); we see that the word "coordinates" comes from seeing $\mathscr{W}_N^*$ as $\mathfrak{D}$-copies of $\mathscr{B}(0, p^{\frac{p-2}{p-1}})$. From now on we systematically write points in the weight space through their coordinates. In particular, for every fixed $\chi \in \mathfrak{D}$, the weights of characters with Nebentypus χ will be points in $\mathscr{B}(0, p^{\frac{p-2}{p-1}})$ rather than in $\mathscr{B}(1, 1)$.

The assumption $(N,p)=1$ allows us to look at the group $\mathrm{Hom}\big((\mathbb{Z}/p)^\times,\mathbb{C}_p^\times\big)$ as a subgroup of $\mathfrak{D}$. It thus makes sense, for each $0\le j\le p-2$, to interpret ω^j as an element of $\mathfrak{D}$. The characters $x\mapsto x^k$, which are accessible with coordinates $(1,k)$, are then the elements of $\mathscr{W}_N(\mathbb{Q}_p)$ which belong to the ω^k-th copy of $\mathscr{B}^*$ and we call them *integral weights* (of trivial Nebentypus).

2.2. p-adic families of modular forms. — We start with a classical eigencuspform $f\in S_{k_0}(\Gamma_1(Np),\varepsilon)$ of weight k_0, level Np and Nebentypus ε, which we factor as product $\varepsilon=\varepsilon_N\omega^{k_0-i}$ for some character ε_N of conductor divisible by N and some $0\le i\le p-1$. Our main reference concerning p-adic modular forms and p-adic families thereof is [Col97b, Part B] as well as [Gou88], in particular Section II.3 for the definition of the U_p-operator.

Definition 2.4. — Let f be a p-adic modular form of level Np which is an eigenvector with respect to the U_p-operator. We define the slope of f to be the p-adic valuation of the U_p-eigenvalue of f. It is a non-negative rational number.

We assume that the slope of f is $0\le\alpha\le k_0-1$. In [Col97b], [Col97a] and [CM98] Coleman and Coleman–Mazur have built a theory of families of p-adic modular forms of slope α interpolating f, a part of which is stated in Theorem 2.5 below and will be crucial for us in this paper. Let us introduce some notation to state Theorem 2.5. Given an element $k_0\in\mathbb{Q}_p$, an integer i and $r<p^{\frac{p-2}{p-1}}\in p^{\mathbb{Q}}$, we denote by $\mathscr{X}_{(k_0,i)}[r]$ the affinoid subspace of the weight space

$$\mathscr{X}_{(k_0,i)}[r_0]:=\{\varepsilon_N\omega^i\}\times\mathscr{B}[k_0,r]\subseteq\mathscr{W}_N^*$$

Let us consider a finite extension $\mathcal{K}$ of $\mathbb{Q}_p$, which plays a role of the field of coefficients of motives associated to cuspforms f in the given Coleman family. We note that $\mathcal{K}$ has nothing to do with the field of definition K, for which we can choose K to be $\mathbb{Q}_p$ which corresponds to of the field of definition of motives associated to cuspforms f in the given Coleman family. From now on, we denote by $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ (resp. $\mathscr{A}_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$) the extension of coefficients $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]}\otimes_{\mathbb{Q}_p}\mathcal{K}$ (resp. $\mathscr{A}_{\mathscr{X}_{(k_0,i)}[r]}\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathcal{K}}$) where $\mathcal{O}_{\mathcal{K}}$ is the ring of integers of $\mathcal{K}$. We have the following result thanks to [Col97b]:

Theorem 2.5 ([Col97b]). — *Suppose that f is a classical normalized cuspidal eigenform, of weight k_0, level $\Gamma_1(Np)$, slope $\alpha<k_0-1$, Nebentypus $\varepsilon=\varepsilon_N\omega^{i-k_0}$ and which is new away from p. In the case $i=0$, suppose moreover that $a^2\neq\varepsilon_N(p)p^{k_0-1}$, where a is the U_p-eigenvalue of f.*

Then, there is a radius $r_0<p^{\frac{p-2}{p-1}}$ lying in $p^{\mathbb{Q}}$ and rigid analytic functions a_n on $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r_0]/\mathcal{K}}$ with some field of coefficient $\mathcal{K}$ for every natural number n such that the following statements hold:

1. *For every integer $k\in\mathscr{X}_{(k_0,i)}[r_0](\mathbb{Q}_p)$ satisfying $k>\alpha+1$ the series $\sum_{n=1}^{\infty}a_n(k)q^n\in\mathcal{K}[\![q]\!]$ coincides with the q-expansion of a classical normalized cuspidal eigenform of level Np, weight k, slope α and character $\varepsilon_N\omega^{i-k}$.*

2. *The series* $\sum_{n=1}^{\infty} a_n(k_0)q^n \in \mathcal{K}[\![q]\!]$ *coincides with the* q*-expansion of* f *at* $k = k_0$.
3. *The space* $\mathscr{X}_{(k_0,i)}[r_0]$ *is* a_p*-small in the sense of* [Kis03, (**5.2**)].

Remark 2.6. — At first glance, it might look better to write $a_n^{(r_0)}$ for the analytic functions appearing in the statement, since the radius r_0 is not uniquely associated to f and these functions might *a priori* depend on its choice. However, we will prove in Corollary 2.8 below that this is not necessary.

Proof. — In [Col97b, pp. 465–467] (and especially along the proof of Corollary B5.7.1 *ibidem*), Coleman attaches to the above f the space $\mathscr{X}_{(k_0,i)}[r_0]$ with a radius $r_0 \in p^{\mathbb{Q}}$ small enough. In [Col97b, pp. 465–467], this space is denoted simply by B and a crucial step is to take a finite étale affinoid algebra $R_{(k_0,i)}[r_0]$ over $\mathscr{A}_{\mathscr{X}_{(k_0,i)}[r_0]}$ whose associated affinoid space parametrizes families of p'-new forms of slope α. As in the paper [Col97b], we write $X(R_{(k_0,i)}[r_0]) \to \mathscr{X}_{(k_0,i)}[r_0]$ for the affinoid space associated to $R_{(k_0,i)}[r_0]$.

Then the statement is Corollary B5.7.1 in [Col97b] *verbatim*, except for the condition that all a_n be power-bounded and that all forms above be normalized. In Lemma B5.3 *ibidem* it is shown that the Hecke eigenvalues of an overconvergent cuspidal eigenform are bounded by 1 if it is normalized. We thus get the result by observing that $a_1 = 1$, which follows from $a_1 = T(1) = 1$ by the construction given in Theorem B5.7 *ibidem*. As discussed in [Kis03, §5.2] it is always possible to shrink a disk around a point in order to get a smaller one which is a_p-small, and this radius we call r_0. □

Definition 2.7. — Let f be a form as in Theorem 2.5 and let $0 < r \leq r_0$ be smaller than or equal to the radius constructed there. We refer to the rigid functions $\{a_n\}_{n\in\mathbb{N}}$ on $\mathscr{X}_{(k_0,i)}[r_0]$ as *a* Coleman family of slope α and radius r passing through the form f. We refer to the formal power series

$$\sum_{n=1}^{\infty} a_n q^n \in \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}[\![q]\!]$$

as the Fourier expansion of the Coleman family. For each $x \in \mathscr{X}_{(k_0,i)}[r]$, we denote by f_x the overconvergent modular form whose expansion is $\sum a_n(x)q^n$. We also refer to the collection of all these forms as the Coleman family of slope α through the form f.

We consider the subset of accessible weight-characters in $\mathscr{X}_{(k_0,i)}[r]$ as follows:

$$\mathscr{Z}_{(k_0,i)}[r] = \Big(\{\varepsilon_N\omega^i\} \times \mathbb{Z}_{>\alpha+1}\Big) \cap \mathscr{X}_{(k_0,i)}[r] \subseteq \mathscr{W}_N^*(\mathbb{Q}_p). \tag{2.2}$$

By abuse of notation, we write k for elements $(\varepsilon_N\omega^i, k)$ of $\mathscr{Z}_{(k_0,i)}[r]$ and we denote the p-adic Deligne representation attached to f_k in [Del68] by ρ_k.

Consider a form f satisfying the assumption of Theorem 2.5 and a radius r_0 as constructed there.

Corollary 2.8. — *Recall that $k_0 \in \mathbb{Z}$ is the weight of the classical form f_0 which is the base of our given Coleman family. Let r_0 be a radius as constructed in Theorem 2.5 and let $r < r_0$ be a smaller radius lying in $p^{\mathbb{Q}}$. Then the functions*

$$\mathrm{res}^{\mathscr{X}_{(k_0,i)}[r_0]}_{\mathscr{X}_{(k_0,i)}[r]}(a_n) \in \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}} \qquad \text{for } n \geq 1$$

are the Fourier expansion of a Coleman family of slope α and radius r passing through the form f.

Proof. — Follows from the definitions, since for every $k \in \mathscr{Z}_{(k_0,i)}[r](\mathbb{Q}_p)$ the series

$$\sum_{n=1}^{\infty} \mathrm{res}^{\mathscr{X}_{(k_0,i)}[r_0]}_{\mathscr{X}_{(k_0,i)}[r]}(a_n)(k)q^n$$

is the q-expansion of a form of the required type. □

Thanks to the above corollary, we can unambiguously speak about its Fourier coefficients a_n without referring to the radius; observe also that if shrink an a_p-small disk to another one of smaller radius gives maintains a_p-smallness. We recall the following lemma:

Lemma 2.9. — *Recall that $k_0 \in \mathbb{Z}$ is the weight of the classical form f_0 which is the base of our given Coleman family. Let $r \in p^{\mathbb{Q}}$ be a radius satisfying $r \leq r_0$ and $\mathcal{K}$ a finite extension of $\mathbb{Q}_p$. If the function $G \in \mathscr{A}_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ vanishes on $\mathscr{Z}_{(k_0,i)}[r_0]$, it is everywhere zero.*

Proof. — The subset $\mathscr{Z}_{(k_0,i)}[r] \subseteq \mathscr{X}_{(k_0,i)}[r](\mathbb{Q}_p)$ is Zariski-dense in $\mathscr{X}_{(k_0,i)}[r]$ thanks to the second assertion of Lemma 2.2. Hence, the assertion is an immediate consequence of the first assertion of Lemma 2.2. □

2.3. p-adic family of Galois representations. — The main result of this subsection is Theorem 2.12 which produces an *integral* Galois representation with values in $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ – for a given Coleman family and a suitable affinoid $\mathscr{X}$ – that specializes to the Deligne representations attached to the classical eigenforms which belong to the given Coleman family. The most important ingredient is to construct pseudo-representations associated to a given Coleman family: once we have a pseudo-representation over our affinoid algebras, it is a standard argument to recover Galois representations from this pseudo-representation.

Since we do not find a standard reference to construct pseudo-representations over affinoid algebras, we construct them by using a formal structure of Coleman families. The formal structure over $\mathbf{A}^0_{/\mathcal{K}}$ of a given Coleman family constructed here will also play an important role in the proof of the main theorem given in Section 5. Let us fix a Coleman family as in Theorem 2.5 and let $r_0 \in p^{\mathbb{Q}}$ be the radius of this Coleman family which appeared there. Let K be a complete subfield of $\mathbb{C}_p$ and take an element $e_0 \in K$ such that $r_0 = |e_0|$. Define

$$\mathbf{A}^0_K = \mathcal{O}_K\Big[\!\Big[\frac{T-k_0}{e_0}\Big]\!\Big] = \Big\{ \sum_{i=0}^{\infty} c_i \left(\frac{T-k_0}{e_0}\right)^i \;\Big|\; c_i \in \mathcal{O}_K \Big\} \tag{2.3}$$

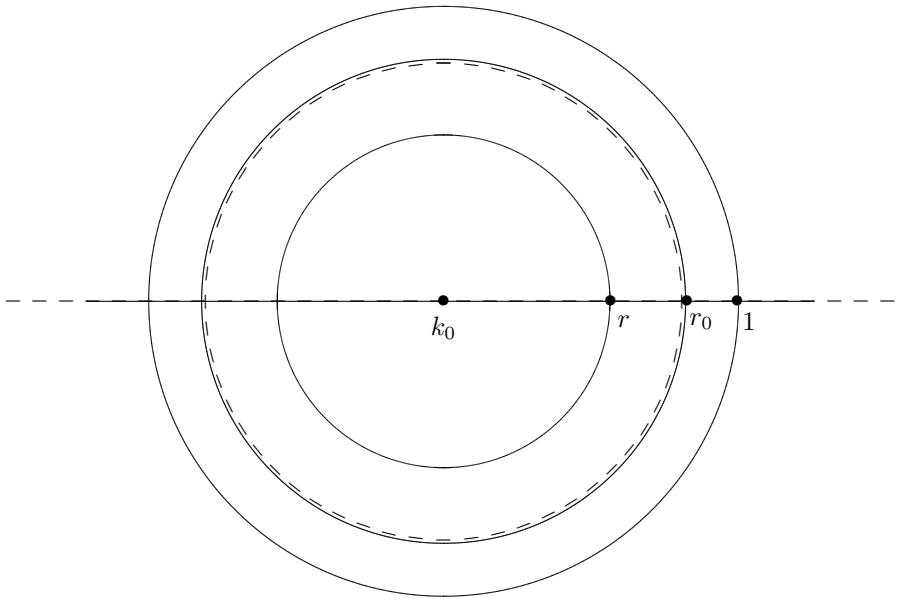

FIGURE 1.

For any $r \in p^{\mathbb{Q}}$ with $r < r_0$, power series in $\mathbf{A}_K^0$ converge on $B[k_0, r]_K$ which is strictly contained in $B(k_0, r_0)_K$ and are there bounded by 1. We consider them as functions on $\mathscr{X}_{(k_0,i)}[r]$.

By restriction, this induces a ring homomorphism $\mathbf{A}_K^0 \xrightarrow{\text{res}} \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_K}$ for any $r \in p^{\mathbb{Q}}$ with $r < r_0$. For each such radius r, we choose $e_r \in \mathbb{C}_p$ such that $|e_r| = r$. By [BGR84, §6.1.5], there is an isomorphism

$$\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_{\mathbb{C}_p}} \cong \Big\{ \sum_{i=0}^{\infty} c_i \left(\frac{T-k_0}{e_r}\right)^i \in \mathcal{O}_{\mathbb{C}_p}\Big[\!\Big[\frac{T-k_0}{e_r}\Big]\!\Big] \;\Big|\; \lim_{i\to\infty} |c_i| = 0 \Big\}.$$

Similarly as above, the inclusion $\{\varepsilon_N \omega^i\} \times \mathscr{B}(k_0, r_0)_K \subset \mathscr{X}_{(k_0,i)}[r_0]_K$ induces a homomorphism $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r_0]_K} \to \mathbf{A}_K^0$ where we have a presentation

$$\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r_0]_K} \cong \Big\{ \sum_{i=0}^{\infty} c_i \left(\frac{T-k_0}{e_0}\right)^i \in \mathcal{O}_K\Big[\!\Big[\frac{T-k_0}{e_0}\Big]\!\Big] \;\Big|\; \lim_{i\to\infty} |c_i| = 0 \Big\}.$$

We have:

$$\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r_0]_K} \to \mathbf{A}_K^0 \xrightarrow{\text{res}} \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_{\mathbb{C}_p}} \tag{2.4}$$

which correspond to the inclusions

$$\mathscr{B}[k_0, r]_{\mathbb{C}_p} \subset \mathscr{B}(k_0, r_0)_K \subset \mathscr{B}[k_0, r_0]_K.$$

In Figure 1, there is a sketch of the radii that occurred so far in our construction: When the radius $r \in p^{\mathbb{Q}}$ with $r < r_0$ tends to r_0, we have

$$\mathbf{A}^0_K \subset \mathbf{A}^0_{\mathbb{C}_p} = \varprojlim_{r \to r_0} \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_{\mathbb{C}_p}} = \bigcap_{r<r_0} \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_{\mathbb{C}_p}}.$$

Here $\mathbf{A}^0_K$ is characterized to be elements in $\mathbf{A}^0_{\mathbb{C}_p}$ which takes values in K at a dense subset in $\mathscr{B}[k_0, r_0]_K$. From now on, when there is no fear of confusion, we denote $\mathbf{A}^0_{\mathbb{Q}_p}$ by $\mathbf{A}^0$ when $K = \mathbb{Q}_p$ dropping the subscript K. For a finite extension $\mathcal{K}$ of $\mathbb{Q}_p$, we denote $\mathbf{A}^0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{K}}$ by $\mathbf{A}^0_{/\mathcal{K}}$. As stated previously, $\mathcal{K}$ plays a role of the field of coefficients and $\mathcal{K}$ has nothing to do with the field of definition K. By the above observation, we have the following proposition:

Proposition 2.10. — *Let us take the same assumptions and notations of Theorem 2.5. Then, there exists a finite extension $\mathcal{K}$ of $\mathbb{Q}_p$ such that, for every natural number n, there is a unique function $A_n \in \mathbf{A}^0{}_{/\mathcal{K}}$ such that the image of A_n via the natural inclusion $\mathbf{A}^0{}_{/\mathcal{K}} \hookrightarrow \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]}{}_{/\mathcal{K}}$ coincides with a_n obtained in Theorem 2.5 on a smaller radius r.*

We remark that $\mathbf{A}^0_{/\mathcal{K}}$ in Proposition 2.10 is isomorphic to the following ring:

$$\mathbf{A}^0{}_{/\mathcal{K}} \cong \mathcal{O}_{\mathcal{K}}\left[\!\left[\frac{T-k_0}{e_0}\right]\!\right] = \left\{ \sum_{i=0}^{\infty} c_i \left(\frac{T-k_0}{e_0}\right)^i \;\middle|\; c_i \in \mathcal{O}_{\mathcal{K}} \right\},$$

where $e_0 \in \mathbb{Q}_p$ is an element with $|e_0| = r_0$.

Proof. — In Theorem 2.5, we obtained a Coleman family over $\mathscr{X}_{(k_0,i)}[r_0]$. By shrinking such a Coleman family to $\mathscr{X}_{(k_0,i)}[r]$ for any $r \in p^{\mathbb{Q}}$ with $r < r_0$ and by taking the limit $r \longrightarrow r_0$, we showed that there exists $A_n \in \mathbf{A}^0{}_{/\mathcal{K}}$ such that the image of A_n via the natural inclusion $\mathbf{A}^0_{/\mathcal{K}} \hookrightarrow \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]}{}_{/\mathcal{K}}$ coincides with a_n obtained in Theorem 2.5. This completes the proof of the proposition. □

Based on the existence of the formal structure $\mathbf{A}^0_{/\mathcal{K}}$, we introduce the notion of 2-dimensional pseudo-representation based on [Wil88]. We remark that other types of pseudo-representations were later introduced by Taylor in [Tay91] and, much more recently, by Chenevier in [Che11] but for our purposes Wiles' approach seems to be the best suited. Given a topological group G and a topological ring R in which 2 is invertible we say that a triple $\pi = (A, D, \Xi)$ of continuous functions

$$A, D : G \longrightarrow R \qquad \Xi : G \times G \longrightarrow R$$

satisfying properties (I)–(IV) in [Wil88, Lemma 2.2.3], is a pseudo-representation.

Given a continuous representation $\rho : G \to \mathrm{GL}_2(R)$, by fixing a basis of R^2 we can write

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \qquad \text{for } \sigma \in G$$

and the triple $\pi_\rho = (A(\sigma) = a(\sigma), D(\sigma) = d(\sigma), \Xi(\sigma, \tau) = b(\sigma)c(\tau))$ is easily checked to be a pseudo-representation; observe that attaching a pseudo-representation to a representation depends on the choice of a basis. Extending the notation introduced in Definition 2.7, we simply denote by π_k the pseudo-representation π_{ρ_k} attached to ρ_k in some chosen basis.

Let $Z_{(k_0,i)}[r_0]$ be the set of points of the formal rigid space which is defined .

$$Z_{(k_0,i)}[r_0] = \mathscr{B}(k_0, r_0) \cap \mathscr{Z}_{(k_0,i)}[r_0] \tag{2.5}$$

where $\mathscr{Z}_{(k_0,i)}[r_0]$ is the set defined at (2.2). We apply this discussion to the ring $R = \mathbf{A}^0_{/\mathcal{K}}$ defined in (2.3) to find:

Proposition 2.11. — *Let S be the finite set of primes of $\mathbb{Q}$ consisting of the primes $\{\ell : \ell \mid N\}$, p and ∞, $G_{\mathbb{Q},S}$ the Galois group of the maximal extension unramified outside S over $\mathbb{Q}$. Then, there exists a continuous pseudo-representation*

$$\pi = (A, D, \Xi) \colon G_{\mathbb{Q},S} \longrightarrow \mathbf{A}^0_{/\mathcal{K}}$$

interpolating the pseudo-representations π_k attached to members of the Coleman family of slope α through f. In other words, for each $k \in Z_{(k_0,i)}[r_0]$, the evaluation $\mathrm{ev}_k \circ \pi = (\mathrm{ev}_k \circ A, \mathrm{ev}_k \circ D, \mathrm{ev}_k \circ \Xi)$ coincides with the pseudo representation π_k.

Proof. — The argument of the pseudo representation of rank two à la Wiles is more or less standard and the proof goes in a quite parallel manner as the proof given in the text book [Hid93, §7.5]. So we only give an outline.

Let us choose a complex conjugation $c \in G_{\mathbb{Q},S}$. For each $k \in Z_{(k_0,i)}[r_0]$, we fix a a basis for the representation ρ_k so that c is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Following [Wil88, Lemma 2.3.3], we define functions

$$A_{(k)}, D_{(k)} : G_{\mathbb{Q},S} \longrightarrow \mathcal{K}$$

by the matrix representation

$$\rho_k(g) = \begin{pmatrix} A_{(k)}(g) & B_{(k)}(g) \\ C_{(k)}(g) & D_{(k)}(g) \end{pmatrix}.$$

We also set $\Xi_{(k)} : G_{\mathbb{Q},S} \times G_{\mathbb{Q},S} \longrightarrow \mathcal{K}$ by $\Xi_{(k)}(\gamma_1, \gamma_2) = A_{(k)}(\gamma_1\gamma_2) - A_{(k)}(\gamma_1)A_{(k)}(\gamma_2)$. The functions $A_{(k)}, D_{(k)}, \Xi_k$ are continuous for every $k \in Z_{(k_0,i)}[r_0]$ thanks to continuity of the Deligne representation ρ_k.

For any $k \in Z_{(k_0,i)}[r_0]$, we denote by $P_k \subset \mathbf{A}^0_{/\mathcal{K}}$ the kernel of the evaluation map $\mathbf{A}^0_{/\mathcal{K}} \longrightarrow \mathcal{K}$ at k. Let us denote the function $A_{(k)} + D_{(k)}$ by Tr_{P_k}. Since $Z_{(k_0,i)}[r_0]$ is a countable set, we give a numbering

$$Z_{(k_0,i)}[r_0] = \{k_1, k_2, \ldots, k_s, \ldots\}.$$

For $k_1, k_2 \in Z_{(k_0,i)}[r_0]$, we consider the map

$$\mathbf{A}^0_{/\mathcal{K}}/P_{k_1} \oplus \mathbf{A}^0_{/\mathcal{K}}/P_{k_2} \longrightarrow \mathbf{A}^0_{/\mathcal{K}}/(P_{k_1} + P_{k_2}),\ (x, y) \mapsto (x \bmod P_{k_2}) - (y \bmod P_{k_1}) \tag{2.6}$$

whose kernel is isomorphic to $\mathbf{A}^0_{/\mathcal{K}}/(P_{k_1} \cap P_{k_2})$. Let ℓ be a prime number outside S. We have $\mathrm{Tr}_{P_k}(\mathrm{Frob}_\ell) = a_\ell(k)$ for every $k \in Z_{(k_0,i)}[r_0]$ and the Fourier coefficients $a_\ell(k_i)$ glue together when k_i varies. Hence the values $\mathrm{Tr}_{P_k}(\mathrm{Frob}_\ell)$ glue together when k varies, which is true for any prime number ℓ outside S. By the sequence (2.6), we have a continuous function $\mathrm{Tr}_{P_{k_1} \cap P_{k_2}} : G_{\mathbb{Q},S} \longrightarrow \mathbf{A}^0_{/\mathcal{K}}/(P_{k_1} \cap P_{k_2})$ whose value $\mathrm{Tr}_{P_{k_1} \cap P_{k_2}}(\mathrm{Frob}_\ell)$ is congruent to $\mathrm{Tr}_{P_{k_1}}(\mathrm{Frob}_\ell)$ (resp. $\mathrm{Tr}_{P_{k_2}}(\mathrm{Frob}_\ell)$) mod P_{k_1} (resp. mod P_{k_2}) for every prime number ℓ outside S. Since the set of Frobenius

elements is dense in $G_{\mathbb{Q},S}$, value $\mathrm{Tr}_{P_{k_1}\cap P_{k_2}}(\sigma)$ is congruent to $\mathrm{Tr}_{P_{k_1}}(\sigma)$ (resp. $\mathrm{Tr}_{P_{k_2}}(\sigma)$) mod P_{k_1} (resp. mod P_{k_2}) for every $\sigma \in G_{\mathbb{Q},S}$.

By inductive argument, for each natural number s, we have a continuous function

$$\mathrm{Tr}_{P_{k_1}\cap P_{k_2}\cap\ldots\cap P_{k_s}} : G_{\mathbb{Q},S} \longrightarrow \mathbf{A}^0_{/\mathcal{K}}/(P_{k_1}\cap P_{k_2}\cap\ldots\cap P_{k_s})$$

such that we recover the function $\mathrm{Tr}_{P_{k_i}}(\sigma)$ mod P_{k_i} for $i = 1, 2, \ldots, s$. Note that we have $\mathbf{A}^0_{/\mathcal{K}} = \varprojlim_s \mathbf{A}^0_{/\mathcal{K}}/(P_{k_1}\cap P_{k_2}\cap\ldots\cap P_{k_s})$ since $\mathbf{A}^0_{/\mathcal{K}}$ is complete and local. We define the continuous function $\mathrm{Tr} : G_{\mathbb{Q},S} \longrightarrow \mathbf{A}^0_{/\mathcal{K}}$ to be $\varprojlim_s \mathrm{Tr}_{P_{k_1}\cap P_{k_2}\cap\ldots\cap P_{k_s}}$.

By using the function Tr, we define the desired function A and D as follows:

$$A(\sigma) = \frac{\mathrm{Tr}\big(\sigma\big) - \mathrm{Tr}\big(c\cdot\sigma\big)}{2}, \quad D(\sigma) = \frac{\mathrm{Tr}\big(\sigma\big) + \mathrm{Tr}\big(c\cdot\sigma\big)}{2}.$$

Since $\mathbf{A}^0_{/\mathcal{K}}$ is complete and local, a similar argument is done by first evaluating at pairs $(g_1, g_2) = (\mathrm{Frob}_{\ell_1}, \mathrm{Frob}_{\ell_2})$ the value $b(g_1)c(g_2)$ and then observing that the set of these pairs is dense in $G_{\mathbb{Q},S} \times G_{\mathbb{Q},S}$.

