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ON TWO PROBLEMS OF HARDY AND MAHLER

PATRICE PHILIPPON AND PURUSOTTAM RATH

ABSTRACT. It is a classical result of Mahler that for any rational number $\alpha > 1$ which is not an integer and any real $0 < c < 1$, the set of positive integers n such that $\|\alpha^n\| < c^n$ is necessarily finite. Here for any real x, $\|x\|$ denotes the distance from its nearest integer. The problem of classifying all real algebraic numbers greater than one exhibiting the above phenomenon was suggested by Mahler. This was solved by a beautiful work of Corvaja and Zannier. On the other hand, for non-zero real numbers λ and α with $\alpha > 1$, Hardy about a century ago asked

"In what circumstances can it be true that $\|\lambda\alpha^n\| \to 0$ as $n \to \infty$? "

This question is still open in general. In this note, we study its analogue in the context of the problem of Mahler. We first compare and contrast with what is known vis-a-vis the original question of Hardy. We then suggest a number of questions that arise as natural consequences of our investigation. Of these questions, we answer one and offer some insight into others.

1. INTRODUCTION

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}^\times$ denote the set of non-negative integers and positive integers respectively. Further for any real x, $\|x\|$ denotes its distance from its nearest integer. In other words,

$$\|x\| := \min\{|x - m| : m \in \mathbb{Z}\}.$$

The growth of the sequence $\|(3/2)^n\|$ is intricately linked to the famous Waring's problem. For a positive integer k, let $g(k)$ denote the minimum number s such that every positive integer is expressible as a sum of s k-th powers of elements in $\mathbb{N}$. It is known that for $k \geq 6$,

$$g(k) = 2^k + [(3/2)^k] - 2$$

if $\|(3/2)^k\| \geq (3/4)^k$.

This was the motivation for Mahler [10] in 1957 to prove that for any $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, $\alpha > 1$, and any real $0 < c < 1$, the set of $n \in \mathbb{N}$ such that $\|\alpha^n\| < c^n$ is finite. Mahler uses Ridout's theorem which is a p-adic extension of the famous theorem of Roth that algebraic irrationals have irrationality measure two.

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Mahler ends his paper suggesting that it would be of some interest to know which algebraic numbers have the same property as the non integral rationals, that is to characterise real algebraic numbers $\alpha > 1$ such that for any real $0 < c < 1$, the set of $n \in \mathbb{N}$ such that $\|\alpha^n\| < c^n$ is finite.

This was answered completely by Corvaja and Zannier [3] who proved the following:

Theorem 1. (Corvaja and Zannier) *Let $\alpha > 1$ be a real algebraic number. Then for some $0 < c < 1$, $\|\alpha^n\| < c^n$ for infinitely many $n \in \mathbb{N}$ if and only if there exists a positive integer s such that α^s is a PV number. In particular, α is an algebraic integer.*

We recall that a PV number (for Pisot-Vijayaraghavan number) is a real algebraic integer α such that:

- $\alpha > 1$;
- all its other conjugates (over $\mathbb{Q}$) have absolute values strictly less than 1.

On the other hand, for non-zero real numbers λ and α with $\alpha > 1$, Hardy [7] about a century ago asked

"In what circumstances can it be true that $\|\lambda\alpha^n\| \to 0$ as $n \to \infty$? "

This question is still open in general. However, when α is further assumed to be algebraic, then the question is easier and can be settled as was shown by Hardy himself. More precisely, one has (Theorem A in [7]):

Theorem 2. (Hardy) *Let $\alpha > 1$ be an algebraic real number and λ be a non-zero real number. Suppose that $\|\lambda\alpha^n\| \to 0$ as $n \to \infty$. Then α is a PV number, hence an algebraic integer. Further in this case, λ necessarily lies in $\mathbb{Q}(\alpha)$ and hence is algebraic.*

In this note, we consider the analogue of this question in the context of the problem of Mahler. More precisely, consider the sets

$$H := \{(\lambda, \alpha) \in \mathbb{R}^\times \times (1, \infty) : \ \|\lambda\alpha^n\| \to 0 \text{ as } n \to \infty\}$$

and

$$M := \{(\lambda, \alpha) \in \mathbb{R}^\times \times (1, \infty) : \exists\, 0 < c < 1 \text{ such that } \|\lambda\alpha^n\| < c^n \text{ for infinitely many } n \in \mathbb{N}\}.$$

The goal of our work is to compare and contrast these two sets lying in $\mathbb{R}^2$. Sometimes we refer to these sets as the Hardy set and the Mahler set respectively.

To start with, we note that it is known that the set H is countable. See for instance, the pretty book of Salem [12, Chap.1, §4].

On the other hand, the set M is uncountable. In fact, for any sequence $(\epsilon_n)_{n\in\mathbb{N}}$ in $\{0, 1\}$ not identically 0, set $\lambda = \sum_{n=0}^{\infty} \frac{\epsilon_n}{10^{n!}}$, then $(\lambda, 10)$ lies in M.

We also note that Corvaja and Zannier in their paper construct a transcendental real number $\alpha > 1$ such that $(1, \alpha)$ lies in M. Further, it follows from a work of Bugeaud and Dubickas [2] that there are uncountably many $\alpha > 1$ such that $(1, \alpha)$ lies in M.

In this context, one has the following folklore conjecture, which proposes an answer to Hardy's question.

Conjecture 3. *If $(\lambda, \alpha) \in H$, then α is a PV number and λ lies in $\mathbb{Q}(\alpha)$.*

Clearly, this no longer holds for the set M as there are transcendental numbers $\alpha > 1$ such that $(1, \alpha)$ lie in M.

More interestingly, even if $(\lambda, \alpha) \in M$ with α algebraic, it does not imply that λ is algebraic. For instance, as remarked before, $(\lambda, 10) \in M$ with λ the Liouville number $\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$. This is in contrast to Theorem 2.

Remark 4. *More generally, when $\alpha > 1$ is an integer and λ is a non zero real number, $(\lambda, \alpha) \in M$ if and only if for some $\varepsilon > 0$, the sequence of digits of the expansion of λ in base α contains infinitely many blocks of the form $x_n \cdots x_{n+[\varepsilon n]}$ consisting only of zeros or only of $(\alpha - 1)$'s.*

These observations give rise to a number of questions.

Questions:

1. What can be said about $(\lambda, \alpha) \in M$ where both λ and α are assumed to be algebraic? Can one have a theorem similar to Theorem 2 in this case?

2. Is it possible to derive some diophantine characterisation of transcendental numbers λ such that $(\lambda, \alpha) \in M$ for some algebraic α?

3. Is it true that $(1, \alpha) \in H$ implies that $(1, \alpha) \in M$? This is a consequence of the following.

4. Is it true that $H \subset M$?

5. What is the Hausdorff dimension of the set M?

As introduced in [3], a pseudo-PV number is an algebraic number $\alpha \in \mathbb{C}$ such that:

- $|\alpha| > 1$,
- all its other conjugates have absolute values strictly less than 1;
- $Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) \in \mathbb{Z}$.

Note that a pseudo-PV number is necessarily real; a positive pseudo-PV number which is an algebraic integer is simply a usual PV number.

Here is our first theorem.

Theorem 5. *Let λ and α be algebraic numbers such that $(\lambda, \alpha) \in M$. Then there exists a positive integer s such that α^s is a PV number and hence an algebraic integer.*

Furthermore for any $h \in \mathbb{N}^{\times}$, there are infinitely many $n \in \mathbb{N}$ such that $\lambda\alpha^n$ lies in $\mathbb{Q}(\alpha^h)$, is a pseudo-PV number whose trace is non-zero and $\|\lambda\alpha^n\| < c^n$ for some $0 < c < 1$.

Proof of this theorem builds upon the techniques developed by Corvaja and Zannier. We also need to prove some further diophantine results on behaviour of powers of Salem numbers (Proposition 16) proof of which requires the p-adic subspace theorem.

Recall that a Salem number is a real algebraic integer α such that

- $\alpha > 1$,
- all its other conjugates have absolute values at most 1;
- at least one conjugate has absolute value equal to 1.

The distribution of exponential sequences $(\|\alpha^n\|)_{n\in\mathbb{N}}$ is rather mysterious and the few cases where one has some information is when α is an algebraic integer and $Tr_{k/\mathbb{Q}}(\alpha^n)$, with $k = \mathbb{Q}(\alpha)$, is close to the nearest integer of α^n. This motivates the study of algebraic numbers α such that $Tr_{k/\mathbb{Q}}(\alpha^n) \in \mathbb{Z}$ for n lying in suitable subsets I of $\mathbb{N}$.

It is not difficult to see that if $Tr_{k/\mathbb{Q}}(\alpha^n)$ is an integer for all $n \in \mathbb{N}$, α is an algebraic integer. For, the complementary module L' of the lattice L generated by $1, \alpha, \cdots, \alpha^d$ is a finitely generated $\mathbb{Z}$-module and the hypothesis above implies that L' contains the ring $\mathbb{Z}[\alpha]$. Hence α is necessarily an algebraic integer.

On the other hand, one has the following nice result of Bart de Smit [4].

Theorem 6. (Bart de Smit) *Let α be an algebraic number of degree d such that $Tr_{k/\mathbb{Q}}(\alpha^i)$ is an integer for all natural numbers i with $1 \leq i \leq d + d\log_2 d$. Then α is an algebraic integer.*

The example $\alpha = \frac{1}{\sqrt{2}}$ shows that the above bound is optimal.

