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Theoretical Performance of Low Rank Adaptive Filters in Gaussian Context in the Large Dimensional Regime

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Abstract—This paper addresses the problem of deriving the asymptotic performance of adaptive Low Rank (LR) filters used in target detection embedded in a disturbance composed of a LR Gaussian noise plus a white Gaussian noise. In this context, we use the Signal to Interference to Noise Ratio (SINR) loss as performance measure which is a function of the estimated projector onto the LR noise subspace. However, although the SINR loss can be determined through Monte-Carlo simulations or real data, this process remains quite time consuming. Thus, this paper proposes to predict the SINR loss behavior in order to not depend on the data anymore and be quicker. To derive this theoretical result, previous works used a restrictive hypothesis assuming that the target is orthogonal to the LR noise. In this paper, we propose to derive this theoretical performance by relaxing this hypothesis and using Random Matrix Theory (RMT) tools. These tools will be used to present the convergences of simple quadratic forms and perform new RMT convergences of structured quadratic forms and SINR loss in the large dimensional regime, i.e. the size and the number of the data tend to infinity at the same rate. We show through simulations the interest of our approach compared to the previous works when the restrictive hypothesis is no longer verified.

Index Terms—Low Rank SINR loss, Random Matrix Theory, Adaptive Filtering, Quadratic Forms convergence, Spiked model

I. INTRODUCTION

In array processing, the covariance matrix \mathbf{R} of the data is widely involved for main applications as filtering [1], [2], radar/sonar detection [3] or localization [4], [5]. However, when the disturbance in the data is composed of the sum of a Low Rank (LR) correlated noise and a White Gaussian Noise (WGN), the covariance matrix is often replaced by the projector onto the LR noise subspace $\mathbf{\Pi}_c$ [6]–[9]. In practice, the projector onto the LR noise subspace (and the covariance matrix) is generally unknown and an estimate is consequently required to perform the different processing. This estimation procedure is based on the so-called secondary data assumed to be independent and to share the same distribution. Then, the true projector is replaced by the estimated one in order to obtain an adaptive processing. An important issue is then to derive the theoretical performance of the adaptive processing as a function of the number of secondary data K . The processing based on the covariance matrix has been widely

studied and led to many theoretical results in filtering [1] and detection [10]–[13]. For example, for classical adaptive processing, $K = 2m$ secondary data (where m is the data size) are required to ensure good performance of the adaptive filtering, i.e. a 3dB loss of the output Signal to Interference plus Noise Ratio (SINR) compared to optimal filtering [1]. For LR processing, some results has been obtained especially in filtering [6], [14]–[16] and localization [17]. Similarly, in LR filtering, the number K of secondary data required to ensure good performance of the adaptive filtering is equal to $2r$ (where r is the rank of the LR noise subspace) [6], [14].

These last results are obtained from the theoretical study of the SINR loss. More precisely, in [14], [16], the derivation of the theoretical results is based on the hypothesis that the steering vector is orthogonal to the LR noise subspace. Nevertheless, even if the result seems to be close to the simulated one, when the hypothesis is no longer valid anymore [18], it is impossible with traditional techniques of [14], [16] to obtain a theoretical performance as a function of the distance between the steering vector and the LR noise subspace. Since, in practice, this dependence is essential to predict the performance of the adaptive filtering, we propose in this paper to derive the theoretical SINR loss, for a disturbance composed of a LR Gaussian noise and a WGN, as a function of K and the distance between the steering vector and the LR noise subspace. The proposed approach is based on the study of the SINR loss structure.

The SINR loss (resp. LR SINR loss) is composed of a simple Quadratic Form (QF) in the numerator, $\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{s}_2$ (resp. $\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2$) and a structured QF in the denominator $\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{R} \hat{\mathbf{R}}^{-1} \mathbf{s}_2$ (resp. $\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{R} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2$). These recent years, the simple QFs (numerator) have been broadly studied [19]–[22] using Random Matrix Theory (RMT) tools contrary to structured QFs (denominator). RMT tools have also been used in array processing to improve the MUSIC algorithm [23], [24] and in matched subspace detection [25], [26] where the rank r is unknown. The principle is to examine the spectral behavior of $\hat{\mathbf{R}}$ by RMT to obtain their convergences, performance and asymptotic distribution when K tends to infinity and when both the data size m and K tend to infinity at the same ratio, i.e. $m/K \rightarrow c \in]0, +\infty[$, for different models of $\hat{\mathbf{R}}$ of the observed data as in [19], [20], [23], [22] and [21]. Therefore, inspired by these works, we propose in this paper to summarize the convergences of the structured QFs $\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{R} \hat{\mathbf{R}}^{-1} \mathbf{s}_2$ and study those of $\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{R} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2$:

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when 1) $K \rightarrow \infty$ with a fixed m and when 2) $m, K \rightarrow \infty$ at the same ratio under the most appropriated model for our data and with the rank assumed to be known. From [27], [28], the *spiked* model has proved to be the more appropriated one to our knowledge. This model, introduced by [29] (also studied in [30], [31] from an eigenvector point of view) considers that the multiplicity of the eigenvalues corresponding to the signal (the LR noise in this article) is fixed for all m and leads to the SPIKE-MUSIC estimator [32] of $s_1^H \hat{\Pi} s_2$. Then, the new results are validated through numerical simulations. From these new theoretical convergences, the paper derives the convergence of the LR SINR loss for LR filters. The new theoretical SINR losses depend on the number of secondary data K but also on the distance between the steering vector and the LR noise subspace. This work is partially related to those of [33]–[36] and [37] which uses the RMT tools to derive the theoretical SINR loss or SNR performance in a full rank context (previously defined as classical).

Finally, this theoretical LR SINR loss is validated in a jamming application context where the purpose is to detect a target thanks to a Uniform Linear Antenna (ULA) composed of m sensors despite the presence of jamming. The response of the jamming is composed of signals similar to the target response. This problem is very similar to the well-known Space Time Adaptive Processing (STAP) introduced in [2]. The results show the interest of our approach with respect to other theoretical results [6], [14]–[16] in particular when the target is close to the jammer.

The paper is organized as follows. Section II presents the received data model, the adaptive filters and the corresponding SINR losses. Section III summarizes the existing studies on the *simple* QFs $s_1^H \hat{\mathbf{R}} s_2$ and $s_1^H \hat{\Pi} s_2$, and exposes the covariance matrix model, the *spiked* model. Section IV gives the theoretical contribution the paper with the convergences of the *structured* QFs $s_1^H \hat{\Pi}_c^\perp \mathbf{B} \hat{\Pi}_c^\perp s_2$ and $s_1^H \hat{\Pi}_c^\perp \mathbf{R} \hat{\Pi}_c^\perp s_2$ and the convergences of the LR SINR loss. The results are finally applied on a jamming application in Section V.

Notations: The following conventions are adopted. An italic letter stands for a scalar quantity, boldface lowercase (uppercase) characters stand for vectors (matrices) and $(\cdot)^H$ stands for the conjugate transpose. \mathbf{I}_N is the $N \times N$ identity matrix, $\text{tr}(\cdot)$ denotes the trace operator and $\text{diag}(\cdot)$ denotes the diagonalization operator such as $(\mathbf{A})_{i,i} = (\text{diag}(\mathbf{a}))_{i,i} = (\mathbf{a})_i$ and $(\mathbf{A})_{i,j} = 0$ if $i \neq j$. $\#\{\mathcal{A}\}$ denotes the cardinality of the set \mathcal{A} . $\llbracket a, b \rrbracket$ is the set defined by $\{x \in \mathbb{Z} : a \leq x \leq b, \forall (a, b) \in \mathbb{Z}^2\}$. $\mathbf{O}_{n \times N}$ is a $n \times N$ matrix full of 0. The abbreviations iid and a.s. stem for *independent and identically distributed* and almost surely respectively.

II. PROBLEM STATEMENT

The aim of the problem is to filter the received observation vector $\mathbf{x} \in \mathbb{C}^{m \times 1}$ in order to whiten the noise without mitigating an eventual complex signal of interest \mathbf{d} (typically a target in radar processing). In this paper, \mathbf{d} will be a target response and is equal to $\alpha \mathbf{a}(\Theta)$ where α is an unknown complex deterministic parameter (generally corresponding to the target amplitude), $\mathbf{a}(\Theta)$ is the steering vector and Θ

is an unknown deterministic vector containing the different parameters of the target (e.g. the localization, the velocity, the Angle of Arrival (AoA), etc.). In the remainder of the article, in order to simplify the notations, Θ will be omitted of the steering vector which will simply be denoted as \mathbf{a} . If necessary, the original notation will be taken.

This section will first introduce the data model. Then, the filters, adaptive filters and the quantity characterizing their performance, the SINR loss, will be defined.