We thus obtain a continuous function

$$\Xi\colon G_{\mathbb{Q},S} \times G_{\mathbb{Q},S} \longrightarrow \mathbf{A}^0_{/\mathcal{K}}$$

which recovers the function $\Xi_{(k)}$ taking the reduction modulo P_k of the function Ξ for every $k \in Z_{(k_0,i)}[r_0]$.

Setting $\pi := (A, D, \Xi)$, we need to check that they verify properties (II)-(IV) of [Wil88] by using the fact that $Z_{(k_0,i)}[r_0]$ is dense in the reduced $\mathscr{X}$. First, let us verify the property (IV), namely we verify that for each $g_1, g_2, h_1, h_2 \in G_{\mathbb{Q},S}$ it holds

$$\Xi(g_1, g_2)\Xi(h_1, h_2) - \Xi(g_1, h_2)\Xi(h_1, g_2) = 0\ . \tag{2.7}$$

We need to check that the function $\Xi(g_1, g_2)\Xi(h_1, h_2) - \Xi(g_1, h_2)\Xi(h_1, g_2)$ vanishes identically on $\mathscr{X}$. Since the above-mentioned property (IV) holds for the pseudorepresentation $\pi_k = (A_{(k)}, D_{(k)}, \Xi_k)$, we have

$$\begin{aligned}\big(\Xi(g_1, g_2)\Xi(h_1, h_2) - \Xi(g_1, h_2)\Xi(h_1, g_2)\big)(k)\\ = \Xi_k(g_1, g_2)\Xi_k(h_1, h_2) - \Xi_k(g_1, h_2)\Xi_k(h_1, g_2)\end{aligned}$$

at each point k in the dense subset $Z_{(k_0,i)}[r_0]$. This proves the desired vanishing of (2.7) and the same argument holds for other properties (II) and (III). This completes the proof. □

Theorem 2.12 below, which is the main result of this section, shows that the pseudorepresentation just constructed comes from a true representation.

Theorem 2.12. — *Let us take the same assumptions and notations as Proposition 2.11. There exists a free $\mathbf{A}^0_{/\mathcal{K}}$-module $\mathbf{T}$ of rank two with a continuous $G_{\mathbb{Q},S}$-action such that the representation*

$$\boldsymbol{\rho}\colon G_{\mathbb{Q},S} \longrightarrow \mathrm{Aut}_{\mathbf{A}^0_{/\mathcal{K}}}(\mathbf{T})$$

satisfies $\pi_{\boldsymbol{\rho}} = \pi$ (in a suitable basis). In particular, $\boldsymbol{\rho}$ modulo P_k is isomorphic to a lattice of ρ_k for all $k \in Z_{(k_0,i)}[r_0]$.

The following proof mainly relies on [Wil88, Lemma 2.2.3], see also [Hid93, Proposition 1, §7.5].

Proof. — Start with a radius r_0 as in Theorem 2.5 and set $r = r_0|\varpi|$. Let $\pi = (A, D, \Xi)$ be the $\mathbf{A}^0_{/\mathcal{K}}$-valued pseudo-representation constructed in Proposition 2.11. There exists a pair of elements $\sigma, \tau \in G_{\mathbb{Q},S}$ such that $\Xi(\sigma,\tau)(k_0) \neq 0$. If not, the diagonal Galois representation $g \mapsto A(g)(k_0) \oplus D(g)(k_0)$ would have the same trace and determinant as the representation ρ_k, which contradicts to Ribet's Theorem 2.3 of [Rib77] saying that ρ_k be irreducible (see in [Wil88, Lemma 2.2.3] or [Hid89, Proposition 1.1]). From now on, let us fix a pair σ, τ such that $\Xi(\sigma,\tau)(k_0) \in \mathcal{K}$ is of minimal valuation, say $\mu \in \mathbb{Q}$. The element $\Xi(\sigma,\tau)$ can be decomposed as $\Xi(\sigma,\tau) = p^\mu V^{-1}$ with $V \in (\mathbf{A}^0_{/\mathcal{K}})^\times$. As in [Hid93, Proposition 1], we check that the map

$$\boldsymbol{\rho}\colon g \mapsto \begin{pmatrix} A(g) & \Xi(g,\tau)Vp^{-\mu} \\ \Xi(\sigma,g) & D(g) \end{pmatrix}$$

is multiplicative, sends $1 \in G_{\mathbb{Q},S}$ to $\mathbf{Id}_2 \in M_2(\mathbf{A}^0_{/\mathcal{K}})$ and takes values in $M_2(\mathbf{A}^0_{/\mathcal{K}} \otimes \mathbb{Q}_p)$. Hence we have a continuous group homomorphism $\boldsymbol{\rho}\colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbf{A}^0_{/\mathcal{K}} \otimes \mathbb{Q}_p)$. We want to produce a finitely generated $\mathbf{A}^0_{/\mathcal{K}}$-submodule of $\mathbf{A}^0_{/\mathcal{K}} \otimes \mathbb{Q}_p$ which is $\boldsymbol{\rho}$-stable, we follow the proof of continuity of [Hid89, Proposition 1.1]. Namely, define $\mathfrak{J} \subseteq \mathbf{A}^0_{/\mathcal{K}}$ to be the ideal generated by all lower-left entries $\Xi(\sigma,g)$ for $g \in G_{\mathbb{Q},S}$ and let $\mathbf{T}'$ be the $\mathbf{A}^0_{/\mathcal{K}}$-submodule of $(\mathbf{A}^0_{/\mathcal{K}})^{\oplus 2} \subset (\mathbf{A}^0_{/\mathcal{K}} \otimes \mathbb{Q}_p)^{\oplus 2}$ generated by all vectors (x,y) with $x \in \mathbf{A}^0_{/\mathcal{K}}, y \in \mathfrak{J}$. The $\mathbf{A}^0_{/\mathcal{K}}$-module $\mathbf{T}'$ is finitely generated since $\mathbf{A}^0_{/\mathcal{K}}$ is noetherian and is p-torsion free since so is $(\mathbf{A}^0_{/\mathcal{K}} \otimes \mathbb{Q}_p)^{\oplus 2}$. Moreover, given ${}^t(x,y) \in \mathbf{T}'$ and $g \in G_{\mathbb{Q},S}$, we have

$$\boldsymbol{\rho}(g) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(g)x + \Xi(g,\tau)Vp^{-\mu}y \\ \Xi(\sigma,g)x + D(g)y \end{pmatrix} \in \mathbf{T}'$$

by the definition of $\mathfrak{J}$ which ensures $\Xi(\sigma,g)x \in \mathfrak{J}$ and by the definition of μ which ensures $\Xi(g,\tau)Vp^{-\mu} \in \mathbf{A}^0_{/\mathcal{K}}$. Thus, it follows that $\mathbf{T}'$ is Galois stable. Finally, $\mathbf{T}' \otimes \mathbb{Q}_p = (\mathbf{A}^0 \otimes \mathbb{Q}_p)^{\oplus 2}$ because $\mathfrak{J}$ contains the element $\Xi(\sigma,\tau) = p^{-\mu}V$ which becomes a unit after inverting p. Let us define $\mathbf{T}$ to be the double dual $\mathrm{Hom}_{\mathbf{A}^0}(\mathrm{Hom}_{\mathbf{A}^0}(\mathbf{T}', \mathbf{A}^0_{/\mathcal{K}}), \mathbf{A}^0_{/\mathcal{K}})$. We have a canonical $G_{\mathbb{Q},S}$-equivariant $\mathbf{A}^0_{/\mathcal{K}}$-linear injection $\mathbf{T}' \hookrightarrow \mathbf{T}$ with finite cokernel. Since $\mathbf{T}$ is a finitely generated reflexive module over a regular local ring $\mathbf{A}^0_{/\mathcal{K}}$ of Krull dimension two, $\mathbf{T}$ is a free $\mathbf{A}^0_{/\mathcal{K}}$-module of finite rank. Since $\mathbf{T} \otimes \mathbb{Q}_p$ is free of rank two over $\mathbf{A}^0_{/\mathcal{K}} \otimes \mathbb{Q}_p$, the rank of $\mathbf{T}$ over $\mathbf{A}^0_{/\mathcal{K}}$ is two.

As for the last property, recall that an irreducible representation of $G_{\mathbb{Q},S}$ with values in a finite extension of $\mathbb{Q}_p$ is uniquely determined by its trace and determinant. Thus $\mathbf{T}/P_k\mathbf{T}$ is isomorphic to a lattice of the Galois drepresentation ρ_k. □

Note that $\iota : \mathbf{A}^0_{/\mathcal{K}} \hookrightarrow \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ is continuous. In fact, $\mathbf{A}^0_{/\mathcal{K}}$ is local with the maximal ideal $(\frac{T-k_0}{e_0}, \varpi)$ and $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ is endowed with the ϖ-adic topology. To

show that ι is continuous, we need to show that $\iota^{-1}\Big(\varpi\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}\Big)$ contains the maximal ideal $(\frac{T-k_0}{e_0},\varpi)$ of $\mathbf{A}^0_{/\mathcal{K}}$. In fact, this implies that $\iota^{-1}(\varpi^n\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}})$ contains $(\frac{T-k_0}{e_0},\varpi)^n$ for every n. The uniformizer ϖ is clearly contained in $\iota^{-1}(\varpi\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}})$ and $\frac{T-k_0}{e_0}$ is also contained in $\iota^{-1}(\varpi\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}})$ since we have $\frac{T-k_0}{e_0}=\varpi\frac{T-k_0}{e_0\varpi}\in\varpi\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$. By extending the coefficient of the result obtained over $\mathbf{A}^0_{/\mathcal{K}}$ in the above theorem to $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$, we obtain the following corollary:

Corollary 2.13. — *Under the same assumptions and notations as Theorem 2.5 and Proposition 2.11, there exists a free $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$-module $\mathscr{T}$ of rank two with continuous $G_{\mathbb{Q},S}$-action such that the representation*

$$\boldsymbol{\rho}\colon G_{\mathbb{Q},S}\longrightarrow \mathrm{Aut}_{\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}}(\mathscr{T})$$

satisfies $\pi_{\boldsymbol{\rho}}=\pi$ (in a suitable basis). In particular, $\boldsymbol{\rho}$ modulo $\mathfrak{m}_k$ is isomorphic to ρ_k for all $k\in Z_{(k_0,i)}[r_0]$ where $\mathfrak{m}_k$ is the unique maximal ideal of $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ such that $\mathfrak{m}_k\cap\mathbf{A}^0_{/\mathcal{K}}=P_k$.

We stress that every finitely generated module over an affinoid algebra $\mathscr{A}$ will always implicitly be endowed with the quotient topology induced by any finite presentation (which is independent of the presentation, see [BGR84, Proposition 3.7.3/3]) and that every finitely generated, torsion-free module $\mathscr{M}$ over a power-bounded affinoid algebra $\mathscr{A}^0$ will be endowed with the subspace topology induced by its injection into the finitely generated $\mathscr{A}^0[\frac{1}{p}]$-module $\mathscr{M}\otimes\mathbb{Q}_p$.

3. Families of crystals

In this subsection, we define the main relevant module $\mathcal{D}$ interpolating the crystals $D_{\mathrm{cris}}(V_x)$ when V_x varies in our Coleman family, where we write V_x for the Galois representation $\mathscr{V}/\mathfrak{m}_x\mathscr{V}$ at the maximal ideal $\mathfrak{m}_x$ of $\mathscr{A}_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ corresponding to $x\in Z_{(k_0,i)}[r_0]$. This module will serve as domain of definition for our big exponential map which will be constructed in our main theorem.

3.1. Preparation on topological tensor products and lattices. — We need some results from the theory of nonarchimedean functional analysis, for which our basic reference is [Sch02]. In what follows, we only consider $\mathbb{Q}_p$-Banach spaces or, more generally, locally convex topological $\mathbb{Q}_p$-vector spaces, and we sometimes simply refer to them as Banach spaces or locally convex spaces, dropping the base-field $\mathbb{Q}_p$. Given a locally convex space V and a lattice M in V, by which we mean a $\mathbb{Z}_p$-module such that $M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=V$, we can consider its corresponding *gauge*:

$$p_M\colon V\longrightarrow\mathbb{R}_{\geq 0},\quad v\longmapsto\inf\Big\{\,|a|\;\Big|\;a\in\mathbb{Q}_p,\,v\in aM\Big\}.$$

Observe that in the above definition the choice of the base field is crucial. In particular, for every lattice $M \subseteq V$, the semi-norm p_M satisfies $p_M(V) = |\mathbb{Q}_p|$. Given two locally convex spaces V and W, we write $W \otimes_{\mathbb{Q}_p} W$ for the projective tensor product, as in [Sch02, §17.B], and we endow it with the tensor product topology; similarly, we denote by $V\widehat{\otimes}_{\mathbb{Q}_p}W$ the Hausdorff completion of the projective tensor product. As explained after [Sch02, Proposition 17.6], when V and W are Banach spaces, $V \otimes_{\mathbb{Q}_p} W$ is still a metric space whose metric can be defined by the tensor product norm $\| \ \|_{\otimes}$ and its Hausdorff completion $(V\widehat{\otimes}_{\mathbb{Q}_p}W, \| \ \|_{\widehat{\otimes}})$ is again Banach. When we speak of an *isomorpism* for two Banach spaces, we mean an isometric linear bijection.

We use the notation $-\widehat{\otimes}_{\mathbb{Z}_p}-$ to denote the completed tensor product of separated $\mathbb{Z}_p$-modules with respect to their p-adic topologies as defined in [Gro60, Chap. 0, §7.7.1]. That is, we have

$$M\widehat{\otimes}_{\mathbb{Z}_p}N = \varprojlim_{n,m} M/p^m M \otimes_{\mathbb{Z}_p} N/p^n N$$

for separated $\mathbb{Z}_p$-modules M, N. We prepare some basics for later use:

Proposition 3.1. — *1. Let $(V, \| \ \|)$ be a normed locally convex space and let V^0 be its unit ball, namely*

$$V^0 = \{v \in V \mid \|v\| \leq 1\} \ .$$

Then the gauge p_{V^0} coincides with $\| \ \|$ and we have $V^0 = \{v \in V \mid p_{V^0}(v) \leq 1\}$.

2. Let $(V, \| \ \|)$ be a Banach space such that $\|V\| = |\mathbb{Q}_p|$. Then the functor $-\widehat{\otimes}_{\mathbb{Q}_p}V$ on the category of Banach space is exact.

3. Let $(V, \| \ \|_V)$ and $(W, \| \ \|_W)$ be Banach spaces such that $\|V\|_V = \|W\|_W = |\mathbb{Q}_p|$, and let V^0, W^0 be the corresponding unit balls. Then $V^0\widehat{\otimes}_{\mathbb{Z}_p}W^0$ is the unit ball of the Banach space $V\widehat{\otimes}_{\mathbb{Q}_p}W$.

Proof. — Let us prove the first assertion. Recall that [Sch02, Lemma 2.2 (ii)] states that there are inequalities

$$c_0 \cdot p_{(V \otimes_{\mathbb{Q}_p} E)^0} \leq \| \ \| \leq p_{(V \otimes_{\mathbb{Q}_p} E)^0}$$

with $c_0 = \sup\Big\{ |b| \ \Big| \ b \in E, |b| < 1 \Big\}$ for any finite extension of E of $\mathbb{Q}_p$. Because of our convention that all locally convex spaces be defined over the discretely-valued $\mathbb{Q}_p$, we find $c_0 = 1$. Since the norms $\| \ \|$ and p_{V^0} are equal, it is obvious that also the relative unit balls coincide.

Let us prove the second assertion. Concerning the exactness of $-\widehat{\otimes}_{\mathbb{Q}_p}V$ for a Banach space V, the fact that $\mathbb{Q}_p$ is discretely valued allows us to combine Remark 10.2 and Proposition 10.5 of [Sch02] to see that every exact sequence of Banach spaces splits isometrically. This implies the exactness of the functor $-\widehat{\otimes}_{\mathbb{Q}_p}V$.

Let us pass to the proof of the third assertion. We first consider the normed locally convex space $(V \otimes_{\mathbb{Q}_p} W, \| \ \|_{\otimes})$ and we claim that $V^0 \otimes_{\mathbb{Z}_p} W^0$ is the unit ball in $V \otimes_{\mathbb{Q}_p} W$. Let us deduce our conclusion assuming this claim. If a sequence of vectors $x_i \in V \otimes_{\mathbb{Q}_p} W$ tends to a vector $x \in V\widehat{\otimes}_{\mathbb{Q}_p}W$ such that $\|x\|_{\widehat{\otimes}} \leq 1$, then $\|x_i\|_{\otimes} \leq 1$ for all i large enough, since $\|V\widehat{\otimes}_{\mathbb{Q}_p}W\|_{\widehat{\otimes}} = |\mathbb{Q}_p| = p^{\mathbb{Z}} \cup \{0\}$. In other words, a vector

in $V\widehat{\otimes}_{\mathbb{Q}_p}W$ has norm bounded by 1 if and only if it lies in the closure $V^0\widehat{\otimes}_{\mathbb{Z}_p}W^0$ of $V^0\otimes_{\mathbb{Z}_p}W^0$, which is what we want. Thus, it only remains for us to prove the claim, namely

$$V^0\otimes_{\mathbb{Z}_p}W^0=(V\otimes_{\mathbb{Q}_p}W,\|\ \|_\otimes)^0\ . \tag{3.1}$$

Recall the definition of $\|\ \|_\otimes$ as follows:

$$\|u\|_\otimes=\inf\left\{\max\{\|v_i\|_V\cdot\|w_i\|_W\}_{i\le i\le r}\ \middle|\ u=\sum_{i=1}^r v_i\otimes w_i, v_i\in V, w_i\in W\right\}.$$

Clearly, any vector $u\in V^0\otimes_{\mathbb{Z}_p}W^0$ satisfies $\|u\|_\otimes\le 1$ and we pass to the non-trivial inclusion in (3.1). So, pick u in $(V\otimes_{\mathbb{Q}_p}W)^0$ and chose an expression $u=\sum_{1\le i\le r}v_i\otimes w_i$ such that

$$\max\{\|v_i\|\ \|w_i\|\}_{i\le i\le r}\le 1\ .$$

where we suppress the subscripts V,W from the norms for simplicity. By assumption, we can write $\|v_i\|=p^{a_i}$ and $\|w_i\|=p^{b_i}$ such that $a_i+b_i\le 0$ for all $1\le i\le r$. Then $\|p^{a_i}v_i\|=1$ and $\|p^{-a_i}w_i\|=p^{a_i+b_i}\le 1$, namely $p^{a_i}v_i\otimes p^{-a_i}w_i\in V^0\otimes_{\mathbb{Z}_p}W^0$ and

$$u=\sum_{i=1}^r v_i\otimes w_i=\sum_{i=1}^r p^{a_i}v_i\otimes p^{-a_i}w_i\in V^0\otimes_{\mathbb{Z}_p}W^0\ .$$

establishing (3.1). □

In the end of this subsection, we recall some basic facts on lattices and norms which are needed later. Let A be any p-adically separated $\mathbb{Z}_p$-module: we can construct the $\mathbb{Q}_p$-vector space $V_A:=A\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and endow it with the topology induced by the family of lattices $\{p^jA\}_{j\in\mathbb{Z}}$ or, equivalently, the topology induced by p_A. This turns V_A into a locally convex space, which is Hausdorff thanks to the second assertion of [**Sch02**, Proposition 4.6] and p_A is a norm on it thanks to the third assertion of the same proposition. Moreover, V_A is Banach if and only if A is p-adically complete, as follows from [**Sch02**, Lemma 7.17, iii.]. Finally, A is the unit ball inside (V_A,p_A) by definition of p_A.

3.2. p-adic families of crystals for $\mathscr{T}$. — Write A_{cris} and $B^+_{\mathrm{cris}}=A_{\mathrm{cris}}[\frac{1}{p}]$ for Fontaine's period rings (see [**Fon94**]). We want to discuss a structure of Banach space on B^+_{cris}. First of all, Fontaine proves in Section **2.3.3** *ibid.* that A_{cris} is $\mathbb{Z}_p$-module which is p-adically complete and separated and defines B^+_{cris} to be the locally convex space $V_{A_{\mathrm{cris}}}$ in the notation of the previous paragraph. It therefore inherits a structure of a $\mathbb{Q}_p$-Banach space for which A_{cris} is the unit ball and we denote the gauge $p_{A_{\mathrm{cris}}}$ by $\|\ \|_{\mathrm{cris}}$.

Remark 3.2. — In the previous paragraph we insist that B^+_{cris} is a Banach space and A_{cris} is its unit ball ignoring the relation of their topology with the ring structure. This is because, as observed in [**Col98**, Sect. III.2], division by elements in B^+_{cris} is not well-behaved with respect to the natural topology; we could use $A_{\max}$ and $B^+_{\max}$ instead, but since we do not need this and the substitution would require us to modify

some of the arguments in Sections 3 and 4 of [Kis03], we prefer to stick to cris-rings instead of max-rings.

Definition 3.3. — For every $\mathbb{Q}_p$-Banach space V endowed with a continuous action of $G_{\mathbb{Q}_p}$ we set $D^+_{\mathrm{cris}}(V) = H^0(G_{\mathbb{Q}_p}, V\widehat{\otimes}_{\mathbb{Q}_p} B^+_{\mathrm{cris}})$. For every complete $\mathbb{Z}_p$-module T endowed with a continuous action of $G_{\mathbb{Q}_p}$ we set $D^0_{\mathrm{cris}}(T) = H^0(G_{\mathbb{Q}_p}, T\widehat{\otimes}_{\mathbb{Z}_p} A_{\mathrm{cris}})$. Both $D^+_{\mathrm{cris}}(V)$ and $D^0_{\mathrm{cris}}(T)$ are endowed with an operator φ induced by the Frobenius acting on B^+_{cris}.

We recall that, when we take a field of definition K, we have the relation of the base algebras

$$\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r_0]_K} \subset \mathbf{A}^0_K \subset \mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_K}$$

where we have $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r_0]_K} = \mathcal{O}_K\left\langle \frac{T-k_0}{e_0}\right\rangle$, $\mathbf{A}^0_K = \mathcal{O}_K\left[\!\left[\frac{T-k_0}{e_0}\right]\!\right]$ and $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_K} = \mathcal{O}_K\left\langle \frac{T-k_0}{e_0\varpi}\right\rangle$. As in the previous setting, we choose K to be $\mathbb{Q}_p$ and denote $\mathbf{A}^0_{\mathbb{Q}_p}$ by $\mathbf{A}^0$ when there is no fear of confusion. To relate our setting here to the setting of Introduction, we denote $\mathscr{X}_{(k_0,i)}[r]_{\mathbb{Q}_p}$ above by $\mathscr{X}$ and $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]_{\mathbb{Q}_p}}$ by $\mathscr{A}^0_{\mathscr{X}}$. On the other hand, we take a finite extension $\mathcal{K}$ of $\mathbb{Q}_p$ which has nothing to do with K above and which plays a role of the field of coefficients. We use the notation $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$ (resp. $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$) to mean $\mathscr{A}_{\mathscr{X}} \otimes_{\mathbb{Q}_p} \mathcal{K}$ (resp. $\mathscr{A}^0_{\mathscr{X}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{K}}$). Since $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ is isomorphic to $\mathcal{O}_{/\mathcal{K}}\langle T\rangle$, the Gauß norm and the p-adic norm coincide on both $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$ and $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$. By [BGR84, Corollary 5.1.4/6] they also coincide with the sup-norm. It follows that, for all $G \in \mathscr{A}_{\mathscr{X}/\mathcal{K}}$, we have

$$\|G\|_{\text{Gauß}} = \|G\|_{\sup} = p^k \text{ if and only if } p^k G \in \mathscr{A}^0_{\mathscr{X}/\mathcal{K}} \text{ and } p^{k-1}G \notin \mathscr{A}^0_{\mathscr{X}/\mathcal{K}}. \tag{3.2}$$

For the Galois representation $\mathscr{T}$ obtained in Corollary 2.13, we denote the extension $\mathscr{T} \otimes_{\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}} \mathscr{A}_{\mathscr{X}/\mathcal{K}}$ by $\mathscr{V}$. We define

$$\mathcal{D} = D^+_{\mathrm{cris}}(\mathscr{V})^{\varphi=a_p}, \tag{3.3}$$

which is a $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module. We also define

$$\mathcal{D}^0 = D^0_{\mathrm{cris}}(\mathscr{T})^{\varphi=a_p} \tag{3.4}$$

Since $\mathscr{T}$ and A_{cris} are submodules of $\mathscr{V}$ and B^+_{cris}, respectively, we regard $\mathcal{D}^0$ as a submodule of $\mathcal{D}$. By (3.2), the Gauß norm on $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ coincides with the p-adic norm, and $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ is separated and complete with respect to it. It follows that $\mathscr{T}$, which is finite and free over $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$, can be endowed with the p-adic topology with respect to which it is separated and complete (see also comments after Corollary 2.13). By the discussion after Proposition 3.1 above, $\mathscr{V}$ is a Banach space when endowed with the norm $p_{\mathscr{T}}$ for which $\mathscr{T}$ is the unit ball. From now on, we regard $\mathscr{V}$ as a Banach space with the norm $p_{\mathscr{T}} := \| \ \|_{\mathscr{V}}$.

Lemma 3.4. — *We have the following statements for the module $\mathcal{D}$ defined above.*

1. *The module $\mathcal{D}$ is non-zero.*

2. *With the norm induced as a subspace of* $B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V}$, $\mathcal{D}$ *is a Banach space whose unit ball is* $\mathcal{D}^0$. *In particular, the restriction to* $\mathcal{D}$ *of the tensor product norm coincides with the gauge* $p^0_{\mathcal{D}}$ *and* $\mathcal{D} = V^0_{\mathcal{D}}$.

Proof. — We start by showing that [Kis03, Corollary 5.16] implies $\mathcal{D} \neq 0$. We apply this result by taking X, Y, $\mathcal{R}$ and M in [Kis03, Corollary 5.16] to be $X = \mathscr{X}$, $Y = a_p$, $\mathcal{R} = \mathscr{A}_{\mathscr{X}}$ so that $\mathrm{Sp}(\mathcal{R}) = \mathscr{X}$ and $M = \mathscr{V}$. In order to apply the Corollary, we first need to check that $\mathscr{X}_{fs} = \mathscr{X}$. For this, we observe that Kisin proves in [Kis03, proof of Theorem 6.4, p. 408] that $\tilde{\mathcal{C}}_{fs} = \tilde{\mathcal{C}}$. Since we have $\mathscr{X} \subseteq \tilde{\mathcal{C}}$ by construction, the final statement of [Kis03, Proposition 5.4] applies to ensure $\mathscr{X}_{fs} = \mathscr{X}$.