On the other hand, a minor modification of the works of Corvaja and Zannier yields the following:

Theorem 7. *Let α be an algebraic number such that $Tr_{k/\mathbb{Q}}(\alpha^n)$ is a non-zero integer for infinitely many $n \in \mathbb{N}$. Then α is necessarily an algebraic integer.*

This is an immediate consequence of Lemma 14 in Section 2.

Note that in the above theorem, the hypothesis that $Tr_{k/\mathbb{Q}}(\alpha^n)$ is non zero is necessary. For $\alpha = \frac{1}{\sqrt{2}}$ satisfies $Tr(\alpha^n) = 0$ for all odd n. However, the example of $\alpha = \frac{1}{\sqrt{2}}$ is not generic. More precisely, if α is a real algebraic number such that $Tr_{k/\mathbb{Q}}(\alpha^n) = 0$ for infinitely many $n \in \mathbb{N}$, then α is not necessarily the root of a rational number. Here is an example.

Example 8. *The roots of the polynomial $X^4 - 6X^2 + 4$ are the real algebraic numbers $\pm\frac{1\pm\sqrt{5}}{\sqrt{2}}$. In particular, the splitting field of this polynomial is real and has only two roots of unity ± 1. Set $\alpha = \frac{1+\sqrt{5}}{\sqrt{2}}$ the largest root, we check $\alpha \notin \mathbb{Q}(\alpha^2)$ and $Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{2m+1}) = 0$ for $m \in \mathbb{N}$. However, α is not a root of a rational number.*

In this context, we have the following theorem.

Theorem 9. *Let α be a real, positive algebraic number, h be the order of the torsion group of the splitting field of the minimal polynomial of α. Then, $Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^n) = 0$ for infinitely many $n \in \mathbb{N}$ if and only if α does not belong to the field $\mathbb{Q}(\alpha^h)$.*

We also derive the following theorem which works for more general algebraic numbers.

Theorem 10. *Let α be a nonzero algebraic number, h be the order of the torsion group of the splitting field of the minimal polynomial of α and ζ a primitive h-th root of unity. Then, $Tr_{\mathbb{Q}(\alpha,\zeta)/\mathbb{Q}(\zeta)}(\alpha^n) = 0$ for infinitely many $n \in \mathbb{N}$ if and only if α does not belong to the field $\mathbb{Q}(\alpha^h, \zeta)$.*

Proof of these results are given in the penultimate section (Section 4) of the paper.

The final theme of the paper which constitutes the last section of our work is devoted to a more careful study of the algebraic elements in the Hardy and Mahler sets. For instance, we give the following more refined description of the pairs of algebraic numbers in the Hardy set H.

Theorem 11. *The elements of the set $H \cap \overline{\mathbb{Q}}^2$ precisely consists of all the pairs (λ, α) where α is a PV number and $\lambda \in \frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha, \frac{1}{\alpha}\right]$ with $P_\alpha \in \mathbb{Z}[X]$ the minimal polynomial of α over $\mathbb{Z}$.*

As for algebraic elements in the Mahler set M, we can show the following.

Theorem 12. *We have $(\lambda, \alpha) \in M \cap \overline{\mathbb{Q}}^2$ if and only if there exists integers s and $0 \leq t < s$ such that α^s is a PV number and λ belongs to $\frac{1}{\alpha^t P'_{\alpha^s}(\alpha^s)}\mathbb{Z}\left[\alpha^s, \frac{1}{\alpha^s}\right]$ with $P_{\alpha^s} \in \mathbb{Z}[X]$ the minimal polynomial of α^s over $\mathbb{Z}$.*

Remark 13. *The above theorems allow us to derive a result (see Corollary 31) which is in the direction of the fourth question. Namely, we show that $H \cap \overline{\mathbb{Q}}^2 \subset M$, which reduces the fourth question to Conjecture 3.*

2. Intermediate results

We fix the notion of height of a non-zero algebraic number which we shall be working with. For any number field K, let M_K be the set of all inequivalent places of K. The corresponding absolute values $|\cdot|_v$ are normalized so that $|\cdot|_v^{\frac{[K:\mathbb{Q}]}{[K_v:\mathbb{Q}_v]}}$ extends the usual archimedean or p-adic absolute value of $\mathbb{Q}$. Thus, the product formula for $\alpha \in K^\times$ reads $\prod_{v\in M_K} |\alpha|_v = 1$ and the height $H(\alpha)$, defined as

$$H(\alpha) := \prod_{v\in M_K} \max\{1, |\alpha|_v\},$$

is unambiguous and does not depend on the choice of K containing α.
Similarly, for an integer $n > 1$ and a non zero vector $\overline{\alpha} = (\alpha_1, \cdots, \alpha_n) \in K^n$, we set

$$H(\overline{\alpha}) := \prod_{v\in M_K} \max\{|\alpha_1|_v, \cdots, |\alpha_n|_v\}.$$

We will need the following lemma, which extends and improves [3, Lemma 4]:

Lemma 14. *Let λ and α be algebraic numbers. Let $k := \mathbb{Q}(\lambda, \alpha)$ and suppose that the trace $Tr_{k/\mathbb{Q}}(\lambda\alpha^n)$ is a non-zero integer for infinitely many n. Then α is necessarily an algebraic integer.*

Proof. The proof follows from that of [3, Lemma 4] with some minor modifications. We just need to work with the field $k = \mathbb{Q}(\lambda, \alpha)$. Finally, an extra argument ensures that α cannot be the root of a (non-integral) rational number and hence must be an algebraic integer.

Let K be the Galois closure of the extension $k/\mathbb{Q}$ and h the order of the torsion group of $K^\times$. Let $I \subset \mathbb{N}$ be the set of exponents such that $Tr_{k/\mathbb{Q}}(\lambda\alpha^n)$ is a non-zero integer. Since I is infinite, there exists an integer $c \in \{0, \dots, h-1\}$ and an infinite set $J \subset \mathbb{N}$ such that $c + hm \in I$ for all $m \in J$. Let $d = [\mathbb{Q}(\alpha^h) : \mathbb{Q}]$.

We first deal with the case $d = 1$. Then $\alpha^h = \frac{a}{b}$ where a and b are co-prime integers. We can write

$$Tr_{k/\mathbb{Q}}(\lambda\alpha^{c+hm}) = Tr_{k/\mathbb{Q}}(\lambda\alpha^c)\left(\frac{a}{b}\right)^m.$$

For $m \in J$ we have $c + hm \in I$, thus $Tr_{k/\mathbb{Q}}(\lambda\alpha^{c+hm})$ is a non-zero integer by hypothesis and $Tr_{k/\mathbb{Q}}(\lambda\alpha^c)$ is a fixed rational number. The above equality implies, as m tends to infinity in J, $b = 1$ and α is an algebraic integer (in fact a root of a rational integer).

In the case $d > 1$, let $\sigma_1, \dots, \sigma_d$ be a complete set of representatives of $\mathrm{Gal}(K/\mathbb{Q})$ modulo the subgroup fixing $\mathbb{Q}(\alpha^h)$. For $i = 1, \dots, d$, let T_i be a complete set of representatives of $\mathrm{Gal}(K/\mathbb{Q})$ modulo the subgroup fixing k that coincides with σ_i modulo the subgroup fixing $\mathbb{Q}(\alpha^h)$. In particular, $T_1 \cup \dots \cup T_d$ is a complete set of representatives of $\mathrm{Gal}(K/\mathbb{Q})$ modulo the subgroup fixing k and we can write

$$Tr_{k/\mathbb{Q}}(\lambda\alpha^{c+hm}) = \sum_{i=1}^{d}\left(\sum_{\tau\in T_i}\tau(\lambda\alpha^c)\right)\sigma_i(\alpha^{hm}) = A_1\sigma_1(\alpha^{hm}) + \dots + A_d\sigma_d(\alpha^{hm}),$$

where we have set $A_i = \sum_{\tau\in T_i}\tau(\lambda\alpha^c)$. We now proceed by contradiction, assuming $|\alpha|_v > 1$ for some finite place v of k. For $m \in J$ we have $c + hm \in I$, thus $Tr_{k/\mathbb{Q}}(\lambda\alpha^{c+hm})$ is a non-zero integer by hypothesis, and the coefficients A_i cannot be all zero. Furthermore, for $\varepsilon < \frac{\log|\alpha|_v}{\log H(\alpha)}$ and $m \in J$ sufficiently large we have

$$|A_1\sigma_1(\alpha^{hm}) + \dots + A_d\sigma_d(\alpha^{hm})|_v = |Tr_{k/\mathbb{Q}}(\lambda\alpha^{c+hm})|_v \le 1 < |\alpha^{hm}|_v H(\alpha^{hm})^{-\varepsilon}.$$

Applying Lemma 1 of [3] with $\Xi = \{\alpha^{hm}; m \in J\}$ and S a suitable Galois set of places of K such that α is an S-unit, we obtain a non-trivial equation satisfied by infinitely many $m \in J$

$$a_1\sigma_1(\alpha^h)^m + \dots + a_d\sigma_d(\alpha^h)^m = 0, \quad a_i \in K.$$

We may then apply Skolem-Mahler-Lech's theorem [14, Corollary 7.2, page 193] which entails that

$$\frac{\sigma_i(\alpha^h)}{\sigma_j(\alpha^h)} = \left(\frac{\sigma_i(\alpha)}{\sigma_j(\alpha)}\right)^h$$

is a root of unity for two distinct indices $i \neq j$. But this ratio must then be 1 because $\frac{\sigma_i(\alpha)}{\sigma_j(\alpha)}$ is a root of unity in K and h is the exponent of the torsion group of $K^\times$. This implies that σ_i and σ_j coincide on $\mathbb{Q}(\alpha^h)$, contradicting their definition. This contradiction shows that $|\alpha|_v \leq 1$ for all finite place v of k, hence α is an algebraic integer. □

We note that in the above theorem, a priori there need not be any relation between the algebraic numbers λ and α. In particular, λ need not be in $\mathbb{Q}(\alpha)$.