A. Data model

The observation vector can be written as:

$$\mathbf{x} = \mathbf{d} + \mathbf{c} + \mathbf{b} \quad (1)$$

where $\mathbf{c} + \mathbf{b}$ is the noise that has to be whitened. $\mathbf{b} \in \mathbb{C}^{m \times 1} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}_m)$ is an Additive WGN (AWGN) and $\mathbf{c} \in \mathbb{C}^{m \times 1}$ is a LR Gaussian noise modeled by a zero-mean complex Gaussian vector with a normalized covariance matrix \mathbf{C} ($\text{tr}(\mathbf{C}) = m$), i.e. $\mathbf{c} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C})$. Consequently, the covariance matrix of the noise $\mathbf{c} + \mathbf{b}$ can be written as $\mathbf{R} = \mathbf{C} + \sigma^2 \mathbf{I}_m \in \mathbb{C}^{m \times m}$. Moreover, considering a LR Gaussian noise, one has $\text{rank}(\mathbf{C}) = r \ll m$ and hence, the eigendecomposition of \mathbf{C} is:

$$\mathbf{C} = \sum_{i=1}^r \omega_i \mathbf{u}_i \mathbf{u}_i^H \quad (2)$$

where ω_i and \mathbf{u}_i , $i \in \llbracket 1; r \rrbracket$ are the non-zero eigenvalues and the associated eigenvectors of \mathbf{C} respectively, unknown in practice. This leads to:

$$\mathbf{R} = \sum_{i=1}^m \lambda_i \mathbf{u}_i \mathbf{u}_i^H \quad (3)$$

where λ_i and \mathbf{u}_i , $i \in \llbracket 1, m \rrbracket$ are the eigenvalues and the associated eigenvectors of \mathbf{R} respectively with $\lambda_1 = \omega_1 + \sigma^2 > \dots > \lambda_r = \omega_r + \sigma^2 > \lambda_{r+1} = \dots = \lambda_m = \sigma^2$. Then, the projector onto the LR Gaussian noise subspace Π_c and the projector onto the orthogonal subspace to the LR Gaussian noise subspace Π_c^\perp are defined as follows:

$$\begin{cases} \Pi_c = \sum_{i=1}^r \mathbf{u}_i \mathbf{u}_i^H \\ \Pi_c^\perp = \mathbf{I}_m - \Pi_c = \sum_{i=r+1}^m \mathbf{u}_i \mathbf{u}_i^H \end{cases} \quad (4)$$

However, in practice, the covariance matrix \mathbf{R} of the noise is unknown. Consequently, it is traditionally estimated with the Sample Covariance Matrix (SCM) which is computed from K iid secondary data $\mathbf{x}_k = \mathbf{c}_k + \mathbf{b}_k$, $k \in \llbracket 1, K \rrbracket$ with $\mathbf{c}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{C})$ and $\mathbf{b}_k \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}_m)$, and can be written as:

$$\hat{\mathbf{R}} = \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k \mathbf{x}_k^H = \sum_{i=1}^m \hat{\lambda}_i \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H \quad (5)$$

where $\hat{\lambda}_i$ and $\hat{\mathbf{u}}_i$, $i \in \llbracket 1, m \rrbracket$ are the eigenvalues and the eigenvectors of $\hat{\mathbf{R}}$ respectively with $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \dots \geq \hat{\lambda}_m$. For simplicity purposes, we set $\sigma^2 = 1$. Finally, the traditional projectors estimators based on the SCM are:

$$\begin{cases} \hat{\Pi}_c = \sum_{i=1}^r \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H \\ \hat{\Pi}_c^\perp = \mathbf{I}_m - \hat{\Pi}_c = \sum_{i=r+1}^m \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H, \end{cases} \quad (6)$$

B. Adaptive filters

A filtering preprocessing on the observation vector \mathbf{x} (Eq.(1)) is first done with the filter \mathbf{w} in order to whiten the received signal: $p = \mathbf{w}^H \mathbf{x}$. The filter maximizing the SINR is given by:

$$\mathbf{w}_{\text{opt}} = \mathbf{R}^{-1} \mathbf{a} \quad (7)$$

Since, in practice, the covariance matrix \mathbf{R} of the noise is unknown, the estimated optimal filter or adaptive filter (sub-optimal) is:

$$\hat{\mathbf{w}} = \hat{\mathbf{R}}^{-1} \mathbf{a} \quad (8)$$

In the case where one would benefit of the LR structure of the noise, one should use the optimal LR filter, based on the fact that $\mathbf{\Pi}_c^\perp$ is the best rank r approximation of \mathbf{R}^{-1} , which is defined by [6]:

$$\mathbf{w}_{\text{LR}} = \mathbf{\Pi}_c^\perp \mathbf{a} \quad (9)$$

Since, in practice, the projector is not known and is estimated from the SCM, the estimated optimal filter or adaptive filter (sub-optimal) is:

$$\hat{\mathbf{w}}_{\text{LR}} = \hat{\mathbf{\Pi}}_c^\perp \mathbf{a} \quad (10)$$

C. SINR Loss

Then, we define the SINR Loss. In order to characterize the performance of the estimated filters, the SINR loss compares the SINR at the output of the filter to the maximum SINR:

$$\hat{\rho} = \frac{\text{SINR}_{\text{out}}}{\text{SINR}_{\text{max}}} = \frac{|\hat{\mathbf{w}}^H \mathbf{d}|^2}{(\hat{\mathbf{w}}^H \mathbf{R} \hat{\mathbf{w}})(\mathbf{d}^H \mathbf{R}^{-1} \mathbf{d})} \quad (11)$$

$$= \frac{|\mathbf{a}^H \hat{\mathbf{R}}^{-1} \mathbf{a}|^2}{(\mathbf{a}^H \hat{\mathbf{R}}^{-1} \mathbf{R} \hat{\mathbf{R}}^{-1} \mathbf{a})(\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})} \quad (12)$$

If $\hat{\mathbf{w}} = \mathbf{w}_{\text{opt}}$, the SINR loss is maximum and is equal to 1. When we consider the LR structure of the noise, the theoretical SINR loss can be written as:

$$\rho_{\text{LR}} = \frac{|\mathbf{w}_{\text{LR}}^H \mathbf{d}|^2}{(\mathbf{w}_{\text{LR}}^H \mathbf{R} \mathbf{w}_{\text{LR}})(\mathbf{d}^H \mathbf{R}^{-1} \mathbf{d})} \quad (13)$$

$$= \frac{|\mathbf{a}^H \mathbf{\Pi}_c^\perp \mathbf{a}|^2}{(\mathbf{a}^H \mathbf{\Pi}_c^\perp \mathbf{R} \mathbf{\Pi}_c^\perp \mathbf{a})(\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})} \quad (14)$$

Finally, the SINR loss corresponding to the adaptive filter in Eq.(10) is defined from Eq.(14) as:

$$\hat{\rho}_{\text{LR}} = \rho_{\text{LR}} \Big|_{\mathbf{\Pi}_c^\perp = \hat{\mathbf{\Pi}}_c^\perp} \quad (15)$$

Since we are interested in the performance of the filters, we would like to obtain the theoretical behavior of the SINR losses. Some asymptotic studies on the SINR loss in LR Gaussian context have already been done [14], [16]. In [14], [16], the theoretical result is derived by using the assumption that the steering vector is orthogonal to the LR noise and, in this case, [16] obtained an approximation of the expectation of the SINR loss $\hat{\rho}_{\text{LR}}$. However, this assumption is not always verified, not highly relevant and is a restrictive hypothesis in real cases. We consequently propose to relax it and study the convergence of the SINR loss using RMT tools through the study of the nominators and denominators. Indeed, one can already note that the numerators are *simple* QFs whose

convergences were widely considered in RMT. However, the denominators contain more elaborated QFs which were not tackled in RMT in LR context yet and will be the object of Sec.IV.

III. RANDOM MATRIX THEORY TOOLS

This section is dedicated to the introduction of classical results from the RMT for the study of the convergence of QFs. This theory and the convergences are based on the behavior of the eigenvalues of the SCM when $m, K \rightarrow \infty$ at the same rate, i.e. $m/K \rightarrow c \in]0, +\infty)$. In order to simplify the notations, we will abusively note $c = m/K$.

The useful tools for the study of the eigenvalues behavior and the assumptions to the different convergences will be first presented. Secondly, the section will expose the data model, the *spiked* model [22]. Finally, the useful convergences of *simple* QFs ($\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{s}_2$, $\mathbf{s}_1^H \hat{\mathbf{\Pi}} \mathbf{s}_2$) will be introduced.

A. Preliminaries

The asymptotic behavior of the eigenvalues when $m, K \rightarrow \infty$ at the same rate is described through the convergence of their associated empirical Cumulative Distribution Function (CDF) $\hat{F}_m(x)$ or their empirical Probability Density Function (PDF) $\hat{f}_m(x)$ ¹. The asymptotic PDF $f_m(x)$ will allow us to characterize the studied data model. The empirical CDF of the sample eigenvalues of $\hat{\mathbf{R}}$ can be defined as:

$$\hat{F}_m(x) = \frac{1}{m} \# \left\{ k : \hat{\lambda}_k \leq x \right\} \quad (16)$$

However, in practice, the asymptotic characterization of $\hat{F}_m(x)$ is too hard. Consequently, one prefers to study the convergence of the Stieltjes transform ($\mathcal{ST}[\cdot]$) of $\hat{F}_m(x)$:

$$\hat{b}_m(z) = \mathcal{ST} \left[\hat{F}_m(x) \right] = \int_{\mathbb{R}} \frac{1}{x - z} d\hat{F}_m(x) \quad (17)$$

$$= \frac{1}{m} \sum_{i=1}^m \frac{1}{\hat{\lambda}_i - z} = \frac{1}{m} \text{tr} \left[(\hat{\mathbf{R}} - z \mathbf{I}_m)^{-1} \right] \quad (18)$$

with $z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im\{z\} > 0\}$ and which almost surely converges to $\bar{b}_m(z)$. It is interesting to note that the PDF can thus be retrieve from the Stieltjes transform of its CDF:

$$\hat{f}_m(x) = \lim_{\Im\{z\} \rightarrow 0^+} \frac{1}{\pi} \Im \left[\hat{b}_m(z) \right] \quad (19)$$

with $x \in \mathbb{R}$. In other words, the characterization of $\hat{f}_m(x)$ (resp. $f_m(x)$) can be obtained from $\hat{b}_m(z)$ (resp. $\bar{b}_m(z)$). Then, to prove the convergences, we assume the following standard hypotheses.