Now, let us denote by $\mathscr{V}^*$ the $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-linear dual $\mathrm{Hom}_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}}(\mathscr{V}, \mathscr{A}_{\mathscr{X}/\mathcal{K}})$. Then, for a fixed natural number k, we define $H \subseteq \mathscr{A}_{\mathscr{X}/\mathcal{K}}$ to be the smallest ideal such that every Galois-equivariant $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$-linear map

$$h\colon \mathscr{V}^* \to \left(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\right)\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{A}_{\mathscr{X}/\mathcal{K}} \tag{3.5}$$

factors through $\left(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\right)\widehat{\otimes}_{\mathbb{Q}_p}H$. The assertion (2) of [Kis03, Corollary 5.16] states that $\mathrm{Sp}(\mathscr{A}_{\mathscr{X}/\mathcal{K}}) \setminus V(H)$ is scheme-theoretically dense in $\mathrm{Sp}(\mathscr{A}_{\mathscr{X}/\mathcal{K}})$ if $k \gg 0$. In particular, we have $H \neq 0$ and there are non-zero Galois-equivariant, $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-linear maps h as above. Now, the assertion (1) of [Kis03, Corollary 5.16] states that each map (3.5) factors through $(B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}/\mathcal{K}})^{\varphi=a_p}$ for k sufficiently large. Now, let us suppose that $H^0\Big(G_{\mathbb{Q}_p}, \mathrm{Hom}_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}}\big(\mathscr{V}^*, (B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}/\mathcal{K}})^{\varphi=a_p}\big)\Big) = 0$. Then, for k sufficiently large so that each map as in (3.5) factors through $(B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}/\mathcal{K}})^{\varphi=a_p}$, H must be zero, which contradicts to the scheme-theoretical density of $\mathrm{Sp}(\mathscr{A}_{\mathscr{X}/\mathcal{K}}) \setminus V(H)$. Thus we showed that

$$H^0\Big(G_{\mathbb{Q}_p}, \mathrm{Hom}_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}}\big(\mathscr{V}^*, (B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}/K})^{\varphi=a_p}\big)\Big) \neq 0. \tag{3.6}$$

Since $\mathscr{V}$ is free of rank two over $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$ by construction, $\mathscr{V}$ is isomorphic to the double $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-linear dual $\mathscr{V}^{**}$ of $\mathscr{V}$. Hence we have the following equalities:

$$\begin{aligned}
\mathcal{D} &= H^0\big(G_{\mathbb{Q}_p}, \mathscr{V}\widehat{\otimes}B^+_{\mathrm{cris}}\big)^{\varphi=a_p}\\
&= H^0\big(G_{\mathbb{Q}_p}, (\mathscr{V}^{**}\widehat{\otimes}B^+_{\mathrm{cris}})^{\varphi=a_p}\big)\\
&= H^0\Big(G_{\mathbb{Q}_p}, \big(\mathrm{Hom}_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}}(\mathscr{V}^*, \mathscr{A}_{\mathscr{X}/\mathcal{K}})\widehat{\otimes}B^+_{\mathrm{cris}}\big)^{\varphi=a_p}\Big)\\
&= H^0\Big(G_{\mathbb{Q}_p}, \big(\mathrm{Hom}_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}}(\mathscr{V}^*, B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}})\big)^{\varphi=a_p}\Big)\\
&= H^0\Big(G_{\mathbb{Q}_p}, \mathrm{Hom}_{\mathscr{A}_{\mathscr{X}}}\big(\mathscr{V}^*, (B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}/\mathcal{K}})^{\varphi=a_p}\big)\Big).
\end{aligned}$$

For the last equality, observe that since $\mathscr{V}^*$ has no φ-action, φ acts on a homomorphism by a_p precisely when it takes values in $(B^+_{\mathrm{cris}}\widehat{\otimes}\mathscr{A}_{\mathscr{X}/\mathcal{K}})^{\varphi=a_p}$. By (3.6), we conclude $\mathcal{D} \neq 0$, which completes the proof of the first assertion.

Now we pass to the second assertion on the Banach properties of $\mathcal{D}$. By the discussion preceding Proposition 3.1, $B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V}$ is a Banach space when endowed with the tensor product norm. Moreover, the third assertion of Proposition 3.1 shows that $A_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Z}_p}\mathscr{T}$ is the unit ball in $B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V}$. Both taking $G_{\mathbb{Q}_p}$-invariants and restricting

to the φ-eigenspace of eigenvalue a_p define closed, and hence complete, subspaces. In particular, $\mathcal{D}$ is Banach and we denote by $\| \ \|_{\mathcal{D}}$ the restriction to $\mathcal{D}$ of the tensor product norm. Note that we have $\|x\|_{\mathcal{D}} \leq 1$ if and only if we have simultaneously, $\|x\|_{\mathscr{V}\otimes B^+_{\mathrm{cris}}} \leq 1$, $g(x) = x$ for all $g \in G_{\mathbb{Q}_p}$ and $\varphi(x) = a_p x$. Thus, by third assertion of Proposition 3.1, we have $\|x\|_{\mathcal{D}} \leq 1$ if and only if we have $x \in \mathcal{D}^0$. This shows that $\mathcal{D}^0$ is the unit ball in $\mathcal{D}$. Finally, [Sch02, Proposition 4.11] shows that the topology of $\mathcal{D}$ can be defined by $\mathcal{D}^0$ because the unit ball is open and bounded in a Banach space and the first assertion of Proposition 3.1 implies that $p^0_{\mathcal{D}} = \| \ \|_{\mathcal{D}}$; the final statement $\mathcal{D} = V^0_{\mathcal{D}}$ is simply the definition of the Banach space $V^0_{\mathcal{D}}$ recalled before Proposition 3.1. □

We always endow $\mathcal{D}$ with a Banach structure making $\mathcal{D}^0$ the unit ball in it, as constructed during the proof of the above lemma. We can now state the main result of this section, which heavily relies on the work [Kis03] of Kisin.

Theorem 3.5. — *We have the following statements for the module $\mathcal{D}^0$ and $\mathcal{D}$ defined previously.*

1. *The modules $\mathcal{D}^0$ and $\mathcal{D}$ are free modules of rank one over $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ and $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$, respectively.*
2. *Under the assumption* (DoubRt)*, the vector space $\mathcal{D}/\mathfrak{m}_x\mathcal{D}$ coincides with $D^+_{\mathrm{cris}}(V_x)^{\varphi=a_p(x)}$ for all $x \in Z_{(k_0,i)}[r_0]$.*

Proof. — By [Kis03, Proposition 2.5] (with $K = E = \mathbb{Q}_p$ and $M = \mathscr{V}$), the $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module $H^0(\mathbb{Q}_p, B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \widehat{\otimes} \mathscr{V})$ is finitely generated for all $j \geq 0$. Since $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$ is a_p-small, we can invoke [Kis03, Corollary 3.7]. By tensoring the injection (3.7.1) of [Kis03] with the free $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module $\mathscr{V}$, we have a Galois equivariant injection

$$\left(B^+_{\mathrm{cris}} \widehat{\otimes} \mathscr{V}\right)^{\varphi=a_p} \hookrightarrow \left(B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}}\right) \widehat{\otimes} \mathscr{V} \tag{3.7}$$

for sufficiently large j thanks to the second assertion of Proposition 3.1. Taking Galois invariants of this injection, $\mathcal{D}$ is a submodule of the finitely generated $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module $H^0(\mathbb{Q}_p, B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \widehat{\otimes} \mathscr{V})$. Therefore $\mathcal{D}$ is a finitely generated $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module because $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$ is noetherian. We next claim that the module $\mathcal{D}$ is also torsion-free over $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$. In order to check this, we recall that we have an $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-linear inclusion $\mathcal{D} \hookrightarrow \mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}$: we thus need only to check that $\mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}$ is $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-torsion-free. Since flatness implies torsion-freeness over every commutative ring, we would rather check that $\mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}$ is $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-flat. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{3.8}$$

be a short exact sequence of $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module. Since every $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module is an injective limit of finitely generated $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-modules and the tensor product commutes with injective limits, we can assume that M'' is a finitely generated $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module. Taking the tensor product of (3.8) with $\mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}$ over $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$, we obtain

$$M' \otimes_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}} (\mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}) \longrightarrow M \otimes_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}} (\mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}) \longrightarrow M'' \otimes_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}} (\mathscr{V} \widehat{\otimes} B^+_{\mathrm{cris}}) \longrightarrow 0.$$

Since $\mathscr{V}$ is free of rank two over $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$, the above sequence is simply

$$M'^{\oplus 2}\,\widehat{\otimes} B^+_{\mathrm{cris}} \longrightarrow M^{\oplus 2}\,\widehat{\otimes} B^+_{\mathrm{cris}} \longrightarrow M''^{\oplus 2}\,\widehat{\otimes} B^+_{\mathrm{cris}} \longrightarrow 0 \tag{3.9}$$

which is the application of the functor $-\widehat{\otimes} B^+_{\mathrm{cris}}$ to the short exact sequence (3.8) of $\mathbb{Q}_p$-vector spaces. Having assumed that M'' is finitely generated, it follows from [BGR84, §3.7, Proposition 3] that $M''^{\oplus 2}$ is a $\mathbb{Q}_p$-Banach space. By the second assertion of Proposition 3.1, the functor $-\widehat{\otimes} B^+_{\mathrm{cris}}$ is exact restricted to $\mathbb{Q}_p$-Banach spaces, so the first arrow in (3.9) must be injective. The flatness of $\mathscr{V}\widehat{\otimes} B^+_{\mathrm{cris}}$ follows.

Hence, $\mathcal{D}$ is a free $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module of finite rank. Moreover, the Banach space $(\mathcal{D}, \|\ \|_{\mathcal{D}})$ is a cartesian $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module. In order to show this, we need to check that we have $p^0_{\mathcal{D}}(Gv) = \|G\|_{\sup} \cdot p^0_{\mathcal{D}}(v)$ for all $v \in \mathcal{D}$ and all $G \in \mathscr{A}_{\mathscr{X}/\mathcal{K}}$ thanks to Lemma 3.4, which follows from the definition of the gauge $p^0_{\mathcal{D}}$ using (3.2). It follows from the equality $(\mathcal{D}, \|\ \|_{\mathcal{D}})^0 = \mathcal{D}^0$ of Lemma 3.4 that $\mathcal{D}^0$ is a free $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$-module of rank equal to $\mathrm{rk}_{\mathscr{A}_{\mathscr{X}/\mathcal{K}}}(\mathcal{D})$.

Fix now $x \in \mathscr{X}$ and let $\mathfrak{m}_x$ be the corresponding maximal ideal of $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$. By [BGR84, Proposition 7.1.1/3], this ideal is principal and we fix a generator π_x, insisting that it lie in $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$. The $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module $\mathscr{V}$ is free, so multiplication by π_x is injective and by taking complete tensor product of the tautological exact sequence given by multiplication by π_x with the Banach space B^+_{cris}, we find the short exact sequence by the second assertion of Proposition 3.1:

$$0 \to B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V} \xrightarrow{\pi_x\otimes 1} B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V} \to B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_x \to 0\,.$$

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V} \arrow[r, "\pi_x\otimes 1"] \arrow[d, "\varphi-a_p"] & B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V} \arrow[r] \arrow[d, "\varphi-a_p"] & B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_x \arrow[r] \arrow[d, "\varphi-a_p(x)"] & 0 \\
0 \arrow[r] & B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V} \arrow[r, "\pi_x\otimes 1"] & B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V} \arrow[r] & B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_x \arrow[r] & 0\,.
\end{tikzcd}$$

By the snake lemma, we deduce a left exact sequence

$$0 \longrightarrow (B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V})^{\varphi=a_p} \longrightarrow (B^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{V})^{\varphi=a_p} \longrightarrow (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_x)^{\varphi=a_p(x)}.$$

By taking $G_{\mathbb{Q}_p}$-invariants of this short exact sequence, we find an exact sequence:

$$0 \longrightarrow \mathcal{D} \xrightarrow{\pi_x} \mathcal{D} \longrightarrow D^+_{\mathrm{cris}}(V_x)^{\varphi=a_p(x)}$$

hence an injection:

$$\mathcal{D}/\mathfrak{m}_x\mathcal{D} \hookrightarrow D^+_{\mathrm{cris}}(V_x)^{\varphi=a_p(x)}. \tag{3.10}$$

By Lemma 3.4, $\mathcal{D}$ is a non-trivial finitely generated module over $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$. Since π_x is not a unit in $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$, the $\mathscr{A}_{\mathscr{X}/\mathcal{K}}/\mathfrak{m}_x$-vector space $\mathcal{D}/\mathfrak{m}_x\mathcal{D}$ is non-zero. On the other hand, the dimension of the $\mathscr{A}_{\mathscr{X}/\mathcal{K}}/\mathfrak{m}_x$-vector space $D^+_{\mathrm{cris}}(V_x)$ is at most two and it is equal to two only when V_x is crystalline. If the weight k_x of the overconvergent form f_x is not an integer, the Hodge–Tate weight of the determinant representation $\wedge V_x$ is not an integer. Hence $\wedge V_x$ is not crystalline. Consequently, ρ_x is not crystalline if

k_x is not an integer. Using (3.10) for $k_x \notin \mathbb{Z}$, we conclude that the rank of the free $\mathscr{A}_{\mathscr{X}/\mathcal{K}}$-module $\mathcal{D}$ is at most one, which completes the proof of the first assertion.

For $k_x \notin \mathbb{Z}$, $D^+_{\mathrm{cris}}(V_x)^{\varphi=a_p(x)} \subseteq D^+_{\mathrm{cris}}(V_x)$ is of dimension one, which implies that (3.10) is an isomorphism. Now, suppose that the weight k_x of the overconvergent form f_x is an integer. If ρ_x is not crystalline, (3.10) must be an isomorphism by the same reason as above. Even if k_x is an integer and ρ_x is crystalline, $D^+_{\mathrm{cris}}(V_x)^{\varphi=a_p(x)}$ is of dimension one unless the characteristic polynomial for φ has a double root equal to $a_p(x)$. Thus (3.10) is an isomorphism since $a_p(x)$ is not a double root of the characteristic polynomial of φ acting on $D^+_{\mathrm{cris}}(V_x)$. This completes the proof of the second assertion. □

We have the following corollary of Theorem 3.5:

Corollary 3.6. — *Under the same assumptions as Theorem 3.5, we have a free $\mathbf{A}^0_{/\mathcal{K}}$-module $\mathbf{D}^0$ of rank one which satisfies the following properties.*

(i) *We have a natural $\mathbf{A}^0_{/\mathcal{K}}$-linear injection $\mathbf{D}^0 \hookrightarrow \mathcal{D}^0$ and we recover $\mathcal{D}^0$ from $\mathbf{D}^0$ in the sense that $\mathbf{D}^0 \otimes_{\mathbf{A}^0_{/\mathcal{K}}} \mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ is canonically isomorphic to $\mathcal{D}^0$.*

(ii) *At each $x \in Z_{(k_0,i)}[r_0]$, the natural map $\mathbf{D}^0/(\mathfrak{m}_x \cap \mathbf{A}^0_{/\mathcal{K}})\mathbf{D}^0 \longrightarrow \mathcal{D}^0/(\mathfrak{m}_x \cap \mathscr{A}^0_{\mathscr{X}/\mathcal{K}})\mathcal{D}^0$ is an isomorphism and the image of the specialization map*

$$x : \mathcal{D}^0 \longrightarrow D_x = D_{\mathrm{cris}}(V_x)^{\varphi=a_p(x)}.$$

gives a lattice of D_x.

Proof. — We apply the same arguments as those employed in §2.3 to construct a formal $\mathbf{A}^0_{/\mathcal{K}}$-structure of a given Coleman family over $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$. For each radius r in (2.4), we have a free crystal over $\mathscr{A}^0_{\mathscr{X}_{(k_0,i)}[r]/\mathcal{K}}$ obtained in Theorem 3.5. Then, we obtain a formal $\mathbf{A}^0_{/\mathcal{K}}$-structure of a given family of crystals over $\mathscr{A}^0_{\mathscr{X}/\mathcal{K}}$ by taking the limit with respect to r. This completes the proof. □

4. Functions of bounded logarithmic order

4.1. Valuations on functions with logarithmic growth. — In this section, we prepare basic materials on certain modules of functions defined on the disk $B(0,1)_{\mathbb{C}_p}$ over $\mathbb{C}_p$.

We normalize the valuation ord on $\mathbb{C}_p$ by $\mathrm{ord}(p) = 1$. We write $\lfloor x \rfloor$ for the largest integer equal to or smaller than x, which is sometimes called the floor function. We denote by ln the natural logarithm in base e. Our main reference for this section is [Col10].

Definition 4.1. — For $i \in \mathbb{Z}_{\geq 1}$ we set $\ell(i)$ to be the smallest integer j satisfying $p^j > i$. Equivalently, we define as follows for $i \geq 1$:

$$\ell(i) = \left\lfloor \frac{\ln(i)}{\ln(p)} \right\rfloor + 1.$$

For $i = 0$, we set $\ell(0) = 0$.

We now introduce an important class of power series in $\mathbb{Q}_p[\![X]\!]$ as follows:

Definition 4.2. — For $h \geq 0$, we define

$$\mathcal{H}_h = \left\{ \sum_{i=0}^{+\infty} a_i X^i \in \mathbb{Q}_p[\![X]\!] \;\middle|\; \inf\left\{ \operatorname{ord}(a_i) + h\ell(i) \mid i \in \mathbb{Z}_{\geq 0} \right\} > -\infty \right\}$$

and call elements of $\mathcal{H}_h$ power series of logarithmic order h.

Since the denominators of the coefficients of elements in $\mathcal{H}_h$ cannot grow faster than $h\ell(i)$, which has logarithmic growth, we see that for every $h \in \mathbb{Z}$, series in $\mathcal{H}_h$ converge on the open unit disk $B(0,1)_{\mathbb{C}_p}$ inside $\mathbb{C}_p$.

Definition 4.3. — Let V be a $\mathbb{Q}_p$-vector space. A function $v : V \to \mathbb{R} \cup \{+\infty\}$ is called a valuation if

(i) $v(x) = +\infty$ if and only if $x = 0$;
(ii) $v(x+y) \geq \min\big(v(x), v(y)\big)$ for every $x, y \in V$;
(iii) $v(\lambda \cdot x) = \operatorname{ord}(\lambda) + v(x)$ for every $\lambda \in \mathbb{Q}_p$ and every $x \in V$.

We say that two valuations v, v' are equivalent if there exist constants C, C' such that $C + v' \leq v \leq C' + v'$. Finally, we call a function v as above satisfying only (ii) and (iii), and such that $v(0) = +\infty$, a semi-valuation.

We want to endow $\mathcal{H}_h$ and $\mathcal{H}_h^+$ with some topological structure. In fact, there are two choices: one can either use one of the two equivalent valuations v_h or v'_h defined below.

Definition 4.4. —

1. For every $\nu \in \mathbb{Z}$, we define $B_{\mathrm{ord}}(\nu)$ to be:

$$B_{\mathrm{ord}}(\nu) := B[0, p^{-p^{-\nu}}]_{\mathbb{C}_p}.$$

We recall that, as defined in Section 2.1, $B[0, p^{-p^{-\nu}}]_{\mathbb{C}_p}$ is the closed ball in $\mathbb{C}_p$:

$$B[0, p^{-p^{-\nu}}]_{\mathbb{C}_p} = \left\{ x \in \mathbb{C}_p \;\middle|\; |x| \leq p^{-p^{-\nu}} \right\} = \left\{ x \in \mathbb{C}_p \mid \operatorname{ord}(x) \geq p^{-\nu} \right\}.$$

2. We define the $\mathbb{Q}_p$-subspace $\operatorname{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ of $\mathbb{Q}_p[\![X]\!]$ to be:

$$\operatorname{Conv}\big(B(0,1)_{\mathbb{C}_p}\big) := \left\{ F(X) \in \mathbb{Q}_p[\![X]\!] \;\middle|\; F(X) \text{ is convergent on } B(0,1)_{\mathbb{C}_p} \right\}$$

which we endow with the topology defined by the family of semi-valuations

$$q_\nu(F) := \inf\left\{ \operatorname{ord}\big(F(z)\big) \right\}_{z \in B_{\mathrm{ord}}(\nu)}$$

for $\nu \in \mathbb{Z}$.

With the above topology, $\operatorname{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ is a Fréchet space, as proven for instance in [Kra46]. Let h be arbitrary non-negative integer. As explained after Definition 4.2, any $F(X) \in \mathcal{H}_h$ is convergent on $B(0,1)_{\mathbb{C}_p}$ and we regard $\mathcal{H}_h$ as a $\mathbb{Q}_p$-subspace of $\operatorname{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$. When we do not mention the topology on $\mathcal{H}_h$ explicitly, we consider $\mathcal{H}_h \subseteq \operatorname{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ endowed with the subspace topology induced from the family of q_ν's.

Definition 4.5. — 1. We define the function v_h on $\mathbb{Q}_p[\![X]\!]$ with values in $\mathbb{R}\cup\{\pm\infty\}$ by assigning $v_h(F)$ for $F=\sum_{i=0}^{+\infty}a_iX^i\in\mathbb{Q}_p[\![X]\!]$ as follows:

$$v_h(F)=\inf\left\{\mathrm{ord}(a_i)+h\ell(i)\;\middle|\; i\in\mathbb{Z}_{\geq 0}\right\}.$$

2. Let us fix $h\in\mathbb{Z}_{\geq 0}$. We define the function v'_h on $\mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ with values in $\mathbb{R}\cup\{\pm\infty\}$ by assigning $v'_h(F)$ for $F=\sum_{i=0}^{+\infty}a_iX^i\in\mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ as follows:

$$v'_h(F)=\inf\left\{\inf\{\mathrm{ord}(F(z))\mid z\in\mathbb{C}_p,\,\mathrm{ord}(z)\geq p^{-\nu}\}+h\nu\;\middle|\;\nu\in\mathbb{Z}_{\geq 1}\right\}.$$

By definition, v_h is a valuation on $\mathcal{H}_h$ and we show in Proposition 4.10 below that the same is true for v'_h. The internal inf in the definition of v'_h is nothing but the inf-valuation for analytic functions of order ν, namely those converging on every closed disk $B[0,p^{-p^{-\nu}}]_{\mathbb{C}_p}$. We remark that this definition of v'_h differs slightly from Colmez' in the normalization.

Using the function v_h, we can define an "integral structure" as follows:

Definition 4.6. — Given an integer $h\in\mathbb{Z}_{\geq 0}$, we put

$$\mathcal{H}_h^+=\left\{F\in\mathcal{H}_h\middle|v_h(F)\geq 0\right\}.$$

By definition, $\mathcal{H}_0$ is the ring of power series with bounded denominators and $\mathcal{H}_0^+$ is the ring of power series all of whose coefficients are integral. Note that we have

$$\mathcal{H}_0=\mathbb{Z}_p[\![X]\!]\otimes\mathbb{Q}_p\supset\mathbb{Z}_p[\![X]\!]=\mathcal{H}_0^+$$

and the function v_0 coincides with the p-adic valuation. That is, for $F\in\mathbb{Z}_p[\![X]\!]\otimes\mathbb{Q}_p$, $v_0(F)$ is the largest integer i such that $p^{-i}F\in\mathbb{Z}_p[\![X]\!]$.

Remark 4.7. — The above observation shows that $\mathcal{H}_h^+$ (resp. $\mathcal{H}_h$) is an $\mathcal{H}_0^+$-module (resp. $\mathcal{H}_0$-module), but it is not a ring unless $h=0$. See Lemma 4.11 for instances to see that $\mathcal{H}_h^+$ is not a ring for $h>0$. In particular, the equality $v_h(F\cdot G)=v_h(F)+v_h(G)$ does not hold in general, but we have the following Lemma.

Lemma 4.8. — *Let $h,h'\geq 0$ be two integers. Given two power series $F=\sum_{i=0}^{+\infty}a_iX^i\in\mathcal{H}_h$ and $G=\sum_{j=0}^{+\infty}b_jX^j\in\mathcal{H}_{h'}$, we have $F\cdot G\in\mathcal{H}_{h+h'}$. Moreover, if $F\in\mathcal{H}_h^+$ and $G\in\mathcal{H}_{h'}^+$, then $F\cdot G\in\mathcal{H}_{h+h'}^+$.*

Proof. — Writing $F\cdot G=\sum_{k=0}^{+\infty}c_kX^k$, we have $v_{h+h'}(F\cdot G)=\inf\{\mathrm{ord}(c_k)+(h+h')\ell(k)\mid k\geq 0\}$. For every $k\in\mathbb{Z}_{\geq 0}$, the presentation $c_k=\sum_{i+j=k}a_ib_j$ implies

$$\mathrm{ord}(c_k)\geq\min\{\mathrm{ord}(a_i)+\mathrm{ord}(b_j)\mid i+j=k\}.$$

We thus obtain

$$\begin{aligned}
&\mathrm{ord}(c_k) + (h+h')\ell(k) \\
&\geq \min\{\mathrm{ord}(a_i) + h\ell(k) + \mathrm{ord}(b_j) + h'\ell(k) \mid i+j=k\} \\
&= \min\{\mathrm{ord}(a_i) + h\ell(i) \mid i \in \mathbb{Z}_{\geq 0}\} + \min\left\{\mathrm{ord}(b_j) + h'\ell(j) \mid j \in \mathbb{Z}_{\geq 0}\right\} \\
&= v_h(F) + v_{h'}(G).
\end{aligned}$$

Therefore we have proven that

$$v_{h+h'}(F \cdot G) \geq v_h(F) + v_{h'}(G). \tag{4.1}$$

The lemma follows. □

Remark 4.9. — The module $\mathcal{H}_h^+$ is not finitely generated as an $\mathcal{H}_0^+$-module when $h \in \mathbb{Z}_{\geq 1}$. Let us consider the following module:

$$\mathcal{H}_h^{\geq d} = \left\{ \sum_{n=0}^{+\infty} a_n X^n \in \mathcal{H}_h^+ \;\middle|\; \mathrm{ord}(a_i) \geq -h\ell(d) \text{ for all } i \geq d \right\}.$$

Each $\mathcal{H}_h^{\geq d}$ is a sub-$\mathcal{H}_0^+$-module of $\mathcal{H}_h^+$ and they form a non-stationary increasing sequence in the sense that, for any $d \in \mathbb{Z}_{\geq 0}$, there exists an integer $d' > d$ such that $\mathcal{H}_h^{\geq d} \subsetneqq \mathcal{H}_h^{\geq d'}$. To see this, observe that for any d there exists an integer $d' > d$ such that $\ell(d) < \ell(d')$. Then we see that the element $p^{-\ell(d')h} X^{d'}$ lies in $\mathcal{H}_h^{\geq d'}$ but not in $\mathcal{H}_h^{\geq d}$. It follows that $\mathcal{H}_h^+$ is not a noetherian $\mathcal{H}_0^+$-module, and hence not finitely generated over the noetherian ring $\mathcal{H}_0^+$.

