



**HAL**  
open science

## Schottky group actions in complex geometry

Christian Miebach, Karl Oeljeklaus

► **To cite this version:**

Christian Miebach, Karl Oeljeklaus. Schottky group actions in complex geometry. Geometric Complex Analysis: In Honor of Kang-Tae Kims 60th Birthday, 2017, Jul 2017, Gyeong-Ju, South Korea. pp.257-268, 10.1007/978-981-13-1672-2\_20 . hal-02083023

**HAL Id: hal-02083023**

**<https://hal.science/hal-02083023v1>**

Submitted on 8 Apr 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SCHOTTKY GROUP ACTIONS IN COMPLEX GEOMETRY

CHRISTIAN MIEBACH AND KARL OELJEKLAUS

*Dedicated to Kang-Tae Kim at the occasion of his 60th birthday*

ABSTRACT. This paper discusses some recent progress on Schottky group actions on compact and non-compact complex manifolds.

## 1. INTRODUCTION

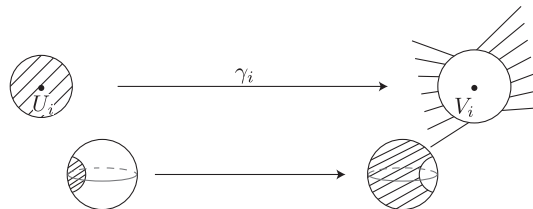
In this article we survey a number of results concerning Schottky group actions on complex manifolds. As an abstract group a Schottky group is isomorphic to the free group  $F_r$  of rank  $r \geq 1$ , and a Schottky group action on the complex manifold  $X$  yields an  $F_r$ -invariant connected open subset  $\mathcal{U}$  of  $X$  on which action of  $F_r$  is free and properly discontinuous. Our motivation for the study of Schottky group actions stems from the fact that the quotient manifolds  $\mathcal{U}/F_r$  have very interesting properties .

Having discussed the historical background of Schottky groups in complex geometry in Section 1, we review the main results of [MO16] in Section 2. Section 3 contains a new result concerning Schottky group actions on the unit ball of  $\mathbb{C}^n$ , which has been obtained independently by Stefan Nemirovski. To the best of our knowledge this is the first time that complex analytic properties of Schottky quotients are discussed in a non-compact setting. In the last section we list a number of open problems in this circle of ideas.

## 2. HISTORICAL REMARKS ON SCHOTTKY ACTIONS

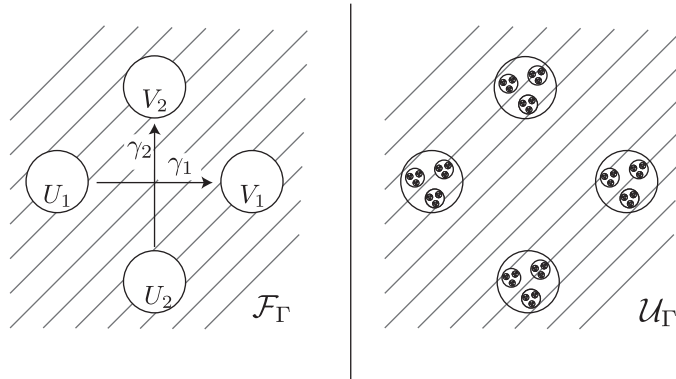
In 1877 Schottky constructed holomorphic actions of free groups on the Riemann sphere in the following way: Take  $(U_1, V_1), \dots, (U_r, V_r)$  to be  $r$  pairs of open subsets of  $\mathbb{P}_1$  bounded by Jordan curves, such that  $\overline{U_1}, \overline{V_1}, \dots, \overline{U_r}, \overline{V_r}$  are pairwise disjoint and such that there are loxodromic elements  $\gamma_1, \dots, \gamma_r \in \text{Aut}(\mathbb{P}_1)$  with

$$\gamma_i(U_i) = \mathbb{P}_1 \setminus \overline{V_i}, \quad i = 1, \dots, r.$$



The *Schottky group*  $\Gamma := \langle \gamma_1, \dots, \gamma_r \rangle \subset \mathrm{PSL}(2, \mathbb{C})$  is free and non-abelian. Define

$$\mathcal{F}_\Gamma := \mathbb{P}_1 \setminus \bigcup_{i=1}^r (U_i \cup V_i) \text{ and } \mathcal{U}_\Gamma := \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{F}_\Gamma).$$