(As1) \mathbf{R} has uniformly bounded spectral norm $\forall m \in \mathbb{N}^*$, i.e. $\forall i \in \llbracket 1, m \rrbracket$, $\lambda_i < \infty$.

(As2) The vectors $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{C}^{m \times 1}$ used in the QFs (here $\mathbf{a}(\Theta)$ and \mathbf{x}) have uniformly bounded Euclidean norm $\forall m \in \mathbb{N}^*$.

(As3) Let $\mathbf{Y} \in \mathbb{C}^{m \times K}$ having iid entries $y_{ij} \sim \mathcal{CN}(0, 1)$. The probability law of \mathbf{Y} is invariant by left multiplication by a deterministic unitary matrix.

¹One can show that under **(As1,As3)** described later, $\hat{f}_m(x)$ a.s. converges towards a nonrandom PDF $f(x)$ with a compact support.

In that case, the following properties are verified. The eigenvalues empirical PDF of $\frac{1}{K} \mathbf{Y} \mathbf{Y}^H$ a.s. converges to the Marčenko-Pastur distribution [38] with support $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2]$ and the maximum (resp. minimum) eigenvalue of $\frac{1}{K} \mathbf{Y} \mathbf{Y}^H$ a.s. tends to $(1 + \sqrt{c})^2$ (resp. to $(1 - \sqrt{c})^2$).

B. Covariance matrix models and convergence of eigenvalues

We first expose the considered data model and, then, the eigenvalues behavior of the SCM. The SCM can be written as $\hat{\mathbf{R}} = \frac{1}{K} \mathbf{X} \mathbf{X}^H$ with:

$$\mathbf{X} = \mathbf{R}^{1/2} \mathbf{Y} = (\mathbf{I}_m + \mathbf{P})^{1/2} \mathbf{Y} \quad (20)$$

and $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_K]$. $\mathbf{R}^{1/2}$ is the $m \times m$ Hermitian positive definite square root of the true covariance matrix. The matrix \mathbf{P} is the rank r perturbation matrix and can be eigendecomposed as $\mathbf{P} = \mathbf{U}_s \mathbf{\Omega} \mathbf{U}_s^H = \sum_{i=1}^r \omega_i \mathbf{U}_i \mathbf{U}_i^H$ with:

$$\mathbf{\Omega} = \begin{bmatrix} \omega_1 \mathbf{I}_{\mathcal{K}_1} & & \\ & \ddots & \\ & & \omega_r \mathbf{I}_{\mathcal{K}_r} \end{bmatrix} \quad (21)$$

where $\mathbf{U}_s = [\mathbf{U}_1 \dots \mathbf{U}_r]$ and $r + 1$ is the number of distinct eigenvalues of \mathbf{R} . Moreover, $\mathbf{U}_i \in \mathbb{C}^{m \times \mathcal{K}_i}$ where \mathcal{K}_i is the multiplicity of ω_i . \mathcal{K}_i is fixed $\forall i \in \llbracket 1, r \rrbracket$ and does not increase with m , i.e. $\mathcal{K}_i/m \xrightarrow{m, K \rightarrow \infty} 0^+$, $\forall i \in \llbracket 1, r \rrbracket$. For simplicity and in order to fit with our data model, \mathcal{K}_i will be set to 1 for all $i \in \llbracket 1, r \rrbracket$. Consequently, we have $\text{rank}(\mathbf{\Omega}) = \sum_{i \in \llbracket 1, r \rrbracket} \mathcal{K}_i = r$ and $\mathcal{K}_{r+1} = m - r$.

Hence, the covariance matrix in Eq.(3) can be rewritten as:

$$\mathbf{R} = \sum_{i=1}^{r+1} \lambda_i \mathbf{U}_i \mathbf{U}_i^H = \mathbf{U}_{r+1} \mathbf{U}_{r+1}^H + \sum_{i=1}^r \lambda_i \mathbf{u}_i \mathbf{u}_i^H \quad (22)$$

where λ_i and \mathbf{U}_i are the eigenvalues and the associated subspaces of \mathbf{R} respectively, with $\lambda_1 = 1 + \omega_1 > \dots > \lambda_r = 1 + \omega_r > \lambda_{r+1} = 1$ and $\mathbf{U}_{r+1} = [\mathbf{u}_{r+1} \dots \mathbf{u}_m]$. In other words, the model specifies that only a few eigenvalues of the LR noise component c are non-unit (and do not contribute to the noise \mathbf{b} unit-eigenvalues associated to λ_{r+1}) and fixed.

This model leads to a specific asymptotic eigenvalues PDF of \mathbf{R} as detailed hereafter. The convergence of the eigenvalues is addressed through the convergence of the Stieltjes transform of the eigenvalues CDF. The asymptotic eigenvalue behavior of $\hat{\mathbf{R}}$ for the *spiked* model was introduced by Johnstone [29] and its eigenvalue behavior was studied in [39]. In order to derive it, [39] exploited the specific expression given in Eq.(20). Then, [22] introduced the final assumption (*separation condition*) under which the following convergences are given.

(As4.S) The eigenvalues of \mathbf{P} satisfy the *separation condition*, i.e. $|\omega_i| > \sqrt{c}$ for all $i \in \llbracket 1, r \rrbracket$.

Thus, under **(As1-As3, As4.S)**, we have:

$$\hat{f}_m(x) \xrightarrow[m/K \rightarrow c < \infty]{m, K \rightarrow \infty} f(x) \quad (23)$$

where $f(x)$ is the Marčenko-Pastur law:

$$f(x) = \begin{cases} (1 - \frac{1}{c}), & \text{if } x = 0 \text{ and } c > 1 \\ \frac{1}{2\pi cx} \sqrt{(\lambda_- - x)(x - \lambda_+)}, & \text{if } x \in]\lambda_-, \lambda_+[\\ 0, & \text{otherwise} \end{cases} \quad (24)$$

with $\lambda_- = (1 - \sqrt{c})^2$ and $\lambda_+ = (1 + \sqrt{c})^2$. However, it is essential to note that, for all $i \in \llbracket 1, r \rrbracket$:

$$\hat{\lambda}_{j \in \mathcal{M}_i} \xrightarrow[m/K \rightarrow c < \infty]{m, K \rightarrow \infty, \text{ a.s.}} \tau_i = 1 + \omega_i + c \frac{1 + \omega_i}{\omega_i} \quad (25)$$

where \mathcal{M}_i is the set indexes corresponding to the j -th eigenvalue of \mathbf{R} (for example $\mathcal{M}_{r+1} = \{r+1, \dots, m\}$ for λ_{r+1}). Two representations of $\hat{f}_m(x)$ for two different c and a sufficient large m are shown on Fig. 1 when the eigenvalues of \mathbf{R} are 1, 2, 3, and 7. The eigenvalues have the same multiplicity and the eigenvalue 1 corresponds the noise eigenvalue. One can observe that **(As4.S)** is verified if and only if $\tau_r > \lambda_+$. In other words, all the sample eigenvalues corresponding to the non-unit eigenvalues of \mathbf{R} converge to a value τ_i which is outside the support of the Marčenko-Pastur law (“asymptotic” PDF of the “unit” sample eigenvalues). As an illustration, one can notice that, in Fig. 1, for $\hat{f}_m(x)$ plotted for $c = 0.1$, the *separation condition* is verified ($\omega_1 = 6$, $\omega_2 = 2$ and $\omega_3 = 1$ are greater than $\sqrt{c} = 0.316$) and the three non-unit eigenvalues are represented on the PDF and outside the support of the Marčenko-Pastur law by their respective limits $\tau_1 = 7.116$, $\tau_2 = 3.15$ and $\tau_3 = 2.2$. On the contrary, for $\hat{f}_m(x)$ plotted for $c = 1.5$, only the two greatest eigenvalues are represented on the PDF by their respective limits $\tau_1 = 8.75$ and $\tau_2 = 5.25$ while the *separation condition* is not verified for the eigenvalue $\lambda_3 = 2$ ($\omega_3 = 1 < \sqrt{c} = 1.223$). In this case, the sample eigenvalues corresponding to the eigenvalue $\lambda_3 = 2$ belongs to the Marčenko-Pastur law.

C. Convergence of simple quadratic forms

Here, we compare the convergence of two *simple* QFs in two convergence regimes: when $K \rightarrow \infty$ with a fixed m and when $m, K \rightarrow \infty$ at the same rate.