Another interesting, and at first sight strange, feature of $\mathcal{H}_h^+$ is that, when $h > 0$, it is not *factorial* in the sense of [Lu77], by which we mean that there exists elements $F \in p\mathcal{H}_h^+ \cap X\mathcal{H}_h^+$ such that $F \notin pX\mathcal{H}_h^+$. For example, when $h = 1$, we have $\dfrac{X}{p} \in \mathcal{H}_1^+$ since $v_1(X) = 1$, hence we have $X \in p\mathcal{H}_1^+ \cap X\mathcal{H}_1^+$. Nevertheless, we have $X \notin pX\mathcal{H}_1^+$ since this would force $p^{-1} \in \mathcal{H}_1^+$ which is a false statement. Note that $\mathcal{H}_1^+/X\mathcal{H}_1^+$ is *not* torsion-free as a $\mathbb{Z}_p$-module since the class of $\dfrac{X}{p} \in \mathcal{H}_1^+$ gives a non-trivial p-torsion element.

Let us define non-negative constants c_1 and c_2 depending on h as follows:

$$c_1 = \min\left\{ h, -h + \frac{h}{\ln(p)} \left(1 + \ln\left(\frac{\ln(p)}{h}\right)\right) \right\} \tag{4.2}$$

$$c_2 = \max\{h, p-h\}. \tag{4.3}$$

Proposition 4.10. — *Let $h \in \mathbb{Z}_{\geq 0}$.*

1. *For any $F \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$, $v_h(F) > -\infty$ holds if and only if $v'_h(F) > -\infty$ holds.*
2. *The function v'_h is a valuation on $\mathcal{H}_h$ and we have the relation:*

$$c_1 + v_h(F) \leq v'_h(F) \leq c_2 + v_h(F)$$

for any $f \in \mathcal{H}_h$. In particular, the valuation v'_h is equivalent to v_h on $\mathcal{H}_h$.

Proof. — For the proof, we mimic [Col10, Lemme II.1.1] combined with Proposition I.4.3, *ibid.*. For each $i \in \mathbb{Z}_{\geq 1}$, we define the function $\beta_{h,i}$ by

$$\beta_{h,i}(x) = ip^{-x} + hx \tag{4.4}$$

on $\mathbb{R}_{\geq 0}$. Its derivative vanishes at $x_0 = \dfrac{1}{\ln(p)} \ln\left(\frac{i\ln(p)}{h}\right)$ and $\beta_{h,i}$ has the minimal value

$$\beta_{h,i}(x_0) = \frac{h\ln(i)}{\ln(p)} + \frac{h}{\ln(p)}\left(1 + \ln\left(\frac{\ln(p)}{h}\right)\right)$$

at $x = x_0$. We have $h\ell(i) = h\left\lfloor \frac{\ln(i)}{\ln(p)} \right\rfloor + h$ by Definition 4.1. Hence we have the following lower bound for each $i \in \mathbb{Z}_{\geq 1}$:

$$c_1 + h\ell(i) \leq \beta_{h,i}(x_0). \tag{4.5}$$

Since x_0 is the minimal point of the function $\beta_{h,i}(x)$, we see that

$$\beta_{h,i}(x_0) \leq \inf\left\{\beta_{h,i}(\nu) \mid \nu \in \mathbb{Z}_{\geq 1}\right\}. \tag{4.6}$$

By the definition of $\beta_{h,i}$ (see (4.4)) and the definition of ℓ (see Definition 4.1), we see that

$$\inf\left\{\beta_{h,i}(\nu) \mid \nu \in \mathbb{Z}_{\geq 1}\right\} \leq \beta_{h,i}(\ell(i)) \leq p + h\ell(i). \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we obtain:

$$c_1 + h\ell(i) \leq \inf\left\{\beta_{h,i}(\nu) \mid \nu \in \mathbb{Z}_{\geq 1}\right\} \leq p + h\ell(i)$$

for each $i \in \mathbb{Z}_{\geq 1}$.

Now, let us take an element $F = \sum_{i=0}^{+\infty} a_i X^i \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$. Adding $\mathrm{ord}(a_i)$ to the above inequalities we obtain

$$c_1 + \mathrm{ord}(a_i) + h\ell(i) \leq \inf\left\{\beta_{h,i}(\nu) + \mathrm{ord}(a_i) \mid \nu \in \mathbb{Z}_{\geq 1}\right\} \leq c_2 + \mathrm{ord}(a_i) + h\ell(i)$$

for each $i \in \mathbb{Z}_{\geq 1}$. Rewriting the values $\beta_{h,i}(\nu)$ according to the definition of the function $\beta_{h,i}$, we obtain:

$$c_1 + \mathrm{ord}(a_i) + h\ell(i) \leq \inf\left\{\mathrm{ord}(a_i) + ip^{-\nu} + h\nu \;\middle|\; \nu \in \mathbb{Z}_{\geq 1}\right\} \leq c_2 + \mathrm{ord}(a_i) + h\ell(i) \tag{4.8}$$

for each $i \in \mathbb{Z}_{\geq 1}$. Since we have $\ell(0) = 0$ by definition, the same inequality as (4.8) holds true also for $i = 0$. By [Col10, Proposition I.4.3], the infimum of the middle term of (4.8) over $i \in \mathbb{Z}_{\geq 0}$ is equal to $v'_h(F)$. By definition, the infimum of the term $\mathrm{ord}(a_i) + h\ell(i)$ of (4.8) over $i \in \mathbb{Z}_{\geq 0}$ is equal to $v_h(F)$. Hence, taking the infimum over $i \in \mathbb{Z}_{\geq 0}$ of the inequalities (4.8), we obtain the desired inequalities $c_1 + v_h(F) \leq v'_h(F) \leq c_2 + v_h(F)$. This completes the proof. □

Let us define

$$\log(1+X) = -\sum_{i=1}^{+\infty} \frac{(-X)^i}{i}.$$

Lemma 4.11. — *The following statements hold:*

1. *For all $h > 0$, we have $v_h\big(\log(1+X)\big) = h$, hence $\log(1+X) \in \mathcal{H}_h^+$.*

2. *Given an integer $d \geq 1$, we have*

$$v'_h\Big(\log^d(1+X)\Big) > -\infty \quad \textit{if and only if} \quad d \leq h.$$

Further, we have $v'_h\big(\log^d(1+X)\big) = h$ for $d \leq h$.

Proof. — We prove the first assertion. For $i \geq 1$, the i-th coefficient of $\log(1+X)$ is $(-1)^{-i+1}i^{-1}$. We have $\mathrm{ord}(a_i) = -\mathrm{ord}(i)$, while $\mathrm{ord}(a_0) = +\infty$. Hence, we restrict to $i \geq 1$ from now on. From the inequality $\mathrm{ord}(i) \leq \ell(i) - 1$ discussed in Definition 4.1, we find $\mathrm{ord}(a_i) + h\ell(i) = -\mathrm{ord}(i) + h\ell(i) \geq (h-1)\ell(i) + 1$. Since $h > 0$, the function $(h-1)x+1$ on $\mathbb{R}_{\geq 1}$ is non-decreasing and its minimum over $\mathbb{Z}_{\geq 1}$ is attained at $x = 1$ where the value is h; as $\mathrm{ord}(a_1) + h\ell(1) = h$. By Definition 4.5, this completes the proof of $v_h\big(\log(1+X)\big) = h$.

Let us prove the second assertion. Recall that, if $z \in \mathbb{C}_p$ satisfies $\mathrm{ord}(z) = p^{-\nu}$ for some $\nu \in \mathbb{Z}_{\geq 1}$, we have $\mathrm{ord}(\log(1+z)) = 1 - \nu$ (see [DGS94, Proposition II.1.1]). Applying the Maximum Principle, we have:

$$\begin{aligned}\inf\left\{\mathrm{ord}\left(\log(1+z)\right) \;\middle|\; \mathrm{ord}(z) \geq p^{-\nu}\right\} &= \inf\left\{\mathrm{ord}\left(\log(1+z)\right) \;\middle|\; \mathrm{ord}(z) = p^{-\nu}\right\} \\ &= 1 - \nu\end{aligned}$$

for all $\nu \in \mathbb{Z}_{\geq 1}$. Since the map ord is multiplicative, we deduce

$$\inf\left\{\mathrm{ord}\left(\log^d(1+z)\right) \;\middle|\; \mathrm{ord}(z) \geq p^{-\nu}\right\} = d - d\nu. \tag{4.9}$$

By Definition 4.5, we have:

$$v'_h\big(\log^d(1+X)\big) = \inf\left\{d + (h-d)\nu \;\middle|\; \nu \in \mathbb{Z}_{\geq 1}\right\}.$$

When $d > h$, the function $x \mapsto (h-d)x + d$ for $x \in \mathbb{R}_{\geq 1}$ is unbounded from below as $x \to +\infty$. When $d \leq h$, the function has its minimum at $x = 1$, where it equals h. This completes the proof of the second statement. □

Corollary 4.12. — *Let $h \in \mathbb{Z}_{\geq 0}$. The logarithm $\log^d(1+X)$ belongs to $\mathcal{H}_h$ if and only if $d \leq h$. Further, $\log^d(1+X)$ belongs to $p^{h-c_2}\mathcal{H}_h^+ = p^{\min\{0,2h-p\}}\mathcal{H}_h^+$ when $d \leq h$.*

The difference between Lemma 4.11 and Corollary 4.12 is that the integral structure $\mathcal{H}_h^+$ inside $\mathcal{H}_h$ is defined in terms of the valuation v_h instead of v'_h which appears in Lemma 4.11.

Proof. — By Definition 4.2 and Definition 4.5, $\log^d(1+X)$ belongs to $\mathcal{H}_h$ if and only if $v_h(\log^d(1+X)) > -\infty$. Hence the first statement of the corollary follows from the second statement of Lemma 4.11. Let us prove the second statement. By Proposition 4.10 and Lemma 4.11, we have

$$v_h\big(\log^d(1+X)\big) \geq h - c_2.$$

Since we have $c_2 = \max\{h, p-h\}$, we obtain $h - c_2 = \min\{0, 2h-p\}$. This completes the proof. □

In Definition 4.2 we called elements in $\mathcal{H}_h$ power series of "logarithmic order h". Thanks to the above corollary, we can prove the following result which justifies the name. To state it, we recall that given two power series $F, G \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ the notation $F = O(G)$ is defined to be

$$\inf_{\nu \geq 0}\Big\{q_\nu(F) - q_\nu(G)\Big\} > -\infty.$$

Proposition 4.13. — *Let $F \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$. Then $F \in \mathcal{H}_h$ if and only if $F = O\big(\log^h(1+X)\big)$.*

Proof. — By equation (4.9) we know that $q_\nu\big(\log^h(1+X)\big) = h - h\nu$. Therefore, given $F \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ we find

$$\begin{aligned}\inf_{\nu \geq 0}\Big\{q_\nu(F) - q_\nu\big(\log^h(1+X)\big)\Big\} &= \inf_{\nu \geq 0}\Big\{q_\nu(F) - h + h\nu\Big\} \\ &= \inf_{\nu \geq 0}\Big\{q_\nu(F) + h\nu\Big\} - h \\ &= v'_h(F) - h.\end{aligned}$$

It follows that the left-hand side of the above equation is bounded from below if and only if $v_h(F') > -\infty$, which is equivalent to $F \in \mathcal{H}_h$. □

4.2. Topologies on functions with logarithmic growth. — In this section, we continue to study topological structure of $\mathcal{H}_h$ and $\mathcal{H}_h^+$. Our main source about topological $\mathbb{Q}_p$-vector spaces is [Sch02]. Our definition of $\mathcal{H}_h$ and $\mathcal{H}_h^+$ are different from that of [PR94, Section 2], but what follows below offers a refinement of [PR94, Section 2] in the sense that the constants appearing in the calculation are all calculated explicitly even for those whose corresponding constants in [PR94, Section 2] are left ambiguous.

Recall that we consider the topology on $\mathcal{H}_h$ induced by either v_h or v'_h: they induce different metric structures but the same topology, thanks to Proposition 4.10. For our purposes, only the metric structure attached to v_h will play a role. When we want to consider $\mathcal{H}_h$ and $\mathcal{H}_h^+$ endowed with the metric topology induced by v_h we write $(\mathcal{H}_h, v_h)$ and $(\mathcal{H}_h^+, v_h)$, respectively.

Theorem 4.14. — *The space $(\mathcal{H}_h, v_h)$ is a Banach space.*

Proof. — It suffices to show that $\mathcal{H}_h$ is complete with respect to the topology defined by the valuation v_h.

Let us take any Cauchy sequence:

$$\left\{F^{(m)} = \sum_{i=0}^{+\infty} a_i^{(m)} X^i\right\}_{m \geq 1}$$

for the topology defined by v_h. By Definition 4.5, for each $N > 0$, there exists a constant $M(N)$ such that the inequality

$$\inf\Big\{\mathrm{ord}(a_i^{(m)} - a_i^{(m')}) + h\ell(i) \;\Big|\; i \in \mathbb{Z}_{\geq 0}\Big\} > N \tag{4.10}$$

holds for every $m, m' \geq M(N)$. Thus, for every $i \geq 0$, the sequence $\{a_i^{(m)}\}_{m\geq 1}$ is Cauchy in the p-adic topology of $\mathbb{Q}_p$. Writing $a_i = \lim_{m\to+\infty} a_i^{(m)}$, we set

$$F = \sum_{i=0}^{+\infty} a_i X^i \in \mathbb{Q}_p[\![X]\!]$$

and we claim $F \in \mathcal{H}_h$. Since we have $\mathrm{ord}(a_i) \geq \min\{\mathrm{ord}\left(a_i - a_i^{(m)}\right), \mathrm{ord}(a_i^{(m)})\}$ for every $i \geq 0$, it follows by the definition of v_h that

$$\begin{aligned} v_h(F) &= \inf\{\mathrm{ord}(a_i) + h\ell(i)\}_{i\in\mathbb{Z}_{\geq 0}} \\ &\geq \min\left\{\inf\left\{\mathrm{ord}(a_i - a_i^{(m)}) + h\ell(i)\right\}_{i\in\mathbb{Z}_{\geq 0}}, \inf\left\{\mathrm{ord}(a_i^{(m)}) + h\ell(i)\right\}_{i\in\mathbb{Z}_{\geq 0}}\right\} \\ &= \min\left\{\inf\left\{\mathrm{ord}(a_i - a_i^{(m)}) + h\ell(i)\right\}_{i\in\mathbb{Z}_{\geq 0}}, v_h(F^{(m)})\right\}. \end{aligned}$$

By (4.10) and by the assumption that $\{F^{(m)}\}_{m\geq 1}$ is a Cauchy sequence with respect to the topology defined by the valuation v_h, we deduce that $v_h(F) > -\infty$. This completes the proof. □

Remark 4.15. — If $h \geq 1$, the sequence $F^{(m)} = X^m$ converges in $(\mathcal{H}_h, v_h)$. In fact, this is easily checked since $v_h(X_m) = h\ell(m)$ tends to ∞ as $m \to \infty$. In particular, the topology on $\mathcal{H}_h$ induced by v_h is not the p-adic topology since the sequence $F^{(m)} = X^m$ is not convergent in the p-adic topology.

Suppose $h = 0$, so $\mathcal{H}_h^+ = \mathbb{Z}_p[\![X]\!]$. Then the v_h-adic topology on $\mathcal{H}_h^+ = \mathbb{Z}_p[\![X]\!]$ is the p-adic topology and it does not coincide with the (p, X)-adic topology with which $\mathbb{Z}_p[\![X]\!]$ is normally endowed in Iwasawa theory. On the contrary, the Fréchet topology of $\mathcal{H}_0^+$ is precisely the (p, X)-adic one.

When $h \geq 0$, we also see that the restriction of the X-adic topology on $\mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ to the subspace $\mathcal{H}_H$ is not the topology induced by the family $\{q_\nu\}_{\nu\in\mathbb{Z}}$. Indeed, the sequence $F^{(m)} = \{p^m\}$ is Cauchy with respect to every q_ν because $q_\nu(p^m) = m$ and the limit for $\mu \to +\infty$ exists and is 0 in the topology induced by the $\{q_\nu\}$'s: on the other hand, the sequence is not convergent in the X-adic topology.

Proposition 4.16. — *For every $h \in \mathbb{Z}_{\geq 0}$, the Banach topology induced by v_h on $\mathcal{H}_h$ is finer than the compact-open topology induced by the family $\{q_\nu\}$. In other words, the identity map*

$$\mathrm{id}\colon (\mathcal{H}_h, v_h) \longrightarrow \mathcal{H}_h$$

is continuous.

For every $\nu \in \mathbb{Z}$, we consider the lattice

$$L_\nu = \{F \in \mathcal{H}_h \mid q_\nu(F) \geq 0\}. \tag{4.11}$$

Proof. — A basis of the open neighbourhoods of 0 in $\mathcal{H}_h$ is given by the collection of $p^a L_\nu$ for $a, \nu \in \mathbb{Z}$, as in [Sch02, Proposition 4.3]. In order to prove continuity of the linear map id, it is enough, by linearity, to show that $\mathrm{id}^{-1}(L_\nu)$ is open in $(\mathcal{H}_h, v_h)$ for

every $\nu \in \mathbb{Z}$. Since a basis of the topology of $(\mathcal{H}_h, v_h)$ is given by $\{p^b\mathcal{H}_h^+\}_{b\in\mathbb{Z}}$, this openness is equivalent to the existence of $\{b(\nu) \in \mathbb{Z}\}_{\nu\in\mathbb{Z}}$ such that

$$p^{b(\nu)}\mathcal{H}_h^+ \subseteq L_\nu \tag{4.12}$$

holds for each $\nu \in \mathbb{Z}$. By invoking Proposition 4.10, we know that

$$\mathcal{H}_h^+ \subseteq \left\{ F \in \mathcal{H}_h \;\middle|\; \inf\left\{ q_\nu(F) + h\nu \mid \nu \in \mathbb{Z}_{\geq 1} \right\} \geq c_1 \right\}.$$

It follows that each $F \in \mathcal{H}_h^+$ satisfies $q_\nu(F) \geq c_1 - h\nu$ for all ν. Writing $b(\nu) = -\lfloor c_1 - h\nu \rfloor$, we have $p^{b(\nu)}F \in L_\nu$ for every $F \in \mathcal{H}_h^+$, which completes the proof of the proposition. □

Remark 4.17. — The space $\mathcal{H}_h$ equipped with the topology induced by the family $\{q_\nu\}$ is not homeomorphic to $(\mathcal{H}_h, v_h)$. In fact, if the identity map $\mathcal{H}_h$ from $(\mathcal{H}_h, v_h)$ were a homeomorphism, the lattice $\mathcal{H}_h^+$ would be open in $\mathcal{H}_h$ and this would mean that there should exists $\mu \in \mathbb{Z}$ such that

$$L_\mu \subseteq \mathcal{H}_h^+$$

where L_μ is the lattice introduced in the proof of Proposition 4.16. The above inclusion cannot hold, since elements F in $\mathcal{H}_h^+$ verify the bound $q_\nu(F) + h\nu \geq c_1$ for every $\nu > 0$, whereas elements F in L_μ verify $q_\mu(F) > 0$ without any restriction concerning q_ν for $\nu \neq \mu$.

In order to state the results below, we fix an integer $d \geq 0$. We can consider the subspace $X^d\mathcal{H}_h$ of all power series in $X^d\mathbb{Q}_p[\![X]\!] \cap \mathcal{H}_h$ and we can endow it with the subspace topology, which coincides with the topology induced by the family of seminorms $q_\nu|_{X^d\mathcal{H}_h}$ and which turns it into a locally convex space (see [Sch02, Section §5 A]). Similarly, we can consider the restriction $v_h|_{X^d\mathcal{H}_h}$, which is again a valuation and we denote the corresponding normed space simply by $(X^d\mathcal{H}_h, v_h)$.

Proposition 4.18. — *For every $d \geq 0$ the space $(X^d\mathcal{H}_h, v_h)$ is Banach, and $(X^d\mathcal{H}_h^+, v_h)$ is complete.*

Proof. — Let $d \geq 0$. The first statement follows from the fact that $(\mathcal{H}_h, v_h)$ is Banach if $(X^d\mathcal{H}_h, v_h)$ is closed inside $(\mathcal{H}_h, v_h)$. In order to prove that $(X^d\mathcal{H}_h, v_h)$ is closed inside $(\mathcal{H}_h, v_h)$, we define for each $0 \leq j \leq d$ the map

$$\xi_j \colon (\mathcal{H}_h, v_h) \longrightarrow \mathbb{Q}_p$$

which extracts the j-th coefficient, namely $\xi_j(\sum a_i X^i) = a_j$. We claim that ξ_j is continuous; this would imply that the intersection

$$X^d\mathcal{H}_h = \bigcap_{0\leq j\leq d} \ker(\xi_j)$$

is closed inside $(\mathcal{H}_h, v_h)$, completing the proof of the the first statement of the proposition. Let us prove the claim now. Since ξ_j is $\mathbb{Q}_p$-linear it is enough to check that

$\xi_j^{-1}(\mathbb{Z}_p)$ is open in $(\mathcal{H}_h, v_h)$, and to do this we observe that

$$\xi_j^{-1}(\mathbb{Z}_p) = \left\{ \sum_{i=0}^{+\infty} a_i X^i \in \mathcal{H}_h \;\middle|\; \mathrm{ord}(a_j) \geq 0 \right\}.$$

Since every element $F = \sum a_i X^i \in \mathcal{H}_h^+$ satisfies $\mathrm{ord}(a_j) + h\ell(j) \geq 0$, it follows that

$$p^{h\ell(j)}\mathcal{H}_h^+ \subseteq \xi_j^{-1}(\mathbb{Z}_p),$$

which means that ξ_j is continuous and this completes the proof of the first statement.

Let us prove the second statement. The second statement follows from the first if $(X^d\mathcal{H}_h^+, v_h)$ is closed in $(X^d\mathcal{H}_h, v_h)$. First, we remark that the inclusion $X^d\mathcal{H}_h^+ \subseteq X^d\mathcal{H}_h \cap \mathcal{H}_h^+$ is strict, in general. For example, the element $X^d/p^{h\ell(d)}$ is in the latter space but not in the former. Nevertheless, we will prove that the topology induced on $X^d\mathcal{H}_h$ by v_h, for which $X^d\mathcal{H}_h \cap \mathcal{H}_h^+$ is a bounded open lattice, is equivalent to the topology on $X^d\mathcal{H}_h$ for which $X^d\mathcal{H}_h^+$ is a bounded open lattice. In order to prove this, we observe that

$$\begin{aligned} X^d\mathcal{H}_h^+ &= \big\{ X^d \sum_{j=0}^{+\infty} a_j X^j \;\big|\; \mathrm{ord}(a_j) \geq -h\ell(j), \forall\, j \geq 0 \big\} \\ &= \big\{ \sum_{i=d}^{+\infty} b_i X^i \;\big|\; \mathrm{ord}(b_i) \geq -h\ell(i-d), \forall\, j \geq d \big\}. \end{aligned}$$

By definition, we have

$$\ell(i+d) - \ell(i) \leq \frac{\ln(i+d)}{\ln(p)} - \frac{\ln(i)}{\ln(p)} + 2 = \frac{\ln(1+d/i)}{p} + 2.$$

Since the right-hand term is bounded as $i \to +\infty$, we can define a non-negative integer $N(d)$ as follows:

$$N(d) = \max\{\ell(i+d) - \ell(i) \;|\; i \in \mathbb{Z}_{\geq 0}\}.$$

It follows that there are inclusions $p^{hN(d)}(X^d\mathcal{H}_h \cap \mathcal{H}_h^+) \subseteq X^d\mathcal{H}_H^+ \subseteq X^d\mathcal{H}_h \cap \mathcal{H}_h^+$, and the claimed equivalence of two topologies follows. This implies immediately that $(X^d\mathcal{H}_h^+, v_h)$ is closed in $(X^d\mathcal{H}_h, v_h)$, which completes the proof of the second statement. □

4.3. Inverting the operator $1 - \varphi_D \otimes \varphi_H$. — First, we introduce important operators φ_H and D_{CW}. Let us define the endomorphism φ_H of the $\mathbb{Q}_p$-algebra $\mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$:

$$\varphi_H(F(X)) = F\big(\omega(X)\big) \tag{4.13}$$

for every $F(X) \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$, where $\omega(X)$ is the polynomial defined to be $\omega(X) = (1+X)^p - 1$. For the fact that φ_H is well-defined and for some basic properties of φ_H, we refer the reader to Coleman's original paper [Col79] where the field denoted H in ibid. corresponds $\mathbb{Q}_p$ and $\mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ is denoted $H[\![T]\!]_1$. In

order to study eigenvectors of φ_H we also introduce a differential operator of Coates and Wiles. For every $F \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$, we define

$$D_{CW}(F) = (1+X)\frac{dF}{dX}. \tag{4.14}$$

For every $j \in \mathbb{Z}_{\geq 0}$, we let

$$\Delta_j(F) = D^j_{CW}(F)(0). \tag{4.15}$$

Lemma 4.19. — *The following statements hold.*

1. *We have* $\varphi_H\big(\log^i(1+X)\big) = p^i \log^i(1+X)$ *for every* $i \in \mathbb{Z}_{\geq 0}$.
2. *Given* $F \in \mathbb{Q}_p[\![X]\!]$, *we have* $\Delta_i(\varphi_H(F)) = p^i \Delta_i(F)$ *for every* $i \in \mathbb{Z}_{\geq 0}$.
3. *Given* $F \in \mathbb{Q}_p[\![X]\!]$, *the coefficient of* Z^j *in* $F(X)|_{X=\exp(Z)-1}$ *is equal to* $(j!)^{-1}\Delta_j(F)$. *In particular, if* $\Delta_i(F) = 0$ *for every* $i \in \mathbb{Z}_{\geq 0}$, *we have* $F = 0$.
4. *For non-negative integers* i *and* j, *we have*
$$\Delta_i(\log^j(1+X)) = i!\delta_{i,j}$$
where $\delta_{i,j}$ *is the Kronecker delta function.*
5. *Given* $F \in \mathcal{H}^+_h$, *we have*
$$\mathrm{ord}(\Delta_i(F)) \geq -hi$$
for every $i \in \mathbb{Z}_{\geq 0}$.

Proof. — We start with the first assertion. For $i = 0$, the equality is trivially true. For $i = 1$, we have

$$\varphi_H\big(\log(1+X)\big) = \log\big((1+X)^p\big) = p\log(1+X)$$

which follows from the usual formal properties of the logarithm. Since φ_H is an algebra homomorphism, the case with $i > 1$ follows from the above.