Then  $\Gamma$  is a free group of  $r$  generators and it acts freely and properly discontinuously on  $\mathcal{U}_\Gamma$  with  $\mathcal{F}_\Gamma$  as a fundamental domain. Furthermore the quotient  $Q_\Gamma := \mathcal{U}_\Gamma/\Gamma$  is a compact Riemann surface  $R_r$  of genus  $r$  and every such surface  $R_r$  can be obtained this way, [Koe10]. The complement  $\mathbb{P}_1 \setminus \mathcal{U}_\Gamma$  is a Cantor set and  $\mathcal{U}_\Gamma$  is not simply-connected. In 1967 Maskit proved that a subgroup  $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$  is Schottky if and only if it is finitely generated, free, has nonempty domain of discontinuity and all non-trivial elements are conjugate to some  $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  with  $|\lambda| \neq 1$ , see [Ma67]. A natural idea is to generalize this construction to higher dimensions which has been done since the beginning of the 1980s.

In [Nor86] Nori gave an explicit construction of Schottky groups acting on  $\mathbb{P}_N$  for every odd  $N$ . Here a subgroup  $\Gamma$  of  $\mathrm{Aut}(\mathbb{P}_N)$  is called a *Schottky group* if there exist  $2r$  open subsets  $U_1, \dots, U_r, V_1, \dots, V_r$  of  $\mathbb{P}_N$ , being the interiors of their pairwise disjoint topological closures, such that  $\Gamma$  is generated by  $r$  automorphisms  $\gamma_1, \dots, \gamma_r \in \mathrm{Aut}(\mathbb{P}_N)$  which satisfy

$$\gamma_j(U_j) = \mathbb{P}_N \setminus \overline{V_j} \quad \text{for all } 1 \leq j \leq r.$$

In this situation standard arguments show that  $\Gamma$  is freely generated by  $\gamma_1, \dots, \gamma_r$  and hence isomorphic to the free group of rank  $r$ . Defining

$$\mathcal{F}_\Gamma := \mathbb{P}_N \setminus \bigcup_{j=1}^r (U_j \cup V_j) \quad \text{and} \quad \mathcal{U}_\Gamma := \Gamma \cdot \mathcal{F}_\Gamma,$$

it is not hard to show that  $\Gamma$  acts properly on the connected open subset  $\mathcal{U}_\Gamma$  of  $\mathbb{P}_N$  with fundamental domain  $\mathcal{F}_\Gamma$  and that we obtain a compact complex manifold  $Q_\Gamma := \mathcal{U}_\Gamma/\Gamma$  as quotient. Moreover, if  $N > 1$ , then  $\mathcal{U}_\Gamma$  is simply connected and thus the fundamental group of  $Q_\Gamma$  is isomorphic to  $\Gamma$ .

In [SV03] Seade and Verjovsky use inversions at certain hypersurfaces of  $\mathbb{P}_N$  with  $N$  odd in order to construct discrete subgroups of  $\mathrm{Aut}(\mathbb{P}_N)$  which contain Nori's Schottky groups as subgroups of index 2. On the other hand, in [Can08] Cano shows that there are no Schottky groups acting on  $\mathbb{P}_N$  for even  $N$ .

In dimension 3 a more general approach to Schottky quotients has been developed by Ma. Kato. A compact complex 3-fold is of class  $L$  if it contains an open set biholomorphic to a neighborhood of a complex line in the complex projective space  $\mathbb{P}_3$ . This notion was introduced by Ma. Kato (see [Ka82], [Ka85]) who studied manifolds of class  $L$  in great detail in several articles. An important subclass of class  $L$  is given by compact complex manifolds whose universal covering is biholomorphic to a domain in  $\mathbb{P}_3$  containing a complex line. In his paper [Ka89], Kato constructed many examples of such manifolds which he calls “manifolds of Schottky type” using the method of “Klein combination” which produces given two such manifolds a third one still having a domain of  $\mathbb{P}_3$  as universal covering. These spaces are quotients by discrete subgroups of the automorphism group of  $\mathbb{P}_3$ , but these groups are only implicitly constructed. In [Lár98] Lárússon has also contributed to the theory of 3-dimensional manifolds of Schottky type.