We first present the useful convergences of *simple* QFs function of $\hat{\mathbf{R}}$. It is well known that, due to the strong law of large numbers, when $K \rightarrow \infty$ with a fixed m , $\hat{\mathbf{R}} \rightarrow \mathbf{R}$ a.s. [40]. Thus,

$$\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{s}_2 \xrightarrow[m < \infty]{K \rightarrow \infty, \text{ a.s.}} \mathbf{s}_1^H \mathbf{R}^{-1} \mathbf{s}_2 \quad (26)$$

Moreover, when $m, K \rightarrow \infty$ at the same rate [19], [41]:

$$\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{m, K \rightarrow \infty, \text{ a.s.}} (1 - c)^{-1} \mathbf{s}_1^H \mathbf{R}^{-1} \mathbf{s}_2 \quad (27)$$

The useful convergences of *simple* QFs function of $\hat{\mathbf{\Pi}}_c^\perp$ are hereafter presented. As $\hat{\mathbf{R}} \rightarrow \mathbf{R}$ a.s. when $K \rightarrow \infty$ with a fixed m , $\hat{\mathbf{\Pi}}_c^\perp \rightarrow \mathbf{\Pi}_c^\perp$ a.s. [19] in the same convergence regime. Thus:

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m < \infty]{K \rightarrow \infty, \text{ a.s.}} \mathbf{s}_1^H \mathbf{\Pi}_c^\perp \mathbf{s}_2 \quad (28)$$

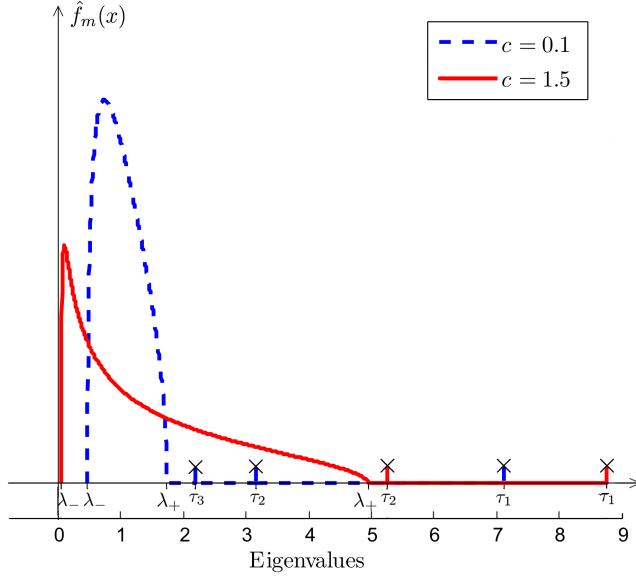


Fig. 1. PDF of the eigenvalues of the SCM with the *spiked* model when the eigenvalues of \mathbf{R} are 1, 2, 3, and 7 with the same multiplicity, where 1 is the noise eigenvalue.

For the convergences in the large dimensional regime ($m, K \rightarrow \infty$ at the same rate), they are presented under (**As1-As3**) and the *separation condition* **As4.S**. [22] showed that, $\forall i \in [1, r]$:

$$\mathbf{s}_1^H \hat{\mathbf{U}}_i \hat{\mathbf{U}}_i^H \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \frac{1 - c\omega_i^{-2}}{1 + c\omega_i^{-1}} \mathbf{s}_1^H \mathbf{U}_i \mathbf{U}_i^H \mathbf{s}_2 \quad (29)$$

where $\omega_i = \lambda_i - 1$ and λ_i is the i -th distinct eigenvalue of \mathbf{R} . Let $\chi_i = \frac{1 - c\omega_i^{-2}}{1 + c\omega_i^{-1}}$. Thus, using the following relationship,

$$\hat{\mathbf{\Pi}}_c^\perp = \mathbf{I}_m - \sum_{i=1}^r \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H \quad (30)$$

one can deduce that with the *spiked* model and in the large dimensional regime:

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \mathbf{s}_1^H \bar{\mathbf{\Pi}}_{c,S}^\perp \mathbf{s}_2 \quad (31)$$

with $\bar{\mathbf{\Pi}}_{c,S}^\perp = \sum_{i=1}^m \psi_i \mathbf{u}_i \mathbf{u}_i^H$ and

$$\psi_i = \begin{cases} 1, & \text{if } i > r \\ 1 - \chi_i, & \text{if } i \leq r \end{cases} \quad (32)$$

To summarize, $\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{s}_2$ is consistent when $K \rightarrow \infty$ with a fixed m and its limits in the large dimensional regime is proportional to the theoretical QF $\mathbf{s}_1^H \mathbf{R}^{-1} \mathbf{s}_2$. However, although $\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2$ is consistent when $K \rightarrow \infty$ with a fixed m , it is no more consistent under the regime of interest i.e. when both $m, K \rightarrow \infty$ at the same rate. More precisely, the limit of $\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2$ in the large dimensional regime is not purely proportional to the theoretical QF $\mathbf{s}_1^H \bar{\mathbf{\Pi}}_c^\perp \mathbf{s}_2$ but a completely different QF.

IV. NEW CONVERGENCE RESULTS

A. Convergence of LR structured quadratic forms

In this section, the convergence of the *structured* QF function of $\hat{\mathbf{\Pi}}_c^\perp$ is analyzed and results in Proposition 1.

Proposition 1: Let \mathbf{B} be a $m \times m$ deterministic complex matrix with a uniformly bounded spectral norm for all m . Then, under (**As1-As3**, **As4.S**) and the *spiked* model,

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{B} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \mathbf{s}_1^H \bar{\mathbf{\Pi}}_{c,S}^\perp \mathbf{B} \bar{\mathbf{\Pi}}_{c,S}^\perp \mathbf{s}_2 \quad (33)$$

where $\bar{\mathbf{\Pi}}_{c,S}^\perp = \sum_{i=1}^m \psi_i \mathbf{u}_i \mathbf{u}_i^H$ with ψ_i defined by Eq.(32). ■

Proof: See Appendix.

Moreover, one can remark that if $\mathbf{B} = \mathbf{R}$, where \mathbf{R} is the covariance matrix as defined in Eq.(5), the following convergence holds:

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{R} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \mathbf{s}_1^H \bar{\mathbf{\Pi}}_{c,S}^\perp \mathbf{R} \bar{\mathbf{\Pi}}_{c,S}^\perp \mathbf{s}_2 \quad (34)$$

A visualization of the convergence of Eq.(33) in terms of Mean Squared Error (MSE) can be found in Fig. 2 when $m, K \rightarrow \infty$ at a fixed ratio. It is compared to the MSE corresponding to the following convergence when $K \rightarrow \infty$ with a fixed m :

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{B} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m < \infty]{\text{a.s.}} \mathbf{s}_1^H \bar{\mathbf{\Pi}}_c^\perp \mathbf{B} \bar{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \quad (35)$$

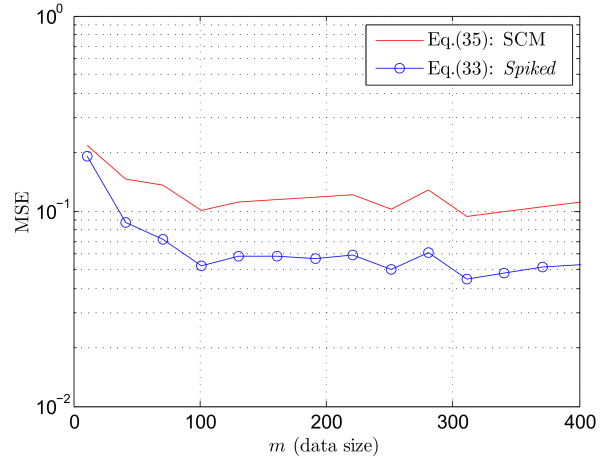


Fig. 2. MSE over 10^3 iterations corresponding to Eq.(33) and Eq.(35) when the eigenvalues of \mathbf{R} are 1, 21, 31, and 71 with the multiplicity $m - 3, 1, 1$ and 1 respectively, $c = 0.1$, $\mathbf{s}_1 = \mathbf{s}_2$ are steering vectors of the LR noise component c and $\mathbf{B} = \mathbf{R}$.

B. Convergence of SINR losses

Now, we provide the convergences of the estimated SINR losses using the convergences previously presented and the following convergence. We recall that, as $\hat{\mathbf{R}} \rightarrow \mathbf{R}$ a.s. when $K \rightarrow \infty$ with a fixed m , one has:

$$\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{R} \hat{\mathbf{R}}^{-1} \mathbf{s}_2 \xrightarrow[m < \infty]{\text{a.s.}} \mathbf{s}_1^H \mathbf{R}^{-1} \mathbf{s}_2 \quad (36)$$

Hence, when $K \rightarrow \infty$ with a fixed m and using Eq.(26), Eq.(36) and the continuous mapping theorem [40]:

$$\hat{\rho} \xrightarrow[m < \infty]{K \rightarrow \infty, \text{a.s.}} \frac{|\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a}|^2}{(\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})(\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})} = 1 \quad (37)$$

And, under (As1-As3), when $m, K \rightarrow \infty$ at the same rate, from [33], [35], [36], we have:

$$\hat{\rho} \xrightarrow[m/K \rightarrow c < \infty]{m, K \rightarrow \infty, \text{a.s.}} \frac{(1-c)|\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a}|^2}{(\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})(\mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})} = 1 - c \quad (38)$$

To summarize, $\mathbf{s}_1^H \hat{\mathbf{R}}^{-1} \mathbf{s}_2$ is consistent when $K \rightarrow \infty$ with a fixed m and its limits in the large dimensional regime is proportional to the theoretical QF $\mathbf{s}_1^H \mathbf{R}^{-1} \mathbf{s}_2$. Thus, the estimated SINR loss $\hat{\rho}$ is consistent when $K \rightarrow \infty$ with m fixed and its limits in the large dimensional regime is proportional to ρ . Consequently, RMT cannot help us to improve the estimation of the theoretical SINR loss as a function of r .