For the second assertion, we first show that

$$D^i_{CW} \circ \varphi_H(F) = p^i \varphi_H \circ D^i_{CW}(F) \tag{4.16}$$

holds for any $F(X) \in \mathbb{Q}_p[\![X]\!]$ and for any natural number i. Noting that $\varphi_H(F(X)) = F(X)|_{X=\omega(X)-1}$, the case for $i = 1$ of (4.16) is checked as follows by the chain rule of derivation:

$$\begin{aligned} D_{CW}\big(F(X)|_{X=\omega(X)-1}\big) &= (1+X)\frac{dF}{dX}(X)\bigg|_{X=\omega(X)-1} \cdot \frac{d\omega(X)}{dX} \\ &= (1+X)\frac{dF}{dX}(X)\bigg|_{X=\omega(X)-1} \cdot p\frac{\omega(X)-1}{(1+X)} \\ &= p(D_{CW}F)(X)|_{X=\omega(X)-1}. \end{aligned}$$

Then, (4.16) follows for any natural number i by induction. Since $\omega(0) = 0$, evaluating (4.16) at $X = 0$ implies the second assertion.

For the third assertion, let us set $G(Z) = F(X)|_{X=\exp(Z)-1}$. we first show that

$$\frac{d^i G}{dZ^i}(Z) = D^i_{CW}(F)(X)\big|_{X=\exp(Z)-1} \tag{4.17}$$

holds for any $F(X) \in \mathbb{Q}_p[\![X]\!]$ and for any natural number i. Note that we have $Z = \log(1+X)$ and $\exp(Z) = 1+X$. The case for $i = 1$ of (4.17) is checked as follows by the chain rule of derivation:

$$\begin{aligned}\frac{dG}{dZ}(Z) &= \frac{dF}{dX}(X)\bigg|_{X=\exp(Z)-1} \cdot \exp(Z) \\ &= \left((1+X)\cdot\frac{dF}{dX}(X)\right)\bigg|_{X=\exp(Z)-1} = D_{CW}(F)(X)|_{X=\exp(Z)-1}\,.\end{aligned}$$

Then, (4.17) follows for any natural number i by induction. It follows that the coefficient of Z^j in $F(X)|_{X=\exp(Z)-1}$ is equal to $(j!)^{-1}\Delta_j(F)$ by evaluating (4.17) at $Z = 0$.

The fourth assertion follows immediately from the third assertion.

For the fifth assertion, let $F(X) \in \mathcal{H}_h^+$ and write

$$F(X) = \sum_{n=0}^{+\infty} a_n X^n.$$

We denote the n-th coefficient of the i times iterated Coates–Wiles derivative of $F(X)$ by $a_n^{(i)}$, that is

$$D^i_{CW}(F)(X) = \sum_{n=0}^{+\infty} a_n^{(i)} X^n.$$

By definition, we see that

$$D_{CW}(F)(X) = \sum_{n=0}^{+\infty} \big(na_n + (n+1)a_{n+1}\big)X^n$$

and therefore, recursively, that $a_n^{(i)} = na_n^{(i-1)} + (n+1)a_{n+1}^{(i-1)}$. From this, it follows that

$$\operatorname{ord}(a_n^{(i)}) \geq \min\big\{na_n^{(i-1)}, (n+1)a_{n+1}^{(i-1)}\big\} \geq \min\big\{a_n^{(i-1)}, a_{n+1}^{(i-1)}\big\}. \tag{4.18}$$

We first evaluate $\operatorname{ord}(a^{(1)})$. Since $F \in \mathcal{H}_h^+$, we have $\operatorname{ord}(a_n^{(0)}) = \operatorname{ord}(a_n) \geq -h\ell(n)$ and $\operatorname{ord}(a_{n+1}^{(0)}) = \operatorname{ord}(a_{n+1}) \geq -h\ell(n+1)$. By (4.18), we have

$$\operatorname{ord}(a_n^{(1)}) \geq \min\big\{a_n^{(0)}, a_{n+1}^{(0)}\big\} \geq -h\ell(n+1).$$

On the other hand, we can show that $\ell(n+1) \leq \ell(n)+1$ by a calculation based on the definition of ℓ. Thus we obtain $\operatorname{ord}(a_n^{(1)}) \geq -h\ell(n+1) \geq -h\ell(n) - h$. By induction with respect to i, we obtain

$$\operatorname{ord}(a_n^{(i)}) \geq -h\ell(n) - hi.$$

Since $\Delta_i(F)$ is the constant term $a_0^{(i)}$ of $D_{CW}(F)(X)$, this completes the proof. □

Proposition 4.20. — *Let $h \in \mathbb{Z}_{\geq 0}$ and let us fix $\nu \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Set*

$$a(n,\nu) = \begin{cases} n - \nu + p^{-\nu} & \text{if } n \geq \nu, \\ p^{n-\nu} & \text{if } n < \nu. \end{cases}$$

For every $d \geq 0$ and for each $F \in X^d \mathcal{H}_h$, we have

$$\begin{aligned} &q_\nu\big(\varphi_H^n(F)\big) \\ &\geq \begin{cases} v_h(F) + da(n,\nu) - \ell(d)h & \text{if } d \geq \dfrac{h}{a(n,\nu)\ln(p)}, \\ v_h(F) + \dfrac{h}{\ln(p)} \ln\big(a(n,\nu)\big) + \dfrac{h}{\ln(p)}\left(1 - \ln\left(\frac{h}{\ln(p)}\right)\right) - h & \text{if } d \leq \dfrac{h}{a(n,\nu)\ln(p)}. \end{cases} \end{aligned} \tag{4.19}$$

For each $F \in \mathcal{H}_h$, we have

$$\begin{aligned} &q_\nu\big(\varphi_H^n(F)\big) \\ &\geq \min\left\{ v_h(F),\ v_h(F) + \frac{h}{\ln(p)} \ln\big(a(n,\nu)\big) + \frac{h}{\ln(p)}\left(1 - \ln\left(\frac{h}{\ln(p)}\right)\right) - h \right\} \end{aligned} \tag{4.20}$$

Proof. — Recall that the equality $\operatorname{ord}\binom{p^n}{i} = n - \operatorname{ord}(i)$ is deduced by observing that for every $k < p^n$ we have $\operatorname{ord}(p^n - k) = \operatorname{ord}(k)$ and by taking p-adic valuations in the expression

$$i \cdot \binom{p^n}{i} = p^n \cdot \frac{(p^n-1)(p^n-2)\cdots\big(p^n-(i-1)\big)}{1\cdot 2\cdots(i-1)}.$$

Since we have

$$\varphi_H^n(X) = (1+X)^{p^n} - 1 = \sum_{i=1}^{p^n} \binom{p^n}{i} X^i,$$

we find that

$$\begin{aligned} q_\nu\big(\varphi_H^n(X)\big) &= \inf\left\{ \operatorname{ord}\left(\sum_{i=1}^{p^n}\binom{p^n}{i} z^i\right) \;\middle|\; z \in B_{\operatorname{ord}}(\nu) \right\} \\ &= \min\left\{ n - \operatorname{ord}(i) + ip^{-\nu} \mid 1 \leq i \leq p^n \right\} \\ &= n - \max\Big\{ \max\left\{ \operatorname{ord}(i) - ip^{-\nu} \mid \operatorname{ord}(i) = s, i \leq p^n \right\} \;\Big|\; 0 \leq s \leq n \Big\}. \end{aligned}$$

The internal max in the above expressions is attained at $i = p^s$, where it takes the value $s - p^{s-\nu}$, hence we have

$$q_\nu\big(\varphi_H^n(X)\big) \geq n - \max\{s - p^{s-\nu} \mid 0 \leq s \leq n\}.$$

In order to compute the maximum, we consider the function $k(x) = x - p^{x-\nu}$, whose derivative $k'(x) = 1 - \ln(p)p^{x-\nu}$ vanishes at $x_0 = \nu - \dfrac{\ln(\ln(p))}{\ln(p)}$. We have

$$k(x_0) = \nu - \frac{\ln(\ln(p)) + 1}{\ln(p)} < \nu.$$

Now, we have $n \geq x_0$ if and only if $n \geq \nu$. In this case, we have $q_\nu\big(\varphi_H^n(X)\big) \geq n - k(x_0) > n - \nu$. Since q_ν takes values in a discrete additive group $p^{-\nu}\mathbb{Z}$, we obtain $q_\nu\big(\varphi_H^n(X)\big) \geq n - \nu + p^{-\nu}$. We thus obtain

$$q_\nu\big(\varphi_H^n(X)\big) \geq \begin{cases} n - \nu + p^{-\nu} & \text{if } n \geq \nu, \\ p^{n-\nu} & \text{if } n < \nu. \end{cases}$$

Since φ_H^n is an algebra homomorphism satisfying $\varphi_H^n(X^i) = \varphi_H^n(X)^i$, we have more generally

$$q_\nu\big(\varphi_H^n(X^i)\big) \geq \begin{cases} i(n - \nu + p^{-\nu}) & \text{if } n \geq \nu, \\ ip^{n-\nu} & \text{if } n < \nu, \end{cases} \tag{4.21}$$

for all $i \geq 0$. Using the notations introduced in the statement of the proposition, we rewrite this as $q_\nu\big(\varphi_H^n(X^i)\big) \geq ia(n,\nu)$ for all n, ν and i.

Let us take $F = \sum_{i=d}^{+\infty} a_i X^i \in X^d \mathcal{H}_h$. By the definition of v_h, it satisfies

$$\begin{aligned} &\operatorname{ord}(a_i) \geq v_h(F) - h\ell(i) && \text{for } i \geq d, \\ &\operatorname{ord}(a_i) = +\infty && \text{for } i < d. \end{aligned}$$

Thus, we have:

$$\begin{aligned} q_\nu\big(\varphi_H^n(F)\big) &\geq \inf\left\{\operatorname{ord}(a_i) + q_\nu\big(\varphi_H^n(X^i)\big) \;\middle|\; i \geq d\right\} \\ &\geq v_h(F) + \inf\left\{ia(n,\nu) - h\ell(i) \;\middle|\; i \geq d\right\} \end{aligned}$$

and we need to study

$$\inf\left\{ia(n,\nu) - h\ell(i) \mid i \geq d\right\}. \tag{4.22}$$

The infimum above is $a(n,\nu)d$ when $h = 0$. When $h > 0$, we introduce the function $k_1(x) = xa(n,\nu) - h\left(\frac{\ln(x)}{\ln(p)} + 1\right)$ defined for $x > 0$ which satisfies $k_1(i) \leq ia(n,\nu) - h\ell(i)$ for every $i > 0$. Its derivative vanishes at $x_0 = \frac{h}{a(n,\nu)\ln(p)}$ where we have

$$\begin{aligned} k_1(x_0) &= \frac{h}{\ln(p)}\left(1 - \ln\left(\frac{h}{a(n,\nu)\ln(p)}\right)\right) - h \\ &= \frac{h}{\ln(p)}\ln\big(a(n,\nu)\big) + \frac{h}{\ln(p)}\left(1 - \ln\left(\frac{h}{\ln(p)}\right)\right) - h. \end{aligned}$$

If we have $d = 0$, the infimum appearing in (4.22) is $\min\{0, k_1(x_0)\}$. If we have $0 < d \leq x_0 = \frac{h}{a(n,\nu)\ln(p)}$, the infimum is at least $k_1(x_0)$. On the other hand, if $d \geq x_0$, the infimum is reached at $i = d$. This completes the proof of the proposition. □

Given a topological $\mathbb{Q}_p$-vector space V we denote by $\mathcal{B}(V)$ the space of continuous linear endomorphisms of V. When V is normed, it coincides with the space of linear bounded operators, as in [Sch02, Proposition 3.1].

Recall that the the valuation v'_h given in Definition 4.4 is equivalent to v_h as shown in Proposition 4.10 and that v'_h is related to the family of semi-norms q_ν as follows:

$$v'_h(F) = \inf\left\{q_\nu(F) + h\nu \;\middle|\; \nu \in \mathbb{Z}_{\geq 1}\right\} \in \mathbb{R} \cup \{\pm\infty\}. \tag{4.23}$$

We will deduce the following corollary from Proposition 4.20.

Corollary 4.21. — *The operator φ_H belongs to $\mathcal{B}(\mathcal{H}_h, v_h)$ and to $\mathcal{B}(\mathcal{H}_h)$. More generally φ_H belongs to $\mathcal{B}(X^d\mathcal{H}_h, v_h)$ and $\mathcal{B}(X^d\mathcal{H}_h)$ for every $d \geq 0$.*

Before we prove the corollary, we define the constant c_3 depending only on h and p as follows:

$$c_3 = \frac{h}{\ln(p)}\left(1 - \ln\left(\frac{h}{\ln(p)}\right)\right). \tag{4.24}$$

Proof. — The statement concerning $(X^d\mathcal{H}_h, v_h)$ (*resp.* concerning $X^d\mathcal{H}_h$) follows from the first statement observing that $X^d\mathcal{H}_h$ is stable under φ_H and that the topology on $(X^d\mathcal{H}_h, v_h)$ (*resp.* on $X^d\mathcal{H}_h$) is induced from that of $(\mathcal{H}_h, v_h)$ (*resp.* from that of $(X^d\mathcal{H}_h)$) for any $d \geq 0$. In the rest of the proof, we will hence only prove the statements relative to $(\mathcal{H}_h, v_h)$ and to $\mathcal{H}_h$, since they imply the statements relative to $(X^d\mathcal{H}_h, v_h)$ and to $X^d\mathcal{H}_h$ for $d \geq 0$.

We first prove continuity of φ_H in the topology induced by v_h. We are going to exploit the relation between the family of semi-norms $\{q_\nu\}_{\nu\geq 1}$ and the valuation v'_h, and then use the fact that v'_h is equivalent to v_h. By the definition of v'_h which we recalled in the equation (4.23), we have:

$$v'_h(\varphi_H(F)) = \inf\{q_\nu(\varphi_H F) + h\nu \mid \nu \geq 1\} \tag{4.25}$$

for every $F \in \mathcal{H}_h$. Since we have $\ln\big(a(1,\nu)\big) = \ln(p) - \nu\ln(p)$, the above inf can be estimated with Proposition 4.20 as follows :

$$\inf\left\{q_\nu(\varphi_H F) + h\nu \;\middle|\; \nu \geq 1\right\} \tag{4.26}$$

$$\geq \inf\left\{v_h(F) + \left[h - h\nu - \frac{h}{\ln(p)}\ln\left(\frac{h}{\ln(p)}\right)\right] + h\nu + \frac{h}{\ln(p)} - h \;\middle|\; \nu \geq 1\right\} \geq v_h(F) + c_3.$$

By combining (4.25) and (4.26), we have

$$v'_h(\varphi_H(F)) \geq v_h(F) + c_3.$$

For every $F \in \mathcal{H}_h$, we have $v_h(\varphi_H F) \geq v'_h(\varphi_H F) - c_2$ by Proposition 4.10. Thus we have

$$v_h(\varphi_H F) \geq v_h(F) + c_3 - c_2$$

and thus $\varphi_H \in \mathcal{B}(\mathcal{H}_h, v_h)$.

We now prove continuity of φ_H in the compact-open topology defined by the collection $\{q_\nu\}_{\nu\in\mathbb{Z}_p}$: we start with a couple of preliminary remarks. First of all, by definition, the collection of balls $B_{\mathrm{ord}}(\nu)$ is increasing for $\nu \to +\infty$, so

$$B_{\mathrm{ord}}(\nu) \subseteq B_{\mathrm{ord}}(\nu') \qquad \text{if } \nu \leq \nu';$$

as a consequence, for all $F \in \mathcal{H}_h$, the corresponding valuations verify:

$$q_\nu(F) = \inf_{z \in B_{\mathrm{ord}}(\nu)} \mathrm{ord}(F(z)) \geq \inf_{z \in B_{\mathrm{ord}}(\nu')} \mathrm{ord}(F(z)) = q_{\nu'}(F) \quad \text{if } \nu \leq \nu'. \tag{4.27}$$

Moreover since $z \in B_{\mathrm{ord}}(\nu)$ if and only if $\mathrm{ord}(z) \geq p^{-\nu}$, we find that, for every $a \in \mathbb{Q}_{>0}$, we have

$$\mathrm{ord}(z) \geq a \Longleftrightarrow z \in B_{\mathrm{ord}}\left(-\frac{\ln a}{\ln p}\right)$$

Now, the relation (4.21) for $i = n = 1$ implies that

$$q_\nu\big(\varphi_H(X)\big) \geq a(1, \nu)$$

or, equivalently, that for every $z \in B_{\mathrm{ord}}(\nu)$ we have $(1+z)^p - 1 \in B_{\mathrm{ord}}(b(\nu))$ where $b(\nu) = -\dfrac{\ln\big(a(1,\nu)\big)}{\ln(p)}$: since every $x \in \mathbb{C}_p$ can be written as $x = (1+x')^p - 1$ with $(1+x') \in \mathbb{C}_p$ being a pth root of $(x+1)$, we have a surjection

$$B_{\mathrm{ord}}\big(\nu\big) \twoheadrightarrow B_{\mathrm{ord}}\big(b(\nu)\big), \qquad z \mapsto (1+z)^p - 1$$

It follows that, for every $F \in \mathcal{H}_h$, we have

$$\begin{aligned} q_\nu(\varphi_H F) &= \inf_{z \in B_{\mathrm{ord}}(\nu)} \mathrm{ord}\big(\varphi_H F(z)\big) = \inf_{z \in B_{\mathrm{ord}}(\nu)} \mathrm{ord}\big(F\big((1+z)^p - 1\big)\big) \\ &= \inf_{z \in B_{\mathrm{ord}}(b(\nu))} \mathrm{ord}\big(F(z)\big) = q_{b(\nu)}(F) \geq q_\nu(F) \end{aligned}$$

thanks to (4.27), because $b(\nu) \leq \nu$ for every $\nu \in \mathbb{Z}_p$: this follows by definition, since

$$b(\nu) = \begin{cases} \nu - 1 & \text{if } \nu > 1 \\ -\frac{1}{\ln p} \ln(1 - \nu + p^{-\nu}) & \text{if } \nu \leq 1 \end{cases}$$

and thus the inequality $b(\nu) \leq \nu$ is obvious if $\nu > 1$ and follows from the relation

$$p^{-b(\nu)} = p^{\frac{\ln(1-\nu+p^{-\nu})}{\ln p}} = 1 - \nu + p^{-\nu}$$

when $\nu \leq 1$. Since the above relation shows that the linear map φ_H is continuous with respect to the compact-open topology defined by the collection $\{q_\nu\}_{\nu \in \mathbb{Z}_p}$, this completes the proof. □

We define the constant c_4 depending only on h and p as follows:

$$c_4 = \min\left\{-\ell(h+1)h, \frac{h}{\ln(p)}\left(1 - \ln\left(\frac{h}{\ln(p)}\right)\right) - h\right\}. \tag{4.28}$$

Recall that we fixed $\mathcal{K}$ a finite extension of $\mathbb{Q}_p$ and we denoted its ring of integers by $\mathcal{O}_{\mathcal{K}}$. We set

$$\mathcal{H}^+_{h/\mathcal{K}} = \mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}_p} \mathcal{H}^+_h \qquad \text{and} \qquad \mathcal{H}_{h/\mathcal{K}} = \mathcal{K} \otimes_{\mathbb{Q}_p} \mathcal{H}_h = \mathbb{Q}_p \otimes \mathcal{H}^+_{h/\mathcal{K}}. \tag{4.29}$$

Proposition 4.22. — *Let $h \in \mathbb{Z}_{\geq 0}$ and let us take $u \in \mathcal{O}^\times_{\mathcal{K}}$. Then, for every $F \in X^{h+1}\mathcal{H}_{h/\mathcal{K}} \cap \mathcal{H}^+_{h/\mathcal{K}}$, the series*

$$\sum_{n=0}^{+\infty} (pu)^{-nh} \varphi_H^n(F) \tag{4.30}$$

converges to an element $\widetilde{F}$ *of* $p^{c_4-c_2}\mathcal{H}^+_{h/\mathcal{K}}$ *with respect to the topology induced by the family* $\{q_\nu\}$.

We set

$$\mathrm{Conv}\big(B(0,1)\big)_{/\mathcal{K}} = \mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}_p} \mathrm{Conv}\big(B(0,1)\big). \tag{4.31}$$

To prove the proposition, we prepare the following lemma:

Lemma 4.23. — *Under the setting of Proposition 4.22, the series (4.30) converges to an element* $\widetilde{F}$ *in* $\mathrm{Conv}\big(B(0,1)\big)_{/\mathcal{K}}$ *with respect to the topology induced by the family* $\{q_\nu\}$.

Proof of Lemma 4.23. — Fix any $\nu \geq 0$. By applying Proposition 4.20 with $d = h+1$, we have

$$q_\nu\big((pu)^{-nh}\varphi_H^n(F)\big) \geq -nh + v_h(F) + \kappa(n,\nu) + c_4 \tag{4.32}$$

for all $n \geq 0$, where $\kappa(n,\nu)$ is defined as

$$\kappa(n,\nu) = \begin{cases} (h+1)(n-\nu+p^{-\nu}) & \text{if} \quad h+1 \geq \dfrac{h}{(n-\nu+p^{-\nu})\ln p} \quad \text{and} \quad n \geq \nu, \\[2ex] \dfrac{h}{\ln p}\ln(n-\nu+p^{-\nu}) & \text{if} \quad h+1 \leq \dfrac{h}{(n-\nu+p^{-\nu})\ln p} \quad \text{and} \quad n \geq \nu, \\[2ex] (h+1)p^{n-\nu} & \text{if} \quad h+1 \geq \dfrac{h}{p^{n-\nu}\ln p} \quad \text{and} \quad n \leq \nu, \\[2ex] h(n-\nu) & \text{if} \quad h+1 \leq \dfrac{h}{p^{n-\nu}\ln p} \quad \text{and} \quad n \leq \nu. \end{cases} \tag{4.33}$$

As we are interested in the convergency of the series, we can assume $n \geq \nu$ so that the third and fourth options can be omitted in the definition of $\kappa(n,\nu)$. The assumption $n \geq \nu$ ensures that $h+1 \geq \dfrac{h}{(n-\nu+p^{-\nu})\ln p}$, hence the second option in (4.33) can be also omitted in the definition of $\kappa(n,\nu)$. Thus we have $\kappa(n,\nu) = (h+1)(n-\nu+p^{-\nu})$ for $n \geq \nu$ and therefore we have

$$\lim_{n\to\infty} -nh + \kappa(n,\nu) = +\infty. \tag{4.34}$$

Combining (4.32) and (4.34) shows that the series in the statement is Cauchy in the q_ν-topology of $\mathrm{Conv}\big(B(0,1)\big)$. As this space is Fréchet with respect to the family of valuations q_ν, this series converges to an element $\widetilde{F}$ and this completes the proof. □

Proof of Proposition 4.22. — Thanks to Lemma 4.23, the limit of the series (4.30) exists as an element $\widetilde{F}$ in $\mathrm{Conv}\big(B(0,1)\big)_{/\mathcal{K}}$. Recall that $\widetilde{F} \in \mathrm{Conv}\big(B(0,1)\big)_{/\mathcal{K}}$ is

contained in the subspace $\mathcal{H}_h \subset \mathrm{Conv}\big(B(0,1)\big)_{/\mathcal{K}}$ if and only if $v_h(\widetilde{F}) > -\infty$. Using Proposition 4.10, it is enough to prove that

$$\inf\left\{q_\nu(\widetilde{F}) + h\nu \;\middle|\; \nu \geq 0\right\} \geq c_4 \tag{4.35}$$

In order to show (4.35), we claim that

$$\kappa(n,\nu) \geq hn - h\nu \quad \text{for every } n \text{ and } \nu. \tag{4.36}$$

For $n \geq \nu$, since we have $\kappa(n,\nu) = (h+1)(n-\nu+p^{-\nu})$ as defined in (4.33), we deduce $\kappa(n,\nu) \geq nh - h\nu$. For $n \leq \nu$, we can omit the first two options in (4.33) and we have

$$\kappa(n,\nu) \geq \min\left\{(h+1)p^{n-\nu}, h(n-\nu)\right\} \geq hn - h\nu.$$

Thus the claim (4.36) was verified. It is now easy to derive (4.35). Indeed, we have

$$q_\nu(\widetilde{F}) + h\nu \geq \min\left\{q_\nu((pu)^{-nh}\varphi_H^n F) + h\nu\right\}_{n\geq 0}$$

and by combining (4.32), (4.36) and the bound $v_h(F) \geq 0$ obtained by the assumption $F \in \mathcal{H}^+_{h/\mathcal{K}}$, we deduce

$$q_\nu(\widetilde{F}) + h\nu \geq c_4$$

for all ν. We thus finish the proof of the proposition. □

Remark 4.24. — Recall that the space $\mathcal{H}_h$ equipped with the topology induced by the family $\{q_\nu\}$ is not homeomorphic to $(\mathcal{H}_h, v_h)$ as discussed in Remark 4.17. We are not sure if the series (4.29) converges with respect to the topology defined by v_h.

Let us fix a non-zero $\lambda \in \overline{\mathbb{Q}_p}$. Given $F \in \mathcal{H}_{h/\mathcal{K}}$, we discuss solutions $\widetilde{F} \in \mathcal{H}_{h/\mathcal{K}}$ of the equation

$$(1 - \lambda\varphi_H)\widetilde{F} = F \tag{$E^h_{F;\lambda}$}$$

as well as the denominators of the solution $\widetilde{F}$. Let us define a constant $c_{h,\lambda} \in \mathbb{Z}$ as follows:

$$c_{h,\lambda} = \min\{0, 2h-p\} - h^2 - \mathrm{ord}(h!) + \min\{-c_\lambda, c_4 - c_2\} \tag{4.37}$$

where $c_\lambda = \max\{\mathrm{ord}(\lambda p^i - 1) \mid 0 \leq i \leq h,\ \lambda p^i \neq 1\}$.