### 3. SCHOTTKY GROUP ACTIONS ON FLAG MANIFOLDS

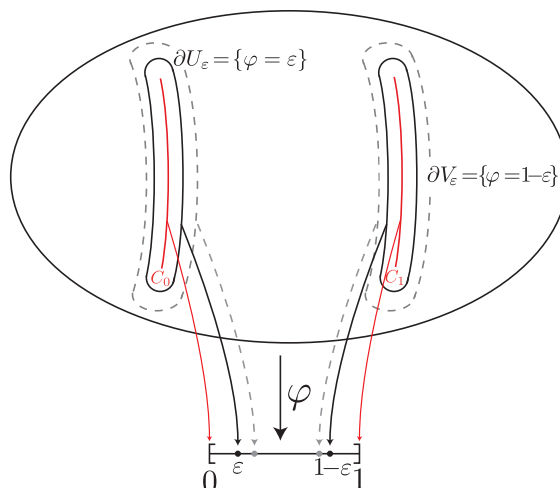
In this section we review the main results from our paper [MO16].

**3.1. Construction of Schottky groups acting on flag manifolds.** Let us recall Nori’s construction of Schottky group actions on an odd-dimensional complex projective space  $\mathbb{P}_{2n+1}$  with  $n \geq 0$ . Writing homogeneous coordinates of  $\mathbb{P}_{2n+1}$  as  $[z : w]$  with  $z = (z_0, \dots, z_n)$  and  $w = (w_0, \dots, w_n)$  we define

$$C_0 := \{[z : w] \in \mathbb{P}_{2n+1}; w = 0\} \quad \text{and} \quad C_1 := \{[z : w] \in \mathbb{P}_{2n+1}; z = 0\}$$

as well as a smooth function  $\varphi: \mathbb{P}_{2n+1} \rightarrow [0, 1]$  by

$$\varphi[z : w] := \frac{\|w\|^2}{\|z\|^2 + \|w\|^2}.$$



Note that  $C_j = \varphi^{-1}(j)$  for  $j = 0, 1$ . For  $0 < \varepsilon < \frac{1}{2}$  we define open neighborhoods of  $C_0, C_1$  by  $U_\varepsilon := \{\varphi < \varepsilon\}$  and  $V_\varepsilon := \{\varphi > 1 - \varepsilon\}$ , respectively. A direct calculation

shows that the automorphism  $\gamma \in \text{Aut}(\mathbb{P}_{2n+1})$  given by

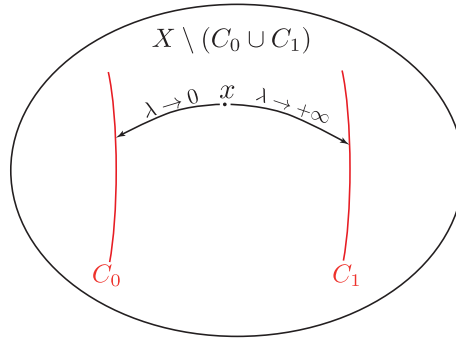
$$\gamma([z : w]) := [z : \lambda w]$$

satisfies  $\gamma(U_\varepsilon) = \mathbb{P}_{2n+1} \setminus \overline{V}_\varepsilon$  if  $|\lambda| = \frac{1-\varepsilon}{\varepsilon}$ . Choosing  $\varepsilon$  sufficiently small and conjugating  $\gamma$  by automorphisms  $f_1, \dots, f_r \in \text{Aut}(\mathbb{P}_{2n+1})$  we obtain Nori's examples of Schottky groups acting on  $\mathbb{P}_{2n+1}$ .

Since Nori and Cano settled the question of existence of Schottky groups acting on complex projective space, the problem that motivated the research leading to [MO16] was to determine all complex flag manifolds that admit Schottky group actions. Here, by flag manifold we mean a complex homogeneous manifold  $X = G/P$  where  $G$  is a connected semisimple complex Lie group and  $P$  is a parabolic subgroup of  $G$ .

From now on we replace complex projective space by an arbitrary complex flag manifold  $X$ . Motivated by Nori's construction we define a *Schottky pair in  $X$*  to be a pair of disjoint connected compact complex submanifolds  $C_0, C_1$  of  $X$  for which there exists a holomorphic  $\mathbb{C}^*$ -action on  $X$  given by the one parameter family  $\{\varphi_\lambda\}_{\lambda \in \mathbb{C}^*}$  of automorphisms of  $X$  such that

- (1) the fixed point set  $X^{\mathbb{C}^*}$  coincides with  $C_0 \cup C_1$  and
- (2)  $\mathbb{C}^*$  acts freely and properly on  $X \setminus (C_0 \cup C_1)$ .