For the SINR loss corresponding to the adaptive LR filters, when $K \rightarrow \infty$ with a fixed m , using Eq.(28), Eq.(35) and the continuous mapping theorem, we have:

$$\hat{\rho}_{\text{LR}} \xrightarrow[m < \infty]{K \rightarrow \infty, \text{a.s.}} \rho_{\text{LR}} \quad (39)$$

where ρ_{LR} is defined by Eq.(14). When $m, K \rightarrow \infty$ at the same rate, we obtain the following convergence:

$$\hat{\rho}_{\text{LR}} \xrightarrow[m/K \rightarrow c < \infty]{m, K \rightarrow \infty, \text{a.s.}} \hat{\rho}_{\text{LR}}^{(S)} = \rho_{\text{LR}} |_{\Pi_c^\perp = \bar{\Pi}_{c,S}^\perp} \neq \rho_{\text{LR}} \quad (40)$$

where Eq.(31), Proposition 1 and the continuous mapping theorem were used to prove Eq.(40). One can observe that, although the traditional estimator of ρ_{LR} is consistent when $K \rightarrow \infty$ with a fixed m , it is no more consistent when $m, K \rightarrow \infty$ at the same rate. It is also important to underline that the new convergence result leads to a more precise approximation of $\hat{\rho}_{\text{LR}}$ than previous works [14], [16]. Indeed, [14], [16] propose an approximation dependent on K and r . In this article, the proposed approximation depends on K (and of course on c) and r as well as on the parameter Θ , which allows us to have a performance measure function of the ‘‘distance’’ between the signal of interest and the jamming (typically $\{\mathbf{u}_i^H \mathbf{a}(\Theta) = g_i(\Theta_{\text{jamming}}, \Theta)\}_{i \in [1,r]} \in [-1, 1]$).

V. SIMULATIONS

A. Parameters

As an illustration of the interest of the RMT utilization in filtering, the jamming application is chosen. The purpose of this application is to detect a target thanks to a ULA composed of m sensors despite the presence of jamming. The response of the jamming, c is composed of signals similar to the target response. In this section, except for the convergences when $m, K \rightarrow \infty$ at the same rate c , we choose $m = 100$ in order to have a large number for the data dimension. Even if, in some basic array processing applications, this number could seem significant, it actually became standard in many applications such as STAP [2], MIMO applications [42], [43],

MIMO-STAP [42], etc. Here, $\Theta = \theta$ where θ is the AoA. The jamming is composed of three synthetic targets with AoA -20° , 0° and 20° and wavelength $l_0 = 0.667\text{m}$. Thus, the jamming (LR noise) has a rank $r = 3$. Then, the AWGN \mathbf{b} power is $\sigma^2 = 1$. Finally, the theoretical covariance matrix of the total noise can be written as $\mathbf{R} = \frac{JNR}{\text{tr}(\mathbf{A})} \mathbf{U} \mathbf{A} \mathbf{U}^H + \sigma^2 \mathbf{I}_m$ with $\mathbf{A} = \text{diag}([6, 2, 1])$ and where JNR is the jamming to noise ratio. $\frac{JNR}{\text{tr}(\mathbf{A})}$ is set at 10dB except for Fig. 4.

In order to validate the *spiked* model as covariance matrix model, we visualize a zoom of the experimental PDF of the eigenvalues of our data without target in Fig. 3 over 5×10^4 Monte-Carlo iterations. We observe a Marčenko-Pastur law around 1 (eigenvalues of the white noise) and Gaussian distributions for the eigenvalues of the jamming, which is consistent to the CLT for the *spiked* model proved in [22]. The *spiked* model is consequently relevant for our data model.

Moreover, in order to verify that the *spiked* model

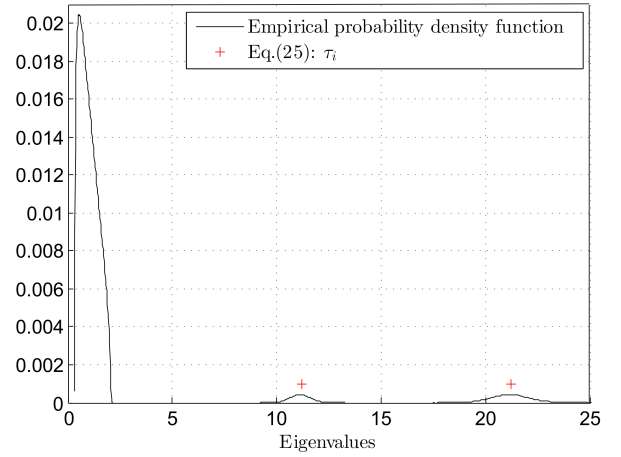


Fig. 3. Zoom of the experimental PDF of jamming plus noise data with $c = 0.2$ and $\frac{JNR}{\text{tr}(\mathbf{A})} = 10\text{dB}$. Due to the zoom, the highest eigenvalue 71 is out of the figure.

is realistic in terms of *separation condition*, Fig. 4 shows $(\omega_r - \sqrt{c})$ as a function of $\frac{JNR}{\text{tr}(\mathbf{A})}$ in dB. We recall that, in order to satisfy the *separation condition*, one should have $\omega_r - \sqrt{c} > 0$. Consequently, we gladly observe that it is satisfied for $\frac{JNR}{\text{tr}(\mathbf{A})} > 6.2\text{dB}$ for the majority of c even if $c > 17$. Indeed, in practice, if the $\frac{JNR}{\text{tr}(\mathbf{A})}$ is lower than 6.2dB, the jamming will not have any effects on the performance. Consequently, the chosen parameters are realistic even if $K = 2r$ (implying $c \simeq 16.7$).

B. Performance of filters

We now observe the performance of filters through the SINR loss. We are first interested in the validation of the convergence of $\hat{\rho}_{\text{LR}}$ in Eq.(40) as $m, K \rightarrow \infty$ at the same rate. This convergence is validated and presented in Fig. 5 in terms of MSE over 10^3 realizations with $c = 3$ for an AoA of the target ($\theta = 50^\circ$) and an AoA of the jamming ($\theta = 20^\circ$).

Fig. 6 shows the visualization of Eq.(14) (blue line with stars), Eq.(15) (blue dashed line), the right side of the convergence in Eq.(40) (green line with circles) and the approximation $\mathbb{E}[\hat{\rho}_{\text{LR}}] \simeq 1 - \frac{r}{K}$ introduced by [14] (black

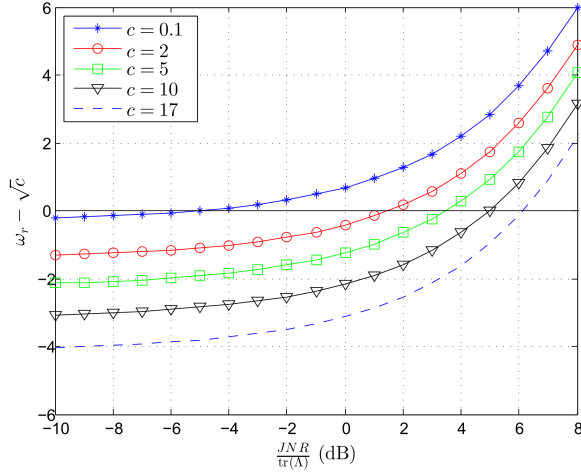


Fig. 4. Separation condition ($\omega_r - \sqrt{c}$) of the *spiked* model for the lowest non-unit eigenvalue as a function of the ratio $\frac{JNR}{\text{tr}(\Lambda)}$ in dB.

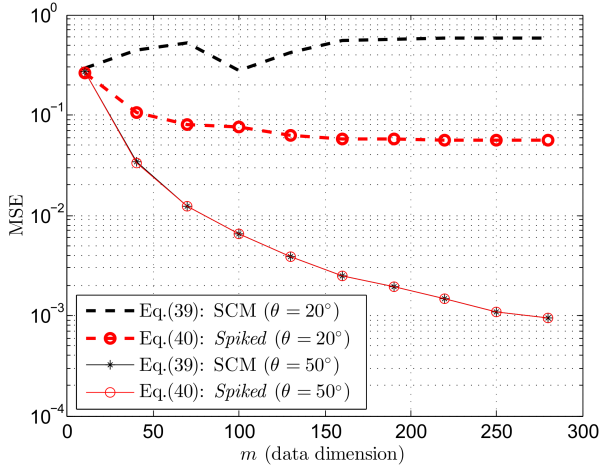


Fig. 5. MSE corresponding to Eq.(39) and Eq.(40) when $m, K \rightarrow \infty$ at a fixed ratio $c = 3$ and $\frac{JNR}{\text{tr}(\Lambda)} = 10\text{dB}$.

line) as a function of K when the target is near from the jamming, i.e. $\theta = 20.5^\circ$. We observe that the *spiked* model and the RMT helps us to obtain a better estimation of $\mathbb{E}[\hat{\rho}_{\text{LR}}]$ than the estimation $\mathbb{E}[\hat{\rho}_{\text{LR}}] \simeq 1 - \frac{r}{K}$ as the curve of $\hat{\rho}_{\text{LR}}^{(S)}$ has the same behavior as the curve of $\hat{\rho}_{\text{LR}}$. Then, similarly, the same equations are visualized as a function of θ in Fig. 7 with $K = 2r$. We observe that, unlike the estimation $1 - r/K$, the RMT with the *spiked* model permits us to obtain a better estimation of $\mathbb{E}[\hat{\rho}_{\text{LR}}]$ as a function of θ and consequently a better approximation of its behavior. Thus, it permits to predict the parameter θ value corresponding to the performance break (here around 21.1°).