Theorem 4.25. — *Let $h \in \mathbb{Z}_{\geq 0}$ be an integer which is equal to or greater than $-\mathrm{ord}(\lambda)$. Then we have the following statements for solutions of the equation ($E^h_{F;\lambda}$) for a given $F \in \mathcal{H}_{h/\mathcal{K}}$.*

1. *The equation ($E^h_{F;\lambda}$) has a solution $\widetilde{F}$ in $\mathcal{H}_{h/\mathcal{K}}$ if one of the following conditions holds:*
   - *(a) We have $\lambda p^i \neq 1$ for every integer i in $[0,h]$.*
   - *(b) There exists an integer j in $[0,h]$ such that we have $\lambda p^j = 1$ and $\Delta_j(F) = 0$.*
2. *In the case (a) of the first assertion, the solution $\widetilde{F}$ is unique.*
3. *In the case (b) of the first assertion, the solution $\widetilde{F}$ is unique modulo $\mathcal{K}\cdot\log^j(1+X)$.*

4. *Assume further that F is contained in $\mathcal{H}^+_{h/\mathcal{K}}$. Then, the unique solution $\widetilde{F}$ of the equation $(E^h_{F;\lambda})$ in the case (a) lies in $p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}$. In the case (b), there exists a solution $\widetilde{F}$ of the equation $(E^h_{F;\lambda})$ which lies in $p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}$.*

Proof. — Since $\mathcal{H}_{h/\mathcal{K}} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{H}^+_{h/\mathcal{K}}$, we may assume that $F \in \mathcal{H}^+_{h/\mathcal{K}}$. We define F_0 to be

$$F_0 = F - \sum_{i=0}^{h} (i!)^{-1}\Delta_i(F)\log^i(1+X).$$

We have $F_0 \in X^{h+1}\mathcal{H}_{h/\mathcal{K}}$ by the third and fourth assertions of Lemma 4.19. By Corollary 4.12, $\log(1+X), \dots, \log^h(1+X)$ are contained in $p^{\min\{0,2h-p\}}\mathcal{H}^+_{h/\mathcal{K}}$. Since F is contained in $\mathcal{H}^+_{h/\mathcal{K}}$, we have $\mathrm{ord}(\Delta_i(F)) \geq -h^2$ for every non-negative number $i \leq h$ by the fifth assertion of Lemma 4.19. Hence, we have $F_0 \in (h!)^{-1}p^{\min\{0,2h-p\}-h^2}\mathcal{H}^+_{h/\mathcal{K}}$. We set

$$\widetilde{F_0} = \sum_{n=0}^{+\infty} \lambda^n \varphi_H^n(F_0)$$

and we claim that the series is convergent to an element in $p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}$ for the constant $c_{h,\lambda}$ given in (4.37). To check the convergence, we note that $\mathrm{ord}(\lambda^n) = n\,\mathrm{ord}(\lambda) = n\big(\mathrm{ord}(\lambda)+h\big)-nh \geq -nh$ since we have $\mathrm{ord}(\lambda)+h \geq 0$ by assumption. By Proposition 4.22, we have

$$\sum_{n=0}^{\infty} (pu)^{-nh}\varphi_H^n(F_0) \in p^{c_4-c_2+\min\{0,2h-p\}-h^2-\mathrm{ord}(h!)}\mathcal{H}^+_{h/\mathcal{K}}.$$

Thus we obtain $\widetilde{F_0} \in p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}$. By the construction of $\widetilde{F_0}$ and by the continuity of φ_H with respect to the topology induced by $\{q_\nu\}_{\nu\in\mathbb{Z}}$ proven in Corollary 4.21, $\widetilde{F_0}$ satisfies

$$(1-\lambda\varphi_H)\widetilde{F_0} = F_0.$$

In order to complete the proof of the existence of a solution to $(E^h_{F;\lambda})$, we need to prove that there exists $\widetilde{F_{\log}} \in p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}$ such that

$$(1-\lambda\varphi_H)\widetilde{F_{\log}} = \sum_{i=0}^{h} (i!)^{-1}\Delta_i(F)\log^i(1+X), \tag{4.38}$$

which will allow us to set $\widetilde{F} = \widetilde{F_0} + \widetilde{F_{\log}}$. We define

$$\widetilde{F_{\log}} = \sum_{\substack{i=0\\ \lambda p^i \neq 1}}^{h} \frac{(i!)^{-1}\Delta_i(F)}{1-\lambda p^i}\log^i(1+X) \in p^{\min\{0,2h-p\}-h^2-c_\lambda}\mathcal{H}^+_{h/\mathcal{K}} \subseteq p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}.$$

For each integer $0 \leq i \leq h$ satisfying $\lambda p^i \neq 1$, we have

$$(1-\lambda\varphi_H)\left(\frac{(i!)^{-1}\Delta_i(F)}{1-\lambda p^i}\log^i(1+X)\right) = (i!)^{-1}\Delta_i(F)\log^i(1+X),$$

thanks to the first assertion of Lemma 4.19. Now, (4.38) follows by linearity. This proves the first assertion.

For the second assertion, it suffices to show that $E = 0$ is the only solution to the equation $(1 - \lambda\varphi_H)(E) = 0$ with $E \in \mathcal{H}_{h/\mathcal{K}}$. In fact, if we have two solutions $\widetilde{F}, \widetilde{F}'$ to $(E^h_{F;\lambda})$, $E = \widetilde{F} - \widetilde{F}'$ is a solution of the equation $(1 - \lambda\varphi_H)(E) = 0$. Suppose that we have $(1 - \lambda\varphi_H)(E) = 0$. By the second assertion of Lemma 4.19, we have $\Delta_j(E) = \lambda p^j \Delta_j(E)$. Hence if $\lambda p^j \neq 1$ for every $j \geq 1$, we have $E = 0$ by the third assertion of Lemma 4.19. For the third assertion, let $\lambda = p^{-j}$ for some integer $j \in [0, h]$ and let us choose a solution $\widetilde{F}$. By the same argument as the proof of the second assertion, given two solutions $\widetilde{F}$ and $\widetilde{F}'$, the difference $E = \widetilde{F} - \widetilde{F}'$ satisfies $\Delta_i(E) = 0$ for every integer i which is not equal to j. Hence we have $E \in \mathcal{K}\log^j(1+X)$ by Lemma 4.19, which implies the third assertion.

The final assertion also follows immediately by combining three assertions proved above. □

Let D^0 be a free $\mathcal{O}_{\mathcal{K}}$-module of rank one. We consider a $\mathcal{K}$-linear endomorphism φ_D of the one-dimensional $\mathcal{K}$-vector space $D := D^0 \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{K}$ which coincides with multiplication by λ. We write $\varphi_D \otimes \varphi_H$ for the tensor product of the endomorphisms φ_D and φ_H acting on $D \otimes_{\mathbb{Q}_p} \mathcal{H}_h$. Given $\Xi \in D \otimes_{\mathbb{Q}_p} \mathcal{H}_h$, we discuss solutions $\widetilde{\Xi} \in D \otimes_{\mathbb{Q}_p} \mathcal{H}_h$ of the equation:

$$(E^h_{\Xi;\lambda}) \qquad (1 - \varphi_D \otimes \varphi_H)\widetilde{\Xi} = \Xi$$

as well as the denominators of the solution $\widetilde{\Xi} \in D \otimes_{\mathbb{Q}_p} \mathcal{H}_h$.

Corollary 4.26. — *Let $h \in \mathbb{Z}_{\geq 0}$ be an integer which is equal to or greater than $-\mathrm{ord}(\lambda)$. Then we have the following statements for solutions of the equation $(E^h_{\Xi;\lambda})$ for a given $\Xi \in D \otimes_{\mathbb{Q}_p} \mathcal{H}_h$.*

1. *The equation $(E^h_{\Xi;\lambda})$ has a solution $\widetilde{\Xi} \in D \otimes_{\mathbb{Q}_p} \mathcal{H}_h$ if one of the following conditions holds:*
   (a) *We have $\lambda p^i \neq 1$ for every integer i in $[0, h]$.*
   (b) *There exists an integer j in $[0, h]$ such that we have $\lambda p^j = 1$ and $1 \otimes \Delta_j(\Xi) = 0$.*
2. *In the case (a) of the first assertion, the solution $\widetilde{\Xi}$ is unique.*
3. *In the case (b) of the first assertion, the solution $\widetilde{\Xi}$ is unique modulo $\mathcal{K} \cdot e \otimes_{\mathbb{Q}_p} \log(1 + X)^j$ where e is an $\mathcal{O}_{\mathcal{K}}$-basis of D^0.*
4. *Assume further that Ξ is contained in $D^0 \otimes_{\mathbb{Z}_p} \mathcal{H}_h^+$. Then we have the followings:*
   (a) *In the case (a) of the first assertion, the unique solution $\widetilde{\Xi}$ of the equation $(E^h_{\Xi;\lambda})$ lies in $D^0 \otimes_{\mathbb{Z}_p} p^{c_{h,\lambda}}\mathcal{H}_h^+$.*
   (b) *In the case (b) of the first assertion, there exists a solution to equation $(E^h_{\Xi;\lambda})$ that lies in $D^0 \otimes_{\mathbb{Z}_p} p^{c_{h,\lambda}}\mathcal{H}_h^+$.*

Proof. — Fix a $\mathcal{K}$-basis e of D, so that we obtain an isomorphism of $\mathcal{K}$-vector spaces endowed with a linear operator:

$$(D \otimes_{\mathbb{Q}_p} \mathcal{H}_h, \varphi_D \otimes \varphi_H) \xrightarrow{\sim} (\mathcal{H}_{h/\mathcal{K}}, \lambda\varphi_H). \tag{4.39}$$

If we denote the image of Ξ via the isomorphism (4.39) by $F = F_\Xi \in \mathcal{H}_{h/\mathcal{K}}$, we have a natural bijection between the set of solutions of $(E^h_{\Xi;\lambda})$ and the set of solutions of $(E^h_{F;\lambda})$, which proves the first three assertions. Moreover, if $\Xi \in \mathcal{O}_\mathcal{K} \cdot e \otimes_{\mathbb{Z}_p} \mathcal{H}^+_h$, then $F \in \mathcal{H}^+_{h/\mathcal{K}}$ and we can assume $\widetilde{F} \in p^{c_{h,\lambda}}\mathcal{H}^+_{h/\mathcal{K}}$, whence the last claim of the corollary. □

4.4. Interpolation. — We now present another interpretation of functions of logarithmic order h in terms of their "interpolating polynomials", as concisely described in [AV75]. We are grateful to Daniel Barsky for many communications and explanations regarding Amice–Vélu's work: we also thank him for providing us in [Bar13] with a detailed proof of [AV75, Proposition IV.4], which seems missing in the original work.

Though we omit a precise definition, we have a notion of tempered distribution of order h on the space $\mathbb{Z}_p$ for each $h \in \mathbb{Z}_{\geq 0}$, which is denoted by $\mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ (see [Col10, §II. 3] for the definition of $\mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p)$). The space $\mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ is Banach. The space $\mathscr{D}_0(\mathbb{Z}_p, \mathbb{Q}_p)$ is nothing but the space of bounded measures on $\mathbb{Z}_p$ valued in $\mathbb{Q}_p$ and we have extensions of spaces $\mathscr{D}_0(\mathbb{Z}_p, \mathbb{Q}_p) \subset \mathscr{D}_1(\mathbb{Z}_p, \mathbb{Q}_p) \subset \cdots \subset \mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p) \subset \cdots$. The following important result is known.

Theorem 4.27. — *Let μ be a distribution. Then the power series*

$$\mathscr{A}_\mu(X) = \int_{\mathbb{Z}_p} (1+X)^x \mu(x) \tag{4.40}$$

defines an element of $\mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$. *Further, for each $h \in \mathbb{Z}_{\geq 0}$, this induces an isomorphism of Banach spaces*

$$\mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p) \xrightarrow{\sim} \mathcal{H}_h, \ \ \mu \mapsto \mathscr{A}_\mu. \tag{4.41}$$

We refer the reader to [Col10, §II. 2 and §II. 2] for the proof of the above theorem. The maps (4.40) and (4.41) are often called the Amice transform.

On the left-hand side of (4.41), we note that $\mathbb{Z}_p$ contains an open subset $\mathbb{Z}_p^\times$ and the cyclotomic character χ_{cyc} induces a commutative diagram of multiplicative groups

$$\begin{tikzcd} \Gamma_{\mathrm{cyc}} \arrow[r, "\sim", "\chi_{\mathrm{cyc}}"'] \arrow[d, hook] & 1+p\mathbb{Z}_p \arrow[d, hook] \\ G_{\mathrm{cyc}} \arrow[r, "\sim", "\chi_{\mathrm{cyc}}"'] & \mathbb{Z}_p^\times . \end{tikzcd} \tag{4.42}$$

By this canonical identification, we can regard the spaces of tempered distributions of order h on $\mathbb{Z}_p^\times$ (resp. $1+p\mathbb{Z}_p$) which are denoted $\mathscr{D}_h(\mathbb{Z}_p^\times, \mathbb{Q}_p)$ (resp. $\mathscr{D}_h(1+p\mathbb{Z}_p, \mathbb{Q}_p)$) as the spaces of tempered distributions of order h on G_{cyc} (resp. Γ_{cyc}). In this article, we will denote $\mathscr{D}_h(\mathbb{Z}_p^\times, \mathbb{Q}_p)$ (resp. $\mathscr{D}_h(1+p\mathbb{Z}_p, \mathbb{Q}_p)$) by $\mathcal{H}_h(G_{\mathrm{cyc}})$ (resp. $\mathcal{H}_h(\Gamma_{\mathrm{cyc}})$).

We remark that $\Lambda(G_{\mathrm{cyc}}) \otimes \mathbb{Q}_p = \mathcal{H}_0(G_{\mathrm{cyc}})$ and $\Lambda(\Gamma_{\mathrm{cyc}}) \otimes \mathbb{Q}_p = \mathcal{H}_0(\Gamma_{\mathrm{cyc}})$. For $h \in \mathbb{Z}_{\geq 0}$, the spaces $\mathcal{H}_h(G_{\mathrm{cyc}})$ and $\mathcal{H}_h(\Gamma_{\mathrm{cyc}})$ are endowed with the valuation $v_{\mathscr{D}_h}$ defined in [Col10, §II.3], which we denote v_h by abuse of notation. Also, $\mathcal{H}_h(G_{\mathrm{cyc}})$

and $\mathcal{H}_h(\Gamma_{\mathrm{cyc}})$ are Banach spaces whose unit balls with respect to this valuation are $\mathcal{H}_h^+(G_{\mathrm{cyc}})$ and $\mathcal{H}_h^+(\Gamma_{\mathrm{cyc}})$.

Since the groups $\Gamma_{\mathrm{cyc}} \subseteq G_{\mathrm{cyc}}$ are canonically identified as open and closed subsets of $\mathbb{Z}_p$, the restriction of distributions induces inclusions:

$$\mathcal{H}_h(\Gamma_{\mathrm{cyc}}) \hookrightarrow \mathcal{H}_h(G_{\mathrm{cyc}}) \hookrightarrow \mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p)$$

which are isometries of Banach spaces. Let us define an operator $\psi_{\mathscr{D}}$ acting on $\mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ to be the operator of the restriction of a distribution to $p\mathbb{Z}_p = \mathbb{Z}_p \setminus \mathbb{Z}_p^\times$, which is defined explicitly by

$$\int_{\mathbb{Z}_p} f(x)\psi_{\mathscr{D}}\mu(x) = \int_{p\mathbb{Z}_p} f(p^{-1}x)\mu(x).$$

By definition, the operator $\psi_{\mathscr{D}}$ provides an identification

$$\mathcal{H}_h(G_{\mathrm{cyc}}) = \mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p)^{\psi_{\mathscr{D}}=0} = \{\mu \in \mathscr{D}_h(\mathbb{Z}_p, \mathbb{Q}_p) \text{ such that } \psi_{\mathscr{D}}(\mu) = 0\}.$$

Let us pass to the right-hand side of (4.41) now. Recall that we have $\mathcal{H}_h \subset \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big) \subset \mathbb{Q}_p[\![X]\!]$ and that φ_H is the Frobenius operator

$$\varphi_H \colon \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big) \longrightarrow \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$$

defined by $f(X) \mapsto f\big((1+X)^p - 1\big)$ introduced in (4.13).

It follows from [Col79, Theorem 4 and Corollary 5] that there exists a unique map $\mathscr{S} \colon \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big) \to \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ satisfying

$$\big(\varphi_H \circ \mathscr{S}\big)(f) = \sum_{\zeta^p=1} f\big(\zeta(1+X) - 1\big).$$

By [Col79, Lemma 6], we can divide $\mathscr{S}(f)$ by p to obtain an operator $\psi = p^{-1}\mathscr{S}$ satisfying, for every $f(X) \in \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)$ the relations

$$\begin{cases} \big((\varphi_H \circ \psi)f\big)(X) = \frac{1}{p}\sum\limits_{\zeta^p=1} f\big(\zeta(1+X) - 1\big), \\ \psi \circ \varphi_H = 1 \end{cases} \tag{4.43}$$

where the second relation follows by applying φ_H to $\big((\psi \circ \varphi_H)f\big)(X)$, finding

$$\begin{aligned} \big((\varphi_H \circ \psi \circ \varphi_H)f\big)(X) &= \big((\varphi_H \circ \psi)f\big)\big((1+X)^p - 1\big) \\ &= \frac{1}{p}\sum_{\zeta^p=1} f\Big(\big(\zeta(1+X)\big)^p - 1\Big) = (\varphi_H f)(X). \end{aligned}$$

As proven in [Col10, Proposition II.3.1], the isomorphism of Banach spaces (4.41) given by Amice transform transforms the operator $\psi_{\mathscr{D}}$ defined on the ledft-hand side of (4.41) into the operator ψ defined on the right-hand-side of (4.43). In fact, the Amice transform $\mathscr{A}_{\psi_{\mathscr{D}}\mu}$ of $\psi_{\mathscr{D}}\mu$ is the power series

$$\mathscr{A}_{\psi_{\mathscr{D}}\mu}(X) = \int_{\mathbb{Z}_p} (1+X)^x \psi_{\mathscr{D}}\mu(x) = \int_{\mathbb{Z}_p} \mathbf{1}_{p\mathbb{Z}_p} \cdot (1+X)^{p^{-1}x}\mu(x)$$

where $\mathbf{1}_{p\mathbb{Z}_p}$ is the characteristic power series of the open $p\mathbb{Z}_p$ (see [Col10, §II.2]). Now it suffices to observe that

$$\mathbf{1}_{p\mathbb{Z}_p} = \frac{1}{p}\sum_{\zeta^p=1}\zeta^x$$

and therefore

$$\begin{aligned}\mathscr{A}_{\psi_{\mathscr{D}}\mu}\big((1+X)^p-1\big) &= \int_{\mathbb{Z}_p}\mathbf{1}_{p\mathbb{Z}_p}\cdot(1+X)^x\mu(x) = \frac{1}{p}\int_{\mathbb{Z}_p}\sum_{\zeta^p=1}\big((1+X)\zeta\big)^x\mu(x)\\ &= \frac{1}{p}\sum_{\zeta^p=1}\mathscr{A}_\mu\big((1+X)\zeta-1\big) = \big((\varphi_H\circ\psi)\mathscr{A}_\mu\big)(X)\end{aligned}$$

which implies that $\varphi_H\mathscr{A}_{\psi_{\mathscr{D}}\mu}(X) = \varphi_H\big(\psi\mathscr{A}(X)\big)$ and then $\mathscr{A}_{\psi_{\mathscr{D}}\mu}(X) = \psi\mathscr{A}(X)$.

As a consequence, we obtain an isomorphism

$$\mathcal{H}_h(G_{\mathrm{cyc}}) \cong \mathcal{H}_h^{\psi=0} = \mathcal{H}_h \cap \mathrm{Conv}\big(B(0,1)_{\mathbb{C}_p}\big)^{\psi=0}. \tag{4.44}$$

Finally, for a finite extension $\mathcal{K}$ of $\mathbb{Q}_p$, we denote by $\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$ (resp. $\mathcal{H}^+_{h/\mathcal{K}}(G_{\mathrm{cyc}})$) the $\Lambda(G_{\mathrm{cyc}})$-module $\mathcal{K}\otimes_{\mathbb{Q}_p}\mathcal{H}_h(G_{\mathrm{cyc}})$ (resp. $\mathcal{O}_{\mathcal{K}}\otimes_{\mathbb{Z}_p}\mathcal{H}^+_h(G_{\mathrm{cyc}})$). All the discussion and the results above can be generalized by formal arguments to these spaces with coefficients $\mathcal{K}$.

Choosing a topological generator γ_0 of Γ_{cyc}, we have a non-canonical isomorphism of topological $\mathbb{Z}_p$-algebras

$$\Lambda(\Gamma_{\mathrm{cyc}}) \xrightarrow{\sim} \mathbb{Z}_p[\![Y]\!],\ \gamma_0-1\mapsto Y. \tag{4.45}$$

The isomorphism $\Lambda(G_{\mathrm{cyc}})\cong\Lambda(\Gamma_{\mathrm{cyc}})\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]$ induces therefore an isomorphisms

$$\begin{aligned}\mathcal{H}_h(G_{\mathrm{cyc}}) &\cong \mathcal{H}_h(\Gamma_{\mathrm{cyc}})\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]\\ \mathcal{H}^+_h(G_{\mathrm{cyc}}) &\cong \mathcal{H}^+_h(\Gamma_{\mathrm{cyc}})\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times].\end{aligned}$$

Let us choose and fix a topological generator γ_0 of Γ_{cyc} and denote the generator $\chi_{\mathrm{cyc}}(\gamma_0)$ of $1+p\mathbb{Z}_p$ by u. For every $j\in\mathbb{Z}_{\geq 0}$, we set

$$\omega_n^{[j]} = \omega_n^{[j]}(Y) = \big(u^{-j}(1+Y)\big)^{p^n}-1.$$

We sometimes denote $\omega_n^{[j]}$ by ω_n for short. For $l,l'\in\mathbb{Z}$ with $l\leq l'$, we define

$$\Omega_n^{[l,l']} = \Omega_n^{[l,l']}(Y) = \prod_{j=l}^{l'}\omega_n^{[j]}.$$

Via the non-canonical identification $\mathbb{Z}_p[\![Y]\!]\xrightarrow{\sim}\Lambda(\Gamma_{\mathrm{cyc}})$ which is induced by the topological generator γ_0 of Γ_{cyc} sending $1+Y$ to $\chi_{\mathrm{cyc}}(\gamma_0)$, we regard $\omega_n^{[j]}(Y)$ and $\Omega_n^{[l,l']}(Y)$ as elements of $\mathcal{H}_0^+(\Gamma_{\mathrm{cyc}})$. For later use, we remark that for a fixed n and for different $j\neq j'$ the polynomials $\omega_n^{[j]}(Y)$ and $\omega_n^{[j']}(Y)$ have no common factor in $\mathbb{Z}_p[\![Y]\!]$.

Lemma 4.28. — *For every $n \geq 0$, we have*

$$v_h(\omega_n^{[j]}) = \begin{cases} 0 & \text{if} \quad h = 0, \\ n + \min\{\mathrm{ord}(j) + 1, j\} & \text{if} \quad h \geq 1. \end{cases}$$

In particular, for $l, l' \in \mathbb{Z}$ with $l \leq l'$ and for each $h \geq 1$,

$$\lim_{n \to +\infty} v_h(\Omega_n^{[l,l']}) = +\infty.$$

Proof. — Recall that $\Omega_n^{[l,l']}$ was defined as a product of $(l' - l + 1)$ polynomials in $\mathcal{H}_0^+(\Gamma_{\mathrm{cyc}})$. Since we have

$$v_h(F \cdot G) \geq v_h(F) + v_0(G) \qquad \text{for all } F, G \in \mathcal{H}_0^+(\Gamma_{\mathrm{cyc}}),$$

as proven in (4.1), the second statement follows readily from the first one.

Hence we shall prove the first statement in the rest of the proof. We start by observing that $\ell(i) \geq \mathrm{ord}(i) + 1$ by definition.

Since we have

$$\omega_n^{[j]} = \left(u^{-j}(1+Y)\right)^{p^n} - 1 = \sum_{i=1}^{p^n} u^{-jp^n} \binom{p^n}{i} Y^i + (u^{-jp^n} - 1),$$

we recall the following estimation for each $1 \leq i \leq p^n$ in order to compute $v_h(\omega_n^{[j]})$:

$$\mathrm{ord}\left(u^{-jp^n}\binom{p^n}{i}\right) + h\ell(i) = \mathrm{ord}\binom{p^n}{i} + h\ell(i) = n - \mathrm{ord}(i) + h\ell(i).$$

Here, the equality $\mathrm{ord}\binom{p^n}{i} = n - \mathrm{ord}(i)$ is deduced as discussed in the proof of Proposition 4.20.

First, the case $h \leq 0$ is obvious since the v_h-adic valuation is simply the p-adic valuation in this case.

Let us consider the case $h \geq 1$ now. By the definition of v_h, we have

$$v_h(\omega_n^{[j]}) = \min\Big\{\mathrm{ord}(a_0),\ \min\{n - \mathrm{ord}(i) + h\ell(i) \mid 1 \leq i \leq p^n\}\Big\} \tag{4.46}$$

where a_0 is the constant term of $\omega_n^{[j]}$. In order to estimate the internal min, we modify as

$$n - \mathrm{ord}(i) + h\ell(i) = n + (\ell(i) - \mathrm{ord}(i)) + (h-1)\ell(i).$$

Note that we have $\ell(i) - \mathrm{ord}(i) \geq 1$ and $\ell(i) \geq$, which is immediate from the definition. Thus we have

$$n - \mathrm{ord}(i) + h\ell(i) \geq n + h. \tag{4.47}$$

On the other hand, since we have $a_0 = u^{-jp^n} - 1$, we have

$$\mathrm{ord}(a_0) = \mathrm{ord}(-jp^n \log u) = n + \mathrm{ord}(j) + 1. \tag{4.48}$$

By (4.46), (4.46) and (4.48), we have

$$v_h(\omega_n^{[j]}) = n + \min\{\mathrm{ord}(j) + 1, h\}.$$

This completes the proof of the lemma. □

Remark 4.29. — The polynomials $\omega_n^{[j]}$ and $\Omega_n^{[l,l']}$ appear in [AV75, §IV], although they are not called in this way. In the following, we will use some result from Amice–Vélu's paper and we stress that our space $\mathcal{H}_h(\Gamma_{\mathrm{cyc}})$ is closely related to the space B_{h+1} in [AV75], in the sense that $\mathcal{H}_h(\Gamma_{\mathrm{cyc}}) \subseteq \mathbb{Q}_p[\![Y]\!]$ are power series which are $O\big(\log^h(1+Y)\big)$ whereas B_{h+1} are power series in the variable $Y-1$ which are $o\big(\log^{h+1}(1+(Y-1))\big)$. The change of variables $Y \mapsto Y-1$ induces inclusions

$$\ldots \subseteq \mathcal{H}_h(\Gamma_{\mathrm{cyc}}) \subseteq B_{h+1} \subseteq \mathcal{H}_{h+1}(\Gamma_{\mathrm{cyc}}) \subseteq B_{h+2} \subseteq \ldots.$$

(see Lemme IV.2 on [AV75, page 128])

Since the $\Omega_n^{[l,l']}$ are a family of distinguished polynomials of increasing degree, completeness of $\mathcal{H}_0^+(\Gamma_{\mathrm{cyc}}) \cong \mathbb{Z}_p[\![Y]\!]$ in the (p,Y)-adic topology shows that for $l, l' \in \mathbb{Z}$ with $l \leq l'$ there is a ring isomorphism

$$\mathcal{H}_0^+(\Gamma_{\mathrm{cyc}}) \cong \varprojlim_n \mathcal{H}_0^+(\Gamma_{\mathrm{cyc}})/(\Omega_n^{[l,l']}).$$

We want to generalize this isomorphism from $\mathcal{H}_0^+(\Gamma_{\mathrm{cyc}})$ to $\mathcal{H}_h^+(\Gamma_{\mathrm{cyc}})$ with arbitrary $h \geq 0$. The main technical problem is that $\mathcal{H}_h^+(\Gamma_{\mathrm{cyc}})$ is not finitely generated as $\mathcal{H}_0^+(\Gamma_{\mathrm{cyc}})$-module (see Remark 4.9).