Finally, we note the following theorem which we shall need. This is a special case of a deep theorem of Corvaja and Zannier ([3], see Main Theorem).

Theorem 15. *Let Γ be a finitely generated subgroup of $\overline{\mathbb{Q}}^\times$, $\lambda \in \overline{\mathbb{Q}}^\times$ and $\epsilon > 0$ be real. Suppose that the following diophantine inequality*

$$0 < \|\lambda u\| < H(u)^{-\epsilon}$$

has infinitely many solutions $u \in \Gamma$ with $|\lambda u| > 1$. Then all but finitely many such λu are pseudo-PV numbers.

3. Proof of Theorem 5

Proof of Theorem 5 rests on the following intermediate results.

Proposition 16. *Let $\alpha_1, \ldots, \alpha_r$ be non-zero algebraic numbers of modulus 1 and $\lambda_1, \ldots, \lambda_r$ be non-zero algebraic numbers. For $\varepsilon > 0$, assume that there exists an infinite set $I \subset \mathbb{N}^\times$ such that for all $n \in I$, $\lambda_1\alpha_1^n + \cdots + \lambda_r\alpha_r^n$ is a real number and*

$$\|\lambda_1\alpha_1^n + \cdots + \lambda_r\alpha_r^n\| < \mathrm{e}^{-\varepsilon n}.$$

Then there exists algebraic numbers $A_0, A_1, \ldots, A_r$ not all zero such that for infinitely many $n \in I$, one has

$$A_0 + A_1\alpha_1^n + \cdots + A_r\alpha_r^n = 0.$$

Proof. We begin by noting that if one of the $\alpha_1, \ldots, \alpha_r$ is a root of unity, the conclusion holds. So we may assume that none of the $\alpha_1, \ldots, \alpha_r$ is a root of unity.

Let N_n denote the integer closest to $\lambda_1\alpha_1^n + \cdots + \lambda_r\alpha_r^n$ and observe that $|N_n| \leq |\lambda_1| + \cdots + |\lambda_r|$ takes only finitely many values. Thus, there exists an infinite subset $J \subset I$ and an integer λ_0 such that $|\lambda_0 + \lambda_1\alpha_1^n + \cdots + \lambda_r\alpha_r^n| = \|\lambda_1\alpha_1^n + \cdots + \lambda_r\alpha_r^n\| < \mathrm{e}^{-\varepsilon n}$ for $n \in J$. Set $\alpha_0 = 1$.

Let $K \subset \mathbb{C}$ be a Galois number field containing $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r$ and S a finite set of absolute values of K, containing all the archimedean ones and such that $\alpha_1, \ldots, \alpha_r$ are S-units. Let v_0 be the (archimedean) absolute value given by the given inclusion of K in $\mathbb{C}$. If $\lambda_0 \neq 0$ we set $i_0 = 0$ and otherwise $i_0 = 1$. We set

$$L_{v_0,i_0}(\overline{x}) = \lambda_0 x_0 + \cdots + \lambda_r x_r$$

with $\overline{x} = (x_0, \dots, x_r)$ and for $(v, i) \in S \times \{0, \dots, r\}$, $(v, i) \neq (v_0, i_0)$, $L_{v,i}(\overline{x}) = x_i$. Observe that for each v the forms $L_{v,0}, \dots, L_{v,r}$ are linearly independent. Then, with $\overline{\alpha}^n = (\alpha_0^n, \dots, \alpha_r^n)$ and $\|\overline{\alpha}\|_v = \max(|\alpha_0|_v, \dots, |\alpha_r|_v)$, we have

$$\begin{aligned}\prod_{v\in S}\prod_{i=0}^{r}\frac{|L_{v,i}(\overline{\alpha}^n)|_v}{\|\overline{\alpha}^n\|_v} &= \frac{|L_{v_0,i_0}(\overline{\alpha}^n)|_{v_0}}{|\alpha_{i_0}|_{v_0}^n}\prod_{i=0}^{r}\prod_{v\in S}\frac{|\alpha_i|_v^n}{\|\overline{\alpha}\|_v^n}\\ &= \frac{|L_{v_0,i_0}(\overline{\alpha}^n)|}{|\alpha_{i_0}|^n}H(\overline{\alpha})^{-n(r+1)},\end{aligned}$$

since $\prod_{v\in S}|\alpha_i|_v = 1$ by the product formula, and $\prod_{v\in S}\|\overline{\alpha}\|_v = H(\overline{\alpha})$ because the components of $\overline{\alpha}$ are S-units.

For $n \in J$ we further have $|L_{v_0,i_0}(\overline{\alpha}^n)| = \|\lambda_1\alpha_1^n + \cdots + \lambda_r\alpha_r^n\| < \mathrm{e}^{-\varepsilon n}$ and we observe that $|\alpha_{i_0}| = 1$ in any case, thus

$$\prod_{v\in S}\prod_{i=0}^{r}\frac{|L_{v,i}(\overline{\alpha}^n)|_v}{\|\overline{\alpha}^n\|_v} < H(\overline{\alpha}^n)^{-r-1-\varepsilon/\log H(\overline{\alpha})}.$$

The p-adic subspace theorem [13, Chap. V, Thm. 1D′, page 178] then ensures that for all $n \in J$ the point $(\alpha_0^n : \cdots : \alpha_r^n) \in \mathbb{P}_n(K)$ lies in the union of finitely many proper subspaces of $\mathbb{P}_n(K)$. One of these must contain infinitely many points $(\alpha_0^n : \cdots : \alpha_r^n)$ for $n \in J$ and one of its equations can be written as $A_0x_0 + \cdots + A_rx_r = 0$ with $A_0, \dots, A_r \in K$ not all zero. □

Lemma 17. *Let α be a pseudo-PV or a Salem number and let $\alpha = \alpha_1, \dots, \alpha_d$ be its conjugates. For all $A_0, A_1, \dots, A_d \in \overline{\mathbb{Q}}$, not all zero, there are only finitely many $n \in \mathbb{N}$ such that*

$$A_0 + A_1\alpha_1^n + \cdots + A_d\alpha_d^n = 0.$$

Proof. The result is clear if $A_1 = \cdots = A_d = 0$, since then the hypothesis is $A_0 \neq 0$. Otherwise, applying an automorphism σ of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$ sending some α_i with $A_i \neq 0$ to α, we get an equation

$$\alpha^n = -\sigma(A_0/A_i) - \sum_{\substack{1\le j\le d\\ j\neq i}}\sigma(A_j/A_i)\sigma(\alpha_j)^n.$$

But the $\sigma(A_j/A_i)$ are independent of n and the $\sigma(\alpha_j)$, $j \neq i$, are the conjugates of α distinct from α, thus of absolute value bounded by 1. Since $|\alpha| > 1$, the absolute value $|\alpha|^n$ of the left-hand side goes to infinity with n whereas that of the right-hand side remains bounded. Therefore, the equality $A_0 + A_1\alpha_1^n + \cdots + A_d\alpha_d^n = 0$ can hold only for n bounded. □

We now have all the ingredients to prove Theorem 5.

Proof of Theorem 5.

Let λ and α be algebraic numbers such that $(\lambda, \alpha) \in M$. Thus there exists $0 < c < 1$ and infinitely many $n \in \mathbb{N}$ such that $\|\lambda\alpha^n\| < c^n$. If $\|\lambda\alpha^n\| = 0$ for infinitely many n, then α is necessarily the root of an integer and also $\lambda\alpha^n \in \mathbb{Z} \setminus \{0\}$ for infinitely many n.

So we may assume that there are infinitely many n such that $0 < \|\lambda\alpha^n\| < c^n$. Furthermore $|\lambda\alpha^n| > 1$ for n large enough, since $|\alpha| > 1$. Denote $J \subset \mathbb{N}$ the subset of n for which these inequalities hold. There exists $\epsilon > 0$ such that

$$0 < \|\lambda\alpha^n\| < H(\alpha^n)^{-\epsilon}$$

for $n \in J$. Thus, by Theorem 15 with Γ the subgroup generated by α, $\lambda\alpha^n$ is a pseudo-PV number for all n in J except for a possible finite subset. Let $I \subset J$ be the infinite subset such that $\lambda\alpha^n$ is a pseudo-PV number for every $n \in I$. Since $\lambda\alpha^n$ tends to infinity whereas the absolute values of its other conjugates are bounded when n tends to infinity, we may as well define the subset I so that the trace of the pseudo-PV number $\lambda\alpha^n$ is non zero for $n \in I$. Then by Lemma 14, α is necessarily an algebraic integer.