VI. CONCLUSION

In this paper, we proposed new results in random matrix theory with a specific covariance matrix model fitted to our data model: the *spiked* model. Based on this, we studied the convergence of the traditional estimator of the SINR loss in its low rank version when the number of secondary data $K \rightarrow \infty$ with a fixed data dimension m and when $m, K \rightarrow \infty$ at

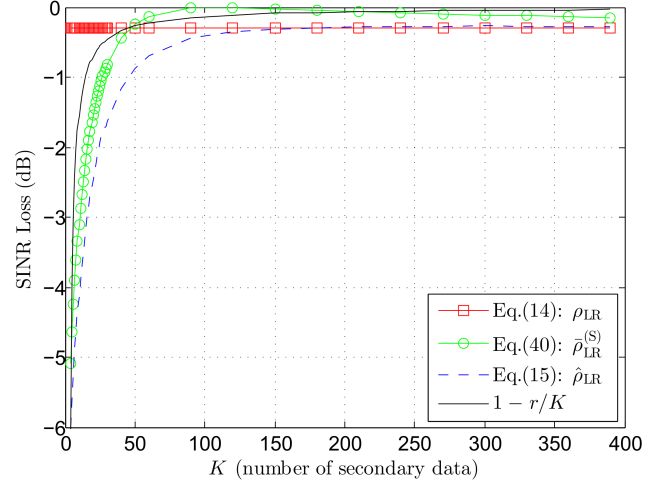


Fig. 6. Visualization of Eq.(14) (red line with squares), Eq.(15) (blue dashed line), the right side of the convergence in Eq.(40) (green line with circles) and the traditional estimation of $\mathbb{E}[\hat{\rho}_{\text{LR}}]$ (black line) as a function of K (over 10^3 realizations) with $\frac{JNR}{\text{tr}(\Lambda)} = 10\text{dB}$, $m = 100$ and $\theta = 20.5^\circ$.

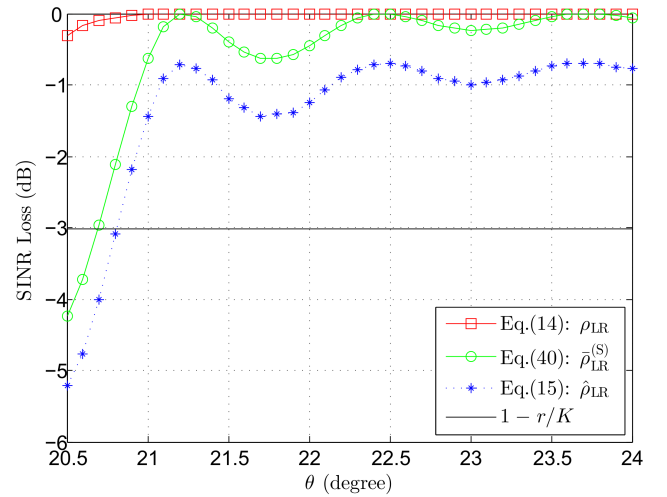


Fig. 7. Visualization of Eq.(14) (red line with squares), Eq.(15) (blue dashed line), the right side of the convergence in Eq.(40) (green line with circles) and the traditional estimation of $\mathbb{E}[\hat{\rho}_{\text{LR}}]$ (black line) as a function of θ (over 10^3 realizations) with $\frac{JNR}{\text{tr}(\Lambda)} = 10\text{dB}$, $m = 100$ and $K = 2r$.

the same rate $c = m/K$ as random matrix theory cannot help us with its full rank version to obtain performance function of c and the distance between the target and the jammers. We observed the low rank version is consistent when $K \rightarrow \infty$ with a fixed m but is not consistent when $m, K \rightarrow \infty$ at the same rate c . Finally, we applied these results to a jamming application. We first observed that the experimental probability density function of the eigenvalues of the covariance matrix of jamming data is relevant with the probability density function of the eigenvalues corresponding to the *spiked* model. Then, we validated the convergence of the SINR loss in its low rank version. We observed that random matrix theory and more precisely the *spiked* model better evaluate the asymptotic performance of the low rank SINR loss corresponding to the adaptive low rank filter, especially when the steering vector parameter is close to the jamming

one. Thus, contrary to previous works, it allows us to predict the filter performance when the signal of interest is close to the low rank noise. Consequently, this new result permits to predict the steering vector parameter value corresponding to the performance break. ■

VII. APPENDIX

The proof is decomposed as follows. We first develop the *structured* QF as a sum of *simple* QFs and base *structured* QF (Subsec. VII-A). In a second time, we formulate the base *structured* QF as a complex integral (Subsec. VII-B) and split it into several integrals (Subsec. VII-C). Then, we determine the deterministic complex integral equivalent of the base *structured* QF (Subsec. VII-D) and its formal expression (Subsec. VII-E). Finally, we use this result to determine the convergence of the *structured* QF in the large dimensional regime (Subsec. VII-F). The regime of convergences in the Appendix, if not precised, is $m, K \rightarrow \infty$ at a fixed ratio c .

A. Development of the structured QF

Let s_1 and s_2 be two deterministic complex vectors and \mathbf{B} be a $m \times m$ deterministic complex matrix with uniformly bounded spectral norm for all m . In order to obtain the convergence of the *structured* QF $s_1^H \hat{\Pi}_c^\perp \mathbf{B} \hat{\Pi}_c^\perp s_2$, one can rewrite, using the notations of Eq.(22) and the *spiked* model, $\hat{\Pi}_c^\perp = \hat{\Pi}_{r+1} = \hat{\mathbf{U}}_{r+1} \hat{\mathbf{U}}_{r+1}^H = \mathbf{I}_m - \sum_{i=1}^r \hat{\Pi}_i$ where $\hat{\mathbf{U}}_{r+1} = [\hat{\mathbf{u}}_{r+1}, \dots, \hat{\mathbf{u}}_m]$, $\hat{\Pi}_i = \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H$, $\forall i \in \llbracket 1, r \rrbracket$ and $\hat{\mathbf{u}}_i$ are the eigenvectors of the SCM. We recall that r is fixed for all m , i.e. $r/m \rightarrow 0^+$ and the notation $\mathbf{U}_s = [\mathbf{u}_1, \dots, \mathbf{u}_r]$. Thus, one can develop the *structured* QF as :

$$\begin{aligned} s_1^H \hat{\Pi}_c^\perp \mathbf{B} \hat{\Pi}_c^\perp s_2 &= s_1^H \left(\mathbf{I}_m - \sum_{i=1}^r \hat{\Pi}_i \right) \mathbf{B} \left(\mathbf{I}_m - \sum_{i=1}^r \hat{\Pi}_i \right) s_2 \quad (41) \\ &= s_1^H \mathbf{B} s_2 - s_1^H \sum_{i=1}^r \hat{\Pi}_i \mathbf{B} s_2 - s_1^H \mathbf{B} \sum_{i=1}^r \hat{\Pi}_i s_2 \\ &\quad + s_1^H \sum_{i=1}^r \hat{\Pi}_i \mathbf{B} \sum_{i=1}^r \hat{\Pi}_i s_2 \quad (42) \\ &= s_1^H \mathbf{B} s_2 - \sum_{i=1}^r \left(s_1^H \hat{\Pi}_i \mathbf{B} s_2 + s_1^H \mathbf{B} \hat{\Pi}_i s_2 \right) \\ &\quad + \sum_{j_1=1}^r s_1^H \hat{\Pi}_{j_1} \mathbf{B} \hat{\Pi}_{j_1} s_2 + \sum_{\substack{j_1, j_2=1 \\ j_1 \neq j_2}}^r s_1^H \hat{\Pi}_{j_1} \mathbf{B} \hat{\Pi}_{j_2} s_2 \quad (43) \end{aligned}$$

B. Formulation of the base structured QF as a complex integral

Remarking that Eq.(43) is a sum of *simple* QFs and base *structured* QFs, we first focus on the convergence of the base *structured* QF $\hat{\eta}(j_1, j_2) = s_1^H \hat{\Pi}_{j_1} \mathbf{B} \hat{\Pi}_{j_2} s_2$, $\{j_1, j_2\} \in \llbracket 1, r \rrbracket^2$. Let us now formulate the base *structured* QF as a complex integral.

Proposition 2: Let \mathbf{B} be a $m \times m$ deterministic complex matrix with a uniformly bounded spectral norm for all m . Then, under (As1-As3, As4.S) and the *spiked* model, $\forall j_1, j_2 \in \llbracket 1, r \rrbracket$, if $\hat{\eta}(j_1, j_2) = s_1^H \hat{\Pi}_{j_1} \mathbf{B} \hat{\Pi}_{j_2} s_2$:

$$\begin{aligned} \hat{\eta}(j_1, j_2) &= \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} s_1^H \left(\hat{\mathbf{R}} - z_1 \mathbf{I}_m \right)^{-1} \\ &\quad \times \mathbf{B} \left(\hat{\mathbf{R}} - z_2 \mathbf{I}_m \right)^{-1} s_2 dz_1 dz_2 \quad (44) \end{aligned}$$

Proof: If $j_1 \neq j_2$, it can be easily shown that $\hat{\eta}(j_1, j_2)$ can be expressed as the following Cauchy integral in two complex variables [44]:

$$A = \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} s_1^H \left(\hat{\mathbf{R}} - z_1 \mathbf{I}_m \right)^{-1} \mathbf{B} \left(\hat{\mathbf{R}} - z_2 \mathbf{I}_m \right)^{-1} s_2 dz_1 dz_2 \quad (45)$$

where C_j^- is a negatively oriented contour encompassing the eigenvalues of $\hat{\mathbf{R}}$ corresponding to the j -th eigenvalue of \mathbf{R} and z_1 and z_2 are independent variables.