Let $\mathcal{Q}_n^{[l,l']}$ be the set of monic polynomials Q in $\mathcal{O}_{\mathcal{K}}[\![Y]\!]$ which divide $\prod_{j=l}^{l'}((1+Y)^{p^n} - (1+p)^{jp^n})$ and let $\mathcal{Q}^{[l,l']} = \cup_{n=1}^{\infty} \mathcal{Q}_n^{[l,l']}$.

We have the following important results of Amice–Vélu:

Proposition 4.30. — *Let $F \in \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$ and let us choose $l, l' \in \mathbb{Z}$ such that $h = l' - l$. Suppose that F is contained in $Q\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$ for every element of $\mathcal{Q}^{[l,l']}$. Then we have $F = 0$.*

Proof. — This proposition is proved by induction with respect to h using [PR94, Lemme 1.3.1]. For the case $h = 0$ (i.e. the case $l = l'$), the proposition follows from the fact that $\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}}) = \Lambda(G_{\mathrm{cyc}}) \otimes_{\mathbb{Z}_p} \mathcal{K}$ and by the Weierstrass Preparation theorem. Now suppose that the proposition holds true for $h-1$. Let us apply [PR94, Lemme 1.3.1] with $f = F$ and $\alpha = l'$. Since F is contained in $Q\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$ for every monic polynomial Q in $\mathcal{O}_{\mathcal{K}}[\![Y]\!]$ which divides $((1+Y)^{p^n} - (1+p)^{l'p^n})$ for some n, [PR94, Lemme 1.3.1] implies that there exists $g \in \mathcal{H}_{h-1/\mathcal{K}}(G_{\mathrm{cyc}})$ such that $F = (l' - \log(1+Y)/\log(1+p))g$. It might seem that the statement of [PR94, Lemme 1.3.1] excludes the case where l' is a p-adic unit. However, as remarked by Perrin-Riou before the proof of [PR94, Lemme 1.3.1], note that we can apply the [PR94, Lemme 1.3.1] for $\alpha \in \mathbb{Z}$ since u^α is always well-defined. By the condition on F, g is contained in $Q\mathcal{H}_{h-1/\mathcal{K}}(G_{\mathrm{cyc}})$ for every element of $\mathcal{Q}^{[l,l'-1]}$. Then we have $g = 0$ by the induction hypothesis. This completes the proof. □

Proposition 4.31. — *Let us choose $l, l' \in \mathbb{Z}$ with $l \leq l'$ and put $h = l' - l$. Let $\{G_{n,j} \in \mathcal{K}[Y]\}_{n \in \mathbb{Z}_{\geq 1}, l \leq j \leq l'}$ be a sequence of polynomials satisfying the following conditions:*

(i) *For each j satisfying $l \leq j \leq l'$, $\|p^{nh} G_{n,j}\|$ is bounded when n varies.*

(ii) *For each j satisfying $l \leq j \leq l'$ and for each $n \in \mathbb{Z}_{\geq 1}$, $G_{n+1,j} - G_{n,j} \equiv 0$ modulo $\omega_n \mathcal{K}[Y]$.*

(iii) *For each j satisfying $l \leq j \leq l'$,*

$$\left\| p^{n(h-j)} \sum_{k=l}^{j} (-1)^{j-k} \binom{j}{k} G_{n,l+k}((1+X)-(1+p)^k) \right\|$$

is bounded when n varies.

Then there exists a unique element $F \in \mathcal{H}_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}})$ such that $F \equiv G_{n,j}$ modulo $\omega_n^{[j]} \mathcal{H}_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}})$ for every $n \in \mathbb{Z}_{\geq 1}$ and for every j satisfying $l \leq j \leq l'$.

Proof. — For the proof of this proposition, we refer to [PR94, Lemme 1.2.2]. We note that the definition of $\mathcal{H}_h$ is different from ours but the proof works in the same way. We also remark that the uniqueness part of the statements in Proposition 4.31 is due to Proposition 4.30. □

5. Construction of families of points

In this section, we will prove our Main Theorem. First, in §5.1, we prepare basic properties of Bloch–Kato exponentials. In Section 5.2, we will prepare some facts on Iwasawa algebra and Galois cohomology with coefficients in Iwasawa modules. Then, in §5.3, we will reduce our main theorem to Theorem 5.8 through two reduction steps. Finally, in §5.4, we integrate what we established in §4 and we give a proof of Theorem 5.8 and thus we finalize the proof of the main theorem.

5.1. Preliminaries on Bloch–Kato exponential. — Recall the fundamental exact sequence from [BK90, Proposition 1.17]

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{i} B_{\mathrm{cris}} \xrightarrow[x \mapsto \left((1-\varphi)x, \overline{x}\right)]{b} B_{\mathrm{cris}} \oplus B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+} \longrightarrow 0 \tag{5.1}$$

on which the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ acts. As explained in [BK90, Remark 1.18], given any finite dimensional $\mathbb{Q}_p$-representation V of $G_{\mathbb{Q}_p}$, the sequence (5.1)$\otimes_{\mathbb{Q}_p} V$:

$$0 \longrightarrow V \longrightarrow (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V) \xrightarrow{b \otimes \mathrm{id}} \left(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right) \oplus (B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V) \longrightarrow 0, \tag{5.2}$$

of complete $\mathbb{Q}_p$-modules admits a continuous section of the surjection $b \otimes \mathrm{id}$. Therefore, by [NSW08, Lemma 2.7.2], it induces a long exact sequence of Galois cohomology.

Definition 5.1. — Let V be a finite dimensional $\mathcal{K}$-vector space endowed with a continuous $\mathcal{K}$-linear action of $G_{\mathbb{Q}_p}$. Write $D_{\mathrm{dR}}(V)$ and $D_{\mathrm{cris}}(V)$ for $(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ and $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ respectively. For each n, we call the $\mathcal{K}$-linear homomorphism:

$$\exp_{V,\mathbb{Q}_p(\mu_{p^n})} : D_{\mathrm{cris}}(V) \oplus \left(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\right) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n}) \longrightarrow H^1(\mathbb{Q}_p(\mu_{p^n}), V) \tag{5.3}$$

obtained as the connecting homomorphism of the short exact sequence (5.2) the Bloch–Kato exponential map for V over $\mathbb{Q}_p(\mu_{p^n})$. Sometimes, we also define by abuse of notation:

$$\exp_{V,\mathbb{Q}_p(\mu_{p^n})} : \big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n}) \longrightarrow H^1(\mathbb{Q}_p(\mu_{p^n}), V) \tag{5.4}$$

the $\mathcal{K}$-linear homomorphism obtained by restricting the map (5.3) to the direct summand $\big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n})$. For any $\mathcal{K}$-subspace D of $D_{\mathrm{dR}}(V)$, we define by abuse of notation:

$$\exp_{V,\mathbb{Q}_p(\mu_{p^n})} : D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n}) \longrightarrow H^1(\mathbb{Q}_p(\mu_{p^n}), V) \tag{5.5}$$

the $\mathcal{K}$-linear homomorphism obtained by composing the map $D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n}) \hookrightarrow D_{\mathrm{dR}}(V) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n}) \twoheadrightarrow \big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n})$ and (5.4). We often denote $\exp_{V,\mathbb{Q}_p}$ by $\exp_V$ when $n = 0$.

Let us recall the following lemma.

Lemma 5.2. — *Let V be a finite dimentional $\mathcal{K}$-vector space with continuous $\mathcal{K}$-linear action of $G_{\mathbb{Q}_p}$. Then, we have the following commutative diagram for every $n \geq 1$:*

$$\begin{tikzcd}
\big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^{n+1}}) \arrow[r, "\exp_{V,\mathbb{Q}_p(\mu_{p^{n+1}})}"] \arrow[d, "\mathrm{id}\otimes\mathrm{Tr}_n^{n+1}"'] & H^1(\mathbb{Q}_p(\mu_{p^{n+1}}), V) \arrow[d, "\mathrm{Cores}_n^{n+1}"] \\
\big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^n}) \arrow[r, "\exp_{V,\mathbb{Q}_p(\mu_{p^n})}"] & H^1(\mathbb{Q}_p(\mu_{p^n}), V).
\end{tikzcd}$$

Proof. — As discussed in [NSW08, page 138], the corestriction map for continuous Galois cohomology enjoys the same properties as the usual one for discrete modules. The statement of the lemma is then a consequence of the commutativity of the diagram on [NSW08, page 48]. □

We choose and fix a basis $\boldsymbol{\zeta}$ of $\mathbb{Z}_p(1)$. We prepare the following lemma to discuss a congruence of Bloch–Kato exponential between different Tate twists.

Lemma 5.3. — *Let V be a finite dimentional $\mathcal{K}$-vector space of dimension greater than one with continuous $\mathcal{K}$-linear action of $G_{\mathbb{Q}_p}$. We take a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_{\mathcal{K}}$-lattice T of V. Suppose that the action of $G_{\mathbb{Q}_p(\mu_p)}$ on $T\varpi T$ is irreducible.*

Then, we have a Tate twist:

$$\begin{aligned} &H^1(\mathbb{Q}_p(\mu_{p^n}), T)/p^m H^1(\mathbb{Q}_p(\mu_{p^n}), T) \\ &\qquad\qquad \underset{\otimes\zeta_{p^m}}{\xrightarrow{\sim}} H^1\big(\mathbb{Q}_p(\mu_{p^n}), T(1)\big)/p^m H^1\big(\mathbb{Q}_p(\mu_{p^n}), T(1)\big), \end{aligned} \tag{5.6}$$

for any pair of integers m, n satisfying $0 \leq m \leq n$ where ζ_{p^m} is the primitive p^m-th root of unity occurring as the m-th coordinate of $\boldsymbol{\zeta}$.

Proof. — The group μ_{p^m} has trivial action of $G_{\mathbb{Q}_p(\mu_{p^n})}$. Thus, when we have $n \geq m$, we have a Tate twist:

$$H^1(\mathbb{Q}_p(\mu_{p^n}), T/p^m T) \underset{\otimes \zeta_{p^m}}{\xrightarrow{\sim}} H^1\big(\mathbb{Q}_p(\mu_{p^n}), T(1)/p^m T(1)\big). \tag{5.7}$$

Note that $H^1(\mathbb{Q}_p(\mu_{p^n}), T)/p^m H^1\big(\mathbb{Q}_p(\mu_{p^n}), T\big)$ is isomorphic to $H^1(\mathbb{Q}_p(\mu_{p^n}), T/p^m T)$ by the assumption of the lemma. Hence, when we have $n \geq m$, (5.7) induces the desired Tate twist (5.6). This completes the proof of the lemma. □

5.2. Preliminaries on commutative algebra and Galois cohomology. — Later, we will need the following lemma:

Proposition 5.4. — *Let M be a module over a ring $R := \mathcal{O}_{\mathcal{K}}[\![X_1, X_2, \ldots, X_m]\!]$ for some non-negative integer m which is free of finite rank s over R with continuous action of G_k for a finite extension k of $\mathbb{Q}_p$. Here $\mathcal{K}$ is another finite extension of $\mathbb{Q}_p$ whose residue field is denoted by $\mathbb{F}$. Assume that $H^0(k, \overline{M}) = H^0(k, \overline{M}^*(1)) = 0$ for $\overline{M} := M/\mathfrak{M}_R M \cong \mathbb{F}^{\oplus s}$ where $\mathfrak{M}_R$ is the maximal ideal of R. Then, $H^1(k, M)$ is free of rank $s \cdot [k : \mathbb{Q}_p]$ over R.*

Proof. — We prove the lemma by induction with respect to m.

When $m = 0$, we have $R = \mathcal{O}_{\mathcal{K}}$. By [Tat76, Proposition (2.3)] and by the assumption $H^0(k, \overline{M}) = 0$, $H^1(k, M)$ is a free R-module. Also, by the local Tate duality theorem and by the assumption $H^0(k, \overline{M}^*(1)) = 0$, we have $H^2(k, \overline{M}) = 0$. Then, since $H^0(k, \overline{M}) = H^2(k, \overline{M}) = 0$, the local Euler–Poincaré characteristic formula implies $H^1(k, \overline{M}) \cong \mathbb{F}^{\oplus s}$. Since $H^2(k, \overline{M}) = 0$, we have $H^2(k, M) = 0$. Hence we have $H^1(k, M)/\varpi H^1(k, M) \cong H^1(k, \overline{M})$, which implies that the rank of $H^1(k, M)$ over R is equal to s by Nakayama lemma. Hence the case $m = 0$ of the lemma holds.

Let us prove the lemma for general m, assuming that the lemma is valid when the number of variables is less than m. Choose a uniformizer ϖ of $\mathcal{K}$ and put $P_i = (1+X_m)-(1+\varpi)^i$ for each $i \in \mathbb{N}$. Let us consider the set of principal ideals $\{(P_i)\}_{i\in\mathbb{Z}}$ and note that $R/(P_i)$ is isomorphic to $\mathcal{O}_{\mathcal{K}}[\![X_1, X_2, \ldots, X_{m-1}]\!]$ for every i. Let us take a free resolution of $H^1(k, M)$ with the minimal possible number of generators q as follows:

$$R^r \longrightarrow R^q \longrightarrow H^1(k, M) \longrightarrow 0. \tag{5.8}$$

Note that the cokernel of the injection

$$H^1(k, M)/(P_i)H^1(k, M) \hookrightarrow H^1(k, M/(P_i)M)$$

obtained by taking the Galois cohomology of

$$0 \longrightarrow M \longrightarrow M \longrightarrow M/(P_i)M \longrightarrow 0$$

is a submodule of $H^2(k, M)$. Since we have $H^2(k, \overline{M}) = 0$ as discussed above, Nakayama lemma implies $H^2(k, M) = 0$. Hence we obtain

$$H^1(k, M)/(P_i)H^1(k, M) \cong H^1(k, M/(P_i)M) \tag{5.9}$$

for every $i \in \mathbb{Z}$. Thus by the induction hypothesis, (5.8) $\otimes_R R/(P_i)$ yields the isomorphism $(R/(P_i))^s \cong H^1(k, M/(P_i)M)$ for every $i \in \mathbb{Z}$. Since there exists an integer i

such that $H^1(k,M)/(P_i)H^1(k,M)$ is generated by s elements over $R/(P_i)$, we have $q=s$ by Nakayama's lemma and the hypothesis of the minimality of q. Also, since all matrix components of the $r\times q$-matrix for the presentation (5.8) is contained in the ideal (P_i) for every i. By $\bigcap_{i\in\mathbb{N}}(P_i)=0$, all matrix components are zero. This means that we must take $r=0$, which completes the proof of the lemma. □

Suppose that M is a finitely generated module over $\mathcal{O}_{\mathcal{K}}[\![Y_1,Y_2]\!]=\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]$. Let us denote by $\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}$ the base extension $\Lambda(\Gamma_{\mathrm{cyc}})\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathcal{K}}$. When we identify $\mathbb{Z}_p[\![Y_2]\!]$ with $\Lambda(\Gamma_{\mathrm{cyc}})$ as in (4.45), we define a formal tensor product $M\widehat{\otimes}_{\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}})$ to be

$$M\widehat{\otimes}_{\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}}):=\varprojlim_n(M/\mathfrak{M}_1^nM)\otimes_{\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}}) \tag{5.10}$$

where $\mathfrak{M}_1$ is the maximal ideal of $\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]=\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}$. Note that M is naturally regarded as a $\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}$-module through an injection $\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}\hookrightarrow\mathcal{O}_{\mathcal{K}}[\![Y_1,Y_2]\!]$.

Suppose further that M is endowed with a structure of $\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]$-module. Then M is regarded as a module over $\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}\cong\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]$. By the isomorphism $\mathcal{H}^+_h(G_{\mathrm{cyc}})_{/\mathcal{K}}\cong\mathcal{H}^+_h(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]$ given after (4.45), we can define

$$M\widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(G_{\mathrm{cyc}}):=\varprojlim_n\Big((M/\mathfrak{M}_1^nM)\otimes_{\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(G_{\mathrm{cyc}})\Big)$$

when M is a finitely generated over $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}$. Further, we define

$$M\widehat{\otimes}_{\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}}):=(M\widehat{\otimes}_{\Lambda(\Gamma_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(\Gamma_{\mathrm{cyc}}))\otimes_{\mathcal{O}_{\mathcal{K}}}\mathcal{K},$$
$$M\widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}}):=(M\widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}}\mathcal{H}^+_{h/\mathcal{K}}(G_{\mathrm{cyc}}))\otimes_{\mathcal{O}_{\mathcal{K}}}\mathcal{K}.$$

We also remark that the definition of the formal tensor product above does not change even if we replace the projective system $\{M/\mathfrak{M}_1^nM\}_n$ by a projective system $\{M/I_nM\}_n$ where I_n is the decreasing set of ideals of $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]$ such that $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]/I_n$ is finitely generated by $\mathcal{O}_{\mathcal{K}}$ and $\cap_nI_n=0$.

Let us endow $\Lambda(G_{\mathrm{cyc}})$ with the structure of a $G_{\mathbb{Q}_p}$-module via the surjection $G_{\mathbb{Q}_p}\twoheadrightarrow G_{\mathrm{cyc}}$. The module $\Lambda(G_{\mathrm{cyc}})$ equipped with this $G_{\mathbb{Q}_p}$-action is denoted by $\Lambda(G_{\mathrm{cyc}})^\sharp$. Note that, for any profinite $\mathbb{Z}_p$-module N, we have

$$H^1\big(\mathbb{Q}_p,N\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp\big)\cong\varprojlim_nH^1\big(\mathbb{Q}_p(\mu_{p^n}),N\big) \tag{5.11}$$

by Shapiro's lemma on Galois cohomology.

Corollary 5.5. — *Let $R=\mathcal{O}_{\mathcal{K}}[\![Y_1,Y_2]\!]=\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]$. Let $\mathcal{P}$ be an infinite subset of $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]$ such that any $P\in\mathcal{P}$ is an irreducible monic polynomial of degree one in $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]$. Let us choose $l,l'\in\mathbb{Z}$ such that $h=l'-l$ and let $\mathcal{Q}^{[l,l']}$ be as defined before Proposition 4.30 for $Y=Y_2$. Let us consider a set of height two ideals of R*

$$\mathcal{I}:=\{(P,Q)\mid P\in\mathcal{P},\,Q\in\mathcal{Q}^{[l,l']}\}.$$

and a free $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]$-module $\mathbf{T}$ of finite rank on which the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ acts continuously. Assume that the residual representation $\mathbf{T}/\mathfrak{M}\mathbf{T}$ is irreducible as $G_{\mathbb{Q}_p}$-module where $\mathfrak{M}$ is the maximal ideal of $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]$. Then

$$H^1(\mathbb{Q}_p, \mathbf{T}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp) \underset{\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}}{\widehat{\otimes}} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$$
$$\longrightarrow \prod_{(P,Q)\in\mathcal{I}} H^1(\mathbb{Q}_p, \mathbf{T}/P\mathbf{T}\otimes_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp/Q\Lambda(G_{\mathrm{cyc}})^\sharp) \underset{\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]/(Q)}{\otimes} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})/Q\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$$

is injective.

Proof. — Recall that $N = H^1(\mathbb{Q}_p, \mathbf{T}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp)$ is a module over $\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!] \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{O}_{\mathcal{K}}[(\mathbb{Z}/p\mathbb{Z})^\times] = \mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}$. For each i satisfying $0 \le i \le p-2$, we denote by $N^{(i)}$ the submodule of N on which the group $(\mathbb{Z}/p\mathbb{Z})^\times$ acts by the character ω^i. We have a character decomposition $N = \bigoplus_{i=0}^{p-2} N^{(i)}$. We remark that $N^{(i)}$ is a module over $\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]$ and that N is free of rank 2 over $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}$ if and only if $N^{(i)}$ is free of rank 2 over $\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]$. Since we assume that the residual representation $\mathbf{T}/\mathfrak{M}\mathbf{T}$ is irreducible as $G_{\mathbb{Q}_p}$-module, for each i satisfying $0 \le i \le p-2$, $N^{(i)}$ is free of finite rank 2 over $\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]$ by applying Proposition 5.4 with $M = N^{(i)}$ and $R = \mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]$.

Similarly, for each $P \in \mathcal{P}$ and for each $Q \in \mathcal{Q}^{[l,l']}$,

$$N_{P,Q} = H^1(\mathbb{Q}_p, (\mathbf{T}/P\mathbf{T})\otimes_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp/Q\Lambda(G_{\mathrm{cyc}})^\sharp)$$
$$\left(\text{resp. } N_Q = H^1(\mathbb{Q}_p, \mathbf{T}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp/Q\Lambda(G_{\mathrm{cyc}})^\sharp)\right)$$

has the character decomposition $N_{P,Q} = \bigoplus_{i=0}^{p-2} N_{P,Q}^{(i)}$ (resp. $N_Q = \bigoplus_{i=0}^{p-2} N_Q^{(i)}$) with respect to the action of $(\mathbb{Z}/p\mathbb{Z})^\times$. As in the above argument, $N_{P,Q}^{(i)}$ (resp. $N_Q^{(i)}$) is free of rank 2 over $\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]/(P,Q)$ (resp. $\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]/(Q)$).

Let us fix a basis v_1, v_2 of $N = H^1(\mathbb{Q}_p, \mathbf{T}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp)$ over $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}$. This allows us to identify $H^1(\mathbb{Q}_p, \mathbf{T}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp) \underset{\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}}{\widehat{\otimes}} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$ with

$$\left(\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!] \underset{\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]}{\widehat{\otimes}} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})\right)^{\oplus 2} = \left(\mathcal{O}_{\mathcal{K}}[\![Y_1]\!] \underset{\mathcal{O}_{\mathcal{K}}}{\widehat{\otimes}} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})\right)^{\oplus 2}.$$

For each $P \in \mathcal{P}$ and for each $Q \in \mathcal{Q}^{[l,l']}$, the images of v_1, v_2 in $N_{P,Q}$ (resp. N_Q) give us a basis of this module over $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}/(P,Q)$ (resp. $\mathcal{O}_{\mathcal{K}}[\![Y_1]\!]\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\Lambda(G_{\mathrm{cyc}})_{/\mathcal{K}}/(Q)$). This allows us to identify $N_{P,Q}$ (resp. N_Q) with

$$\left(\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]/(P,Q) \underset{\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]/(Q)}{\widehat{\otimes}} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})/Q\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})\right)^{\oplus 2},$$
$$\left(\text{resp. } \left(\mathcal{O}_{\mathcal{K}}[\![Y_1, Y_2]\!]/(Q) \underset{\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]/(Q)}{\widehat{\otimes}} \mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})/Q\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})\right)^{\oplus 2}\right).$$

Hence, by Proposition 4.30, an element of $H^1(\mathbb{Q}_p, \mathbf{T}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})^\sharp)\underset{\mathcal{O}_{\mathcal{K}}[\![Y_2]\!]}{\widehat{\otimes}}\mathcal{H}_{h/\mathcal{K}}(G_{\mathrm{cyc}})$ which is zero at all of its specializations in $\mathcal{I}$ must be zero. This completes the proof the injectivity of the map in question. □

5.3. Basic reduction steps for the proof of Main Theorem. — First, we restate our Main Theorem stated in Introduction in a more precise statement with the integral structure, as follows:

Let us recall that we defined a formal $\mathbf{A}^0_{/\mathcal{K}}$-module structure $\mathbf{D}^0$ in $\mathcal{D}^0$ in Corollary 3.6. We denote the specialization $\mathbf{D}^0/(\mathfrak{m}_x\cap\mathbf{A}^0_{/\mathcal{K}})\mathbf{D}^0$ of $\mathbf{D}^0$ at $x\in Z_{(k_0,i)}[r_0]$ by D^0_x for short and we define D_x to be $D^0_x\otimes_{\mathcal{O}_{\mathcal{K}}}\mathcal{K}$. Note that D_x is canonically isomorphic to $D_{\mathrm{cris}}(V_x)^{\varphi=a_p(f_x)}$. Throughout the rest of the paper, $\overline{\rho}: G_{\mathbb{Q}}\longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ denotes the residual representation associated to forms f_x. Note that, thanks to the description of irreducible mod p modular representation by Serre conjecture, $\overline{\rho}$ is irreducible as a representation of $G_{\mathbb{Q}_p}$ if and only if $\overline{\rho}$ is irreducible as a representation of $G_{\mathbb{Q}_p(\mu_p)}$.

Main Theorem is stated as follows.

***Theorem 5.6* (Main Theorem).** — *Let $\mathbf{A}^0_{/\mathcal{K}}$, $\mathbf{T}\cong(\mathbf{A}^0_{/\mathcal{K}})^{\oplus 2}$ and $Z_{(k_0,i)}[r_0]$ be as before. Let us assume the conditions* (NonInt) *and* (DoubRt) *for our Coleman family and let $h\in\mathbb{Z}_{\geq 0}$ be equal to or greater than the slope of our Coleman family $\mathscr{V}$. Assume that $\overline{\rho}$ is irreducible when restricted to $G_{\mathbb{Q}_p(\mu_p)}$.*

Then, we have a unique $\mathbf{A}^0_{/\mathcal{K}}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$-linear big exponential map

$$\mathrm{EXP}_{\mathbf{T}}:\mathbf{D}^0\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})\longrightarrow\left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}),\mathbf{T}\big)\right)\widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})}p^{c_{h,\lambda}}\mathcal{H}^+_h(G_{\mathrm{cyc}})$$

such that, for each $x\in Z_{(k_0,i)}[r_0]$ and for each arithmetic character $\chi^j_{\mathrm{cyc}}\phi\colon G_{\mathrm{cyc}}\longrightarrow\overline{\mathbb{Q}}_p^\times$ with $j\in\mathbb{Z}_{\geq 1}$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{D}^0\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}}) & \xrightarrow{\mathrm{EXP}_{\mathbf{T}}} & \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}),\mathbf{T}\big)\right)\widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})}p^{c_{h,\lambda}}\mathcal{H}^+_h(G_{\mathrm{cyc}}) \\
{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)}\big\downarrow & & \big\downarrow{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)} \\
D_x\otimes D_{\mathrm{dR}}(\chi^j_{\mathrm{cyc}}\phi) & \longrightarrow & H^1(\mathbb{Q}_p, V_x\otimes\chi^j_{\mathrm{cyc}}\phi)
\end{array}$$

where the bottom horizontal map is equal to $(-1)^j(j-1)!E_p(f_x,j,\phi)\exp_{V_x\otimes\chi^j_{\mathrm{cyc}}\phi}$ with $E_p(f_x,j,\phi)$ as defined in (1.1).