Let $h \in \mathbb{N}^\times$ and $n \in I$ such that $n = mh + i$ for $0 \leq i < h$ and $m > \frac{r\log(H(\lambda\alpha^i))}{h\log(\alpha)}$ with $r = [\mathbb{Q}(\alpha^h, \lambda\alpha^i) : \mathbb{Q}(\alpha^h)]$. The identity map on $\mathbb{Q}(\alpha^h)$ gives rise to r different embeddings of $\mathbb{Q}(\alpha^h, \lambda\alpha^i)$ over $\mathbb{Q}(\alpha^h)$ into $\overline{\mathbb{Q}}$. But, with our condition on m, the corresponding r embeddings of $\lambda\alpha^n$ have absolute values greater than 1. Since $\lambda\alpha^n$ is a pseudo-PV number we deduce that $r = 1$ and $\lambda\alpha^n \in \mathbb{Q}(\alpha^h)$ for $n \in I$. In particular, for $h = 1$ we have $\lambda \in \mathbb{Q}(\alpha)$.

Assume a conjugate of α distinct from α has absolute value strictly greater than 1. Write α' and $\lambda' \in \mathbb{Q}(\alpha')$ the corresponding conjugates of α and $\lambda \in \mathbb{Q}(\alpha)$. Since $\lambda\alpha^n$ is a pseudo-PV number for $n \in I$, we must have $\lambda\alpha^n = \lambda'\alpha'^n$ for $n \in I$ large enough. Forming the quotient of two such equalities for $m, n \in I$, $m < n$, we get $\alpha'^{n-m} = \alpha^{n-m}$. Let s be the least common multiple of these exponents $m - n$ when α' runs over all the conjugates of α of absolute value strictly greater than 1. Then α^s has exactly one conjugate of absolute value strictly greater than 1. Since α is an algebraic integer, α^s is either a PV or a Salem number.

Now suppose that s is an integer such that α^s is a Salem number. We have that there exists an $\epsilon > 0$ and infinite set $I \subset \mathbb{N}^\times$ such that $\|\lambda\alpha^n\| < \mathrm{e}^{-\varepsilon n}$ for all $n \in I$. Then there exists $i \in \{0, \ldots, s-1\}$ such that

$$\|\lambda\alpha^{ms+i}\| < \mathrm{e}^{-\varepsilon(ms+i)}$$

for infinitely many m. By our earlier argument, we know that $\lambda\alpha^i \in \mathbb{Q}(\alpha^s)$ is a rational fraction $r(\alpha^s)$ in α^s. Let $\alpha_1, \ldots, \alpha_{2d}$ be the conjugates of α^s of modulus 1, that is all the conjugates except α^s and α^{-s}, and $\lambda_j = r(\alpha_j)$, $j = 1, \ldots, 2d$, be the corresponding conjugates of $\lambda\alpha^i$. If N_m is the integer closest to $\lambda\alpha^{ms+i}$ and $q \in \mathbb{N}^\times$ a denominator of λ, we write

$$|Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(q\lambda\alpha^{ms+i}) - qN_m - q\lambda_1\alpha_1^{ms} - \cdots - q\lambda_{2d}\alpha_{2d}^{ms}| < q\mathrm{e}^{-\varepsilon(ms+i)} + q|r(\alpha^{-s})\alpha^{-ms}| < \mathrm{e}^{-\varepsilon' m}$$

for some $\varepsilon' > 0$ and m large enough. Thus,

$$\|q\lambda_1\alpha_1^{sm} + \cdots + q\lambda_{2d}\alpha_{2d}^{sm}\| < \mathrm{e}^{-\varepsilon' m}$$

for infinitely many m. By Proposition 16, there exists $A_0, \ldots, A_{2d} \in \overline{\mathbb{Q}}$, not all zero, such that

$$A_0 + A_1\alpha_1^{sm} + \cdots + A_{2d}\alpha_{2d}^{sm} = 0$$

for infinitely many m. But by Lemma 17, this is not possible since α^s is a Salem number. This completes the proof of Theorem 5.

4. Proof of Theorems 9 and 10

For an algebraic number α, let $k = \mathbb{Q}(\alpha)$ and let d be the degree of α. We shall need the following lemma for the proof of Theorem 9.

Lemma 18. *Let α be a real, positive algebraic number and $a, h \in \mathbb{N}^\times$, then the following three statements are equivalent:*

1) $Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}(\alpha^h)}(\alpha^a) = 0$;
2) *a is not divisible by* $[\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^h)]$.
3) $\alpha^a \notin \mathbb{Q}(\alpha^h)$;

Proof. Set $L = \mathbb{Q}(\alpha^h) \subset \mathbb{R}$ and let h' be the smallest positive integer such that $\alpha^{h'} \in L$. Obviously $1 \leq h' \leq h$ and for any prime p dividing h', we have $\alpha^{h'} \notin L^p$. Otherwise $\alpha^{h'/p}$, which is the only real, positive p-th root of $\alpha^{h'}$, would belong to L, contradicting the minimality of h'. Furthermore, if $4 \mid h'$ then $\alpha^{h'} \notin -4L^4$, because -4 is not a fourth power in $\mathbb{R}$. It then follows from [9, Chap.VI, §9, Theorem 9.1, page 297], that the polynomial

$$X^{h'} - \alpha^{h'}$$

is irreducible in $L[X]$. Now α is the only real, positive root and its conjugates over L are the numbers $\alpha\xi^i$, $i = 0, \ldots, h'-1$, where ξ is a primitive h'-th root of unity. Thus, h' is the degree of the extension $\mathbb{Q}(\alpha)/L$ and for all $a \in \mathbb{N}^\times$ we have

$$Tr_{\mathbb{Q}(\alpha)/L}(\alpha^a) = \alpha^a \sum_{i=0}^{h'-1} \xi^{ai} = \begin{cases} h'\alpha^a & \text{if } h' \mid a \\ 0 & \text{otherwise} \end{cases}$$

and $Tr_{\mathbb{Q}(\alpha)/L}(\alpha^a) = 0$ if and only if $a \not\equiv 0\ (h')$, proving that the statements 1 and 2 are equivalent.

Now, since $\alpha^{h'} \in L$, the condition $\alpha^a \in L$ is equivalent to[1] $\alpha^{(h',a)} \in L$ and by the minimality of h' this happens if and only if $(h', a) \geq h'$, that is $h' \mid a$. This shows that the statements 2 and 3 are equivalent, ending the proof of the lemma. □

[1]Here and later (m, n) denotes the gcd of m and n.

Proof of Theorem 9.

Let H be the splitting field of the minimal polynomial of α over $\mathbb{Q}$. So h is the order of the torsion group of $H^\times$. Set $L = \mathbb{Q}(\alpha^h)$ and $K = L(\alpha) = \mathbb{Q}(\alpha) \subset H$.

In one direction, assume that $Tr_{K/\mathbb{Q}}(\alpha^n) = 0$ for infinitely many $n \in \mathbb{N}$. Since $\alpha \in K$ we have $Tr_{H/K}(\alpha^n) = [H : K]\alpha^n$ and, since $\mathbb{N}$ is the disjoint union of the congruence classes $a + h\mathbb{N}$ for $a = 0, \ldots, h-1$, we deduce from the hypothesis that

$$Tr_{H/\mathbb{Q}}(\alpha^{a+hm}) = Tr_{K/\mathbb{Q}}\left(Tr_{H/K}(\alpha^{a+hm})\right) = [H : K]Tr_{K/\mathbb{Q}}(\alpha^{a+hm}) = 0$$

for infinitely many $m \in \mathbb{N}$ and some $0 \le a < h$. Let $d = [L : \mathbb{Q}]$ and $\tau_1, \ldots, \tau_d$ be all the embeddings of L in H over $\mathbb{Q}$. We express $Tr_{H/\mathbb{Q}}(\alpha^{a+hm})$ as the m-th term of a linear recurrence sequence:

$$Tr_{H/\mathbb{Q}}(\alpha^{a+hm}) = Tr_{L/\mathbb{Q}}\left(Tr_{H/L}(\alpha^{a+hm})\right) = \sum_{i=1}^{d} \tau_i\left(Tr_{H/L}(\alpha^a)\right)\tau_i(\alpha^h)^m,$$

which vanishes for infinitely many $m \in \mathbb{N}$. Observe that the ratios $\tau_i(\alpha^h)/\tau_j(\alpha^h)$, $i \neq j$, are not roots of unity, because being roots of unity and h power in H they would be 1 but $\tau_i \neq \tau_j$ on L and we must have $\tau_i(\alpha^h) \neq \tau_j(\alpha^h)$. The Skolem-Mahler-Lech's theorem [14, Corollary 7.2, page 193] implies that $Tr_{H/L}(\alpha^a) = 0$ and then $Tr_{K/L}(\alpha^a) = 0$ because

$$Tr_{H/L}(\alpha^a) = Tr_{K/L}\left(Tr_{H/K}(\alpha^a)\right) = [H : K]Tr_{K/L}(\alpha^a).$$

But then Lemma 18 ensures that $\alpha^a \notin \mathbb{Q}(\alpha^h)$ and hence in particular $\alpha \notin \mathbb{Q}(\alpha^h)$.