Then, if $j_1 = j_2 = j \in \llbracket 1, r \rrbracket$, one has:

$$s_1^H \hat{\Pi}_j \mathbf{B} \hat{\Pi}_j s_2 = \frac{1}{2i\pi} \oint_{C_j^-} s_1^H \sum_{n=1}^m \frac{\hat{\mathbf{u}}_n \hat{\mathbf{u}}_n^H \mathbf{B} \hat{\mathbf{u}}_n \hat{\mathbf{u}}_n^H}{\hat{\lambda}_n - z} s_2 dz \quad (46)$$

However, the remaining of the proof is based on the fact that the resolvent $\mathbf{G}(z)$ of the SCM can be found in the complex integral, which is not the case in the previous equation. Consequently, even if Eq.(46) is equivalent to Eq.(44), it will be easier to use Eq.(44). As a consequence, $\forall j_1, j_2 \in \llbracket 1, r \rrbracket$:

$$\begin{aligned} \hat{\eta}(j_1, j_2) &= \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} s_1^H \left(\hat{\mathbf{R}} - z_1 \mathbf{I}_m \right)^{-1} \\ &\quad \times \mathbf{B} \left(\hat{\mathbf{R}} - z_2 \mathbf{I}_m \right)^{-1} s_2 dz_1 dz_2 \quad (47) \end{aligned}$$

C. Development of the complex integral

Next, one want to split the previous line integral into several line integrals where some of them will tend to 0. Thus, from [22], with $k \in \{1, 2\}$, one can write:

$$\begin{aligned} \left(\hat{\mathbf{R}} - z_k \mathbf{I}_m \right)^{-1} &= \left(\mathbf{I}_m + \mathbf{P} \right)^{-1/2} \left[\mathbf{Q}(z_k) - z_k \mathbf{Q}(z_k) \mathbf{U}_s \right. \\ &\quad \times \hat{\mathbf{H}}(z_k)^{-1} \boldsymbol{\Omega} \left(\mathbf{I}_r + \boldsymbol{\Omega} \right)^{-1} \mathbf{U}_s^H \mathbf{Q}(z_k) \left. \right] \\ &\quad \times \left(\mathbf{I}_m + \mathbf{P} \right)^{-1/2} \quad (48) \end{aligned}$$

with

$$\mathbf{Q}(z_k) = \left(\frac{1}{K} \mathbf{Y} \mathbf{Y}^H - z_k \mathbf{I}_m \right)^{-1} \quad (49)$$

$$\hat{\mathbf{H}}(z_k) = \mathbf{I}_r + z_k \boldsymbol{\Omega} \left(\mathbf{I}_r + \boldsymbol{\Omega} \right)^{-1} \mathbf{U}_s^H \mathbf{Q}(z_k) \mathbf{U}_s \quad (50)$$

Then, replacing $\left(\hat{\mathbf{R}} - z_k \mathbf{I}_m \right)^{-1}$ by Eq.(48) in Eq.(44) and developing the obtained result, one obtains:

$$\hat{\eta}(j_1, j_2) = D_1 - D_2 - D_3 + D_4 \quad (51)$$

$$\begin{aligned} \hat{\eta}(j_1, j_2) &= \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} s_1^H \mathbf{E}(z_1) \mathbf{B} \mathbf{E}(z_2) s_2 dz_1 dz_2 \\ &\quad - \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} \left[\hat{\mathbf{e}}_1^H(z_1) \hat{\mathbf{H}}(z_1)^{-1} \boldsymbol{\Omega} \left(\mathbf{I}_r + \boldsymbol{\Omega} \right)^{-1} \right. \\ &\quad \times \mathbf{U}_s^H \mathbf{Q}(z_1) \left(\mathbf{I}_m + \mathbf{P} \right)^{-1/2} \mathbf{B} \left. \right] \mathbf{E}(z_2) s_2 dz_1 dz_2 \\ &\quad - \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} s_1^H \mathbf{E}(z_1) \left[\mathbf{B} \left(\mathbf{I}_m + \mathbf{P} \right)^{-1/2} z_2 \right. \\ &\quad \times \mathbf{Q}(z_2) \mathbf{U}_s \hat{\mathbf{H}}(z_2)^{-1} \hat{\mathbf{e}}_2(z_2) \left. \right] dz_1 dz_2 \\ &\quad + \frac{1}{(2i\pi)^2} \oint_{C_{j_1}^-} \oint_{C_{j_2}^-} \hat{\mathbf{e}}_1^H(z_1) \hat{\mathbf{H}}(z_1)^{-1} \hat{\mathbf{C}}(z_1, z_2) \\ &\quad \times \hat{\mathbf{H}}(z_2)^{-1} \hat{\mathbf{e}}_2(z_2) s_2 dz_1 dz_2 \quad (52) \end{aligned}$$

with

$$\mathbf{E}(z) = (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{Q}(z) (\mathbf{I}_m + \mathbf{P})^{-1/2} \quad (53)$$

$$\hat{\mathbf{e}}_1^H(z) = \mathbf{s}_1^H (\mathbf{I}_m + \mathbf{P})^{-1/2} z \mathbf{Q}(z) \mathbf{U}_s \quad (54)$$

$$\hat{\mathbf{C}}(z_1, z_2) = \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H \mathbf{Q}(z_1) (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{B} \\ \times (\mathbf{I}_m + \mathbf{P})^{-1/2} z_2 \mathbf{Q}(z_2) \mathbf{U}_s \quad (55)$$

$$\hat{\mathbf{e}}_2(z) = \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H \mathbf{Q}(z) (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{s}_2 \quad (56)$$

D. Determination of the deterministic complex integral equivalent

The convergence of the terms D_1 to D_4 has now to be studied. Some of them will tend to 0 and the remainder of the terms will tend to a deterministic integral equivalent.

Proposition 3: Let \mathbf{B} be a $m \times m$ deterministic complex matrix with a uniformly bounded spectral norm for all m . Then, under (As1-As3, As4.S) and the spiked model, $\forall j_1, j_2 \in [1, r]$, $\hat{\eta}(j_1, j_2) - \eta(j_1, j_2) \xrightarrow{a.s.} 0$ with

$$\eta(j_1, j_2) = \frac{1}{(2i\pi)^2} \oint_{\gamma_{j_1}^-} \oint_{\gamma_{j_2}^-} \mathbf{e}_1^H(z_1) \mathbf{H}(z_1)^{-1} \mathbf{C}(z_1, z_2) \\ \times \mathbf{H}(z_2)^{-1} \mathbf{e}_2(z_2) dz_1 dz_2 \quad (57)$$

where γ_j^- is a deterministic negatively oriented circle only enclosing τ_j (cf. Eq.(25)) and

$$\mathbf{H}(z) = \mathbf{I}_r + z \bar{b}_m(z) \Omega (\mathbf{I}_r + \Omega)^{-1} \quad (58)$$

$$\mathbf{e}_1^H(z) = z \bar{b}_m(z) \mathbf{s}_1^H (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{U}_s \quad (59)$$

$$\mathbf{C}(z_1, z_2) = z_2 \bar{b}_m(z_1) \bar{b}_m(z_2) \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H (\mathbf{I}_m + \mathbf{P})^{-1/2} \\ \times \mathbf{B} (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{U}_s \quad (60)$$

$$\mathbf{e}_2(z) = \bar{b}_m(z) \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{s}_2 \quad (61)$$

Proof: The function $\mathbf{E}(z)$ in D_1 , D_2 and D_3 can be rewritten as:

$$\mathbf{E}(z) = (\mathbf{I}_m + \mathbf{P})^{-1/2} \sum_{n=1}^m \frac{\hat{v}_n \hat{v}_n^H}{\hat{\zeta}_n - z} (\mathbf{I}_m + \mathbf{P})^{-1/2} \quad (62)$$

where $\hat{\zeta}_n$ and \hat{v}_n are the eigenvalues and the eigenvectors of $\mathbf{Y}\mathbf{Y}^H/K$ respectively. Thus, $\mathbf{E}(z_1)$ (resp. $\mathbf{E}(z_2)$) has a single simple pole $\hat{\zeta}_n$. As a consequence, $\forall j_1, j_2 \in [1, r]$, $\mathcal{C}_{j_1}^-$ (resp. $\mathcal{C}_{j_2}^-$) does not encompass $\mathbf{E}(z_1)$ (resp. $\mathbf{E}(z_2)$) under (As3, As4.S). Thus, $D_1 = D_2 = D_3 = 0$ and:

$$\hat{\eta}(j_1, j_2) = \frac{1}{(2i\pi)^2} \oint_{\mathcal{C}_{j_1}^-} \oint_{\mathcal{C}_{j_2}^-} \hat{\mathbf{e}}_1^H(z_1) \hat{\mathbf{H}}(z_1)^{-1} \hat{\mathbf{C}}(z_1, z_2) \\ \times \hat{\mathbf{H}}(z_2)^{-1} \hat{\mathbf{e}}_2(z_2) dz_1 dz_2 \quad (63)$$