In this section, we gradually reduce our Main Theorem (Theorem 5.6) to more basic situations.

(Reduction Step I)

Let us reduce Theorem 5.6 to the following theorem.

Theorem 5.7. — *Let* $\mathbf{A}^0_{/\mathcal{K}}$, $\mathbf{T} \cong (\mathbf{A}^0_{/\mathcal{K}})^{\oplus 2}$. *Let us assume the conditions* (NonInt) *and* (DoubRt) *for our Coleman family and let* $h \in \mathbb{Z}_{\geq 0}$ *be equal to or greater than the slope of our Coleman family* $\mathscr{V}$. *Assume that* $\overline{\rho}$ *is irreducible when restricted to* $G_{\mathbb{Q}_p(\mu_p)}$. *We fix an integer* h' *satisfying* $h' \geq h$.

Then, we have a unique $\mathbf{A}^0_{/\mathcal{K}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}})$*-linear big exponential map*

$$\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}} : \mathbf{D}^0 \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) \longrightarrow \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}\big) \right) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}} \mathcal{H}^+_h(G_{\mathrm{cyc}})$$

such that, for each $x \in Z_{(k_0,i)}[r_0]$ *and for each arithmetic character* $\chi^j_{\mathrm{cyc}}\phi \colon G_{\mathrm{cyc}} \longrightarrow \overline{\mathbb{Q}}_p^{\times}$ *with* $j \in [1, 1+h']$, *we have the following commutative diagram:*

$$\begin{array}{ccc}
\mathbf{D}^0 \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) & \longrightarrow & \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}\big) \right) \widehat{\otimes}_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}} \mathcal{H}^+_h(G_{\mathrm{cyc}}) \\
{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)} \big\downarrow & & \big\downarrow {\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)} \\
D_x \otimes D_{\mathrm{dR}}(\chi^j_{\mathrm{cyc}}\phi) & \longrightarrow & H^1(\mathbb{Q}_p, V_x \otimes \chi^j_{\mathrm{cyc}}\phi)
\end{array}$$

where the bottom horizontal map is equal to $(-1)^j (j-1)! E_p(f_x, j, \phi) \exp_{V_x \otimes \chi^j_{\mathrm{cyc}}\phi}$ *with* $E_p(f_x, j, \phi)$ *as defined in* (1.1).

Proof of Theorem 5.6. — Theorem 5.6 is deduced from Theorem 5.7 immediately. In fact, thanks to the uniqueness of the map $\mathrm{EXP}^{[1,1+h']}$ satisfying $h' \geq h$ claimed in Theorem 5.7, we have $\mathrm{EXP}^{[1,1+h]}_{\mathbf{T}} = \mathrm{EXP}^{[1,1+h']}_{\mathbf{T}}$ for any integer $h' \geq h$. Hence, by setting

$$\mathrm{EXP}_{\mathbf{T}} = \mathrm{EXP}^{[1,1+h]}_{\mathbf{T}}$$

$\mathrm{EXP}_{\mathbf{T}}$ satisfies the desired interpolation property of Theorem 5.6 for any $j \in \mathbb{Z}_{\geq 1}$. This completes the proof of Theorem 5.6. □

(Reduction Step II)

From now on, we fix an integer h' satisfying $h' \geq h$. Let $\mathcal{I}$ be the set of height-one ideals in $\mathbf{A}^0_{/\mathcal{K}}$ defined by

$$\mathcal{I} = \Big\{ I = \cap_{i=1}^{s} \mathrm{Ker}\, x_i \;\Big|\; x_1, \ldots, x_s \in Z_{(k_0,i)}[r_0] \text{ all different }, \; k(x_i) - 2 \geq 0 \text{ for } i = 1, \ldots, s \Big\}. \tag{5.12}$$

We equip the set $\mathcal{I}$ with a structure of an ordered set with respect to the inclusion of ideals. Later we will take the inverse limit for this order. For each ideal $I \subset \mathbf{A}^0_{/\mathcal{K}}$ in $\mathcal{I}$, we denote $\mathbf{T}/I\mathbf{T}$ (resp. $\mathbf{D}^0/I\mathbf{D}^0$) by $\mathbf{T}_I$ (resp. $\mathbf{D}^0_I$) for short.

Let us reduce Theorem 5.7 to the following theorem.

Theorem 5.8. — *Let* $\mathbf{A}^0_{/\mathcal{K}}$ *and* $\mathbf{T} \cong (\mathbf{A}^0_{/\mathcal{K}})^{\oplus 2}$ *be as above. Let us assume the conditions* (NonInt) *and* (DoubRt) *for our Coleman family and let* $h \in \mathbb{Z}_{\geq 0}$ *equal to or greater than the slope of our Coleman family* $\mathscr{V}$*. Assume that* $\overline{\rho}$ *is irreducible when restricted to* $G_{\mathbb{Q}_p(\mu_p)}$*. We fix an integer* h' *satisfying* $h' \geq h$*.*

Then, for every ideal $I \subset \mathbf{A}^0_{/\mathcal{K}}$ *in* $\mathcal{I}$*, we have a unique* $(\mathbf{A}^0_{/\mathcal{K}}/I)\otimes_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$*-linear big exponential map*

$$\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_I} : \mathbf{D}^0_I \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) \longrightarrow \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}}\mathcal{H}^+_h(G_{\mathrm{cyc}})$$

such that, for each $x \in Z_{(k_0,i)}[r_0]$ *satisfying* $I \subset \operatorname{Ker} x$ *and for each arithmetic character* $\chi^j_{\mathrm{cyc}}\phi : \Lambda(G_{\mathrm{cyc}}) \longrightarrow \overline{\mathbb{Q}}_p$ *with* $j \in [1, 1+h']$*, we have the following commutative diagram:*

$$\begin{array}{ccc}
\mathbf{D}^0_I \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) & \longrightarrow & \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}}\mathcal{H}^+_h(G_{\mathrm{cyc}}) \\
{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)}\downarrow & & \downarrow{\scriptstyle (x,\chi^j_{\mathrm{cyc}}\phi)} \\
D_x \otimes D_{\mathrm{dR}}(\chi^j_{\mathrm{cyc}}\phi) & \longrightarrow & H^1(\mathbb{Q}_p, V_x \otimes \chi^j_{\mathrm{cyc}}\phi)
\end{array}$$

where the bottom horizontal map is equal to $(-1)^j(j-1)!E_p(f_x, j, \phi)\exp_{V_x\otimes\chi^j_{\mathrm{cyc}}\phi}$ *with* $E_p(f_x, j, \phi)$ *as in* (1.1)*.*

We will prove Theorem 5.7 assuming Theorem 5.8.

Proof of Theorem 5.7. — We now show how to deduce Theorem 5.7 from Theorem 5.8 by taking the projective limit with respect to $I \in \mathcal{I}$.

In fact, by definition, the source $\mathbf{D}^0$ of the Coleman map of Theorem 5.7 is equal to the projective limit with respect to $I \in \mathcal{I}$ of the sources $\mathbf{D}^0_I$ of the Coleman maps of Theorem 5.8. Also, thanks to the assumption that the residual representation $\overline{\rho}$ restricted to $G_{\mathbb{Q}_p(\mu_p)}$ is irreducible, $\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)$ is a free $(\mathbf{A}^0_{/\mathcal{K}}/I)\otimes_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$-module of rank two for every $I \in \mathcal{I}$ by applying Proposition 5.4 over each local component of the semi-local ring $(\mathbf{A}^0_{/\mathcal{K}}/I)\otimes_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$. Similarly, $\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}\big)$ is a free $\mathbf{A}^0_{/\mathcal{K}}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda(G_{\mathrm{cyc}})$-module of rank two

Hence the target of the Coleman map of Theorem 5.7 is equal to the projective limit with respect to $I \in \mathcal{I}$ of the targets of the Coleman maps of Theorem 5.8.

Suppose that we have $I, J \in \mathcal{I}$ such that $J \subset I$. By the uniqueness of the Coleman map satisfying the desired interpolation property, the following diagram must be

commutative:

$$\begin{array}{ccc}
\mathbf{D}^0_J \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) & \xrightarrow{\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_J}} & \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_J\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}}\mathcal{H}^+_h(G_{\mathrm{cyc}}) \\
\text{modulo } I \big\downarrow & & \big\downarrow \text{modulo } I \\
\mathbf{D}^0_I \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) & \xrightarrow[\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_I}]{} & \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}}\mathcal{H}^+_h(G_{\mathrm{cyc}})
\end{array}$$

This means that the maps $\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_I}$ of Theorem 5.8 form a projective system with respect to $I \in \mathcal{I}$. For the existence of the map $\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}}$, we can take the projective limit of the maps the maps $\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_I}$ and the desired map $\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}}$ of Theorem 5.7 is constructed to be

$$\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}} = \varprojlim_{I \in \mathcal{I}} \mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_I}.$$

The uniqueness of the map $\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}}$ follows immediately by applying Corollary 5.5 with $\mathcal{P} = \mathcal{I}$. This completes the proof of Theorem 5.7. □

5.4. Proof of Theorem 5.8. — In the previous section, we reduced our Main Theorem (Theorem 5.6) to Theorem 5.8 through reduction steps I and II. We shall prove Theorem 5.8 in this section and we thus complete the proof of Theorem 5.6.

Throughout the section, we fix an integer h' satisfying $h' \geq h$ and a height-one ideal $I \subset \mathbf{A}^0_{/\mathcal{K}}$ in $\mathcal{I}$. We denote by $d_I \in \mathbf{D}^0_I$ the image of a fixed $\mathbf{A}^0_{/\mathcal{K}}$-basis d of $\mathbf{D}^0$. Recall that $\Lambda(G_{\mathrm{cyc}})$ is identified with $\mathbb{Z}_p[\![X]\!]^{\psi=0}$ via the Amice–Vélu transform (4.44).

Now, we consider any element

$$d_I \otimes g(X) \in \mathbf{D}^0_I \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) = \mathbf{D}^0_I \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{O}_{\mathcal{K}}[\![X]\!]^{\psi=0}.$$

For any integer $j \in [1, 1+h']$ and for any natural number n, we attach an element $T_{n,j}(d_I \otimes g(X)) \in H^1(\mathbb{Q}_p(\mu_{p^n}), V_I(j))$ as follows:

$$T_{n,j}(d_I \otimes g(X)) = (-1)^j \cdot (j-1)! \cdot C_{n,V_I(j)}\big((d_I \otimes e_{-j}) \otimes D^{-j}_{CW}(g(X))\big). \tag{5.13}$$

Here, $C_{n,V_I(j)} :\ \mathbf{D}^0_I(j) \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) \longrightarrow H^1(\mathbb{Q}_p(\mu_{p^n}), V_I(j))$ is given by

$$C_{n,V_I(j)}(d_I \otimes g'(X)) = \exp_{V_I(j),\mathbb{Q}_p(\mu_{p^n})}\big((p\varphi_{\mathbf{D}_I(j)} \otimes \varphi_H)^{-(n+1)}(\widetilde{\Xi})|_{X=\zeta_{p^n}-1}\big)$$

where $\widetilde{\Xi} \in \mathbf{D}^0_I(j) \otimes_{\mathcal{K}} \mathcal{H}^+_{h/\mathcal{K}}$ is an element which satisfies $(1 - \varphi_{\mathbf{D}_I(j)} \otimes \varphi_H)(\widetilde{\Xi}) = d_I \otimes g'(X)$ obtained by Corollary 4.26.

Let j and n be natural numbers. Under the same hypothesis as Theorem 5.8, for any non-zero element $t \in H^1(\mathbb{Q}_p(\mu_{p^n}), V_I(j))$, we define $s_t \in \mathbb{Z}$ to be the smallest integer so that $\varpi^{s_t} t \in H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j))$. We define $\|t\|_{j,n}$ to be $p^{\mathrm{ord}(\varpi)s_t}$. We note that, under the same hypothesis as Theorem 5.8, $H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j))$ has no non-trivial torsion element and $H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j))$ is an $\mathcal{O}_{\mathcal{K}}$-lattice of $H^1(\mathbb{Q}_p(\mu_{p^n}), V_I(j))$.

Recall the following result by Perrin-Riou:

Proposition 5.9. — *Let us assume the same setting and the hypothesis as Theorem 5.8.*

1. *For each positive integer j and for each natural number n, we have*
$$\mathrm{Cores}_n^{n+1}(T_{n+1,j}) = T_{n,j},$$
where Cores_n^{n+1} *is the corestriction map from* $H^1(\mathbb{Q}_p(\mu_{p^{n+1}}), V_I(j))$ *to* $H^1(\mathbb{Q}_p(\mu_{p^n}), V_I(j))$.
2. *For each positive integer j and for each natural number n, we have*
$$\mathrm{Tw}_{\chi_{\mathrm{cyc}}^j}(T_{n,0}) \equiv T_{n,j} \text{ modulo } p^{n+1}H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j)). \tag{5.14}$$
3. *For each positive integer j satisfying $1 \leq j \leq h'$,*
$$\left\| p^{n(h'-j)} \sum_{k=1}^{j} (-1)^{j-k} \binom{j}{k} \mathrm{Tw}_{\chi_{\mathrm{cyc}}^{-k}}(T_{n,j-k}) \right\|_{j,n}$$
is bounded when n varies.

Proof. — The first assertion follows from Lemma 5.2 and the definition of $T_{n,j}$ given in (5.13) by using the fact that $\psi(C_{n,V_I(j)}) = 0$ as well as the argument in the proof of [PR94, Proposition 2.4.2].

As for the second assertion, it suffices to prove a statement as follows:

$$\begin{aligned} &\mathrm{res}_{\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^n})}\big(\mathrm{Tw}_{\chi_{\mathrm{cyc}}^{-j}}(T_{n,0})\big) \\ &\qquad \equiv \mathrm{res}_{\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^n})}\big(T_{n,j}\big) \text{ modulo } p^n H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j)). \end{aligned} \tag{5.15}$$

In fact, we have the following isomorphism by Inflation-Restriction sequence:

$$\begin{aligned} &H^1(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^n}), (\mathbf{T}_I(j)/p^n\mathbf{T}_I(j))^{G_{\mathbb{Q}_p(\mu_{p^\infty})}}) \\ &\cong \mathrm{Ker}\Big[H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j)/p^n\mathbf{T}_I(j)) \xrightarrow{\mathrm{res}_{\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^n})}} H^1(\mathbb{Q}_p(\mu_{p^\infty}), \mathbf{T}_I(j)/p^n\mathbf{T}_I(j))\Big]. \end{aligned}$$

By the assumption of the irreducibility of the residual representation restricted to $G_{\mathbb{Q}_p(\mu_p)}$ and by Nakayama's lemma, we have $(\mathbf{T}_I(j)/p^n\mathbf{T}_I(j))^{G_{\mathbb{Q}_p(\mu_p)}} = 0$ for any n. Since $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^m})/\mathbb{Q}_p(\mu_p)) = G_{\mathbb{Q}_p(\mu_p)}/G_{\mathbb{Q}_p(\mu_{p^m})}$ is a cyclic group of p-power order, we have $(\mathbf{T}_I(j)/p^n\mathbf{T}_I(j))^{G_{\mathbb{Q}_p(\mu_{p^m})}} = 0$ for any m and n. By taking inductive limit with respect to m, we have $(\mathbf{T}_I(j)/p^n\mathbf{T}_I(j))^{G_{\mathbb{Q}_p(\mu_{p^\infty})}} = 0$. Hence the restriction map $\mathrm{res}_{\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^n})}$ in our situation is injective. Thus (5.14) was reduced to (5.15).

In order to prove (5.15), we introduce some notations.

Let R be the perfect ring of characteristic p defined to be the projective limit
$$R := (\mathcal{O}_{\mathbb{C}_p}/(p))_{i=0}^{\infty} = [\mathcal{O}_{\mathbb{C}_p}/(p) \leftarrow \mathcal{O}_{\mathbb{C}_p}/(p) \leftarrow \cdots],$$
where the transition maps for $\underline{r} = (\underline{r}_i) \in R$ are given by $\underline{r}_{i+1}^p = \underline{r}_i$. We take a system of elements $x_i \in \mathbb{C}_p$ so that $x_0 = p$ and $x_{i+1}^p = x_i$ and we define $\underline{p} \in R$ to be an element $\underline{x} = (\underline{x}_i) \in R$ such that $\underline{x}_i \equiv x_i$ modulo p for each $i \geq 0$. We put $\xi := p - [-\underline{p}] \subset W(R)$, where $W(R)$ is the ring of Witt vectors and we denote by square brackets the Teichmüller representative of an element of R inside $W(R)$.

Recall that the ring of p-adic periods A_{cris} is defined to be the p-adic completion of the ring $W(R)[(\frac{\xi^k}{k!})_{k\in\mathbb{N}}]$.

We put $\epsilon \in R$ to be an element $(\zeta_{p^i} \bmod p)_{i=0}^{\infty}$ where $\{\zeta_{p^i} \in \mathbb{C}_p\}_{i\in\mathbb{Z}_{\geq 0}}$ is a fixed norm compatible system of primitive p^i-th roots of unity. For each n, we define β_n to be $([(\zeta_{p^{n+i}} \bmod p)_{i=0}^{\infty}])$. Note that we have

$$\varphi^n_{W(R)}(\beta_n) = ([\epsilon])$$

where $\varphi_{W(R)}$ is the p-power Frobenius map on $W(R)$. For each n, the element $\log(\beta_n)$ is known to be convergent on B_{cris}. We often denote $\log([\epsilon])$ by t. Note that we have $\log^i(\beta_n) = p^{-in}t^i$ for $i \in \mathbb{Z}_{\geq 0}$.

We define an operator $\mathcal{R}_j \colon \mathbf{D}^0_I \otimes \mathbb{Z}_p[\![X]\!]^{\psi=0} \longrightarrow \mathbf{D}^0_I \otimes \mathcal{H}_{h/\mathcal{K}}$ by sending $d_I \otimes g(X) \in \mathbf{D}^0_I \otimes \mathbb{Z}_p[\![X]\!]^{\psi=0}$ to

$$\sum_{i=0}^{j-1}(-1)^i(i!)^{-1}D^i_{CW}(\widetilde{\Xi})\log^i(1+X) \in \mathbf{D}^0_I \otimes \mathcal{H}_{h/\mathcal{K}}. \tag{5.16}$$

where $\widetilde{\Xi} \in \mathbf{D}^0_I \otimes \mathcal{H}_{h/\mathcal{K}}$ is the unique solution of

$$(1 - \varphi_D \otimes \varphi_H)(\widetilde{\Xi}) = d_I \otimes g(X)$$

obtained by Corollary 4.26.

We denote by $\rho_n : \mathcal{H}_{h/\mathcal{K}} \longrightarrow \mathcal{K} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$ the evaluation map defined by:

$$\rho_n(G(X)) = G(X)|_{X=\zeta_{p^n}-1} \text{ for } G(X) \in \mathcal{H}_{h/\mathcal{K}}. \tag{5.17}$$

Also, we denote by $\rho_{n,\mathrm{cris}} : \mathcal{H}_{h/\mathcal{K}} \longrightarrow \mathcal{K} \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0 B_{\mathrm{cris}}$ the map defined by:

$$\rho_{n,\mathrm{cris}}(G(X)) = G(X)|_{X=\beta_n-1} \text{ for } G(X) \in \mathcal{H}_{h/\mathcal{K}}. \tag{5.18}$$

Note that for each $f(X) \in \mathcal{H}_{h/\mathcal{K}}$, $f(X)|_{X=\beta_n-1}$ is convergent and well-defined. By the standard description of the connecting homomorphism of Galois cohomology and by the definition of exponential map of Bloch–Kato given in Definition 5.1, we have the following description of the exponential map:

Lemma 5.10. — *For any non-negative integer n and for any positive integer j, $T_{n,j}(d_I \otimes g(X)) \in H^1_f(\mathbb{Q}_p(\mu_{p^n}), V_I(j))$ given in (5.13) coincides with the cocycle sending $g \in G_{\mathbb{Q}_p(\mu_{p^n})}$ to*

$$(g-1)\big(\rho_{n,\mathrm{cris}}(\mathcal{R}_j(d_I \otimes g(X)))\big).$$

For the proof of the Lemma 5.10, we refer to [**PR94**, Prop. 2.3.6].

Let us return to the proof of the second assertion of Proposition 5.9. There exists an integral version of the fundamental sequence of the ring of p-adic periods as follows (see [**Fon94**, Proposition 5.3.6] for the proof):

$$0 \longrightarrow \mathbb{Z}_p t^{\{j\}} \longrightarrow \mathrm{Fil}^j_p A_{\mathrm{cris}} \xrightarrow{1-\frac{\varphi}{p^j}} A_{\mathrm{cris}} \longrightarrow 0 \tag{5.19}$$

where $t^{\{j\}} = t^{r(j)}(\frac{t^{p-1}}{p})^{[q(j)]}$ and $\mathrm{Fil}^j_p A_{\mathrm{cris}} = \{x \in \mathrm{Fil}^j A_{\mathrm{cris}} \mid \varphi(x) \in p^j A_{\mathrm{cris}}\}$. If we take the Galois cohomology of the exact sequence (5.19) $\otimes_{\mathbb{Z}_p} V_I$ for the Galois group $G_{\mathbb{Q}_p}$, the connection map $D_{\mathrm{cris}}(V_I(j)) \longrightarrow H^1(\mathbb{Q}_p, V_I(j))$ coincides with the

restriction to $D_{\mathrm{cris}}(V_I(j)) \subset D_{\mathrm{dR}}(V_I(j))$ of the Bloch–Kato exponential map (5.4) given in Definition 5.1 for $V = V_I(j)$ and for $n = 0$. Hence the image of $(A_{\mathrm{cris}} \otimes \mathbf{T}_I)^{G_{\mathbb{Q}_p}} \subset D_{\mathrm{dR}}(V_I)$ via the connecting homomorphism of the Galois cohomology of the exact sequence (5.19) $\otimes_{\mathbb{Z}_p} \mathbf{T}_I$ for $G_{\mathbb{Q}_p}$ lands in the module $H^1\big(\mathbb{Q}_p, p^{-[q(j)]}\mathbf{T}_I(j)\big)$.

By Lemma 5.10, in order to show the second statement of Proposition 5.9, it suffices to show that

$$(j-1)!p^{(j-1)(n+1)}\rho_{n,\mathrm{cris}}(\mathcal{R}_j(d_I \otimes g(X))t \\ + j!p^{j(n+1)}\rho_{n,\mathrm{cris}}(\mathcal{R}_{j+1}(d_I \otimes g(X)) \in p^{n+1}A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbf{T}_I.$$

The left-hand side of the above is equivalent to

$$(j-1)!p^{(j-1)(n+1)}\sum_{i=0}^{j-1}(-1)^i(i!)^{-1}p^{-(n+1)i}\rho_{n,\mathrm{cris}}(D^i_{CW}(\widetilde{\Xi}))t^{i+1} \\ + j!p^{j(n+1)}\sum_{i=0}^{j}(-1)^i(i!)^{-1}p^{-(n+1)i}\rho_{n,\mathrm{cris}}(D^i_{CW}(\widetilde{\Xi}))t^i$$

Since we have $D^i_{CW}(\widetilde{\Xi}) \in A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbf{T}_I$, the above term is congruent to

$$(-1)^{j-1}\rho_{n,\mathrm{cris}}(D^i_{CW}(\widetilde{\Xi}))t^j + (-1)^j\rho_{n,\mathrm{cris}}(D^i_{CW}(\widetilde{\Xi}))t^j = 0$$

modulo $p^{n+1}A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbf{T}_I$. This completes the proof of the second assertion. The third assertion is a generalization of the second assertion and the proof goes similarly as [PR94, Lemme 2.4.4] □

Proof of Theorem 5.8. — By the assumption of the irreducibility of the residual representation and by Proposition 5.4, $\varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I(j)) \cong H^1(\mathbb{Q}_p, \mathbf{T}_I(j) \otimes \Lambda(G_{\mathrm{cyc}})^\sharp)$ is free of rank two over $(\mathbf{A}^0_{/\mathcal{K}}/I\mathbf{A}^0_{/\mathcal{K}}) \otimes \Lambda(G_{\mathrm{cyc}})$. As in the proof of Corollary 5.5, we can identify $H^1(\mathbb{Q}_p, \mathbf{T}_I(j) \otimes \Lambda(G_{\mathrm{cyc}})^\sharp)$ with $(\mathbf{A}^0_{/\mathcal{K}}/I\mathbf{A}^0_{/\mathcal{K}}) \otimes \Lambda(G_{\mathrm{cyc}}) \cong (\mathbf{A}^0_{/\mathcal{K}}/I\mathbf{A}^0_{/\mathcal{K}}) \otimes \mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times] \otimes \mathbb{Z}_p[\![Y]\!]$. Further, the operation $\mathrm{Tw}_{\chi_{\mathrm{cyc}}^{-j}}$ on $\Lambda(G_{\mathrm{cyc}})$ corresponds to sending $[a] \otimes F(Y) \in \mathbb{Z}_p[\![Y]\!]$ to $\omega^{-j}(a)[a] \otimes F((1+Y)-(1+p)^j)$ on $\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times] \otimes \mathbb{Z}_p[\![Y]\!]$.

Thanks to Proposition 4.31 and Proposition 5.9, we have a unique $(\mathbf{A}^0_{/\mathcal{K}}/I) \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}})$-linear big exponential map

$$\mathrm{EXP}^{[1,1+h']}_{\mathbf{T}_I} : \mathbf{D}^0_I \otimes_{\mathbb{Z}_p} \Lambda(G_{\mathrm{cyc}}) \longrightarrow \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} \mathcal{H}_h(G_{\mathrm{cyc}})$$

which has the same interpolation property as in the statement of Theorem 5.8.

By Corollary 4.26 4 (a), the denominators of is bounded by $D^0 \otimes_{\mathbb{Z}_p} p^{c_{h,\lambda}} \mathcal{H}_h^+$. Hence the image of $\mathrm{EXP}_{\mathbf{T}_I}^{[1,1+h']}$ lands in

$$\left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} p^{c_{h,\lambda}} \mathcal{H}_h^+(G_{\mathrm{cyc}}) \subset \left(\varprojlim_n H^1\big(\mathbb{Q}_p(\mu_{p^n}), \mathbf{T}_I\big)\right) \otimes_{\Lambda(G_{\mathrm{cyc}})} \mathcal{H}_h(G_{\mathrm{cyc}}).$$

By the definition of $\mathrm{EXP}_{\mathbf{T}_I}^{[1,1+h']}$, the denominator of the image of $\mathrm{EXP}_{\mathbf{T}_I}^{[1,1+h']}$ is bounded by $D^0 \otimes_{\mathbb{Z}_p} p^{c_{h,\lambda}} \mathcal{H}_h^+$. This completes the proof. □

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