In the other direction, the same Lemma 18 shows that if $\alpha \notin \mathbb{Q}(\alpha^h)$, then $Tr_{K/L}(\alpha) = 0$ and

$$Tr_{K/\mathbb{Q}}(\alpha^{1+hm}) = Tr_{L/\mathbb{Q}}\left(\alpha^{hm}Tr_{K/L}(\alpha)\right) = 0$$

for all $m \in \mathbb{N}$. This completes the proof.

Remark 19. *The condition $\alpha \notin \mathbb{Q}(\alpha^h)$ is also equivalent to $\alpha^h \notin \mathbb{Q}(\alpha^h)^h$.*

We now prove the following variant of Lemma 18 which we shall require a little later.

Lemma 20. *Let α be a nonzero algebraic number, $a, h \in \mathbb{N}^\times$ and ζ a primitive h-th root of unity. Then $\alpha^a \notin \mathbb{Q}(\alpha^h, \zeta)$ if and only if $Tr_{\mathbb{Q}(\alpha,\zeta)/\mathbb{Q}(\alpha^h,\zeta)}(\alpha^a) = 0$.*

Furthermore, $Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^a) = 0$ for all integer a not divisible by $[\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}(\alpha^h, \zeta)]$.

Proof. Let $L = \mathbb{Q}(\alpha^h, \zeta)$ and $K = L(\alpha)$, since α is a root of the polynomial

$$X^h - \alpha^h \in L[X]$$

the extension K/L is cyclic of degree $h' \mid h$, the conjugates of α over L are $\alpha\xi^i$, $i = 0, \ldots, h'-1$, where ξ is a primitive h'-th root of unity, and $\alpha^{h'} \in L$, *see* [9, Chap.VI, §6, Theorem 6.2, page

289]. Thus, for all $a \in \mathbb{N}^\times$ we have

$$Tr_{K/L}(\alpha^a) = \alpha^a \sum_{i=0}^{h'-1} \xi^{ai} = \begin{cases} h'\alpha^a & \text{if } h' \mid a \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and $Tr_{K/L}(\alpha^a) = 0$ if and only if $a \not\equiv 0\ (h')$. We observe that h' is the smallest positive integer such that $\alpha^{h'} \in L$, otherwise another application of *ibidem* would lead to a contradiction on the degree of the extension K/L. Now, since $\alpha^{h'} \in L$, the condition $\alpha^a \in L$ is equivalent to $\alpha^{(h',a)} \in L$ and by the minimality of h' this happens if and only if $(h', a) \geq h'$, that is $h' \mid a$.

For the last statement, if a is not divisible by h' then

$$Tr_{K/\mathbb{Q}}(\alpha^a) = Tr_{L/\mathbb{Q}}\left(Tr_{K/L}(\alpha^a)\right) = 0,$$

because $Tr_{K/L}(\alpha^a) = 0$ by (1). But, $Tr_{K/\mathbb{Q}(\alpha)}(\alpha^a) = [K : \mathbb{Q}(\alpha)]\alpha^a$ and we can further write

$$Tr_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^a) = [K : \mathbb{Q}(\alpha)]^{-1} Tr_{K/\mathbb{Q}}(\alpha^a) = 0.$$

□

We end with the proof of Theorem 10 which works for general algebraic numbers.

Proof of Theorem 10.

Let H be the splitting field of the minimal polynomial of α over $\mathbb{Q}$. So that h is the order of the torsion group of $H^\times$ and ζ is a primitive h-th root of unity. Set $H_0 = \mathbb{Q}(\zeta)$, $L = H_0(\alpha^h)$ and $K = L(\alpha) \subset H$.

In one direction, assume $Tr_{K/H_0}(\alpha^n) = 0$ for infinitely many $n \in \mathbb{N}$. Since $\alpha \in K$ we have $Tr_{H/K}(\alpha^n) = [H : K]\alpha^n$ and we deduce from the hypothesis that

$$Tr_{H/H_0}(\alpha^{a+mh}) = Tr_{K/H_0}\left(Tr_{H/K}(\alpha^{a+mh})\right) = [H : K]Tr_{K/H_0}(\alpha^{a+mh}) = 0$$

for infinitely many $m \in \mathbb{N}$ and some $0 \leq a < h$. Let $d = [L : H_0]$ and $\tau_1, \ldots, \tau_d$ be all the embeddings of L in H over H_0. We express $Tr_{H/H_0}(\alpha^{a+mh})$ as the m-th term of a linear recurrence sequence:

$$Tr_{H/H_0}(\alpha^{a+mh}) = Tr_{L/H_0}\left(Tr_{H/L}(\alpha^{a+mh})\right) = \sum_{i=1}^{d} \tau_i\left(Tr_{H/L}(\alpha^a)\right)\tau_i(\alpha^h)^m,$$

which vanishes for infinitely many $m \in \mathbb{N}$. Observe that the ratios $\tau_i(\alpha^h)/\tau_j(\alpha^h)$, $i \neq j$, are not roots of unity, because being roots of unity and h power in H they would be 1, but $\tau_i \neq \tau_j$ on L and, since $\tau_i = \tau_j$ on H_0, we must have $\tau_i(\alpha^h) \neq \tau_j(\alpha^h)$. The Skolem-Mahler-Lech's theorem [14, Corollary 7.2, page 193] implies that $Tr_{H/L}(\alpha^a) = 0$ and then $Tr_{K/L}(\alpha^a) = 0$ because

$$Tr_{H/L}(\alpha^a) = Tr_{K/L}\left(Tr_{H/K}(\alpha^a)\right) = [H : K]Tr_{K/L}(\alpha^a).$$

As before, Lemma 20 ensures that $\alpha^a \notin \mathbb{Q}(\alpha^h, \zeta)$ and, in particular, $\alpha \notin \mathbb{Q}(\alpha^h, \zeta)$.

In the other direction, Lemma 20 also shows that if $\alpha^a \notin \mathbb{Q}(\alpha^h, \zeta)$, then $Tr_{K/L}(\alpha^a) = 0$ and

$$Tr_{K/H_0}(\alpha^{a+mh}) = Tr_{L/H_0}\left(\alpha^{mh} Tr_{K/L}(\alpha^a)\right) = 0$$

for all $m \in \mathbb{N}$. This completes the proof.

5. Periodicity and description of the set $H \cap \overline{\mathbb{Q}}^2$

Let α be an algebraic integer, $P_\alpha \in \mathbb{Z}[X]$ its minimal monic polynomial over $\mathbb{Z}$ and set $k = \mathbb{Q}(\alpha)$. Recall from [8, Chap.III, §1, Cor. to Prop.2] that $\frac{1}{P'_\alpha(\alpha)}\mathbb{Z}[\alpha]$ is the complementary module $\mathbb{Z}[\alpha]'$ of $\mathbb{Z}[\alpha]$, that is the set of elements $y \in \mathbb{Q}(\alpha)$ such that $Tr_{k/\mathbb{Q}}(y\mathbb{Z}[\alpha]) \subset \mathbb{Z}$. Indeed, without the assumption made in the above cited Corollary, its proof shows the desired equality. Thus for $\lambda \in \frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha, \frac{1}{\alpha}\right]$ and n large enough we have $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \in \mathbb{Z}$.

The following lemma can also be found in [6, Lemma 2] and [15, Lemma 2].

Lemma 21. *Let α be an algebraic integer of degree d over $\mathbb{Q}$, $\lambda \in \mathbb{Z}[\alpha]'$ and $b \in \mathbb{N}^\times$, then the sequence $(Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \bmod b)_{n\in\mathbb{N}}$ is ultimately periodic and the period has length $\leq b^d$.*

Proof. Let $t_n \in \mathbb{Z}/b\mathbb{Z}$ be the class of $Tr_{k/\mathbb{Q}}(\lambda\alpha^n)$ modulo b and write

$$P_\alpha(X) = X^d + A_{d-1}X^{d-1} + \cdots + A_0 \in \mathbb{Z}[X]$$

the minimal monic polynomial of α over $\mathbb{Z}$. Let $\overline{A_i}$ denote the class of A_i modulo b. From $Tr_{k/\mathbb{Q}}(P_\alpha(\alpha)\lambda\alpha^n) = 0$ we deduce

$$t_{n+d} = -\overline{A_{d-1}}t_{n+d-1} - \cdots - \overline{A_0}t_n \tag{2}$$

for all $n \in \mathbb{N}$. Since there are finitely many d-tuples of elements of $\mathbb{Z}/b\mathbb{Z}$, at least one must appear twice as blocks in the sequence $(t_n)_{n\in\mathbb{N}}$. Thus there exists natural numbers $m < n$ such that $t_m = t_n, \ldots, t_{m+d-1} = t_{n+d-1}$. It follows inductively from (2) that

$$t_{m+d} = t_{n+d}, t_{m+d+1} = t_{n+d+1}, \ldots$$

and hence the ultimate periodicity of the sequence. The length of the period divides $n - m$ and since there are at most b^d distinct blocks of d elements in $\mathbb{Z}/b\mathbb{Z}$, we have that the period has length at most b^d. □

Remark 22. *If the number α is assumed to be a unit, then equation* (2) *enables one to deduce that the sequence $(t_n)_{n\in\mathbb{N}}$ is purely periodic and in particular, if p denotes the length of the period, that t_{pn} is the class of $Tr_{k/\mathbb{Q}}(\lambda)$ modulo b for all $n \in \mathbb{N}$, see* [15, Lemma 2].