Such a pair is the basic ingredient in order to define a generator of a Schottky group: We may choose suitable open neighborhoods  $U$  of  $C_0$  and  $V$  of  $C_1$  and an automorphism  $\varphi_{\lambda_0}$  such that  $\varphi_{\lambda_0}(U) = X \setminus \overline{V}$ . Hence, we may construct a Schottky group of arbitrarily high rank as soon as we find “many” disjoint Schottky pairs. More precisely, we say that the Schottky pair  $(C_0, C_1)$  is *movable* if for every  $r \geq 1$  there are automorphisms  $f_1, \dots, f_r \in \text{Aut}(X)$  such that  $f_k(C_j)$  are pairwise disjoint for all  $1 \leq k \leq r$  and  $j = 0, 1$ .

In order to construct Seade's and Verjovsky's extended Schottky groups we need an involutive automorphism  $s \in \text{Aut}(X)$  such that  $s(C_0) = C_1$  and  $s \circ \varphi_\lambda = \varphi_{\lambda^{-1}} \circ s$  for all  $\lambda \in \mathbb{C}^*$ . Note that in Nori's examples such an involution is defined by  $s[z : w] = [w : z]$ . In particular, its fixed point set  $X^s$  is the hypersurface

$$H := \{[z : w] \in \mathbb{P}_{2n+1}; \|z\|^2 - \|w\|^2 = 0\}.$$

The crucial observation which eventually leads to the construction of Schottky pairs in certain flag manifolds  $X$  is that  $H$  is an orbit of the non-compact real form  $G_0 = \text{PSU}(n+1, n+1)$  of  $G = \text{Aut}(\mathbb{P}_{2n+1}) \cong \text{PSL}(2n+2, \mathbb{C})$ . Indeed, combining a theorem

of Akhiezer (cf. [Akh77]) with Matsuki duality (cf. [BL02]) we can prove the following, see [MO16, Proposition 3.2].

**Proposition 3.1.** *Let  $X = G/P$  be a complex flag manifold. If there exists a non-compact real form  $G_0$  of  $G$  having a compact orbit of real codimension 1 in  $X$ , then  $X$  admits a Schottky pair.*

Proposition 3.1 suggests the following strategy for the construction of Schottky groups acting on a flag manifold  $X$ . First classify couples  $(G/P, G_0)$  where  $X = G/P$  is a flag manifold and  $G_0$  is a non-compact real form of  $G$  having a compact hypersurface orbit in  $X$ , then determine which of the corresponding Schottky pairs are actually movable in order to produce Schottky groups of arbitrarily high rank acting on  $X$ .

This program leads to the following result, see [MO16, Theorems 4.2 and A.1].

**Theorem 3.2.** *The pairs  $(G/P, G_0)$  of flag manifolds  $X = G/P$  and real forms  $G_0$  of  $G$  having a compact hypersurface orbit in  $X$  are the following:*

- (1)  $G_0 = \mathrm{SU}(p, q)$  acting on  $X = \mathbb{P}_{p+q-1}$ ;
- (2)  $G_0 = \mathrm{Sp}(p, q)$  acting on  $X = \mathbb{P}_{2(p+q)-1}$ ;
- (3)  $G_0 = \mathrm{SU}(1, n)$  acting on the Grassmannian  $X = \mathrm{Gr}_k(\mathbb{C}^{n+1})$  of  $k$ -dimensional linear subspaces in  $\mathbb{C}^{n+1}$ ;
- (4)  $G_0 = \mathrm{SO}^*(2n)$  acting on the quadric  $X = Q_{2n-2} = \{[z : w] \in \mathbb{P}_{2n-1}; z_0 w_0 + \cdots + z_{n-1} w_{n-1} = 0\}$ ;
- (5)  $G_0 = \mathrm{SO}(1, 2n)$  acting on the manifold  $X = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$  of  $n$ -dimensional linear subspaces of  $\mathbb{C}^{2n+1}$  which are isotropic with respect to a non-degenerate symmetric bilinear form on  $\mathbb{C}^{2n+1}$ ;
- (6)  $G_0 = \mathrm{SO}(2, 2n)$  acting on  $X = \mathrm{IGr}_{n+1}(\mathbb{C}^{2n+2})^0$ .