We will then determine a deterministic equivalent of Eq.(63), i.e. its convergence in the large dimensional regime. From lemma 5 of [32] and [22],

$$\hat{\mathbf{H}}(z) \xrightarrow[m/K \rightarrow c < \infty]{a.s.} \mathbf{H}(z) = \mathbf{I}_r + z \bar{b}_m(z) \Omega (\mathbf{I}_r + \Omega)^{-1} \quad (64)$$

$$\hat{\mathbf{e}}_1^H(z) \xrightarrow[m/K \rightarrow c < \infty]{a.s.} \mathbf{e}_1^H(z) = z \bar{b}_m(z) \mathbf{s}_1^H (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{U}_s \quad (65)$$

$$\hat{\mathbf{e}}_2(z) \xrightarrow[m/K \rightarrow c < \infty]{a.s.} \mathbf{e}_2(z) = \bar{b}_m(z) \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{s}_2 \quad (66)$$

Then, the most difficult part concerns the convergence of

$$\hat{\mathbf{C}}(z_1, z_2) = z_2 \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H \mathbf{Q}(z_1) (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{B} \\ \times (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{Q}(z_2) \mathbf{U}_s \quad (67)$$

$$= z_2 \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H \mathbf{Q}(z_1) \mathbf{D}\mathbf{Q}(z_2) \mathbf{U}_s \quad (68)$$

which contains the matrix $m \times m$ complex $\mathbf{Q}(z_1) \mathbf{D}\mathbf{Q}(z_2)$. As the convergence cannot be addressed through classical methods as in [32], we focus on the asymptotic equivalence of matrices. Indeed, it can be proven using Gaussian methods [45], [46] that

$$\mathbf{Q}(z_1) \mathbf{D}\mathbf{Q}(z_2) \longleftrightarrow \bar{b}_m(z_1) \bar{b}_m(z_2) \mathbf{D} \quad (69)$$

where \longleftrightarrow means the asymptotic equivalence of two matrices in the large dimensional regime. However, the proof is a result in itself and therefore is out of scope of this paper. For this reason, it is omitted here. Consequently, considering Eq.(69),

$$\hat{\mathbf{C}}(z_1, z_2) \xrightarrow[m/K \rightarrow c < \infty]{a.s.} \mathbf{C}(z_1, z_2) = z_2 \bar{b}_m(z_1) \bar{b}_m(z_2) \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H \\ \times (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{B} (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{U}_s \quad (70)$$

As a result, $\hat{\eta}(j_1, j_2) - \eta(j_1, j_2) \xrightarrow{a.s.} 0$ with

$$\eta(j_1, j_2) = \frac{1}{(2i\pi)^2} \oint_{\gamma_{j_1}^-} \oint_{\gamma_{j_2}^-} \mathbf{e}_1^H(z_1) \mathbf{H}(z_1)^{-1} \mathbf{C}(z_1, z_2) \\ \times \mathbf{H}(z_2)^{-1} \mathbf{e}_2(z_2) dz_1 dz_2 \quad (71)$$

where γ_j^- is a deterministic negatively oriented circle only enclosing τ_j (cf. Eq.(25)).

E. Determination of the expression of the deterministic equivalent

Let us now find the expression of the deterministic equivalent $\eta(j_1, j_2)$ as a function of the eigenvalues and eigenvectors of the covariance matrix \mathbf{R} .

Proposition 4: Let \mathbf{B} be a $m \times m$ deterministic complex matrix with a uniformly bounded spectral norm for all m . Then, under (As1-As3, As4.S) and the spiked model,

$$\eta(j_1, j_2) = \chi_{j_1} \chi_{j_2} \mathbf{s}_1^H \mathbf{\Pi}_{j_1} \mathbf{B} \mathbf{\Pi}_{j_2} \mathbf{s}_2 \quad (72)$$

with $\chi_j = \frac{1 - c\omega_j^{-2}}{1 + c\omega_j^{-1}}$ and $\{j_1, j_2\} \in [1, r]^2$. ■

Proof: We first decompose Eq.(71) as:

$$\eta(j_1, j_2) = \frac{1}{2i\pi} \oint_{\gamma_{j_1}^-} \mathbf{g}(z_1) dz_1 \mathbf{B} \frac{1}{2i\pi} \oint_{\gamma_{j_1}^-} \tilde{\mathbf{g}}(z_2) dz_2 \quad (73)$$

with

$$\mathbf{g}(z) = \bar{b}_m(z) \mathbf{e}_1^H(z) \mathbf{H}(z)^{-1} \Omega (\mathbf{I}_r + \Omega)^{-1} \mathbf{U}_s^H (\mathbf{I}_m + \mathbf{P})^{-1/2} \quad (74)$$

$$\tilde{\mathbf{g}}(z) = z \bar{b}_m(z) (\mathbf{I}_m + \mathbf{P})^{-1/2} \mathbf{U}_s \mathbf{H}(z)^{-1} \mathbf{e}_2(z) \quad (75)$$

to distinguish the contributions associated to z_1 and z_2 . After an eigendecomposition of Eqs.(74) and (75) and, using [22], one obtains:

$$\mathbf{g}(z_1) = \mathbf{s}_1^H \sum_{l=1}^r \frac{\omega_l \mathbf{\Pi}_l}{(1 + \omega_l)^2} \frac{z_1 \bar{b}_m^2(z_1)}{1 + z_1 \bar{b}_m(z_1)} \frac{\omega_l}{1 + \omega_l} \quad (76)$$

$$\tilde{\mathbf{g}}(z_2) = \sum_{l=1}^r \frac{\omega_l \mathbf{\Pi}_l}{(1 + \omega_l)^2} \frac{z_2 \bar{b}_m^2(z_2)}{1 + z_2 \bar{b}_m(z_2)} \frac{\omega_l}{1 + \omega_l} \mathbf{s}_2 \quad (77)$$

As a result, from the properties of $\bar{b}_m(z)$ and residue calculus (see [22] for further details), one obtains:

$$\eta(j_1, j_2) = \xi(\tau_{j_1}) \xi(\tau_{j_2}) \mathbf{s}_1^H \mathbf{\Pi}_{j_1} \mathbf{B} \mathbf{\Pi}_{j_2} \mathbf{s}_2 \quad (78)$$

with

$$\xi(\tau_j) = \frac{(1 + h(\tau_j)) \bar{b}_m(\tau_j)}{h'(\tau_j)} \quad (79)$$

Finally, the last step consists in expressing $\xi(\tau_j)$ as a function of ω_j . Using Corollary 2 from [22], one expresses $\xi(\tau_j)$ as:

$$\xi(\tau_j) = \chi_j = \frac{1 - c\omega_j^{-2}}{1 + c\omega_j^{-1}} \quad (80)$$

As a consequence,

$$\hat{\eta}(j_1, j_2) \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \eta(j_1, j_2) = \chi_{j_1} \chi_{j_2} \mathbf{s}_1^H \mathbf{\Pi}_{j_1} \mathbf{B} \mathbf{\Pi}_{j_2} \mathbf{s}_2 \quad (81)$$

with $\{j_1, j_2\} \in \llbracket 1, r \rrbracket^2$.

F. Convergence of the structured QF

From the development of the *structured* QF, we recall that the convergences of the *simple* QFs $\mathbf{s}_1^H \hat{\mathbf{\Pi}}_i \mathbf{B} \mathbf{s}_2$ and $\mathbf{s}_1^H \mathbf{B} \hat{\mathbf{\Pi}}_i \mathbf{s}_2$, $\forall i \in \llbracket 1, r \rrbracket$ can be easily determined from [22]:

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_i \mathbf{B} \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \chi_i \mathbf{s}_1^H \mathbf{\Pi}_i \mathbf{B} \mathbf{s}_2 \quad (82)$$

$$\mathbf{s}_1^H \mathbf{B} \hat{\mathbf{\Pi}}_i \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \chi_i \mathbf{s}_1^H \mathbf{B} \mathbf{\Pi}_i \mathbf{s}_2 \quad (83)$$

where χ_i is defined as in Section III.C.

Then, also using Eq.(81) in Eq.(43), one easily obtains:

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{B} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \mathbf{s}_1^H \left[\mathbf{I}_m - \sum_{i=1}^r \chi_i \mathbf{\Pi}_i \right] \mathbf{B} \left[\mathbf{I}_m - \sum_{i=1}^r \chi_i \mathbf{\Pi}_i \right] \mathbf{s}_2 \quad (84)$$

$$\mathbf{s}_1^H \hat{\mathbf{\Pi}}_c^\perp \mathbf{B} \hat{\mathbf{\Pi}}_c^\perp \mathbf{s}_2 \xrightarrow[m/K \rightarrow c < \infty]{\text{a.s.}} \mathbf{s}_1^H \hat{\mathbf{\Pi}}_{c,S}^\perp \mathbf{B} \hat{\mathbf{\Pi}}_{c,S}^\perp \mathbf{s}_2 \quad (85)$$

with $\hat{\mathbf{\Pi}}_{c,S}^\perp = \sum_{i=1}^m \psi_i \mathbf{u}_i \mathbf{u}_i^H$ and

$$\psi_i = \begin{cases} 1, & \text{if } i > r \\ 1 - \chi_i, & \text{if } i \leq r \end{cases} \quad (86)$$

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