Proposition 23. *Let α be a PV number of degree d over $\mathbb{Q}$, $\lambda_0 \in \mathbb{Z}[\alpha]'$ and $b \in \mathbb{N}^\times$. There exists an integer $p \leq b^d$ and*[2] *$i_1, \ldots, i_p \in \{-\lceil b/2\rceil + 1, \ldots, \lfloor b/2\rfloor\}$ such that $Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n) \equiv i_\ell(b)$ and $\left|\left\|\frac{\lambda_0\alpha^n}{b}\right\| - \frac{|i_\ell|}{b}\right| < c^n$ with $\ell = n - p\lfloor n/p\rfloor + 1$, n large enough and some $0 < c < 1$.*

[2]Here and after $\lceil * \rceil$ stands for the least integer larger or equal to $*$ and $\lfloor * \rfloor$ is the largest integer smaller or equal to $*$ (the integer part).

Proof. By Lemma 21, the sequence of classes $t_n \in \mathbb{Z}/b\mathbb{Z}$ modulo b of $Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n)$ is ultimately periodic, with period of length say p. We represent the elements of $\mathbb{Z}/b\mathbb{Z}$ by the integers $-\lceil b/2\rceil+1,\dots,\lfloor b/2\rfloor$ so that the period of the sequence $(t_n)_{n\in\mathbb{N}}$ gives integers $i_1,\dots,i_p$ lying in $\{-\lceil b/2\rceil+1,\dots,\lfloor b/2\rfloor\}$ satisfying $Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n)\equiv i_\ell(b)$ for $\ell=n-p\lfloor n/p\rfloor+1$ and n large. We then observe

$$\left|\frac{\lambda_0\alpha^n}{b}-\frac{Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n)-i_\ell}{b}-\frac{i_\ell}{b}\right|=\left|\frac{\lambda_0\alpha^n}{b}-Tr_{k/\mathbb{Q}}\left(\frac{\lambda_0\alpha^n}{b}\right)\right|<c^n$$

for some $0<c<1$, because α is a PV number. It follows that $\frac{Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n)-i_\ell}{b}$ is the integer closest to $\frac{\lambda_0\alpha^n}{b}$ and $\left\|\frac{\lambda_0\alpha^n}{b}\right\|=\frac{|i_\ell|}{b}+O(c^n)$. □

Remark 24. *Observe that i_1 is the residue modulo b of $Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n)$ for all large n divisible by p. For any $j\in\mathbb{Z}$, replacing λ_0 by $\lambda_0\alpha^j$ simply shifts the sequence $(Tr_{k/\mathbb{Q}}(\lambda_0\alpha^n))_{n\in\mathbb{N}}$ by $|j|$ steps to the left or right according to the sign of j. Thus, with the PV number α and the integer $b\in\mathbb{N}^\times$ fixed, Proposition 23 associates to each $\lambda_0\in\frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha,\frac{1}{\alpha}\right]$, the integer p and the vector $(i_1,\dots,i_p)\in(\mathbb{Z}/b\mathbb{Z})^p$. This map is a homomorphism of $\mathbb{Z}\left[\alpha,\frac{1}{\alpha}\right]$-modules if we define the action of α on $(\mathbb{Z}/b\mathbb{Z})^p$ as the cyclic permutation $\alpha\cdot(i_1,\dots,i_p)=(i_2,\dots,i_p,i_1)$. In particular, the integer p can be chosen independent of λ_0, although for some λ_0 a shorter period may exist.*

More generally for any $\lambda\in\mathbb{Q}(\alpha)$, we choose b to be the least positive integer such that $\lambda_0=b\lambda\in\frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha,\frac{1}{\alpha}\right]$ and consider the integer p and vector $(i_1,\dots,i_p)\in(\mathbb{Z}/b\mathbb{Z})^p$ associated to $b\lambda$. It follows from Proposition 23 that the fractions $\frac{|i_1|}{b},\dots,\frac{|i_p|}{b}$ (lying in $[0,\frac{1}{2}]$) are the limit points of the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$.

Corollary 25. *Let α be a PV number and $\lambda\in\mathbb{Q}(\alpha)$. Then, 0 is a limit point of the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$ if and only if there exists an integer $p\in\mathbb{N}^\times$ and $\ell\in\{1,\dots,p\}$ such that $Tr_{k/\mathbb{Q}}(\lambda\alpha^{np+\ell-1})\in\mathbb{Z}$ for $n\in\mathbb{N}$ large enough or, equivalently, λ belongs to $\frac{1}{\alpha^{\ell-1}P'_{\alpha^p}(\alpha^p)}\mathbb{Z}\left[\alpha^p,\frac{1}{\alpha^p}\right]$.*

And 0 is the unique limit point of the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$ if and only if $\lambda\in\frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha,\frac{1}{\alpha}\right]$.

Proof. Assume 0 is a limit point of the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$. Write $\lambda=\frac{1}{b}f(\alpha)$ with $b\in\mathbb{N}^\times$ and $f\in\mathbb{Z}[X]$. Set $\lambda_0=b\lambda=f(\alpha)$. Since 0 is a limit point of the sequence $\left(\left\|\frac{f(\alpha)\alpha^n}{b}\right\|\right)_{n\in\mathbb{N}}$, it follows that in Proposition 23 some i_ℓ must be 0 and the numbers $Tr_{k/\mathbb{Q}}(f(\alpha)\alpha^{np+\ell-1})$ are divisible by b or, equivalently, $Tr_{k/\mathbb{Q}}(\lambda\alpha^{np+\ell-1})\in\mathbb{Z}$, for n large enough. Thus $\lambda\alpha^{n_0p+\ell-1}\in\frac{1}{P'_{\alpha^p}(\alpha^p)}\mathbb{Z}[\alpha^p]$ for some $n_0\in\mathbb{N}$.

Conversely assume $Tr_{k/\mathbb{Q}}(\lambda\alpha^{np+\ell-1})\in\mathbb{Z}$. Since α is a PV number and n is large enough, $\lambda\alpha^{np+\ell-1}$ is a large real number whereas the sum of its conjugates is of absolute values $<c^n$ for some $0<c<1$. The difference $\|\lambda\alpha^{np+\ell-1}\|=|\lambda\alpha^{np+\ell-1}-Tr_{k/\mathbb{Q}}(\lambda\alpha^{np+\ell-1})|<c^n$ tends to 0 as n goes to ∞ and thus 0 is a limit point of the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$.

If 0 is the unique limit point of the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$, we must have $p=1$ and $i_1=0$ in Proposition 23. Thus $\ell=1$ and $\lambda\alpha^{n_0}$ belongs to the complementary module of $\mathbb{Z}[\alpha]$, for some

n_0. Conversely if $\lambda \in \frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha, \frac{1}{\alpha}\right]$ (i.e. $p = \ell = 1$), then $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \in \mathbb{Z}$ for all n large and the above argument shows that it is the closest integer to $\lambda\alpha^n$, since α is a PV number. Then, $\|\lambda\alpha^n\| = |\lambda\alpha^n - Tr_{k/\mathbb{Q}}(\lambda\alpha^n)| < c^n$ tends to 0 as n goes to ∞. □

Corollary 26. *Let $(\lambda, \alpha) \in H \cap (\mathbb{R}^\times \times \overline{\mathbb{Q}})$, then α is a PV number, $\lambda \in \mathbb{Q}(\alpha)$ and $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \in \mathbb{Z}$ for n large enough.*

Proof. For $(\lambda, \alpha) \in H \cap (\mathbb{R}^\times \times \overline{\mathbb{Q}})$, we know by Theorem 2 that α is a PV number and $\lambda \in \mathbb{Q}(\alpha)$. We may write $\lambda = \frac{1}{b}f(\alpha)$ for some rational integer b and $f \in \mathbb{Z}[X]$. By Proposition 23, the limit points of the sequence $\left(\left\|\frac{f(\alpha)\alpha^n}{b}\right\|\right)_{n\in\mathbb{N}}$ are those rational numbers among $0, 1/b, \ldots, \lfloor b/2\rfloor/b$ for which the numerator is congruent to infinitely many $Tr_{k/\mathbb{Q}}(f(\alpha)\alpha^n)$ modulo b. But, (λ, α) being in H, the sequence converges to 0. Thus, the only possible fraction is 0 and all the numbers $Tr_{k/\mathbb{Q}}(f(\alpha)\alpha^n)$ must be divisible by b and so $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \in \mathbb{Z}$, for n large enough. □

We need to introduce the following notation for the purpose of the next proposition. For any set $S \subset \mathbb{R}$ we define $\|S\| = \{\|\gamma\|; \gamma \in S\} \subset [0, 1/2]$.