The Schottky pairs giving rise to Schottky group actions on  $X$  of arbitrary rank  $r$  are precisely the ones on the odd-dimensional projective space  $\mathbb{P}_{2n+1}$  where  $G_0 = \mathrm{SU}(n+1, n+1)$ , on  $\mathbb{P}_{4n+3}$  where  $G_0 = \mathrm{Sp}(n+1, n+1)$ , on the quadric  $Q_{4n-2}$  where  $G_0 = \mathrm{SO}^*(4n)$  and on the isotropic Grassmannian  $\mathrm{IGr}_n(\mathbb{C}^{2n+1})$  where  $G_0 = \mathrm{SO}(1, 2n)$ .

*Remark.* The movable Schottky pairs obtained by Theorem 3.2 yield Schottky group actions on irreducible flag manifolds<sup>1</sup>. If  $X = G/P$  is one of these flag manifolds, and if  $P'$  is a parabolic subgroup of  $P$ , we obtain a Schottky group action on  $X' = G/P'$  by lifting the one on  $X$ .

In the rest of this subsection we will describe the movable Schottky pairs in greater detail. In particular we will show that in every case there is an involutive automorphism  $s \in \mathrm{Aut}(X)$  such that  $X^s$  is a compact hypersurface orbit of  $G_0$ . Consequently, it is possible to construct discrete subgroups of  $\mathrm{Aut}(X)$  which contain Schottky groups as subgroups of index 2 as it is done in [SV03]. In addition we may extend Kato's notion of complex-analytic connected sum to these cases.

We have already discussed Nori's Schottky groups acting on  $\mathbb{P}_{2n+1}$ . In these cases the Schottky pairs are linked by a holomorphic involution.

<sup>1</sup>A flag manifold  $X = G/P$  is called *irreducible* if  $P$  is a maximal parabolic subgroup of  $G$ , i.e., if  $P$  is not properly contained in any proper parabolic subgroup of  $G$ .

*Example.* Let us consider the case of the quadric  $X = Q_{4n-2}$ , compare [MO16, Section 4.2]. In this case a movable Schottky pair is given by

$$C_0 = \{[z : w] \in Q_{4n-2}; w = 0\} \quad \text{and} \quad C_1 = \{[z : w] \in Q_{4n-2}; z = 0\}.$$

The map defined by  $s[z : w] = [w : z]$  defines a holomorphic involution of  $X = Q_{4n-2}$  whose fixed point set in  $X$  coincides with the compact  $\mathrm{SO}^*(4n)$ -orbit in  $X$ .

*Example.* In the case of the isotropic Grassmannian  $X = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$  a movable Schottky pair is given as follows, see [MO16, Section 4.4]. The set of  $n$ -dimensional isotropic linear subspaces of  $\mathbb{C}^{2n+1}$  which lie in  $\mathbb{C}^{2n} \times \{0\}$  is isomorphic to  $\mathrm{IGr}_n(\mathbb{C}^{2n})$  and has two connected components  $C_0$  and  $C_1$  which form a movable Schottky pair in  $X$ . There is a holomorphic action of the group  $\mathrm{O}(2n, \mathbb{C})$  on  $X$  such that  $C_0 \cup C_1$  is an orbit. Moreover, we may find an element of order 2 in  $\mathrm{O}(2n, \mathbb{C}) \setminus \mathrm{SO}(2n, \mathbb{C})$  which exchanges  $C_0$  and  $C_1$  and therefore yields a holomorphic involution of  $X$  with the desired properties.

**3.2. Analytic and geometric properties of  $Q_\Gamma$ .** Let  $\Gamma \subset G$  be a Schottky group acting on the flag manifold  $X = G/P$ . We suppose throughout the following that  $\dim X = d \geq 2$  and that  $\dim C_j \leq d - 2$ . This implies that the maximal domain  $\mathcal{U}_\Gamma$  on which  $\Gamma$  acts properly is simply connected. Let  $p: \mathcal{U}_\Gamma \rightarrow Q_\Gamma$  be the universal covering of the compact quotient manifold  $Q_\Gamma$ .

The general strategy to investigate holomorphic objects on  $Q_\Gamma$ , such as meromorphic functions or holomorphic differential forms, is to pull them back to  $\mathcal{U}_\Gamma$  and then to use classical extension theorems in order to extend them to the whole of  $X$ , see [MO16, Section 5].