Proposition 27. *Let α be a Salem number of degree d over $\mathbb{Q}$, $\lambda \in \mathbb{Z}[\alpha]'$ and $b \in \mathbb{N}^\times$. There exists an integer $p \le b^d$ and $i_1, \ldots, i_p \in \{-\lceil b/2\rceil + 1, \ldots, \lfloor b/2\rfloor\}$ such that $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \equiv i_\ell(b)$ for $\ell = n - p\lfloor n/p\rfloor + 1$ and any $n \in \mathbb{N}^\times$. Furthermore, let $\alpha = \alpha_1, \alpha^{-1} = \alpha_2, \alpha_3, \ldots, \alpha_d$ be the conjugates of α and $\lambda_{\ell,1}, \ldots, \lambda_{\ell,d}$ be those of $\lambda\alpha^{\ell-1}$ for $\ell \in \{1, \ldots, p\}$. Then the set of limit points of each subsequence $\left(\left\|\frac{\lambda}{b}\alpha^{mp+\ell-1}\right\|\right)_{m\in\mathbb{N}}$ is $\|\mathcal{L}_\ell\|$, where*

$$\mathcal{L}_\ell = \left[\frac{i_\ell}{b} - \sum_{j=3}^{d}\frac{|\lambda_{\ell,j}|}{b}, \frac{i_\ell}{b} + \sum_{j=3}^{d}\frac{|\lambda_{\ell,j}|}{b}\right].$$

Furthermore, an integer closest to $\lambda\alpha^n$ is congruent to $i \in \{-\lceil b/2\rceil + 1, \ldots, \lfloor b/2\rfloor\}$ modulo b for infinitely many n if and only if we have $\left\|\frac{\lambda\alpha^n - i}{b}\right\| \in \left[0, \frac{1}{2b}\right] = \left\|\left[-\frac{1}{2b}, \frac{1}{2b}\right]\right\|$ or equivalently $\left\|\frac{\lambda\alpha^n}{b}\right\| \in \left\|\left[\frac{2i-1}{2b}, \frac{2i+1}{2b}\right]\right\|$, for infinitely many n. In this case $\left(\cup_{\ell=1}^{p}\mathcal{L}_\ell\right) \cap \left\|\left[\frac{2i-1}{2b}, \frac{2i+1}{2b}\right]\right\| \neq \emptyset$.

Proof. The first part of the statement follows from Lemma 21 and Remark 22, which asserts that the sequence of classes $t_n \in \mathbb{Z}/b\mathbb{Z}$ modulo b of $Tr_{k/\mathbb{Q}}(\lambda\alpha^n)$ is purely periodic, with period of length say p. We represent the elements of $\mathbb{Z}/b\mathbb{Z}$ by the integers $-\lceil b/2\rceil + 1, \ldots, \lfloor b/2\rfloor$ so that the period of the sequence $(t_n)_{n\in\mathbb{N}}$ gives $i_1, \ldots, i_p \in \{-\lceil b/2\rceil + 1, \ldots, \lfloor b/2\rfloor\}$ satisfying $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \equiv i_\ell(b)$ for $\ell = n - p\lfloor n/p\rfloor + 1$ and n large. We have

$$\lambda\alpha^{mp+\ell-1} = Tr_{k/\mathbb{Q}}(\lambda\alpha^{mp+\ell-1}) - \sum_{j=3}^{d}\lambda_{\ell,j}\alpha_j^{mp} - \lambda_{\ell,2}\alpha_2^{mp}. \tag{3}$$

Let $\ell \in \{1, \ldots, p\}$, $\rho \in \|\mathcal{L}_\ell\|$ and $\gamma \in \mathcal{L}_\ell$ such that $\rho = \|\gamma\|$. Recall that the conjugates of α other than α and $1/\alpha$ are of the form $\mathrm{e}^{\pm 2\pi\mathrm{i}\theta_j}$, $j = 1, \ldots, (d-2)/2$. Since $1, \theta_1, \ldots, \theta_{(d-2)/2}$ are linearly independent over $\mathbb{Z}$ (see [12, page 32]), it follows from Kronecker's theorem ([12,

Appendix 8], see also [15, Lemma 1]) that there exists an infinite subset $I \subset \mathbb{N}$ such that the sequence $\left(\sum_{j=3}^{d} \lambda_{\ell,j}\alpha_j^{mp}\right)_{m\in I}$ converges to $i_\ell - b\gamma$. Since $\lambda_{\ell,2}\alpha_2^{mp}$ tends to 0 as m goes to ∞ and $Tr_{k/\mathbb{Q}}(\lambda\alpha^{mp+\ell-1}) \equiv i_\ell(b)$, we deduce from (3)

$$\left\|\frac{\lambda}{b}\alpha^{mp+\ell-1}\right\| = \left\|\frac{i_\ell}{b} - \sum_{j=3}^{d}\frac{\lambda_{\ell,j}}{b}\alpha_j^{mp} - \frac{\lambda_{\ell,2}}{b}\alpha_2^{mp}\right\| = \|\gamma + \varepsilon_m\|$$

for $m \in I$ and where $(\varepsilon_m)_{m\in I}$ converges to 0 as $m \in I$ goes to ∞. Thus the limit point of $(\|\gamma + \varepsilon_m\|)_{m\in I}$ is $\|\gamma\| = \rho$.

Conversely, by (3) each term of the series $\left(\left\|\frac{\lambda}{b}\alpha^{mp+\ell-1}\right\|\right)_{m\in\mathbb{N}}$ can be rewritten $(\|\gamma_m\|)_{m\in\mathbb{N}}$ with $\gamma_m = \frac{i_\ell}{b} - \sum_{j=3}^{d}\frac{\lambda_{\ell,j}}{b}\alpha_j^{mp} - \frac{\lambda_{\ell,2}}{b}\alpha_2^{mp} \in \mathbb{R}$, which satisfies $|b\gamma_m - i_\ell| \le \sum_{j=3}^{d}|\lambda_{\ell,j}| + |\lambda_{\ell,2}\alpha_2^{mp}|$. The integer closest to γ_m can take only finitely many values. Thus, if a subsequence $(\|\gamma_m\|)_{m\in I}$ ($I \subset \mathbb{N}$ infinite) converges to a limit ρ, some subsequence $(\gamma_m)_{m\in J}$ ($J \subset I$ infinite) converges to a limit γ. This limit γ satisfies $\|\gamma\| = \rho$ and $|b\gamma - i_\ell| \le \sum_{j=3}^{d}|\lambda_{\ell,j}|$, which shows $\rho \in \|\mathcal{L}_\ell\|$.

If an integer closest to $\lambda\alpha^n$ is written $bN_n + i$, then $\left|\frac{\lambda\alpha^n - i}{b} - N_n\right| \le \frac{1}{2b}$ which entails $\left\|\frac{\lambda\alpha^n - i}{b}\right\| \le \frac{1}{2b}$. In the other direction, if the latter inequality holds, there exists an integer N_n such that $|\lambda\alpha^n - i - bN_n| \le \frac{1}{2}$ which shows that $bN_n + i$ is an integer closest to $\lambda\alpha^n$. If $\left\|\frac{\lambda\alpha^n}{b}\right\| \in \left\|\left[\frac{2i-1}{2b}, \frac{2i+1}{2b}\right]\right\|$ for infinitely many n, the sequence $\left(\left\|\frac{\lambda\alpha^n}{b}\right\|\right)_{n\in\mathbb{N}}$ has a limit point in $\left\|\left[\frac{2i-1}{2b}, \frac{2i+1}{2b}\right]\right\|$ which also belongs to $\cup_{\ell=1}^{p}\|\mathcal{L}_\ell\|$. □

The above result naturally leads to the following problem:

Problem: For a Salem number α, characterise the algebraic numbers $\lambda \in \mathbb{Q}(\alpha)$ such that the sequence $(\|\lambda\alpha^n\|)$ is dense in $[0, 1/2]$.

Remark 28. *The distribution of the sequence of fractional parts of $\lambda\alpha^n/b$ in $[0,1]$ is obtained from that of the difference with the nearest integer, by exchanging the subintervals $[-1/2, 0]$ and $[0, 1/2]$, while the sequence $(\|\alpha^n/b\|)_{n\in\mathbb{N}}$ is the superposition of these two latter intervals head to tail.*

We remind the reader that for a Salem number α, the sequence $(\|\alpha^n\|)_{n\in\mathbb{N}}$ is dense, but not uniformly distributed in $[0, 1/2]$. In fact, when $\lambda = 1$ and $b \le 2d - 4$ the length of each interval $\mathcal{L}_\ell$ is at least 1 and thus the sequence of fractional parts of α^n/b is dense in $[0,1]$, as in [15, Theorem (i)]. *Further, Proposition 27 together with Remark 22 also allows us to recover* [15, Theorem (ii) and (iii)].

However, when $b > 2d - 4$ the behaviour of the sequence $(\|\alpha^n/b\|)_{n\in\mathbb{N}}$ (as well as the sequence of fractional parts) strongly depends on the period $i_1, \ldots, i_p$, which is somewhat mysterious. We give below two contrasting examples illustrating this phenomenon.

Example 29. *Figure 1 shows the distribution of the numbers $\|\lambda\alpha^n\|$ in the interval $[0, 1/2]$ for $n \le 2500$, where α is the Salem number root of the polynomial $x^4 - 25x^3 + x^2 - 25x + 1$ and $\lambda = 1/25$. The period is $0, -2, 0, -2, 0, 4$ of length $p = 6$. This gives the three intervals $[0, 2/25]$, $[0, 4/25]$ and $[2/25, 6/25]$, which cover the whole set of limit points $[0, 6/25]$, with special concentrations on the four values 0, $2/25$, $4/25$ and $6/25$.*

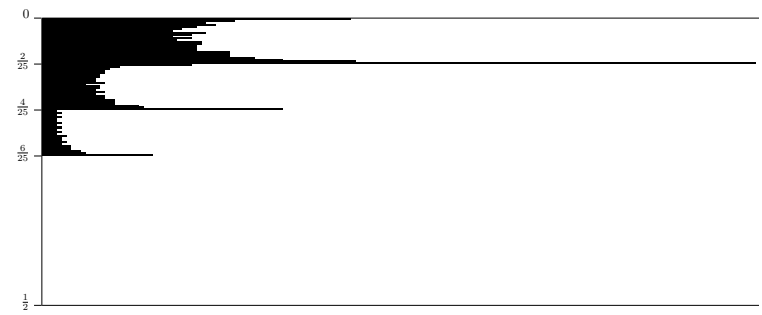

FIGURE 1. Scattering of $\|\lambda\alpha^n\|$ for $\lambda = 1/25$ and α a Salem number, $n \leq 2500$.