Since the fundamental group of  $Q_\Gamma$  is isomorphic to  $\Gamma$ , a free group of rank  $r \geq 1$ , it is well known that  $Q_\Gamma$  cannot be Kähler. Due to the fact that  $\mathcal{U}_\Gamma$  contains many of the rational curves of  $X$ , the quotient manifold  $Q_\Gamma$  is rationally connected. Since holomorphic  $d$ -forms extend from  $\mathcal{U}_\Gamma$  to  $X$ , the Kodaira dimension of  $Q_\Gamma$  is  $-\infty$ .

Moreover, meromorphic functions also extend from  $\mathcal{U}_\Gamma$  to  $X$ . Hence,  $\mathcal{M}(Q_\Gamma)$  can be identified with the set of  $\Gamma$ -invariant rational functions on  $X$ . Since  $\Gamma$ -invariant rational functions on  $X$  are automatically invariant under the Zariski closure of  $\Gamma$  in  $G$ , an application of Rosenlicht's theorem yields the following.

**Theorem 3.3.** *The algebraic dimension  $a(Q_\Gamma)$  coincides with the codimension of a generic  $H$ -orbit in  $X$  where  $H$  denotes the Zariski closure of  $\Gamma$  in  $G$ . In particular,  $a(Q_\Gamma) = 0$  if and only if  $H$  has an open orbit in  $X$ .*

Although a generic Schottky group acting on  $X$  is Zariski dense in  $G$  so that  $Q_\Gamma$  has algebraic dimension 0, we have given examples where  $a(Q_\Gamma)$  is positive for every  $X$  admitting movable Schottky pairs, see [MO16, Section 6.1].

The following assumptions on the dimensions of  $X$  are necessary for us to extend higher cohomology classes from  $\mathcal{U}_\Gamma$ .

**Dimension Assumption 3.4.** *From now on let  $X$  be either  $\mathbb{P}_{2n+1}$  with  $n \geq 3$  or  $Q_{4n+2}$  with  $n \geq 2$  or  $Q_{2n+1}$  with  $n \geq 3$  or  $X_n$  with  $n \geq 4$ .*

**Theorem 3.5.** *Let  $\Gamma$  be a Schottky group of rank  $r$  acting on  $X$  with associated quotient  $\pi: \mathcal{U}_\Gamma \rightarrow Q_\Gamma$ . Then the Picard group  $H^1(Q_\Gamma, \mathcal{O}^*)$  of  $Q_\Gamma$  is isomorphic to  $(\mathbb{C}^*)^r \times \mathbb{Z}$ .*

The factor  $(\mathbb{C}^*)^r$  contains the topologically trivial holomorphic line bundles on  $Q_\Gamma$ , given by a representation of  $\Gamma$  in  $\mathbb{C}^*$ , while the factor  $\mathbb{Z}$  comes from a hyperplane bundle in  $X$ .

In closing we note the following result about the deformation theory of  $Q_\Gamma$ . We observe that  $\Gamma$  acts on the Lie algebra  $\mathfrak{g}$  of  $G$  via the adjoint representation  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$  and we denote by  $\mathfrak{g}^\Gamma$  the subspace of  $\Gamma$ -fixed points.

**Theorem 3.6.** *The Kuranishi space of versal deformations of  $Q_\Gamma$  is smooth at  $Q_\Gamma$  and of complex dimension  $(r - 1) \dim \mathfrak{g} + \dim \mathfrak{g}^\Gamma$ . Moreover, the automorphism group  $\text{Aut}(Q_\Gamma)$  admits as Lie algebra  $\mathfrak{g}^\Gamma$ .*

Note that the automorphism group of  $Q_\Gamma$  can be rather large. For example, it is even possible that  $Q_\Gamma$  is almost homogeneous, see [MO16, Examples 6.4 and 6.5].

#### 4. SCHOTTKY GROUPS ACTING ON THE UNIT BALL

In this section we consider Schottky subgroups  $\Gamma$  of the automorphism group of the unit ball  $\mathbb{B}_n$  in  $\mathbb{C}^n$  and show that the quotient manifolds  $\mathbb{B}_n/\Gamma$  are Stein.