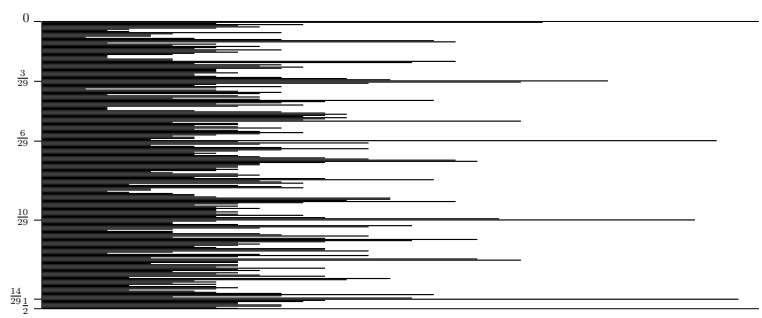

FIGURE 2. Scattering of $\|\lambda\alpha^n\|$ for $\lambda = 1/29$ and α a Salem number, $n \leq 2500$.

Similar pictures are obtained for $\lambda = 1/24$ and $1/26$, but in general the distribution of the numbers $\|\lambda\alpha^n\|$ is dense in $[0, 1/2]$ (although not uniformly), as in the next example.

Example 30. *Figure 2 shows the distribution of the numbers $\|\lambda\alpha^n\|$ in the interval $[0, 1/2]$ for $n \leq 2500$, where α is again the Salem number root of the polynomial $x^4 - 25x^3 + x^2 - 25x + 1$ and $\lambda = 1/29$. The period involves all the 29 integers between -14 and 14. This gives 15 intervals $[0, 2/29]$, $[0, 3/29]$, $[(i-2)/29, (i+2)/29]$, $i = 2, \ldots, 14$, which cover the whole interval $[0, 1/2]$, with several special concentrations.*

We refer to the interested reader the papers [1] and [5] where other aspects of distribution of powers of Salem numbers is studied. See also the papers [2] and [11] where the set of limit points of the sequences $\{\|\alpha^n\|^{\frac{1}{n}}\}$ is investigated.

We will now use these periodicity properties to prove Theorem 11 which refines the conclusion of Hardy's Theorem 2.

Proof of Theorem 11.

When α is a PV number and n is large enough, then $\lambda\alpha^n$ is a large real number whereas the sum of its conjugates is of absolute values $< c^n$ for some $0 < c < 1$. Hence we have

that the difference $|\lambda\alpha^n - Tr_{k/\mathbb{Q}}(\lambda\alpha^n)| < c^n$ tends to 0 as $n \to \infty$. Further, if $\lambda\alpha^n$ belongs to the complementary module of $\mathbb{Z}[\alpha]$ for n large enough, then $Tr_{k/\mathbb{Q}}(\lambda\alpha^n) \in \mathbb{Z}$ and hence $(\lambda, \alpha) \in H \cap \overline{\mathbb{Q}}^2$.

In the other direction, it follows from Corollary 26 that if $(\lambda, \alpha) \in H \cap \overline{\mathbb{Q}}^2$, then α is a PV number, $\lambda \in \mathbb{Q}(\alpha)$ and $\lambda\alpha^n$ belongs to the complementary module of $\mathbb{Z}[\alpha]$ for n large enough. This latter condition can be rewritten $\lambda \in \mathbb{Z}[\alpha]' \left[\frac{1}{\alpha}\right] = \frac{1}{P'_\alpha(\alpha)}\mathbb{Z}\left[\alpha, \frac{1}{\alpha}\right]$. □

We now give the proof of Theorem 12 which is an analogus result for Mahler sets.

Proof of Theorem 12.

If $(\lambda, \alpha) \in M \cap \overline{\mathbb{Q}}^2$, then $\|\lambda\alpha^n\| < c^n$ for infinitely many n. By Theorem 5, we know that α^{s_0} is a PV number for some $s_0 \in \mathbb{N}^\times$. Also because $(\lambda, \alpha) \in M$, there exists an integer $0 \le i < s_0$ such that 0 is a limit point of the sequence $\left(\|\lambda\alpha^{ns_0+i}\|\right)_{n\in\mathbb{N}}$. It follows from Corollary 25 that there exists integers $1 \le \ell \le p$ such that $\lambda\alpha^i \in \frac{1}{\alpha^{s_0(\ell-1)}P'_{\alpha^{ps_0}}(\alpha^{ps_0})}\mathbb{Z}\left[\alpha^{ps_0}, \frac{1}{\alpha^{ps_0}}\right]$. Setting $s = ps_0$ and $t = s_0(\ell-1) + i < \ell s_0 \le ps_0 = s$ proves the assertion, because $\alpha^s = \alpha^{ps_0}$ is again a PV number.

Conversely, if $\lambda\alpha^t \in \frac{1}{P'_{\alpha^s}(\alpha^s)}\mathbb{Z}\left[\alpha^s, \frac{1}{\alpha^s}\right]$, then for n large enough $Tr_{k/\mathbb{Q}}(\lambda\alpha^{ns+t}) \in \mathbb{Z}$. But, since α^s is a PV number, $\|\lambda\alpha^{ns+t}\| = \left|\lambda\alpha^{ns+t} - Tr_{k/\mathbb{Q}}(\lambda\alpha^{ns+t})\right| < c^n$ for n large enough and thus $(\lambda, \alpha) \in M \cap \overline{\mathbb{Q}}^2$. □

For $s \in \mathbb{N}^\times$ and real $\alpha > 1$, let $\alpha^{1/s}$ denote the unique real number $\beta > 1$ such that $\beta^s = \alpha$. We now have the following corollary which is a result in the direction of the fourth question indicated in the introduction (see page 3).

Corollary 31. *The following are true.*
1) $(\lambda, \alpha) \in M \cap \overline{\mathbb{Q}}^2$ if and only if there exists $s \in \mathbb{N}^\times$ and $0 \le t < s$ such that $(\lambda\alpha^t, \alpha^s) \in H \cap \overline{\mathbb{Q}}^2$.
2) If $(\lambda, \alpha) \in H \cap \overline{\mathbb{Q}}^2$, then for all $s \in \mathbb{N}^\times$ and $0 \le t < s$, we have $(\lambda\alpha^{-t/s}, \alpha^{1/s}) \in M \cap \overline{\mathbb{Q}}^2$.
3) For any $s \in \mathbb{N}^\times$, $(\lambda, \alpha) \in H$ if and only if $(\lambda\alpha^t, \alpha^s) \in H$ for every $0 \le t < s$.
4) For any $s \in \mathbb{N}^\times$, $(\lambda, \alpha) \in M$ if and only if there exists $0 \le t < s$ such that $(\lambda\alpha^t, \alpha^s) \in M$.

Proof. Here is the proof of the above statements.

1) By Theorem 12, (λ, α) belongs to $M \cap \overline{\mathbb{Q}}^2$ if and only if there exists integers $0 \le t < s$ such that α^s is a PV number and $\lambda\alpha^t \in \frac{1}{P'_{\alpha^s}(\alpha^s)}\mathbb{Z}\left[\alpha^s, \frac{1}{\alpha^s}\right]$, but these are exactly the conditions in Theorem 11 for $(\lambda\alpha^t, \alpha^s)$ to belong to $H \cap \overline{\mathbb{Q}}^2$.

2) It follows from the previous assertion. For $(\lambda, \alpha) \in H \cap \overline{\mathbb{Q}}^2$ and integers s, t with $0 \le t < s$, let $(\mu, \beta) = (\lambda\alpha^{-t/s}, \alpha^{1/s})$. Then $(\mu\beta^t, \beta^s) = (\lambda, \alpha)$. By the reverse implication in the above proposition, we get $(\mu, \beta) \in M \cap \overline{\mathbb{Q}}^2$.

3) This follows from the definition of H since for $s \in \mathbb{N}^\times$, each of the following s subsequences

$$(\|\lambda\alpha^{ms+t}\|)_{m\in\mathbb{N}}, \qquad 0 \le t < s$$

converges to 0 if and only if the sequence $(\|\lambda\alpha^n\|)_{n\in\mathbb{N}}$ converges to 0.

4) For any $s \in \mathbb{N}^\times$ and $(\lambda, \alpha) \in M$, there exists an integer $0 \le t < s$ such that $\|\lambda\alpha^{ms+t}\| < c^m$ for infinitely many $m \in \mathbb{N}$ and thus $(\lambda\alpha^t, \alpha^s) \in M$. Reciprocally, if there exists an integer $0 \le t < s$ such that $(\lambda\alpha^t, \alpha^s) \in M$, then $\|\lambda\alpha^{ms+t}\| < c^m$ for infinitely many $m \in \mathbb{N}$ and hence $(\lambda, \alpha) \in M$. □

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