**4.1. Construction of Schottky actions.** Let  $\mathbb{B}_n$  denote the unit ball of  $\mathbb{C}^n$ . Recall that every holomorphic automorphism of  $\mathbb{B}_n$  extends to the boundary  $\partial\mathbb{B}_n = S^{2n-1}$  and is called *elliptic* or *parabolic* or *hyperbolic* if it has a fixed point in  $\mathbb{B}_n$  or exactly one or exactly two fixed points in  $\partial\mathbb{B}_n$ , respectively, see [Gol99, Section 6.2.1], p. 203.

Let  $\{\varphi_t\}_{t \in \mathbb{R}}$  be a one parameter group of hyperbolic automorphisms of  $\mathbb{B}_n$ . Then, for every  $z \in \mathbb{B}_n$ , we have

$$\lim_{t \rightarrow \pm\infty} \varphi_t(z) = x_\pm \in \partial\mathbb{B}_n$$

where  $x_\pm$  are the two joint fixed points of  $\{\varphi_t\}$ . Consequently, we may choose open neighborhoods  $U_\pm$  of  $x_\pm$  in  $\partial\mathbb{B}_n$  and  $t_0 \in \mathbb{R}$  such that

$$\varphi_{t_0}(U_-) = \partial\mathbb{B}_n \setminus \overline{U_+}.$$

This shows that we can construct a Schottky group action on the unit sphere  $S^{2n-1}$  (compare [MO16, Remark 4.6]) that extends to a holomorphic group action on the unit ball.

*Remark.* Such a construction can also be found in [KLP16]. Note that there are also Schottky groups containing parabolic elements, see [Li06].

As a consequence we obtain a discrete subgroup  $\Gamma$  of  $\text{Aut}(\mathbb{B}_n)$  which is freely generated by  $r$  hyperbolic automorphisms for every  $r \geq 1$ . We define its limit set  $\Lambda_\Gamma \subset \partial\mathbb{B}_n$  as the set of accumulation points of  $\Gamma \cdot z_0$  in  $\partial\mathbb{B}_n$  for  $z_0 \in \mathbb{B}_n$ . Note that this set is independent of the choice of  $z_0 \in \mathbb{B}_n$ . Moreover,  $\Gamma$  acts freely and properly on  $\mathbb{B}_n \cup \Omega_\Gamma$  where  $\Omega_\Gamma := \partial\mathbb{B}_n \setminus \Lambda_\Gamma$  is the discontinuity domain of  $\Gamma$ , see [Bow95, Proposition 3.2.6].

**4.2. Schottky quotients are Stein.** Let  $X := \mathbb{B}_n/\Gamma$  and  $\partial X := \Omega_\Gamma/\Gamma$  be the corresponding quotient manifolds. Since  $\partial X$  is a compact spherical CR manifold, the union  $\overline{X} := X \cup \partial X$  is a compact complex manifold with strictly pseudoconvex boundary  $\partial X$ . We shall see that  $X$  is a Stein manifold.



*Remark.* Note that  $\Gamma$  is geometrically finite in the sense of [Bow95], that generically  $\Gamma$  is Zariski-dense in the real algebraic group  $\text{Aut}(\mathbb{B}_n)$  and that  $\Lambda_\Gamma$  is a Cantor set in  $\partial\mathbb{B}_n$ .

**Lemma 4.1.** *The quotient manifold  $X$  is strongly pseudoconvex. In particular,  $X$  is a proper modification of a Stein space and therefore holomorphically convex.*

*Proof.* Every point  $x \in \partial X$  has an open neighborhood  $V \subset \overline{X}$  that is biholomorphically equivalent to an open neighborhood  $U$  of some boundary point of  $\mathbb{B}_n$ . Since  $\partial\mathbb{B}_n$  is smooth and strictly pseudoconvex, there exists a strictly plurisubharmonic function  $\rho_V \in \mathcal{C}^\infty(V)$  such that  $\lim_{x \rightarrow \partial X \cap V} \rho_V(x) = \infty$ . Using a smooth partition of unity, we may find a function  $\rho \in \mathcal{C}^\infty(X)$  which is strictly plurisubharmonic near  $\partial X$  such that  $\lim_{x \rightarrow \partial X} \rho(x) = \infty$  holds. Since  $\overline{X}$  is compact,  $\rho$  is an exhaustion function on  $X$  which is strictly plurisubharmonic outside some compact subset of  $X$ , i.e.,  $X$  is strongly pseudoconvex.  $\square$

In order to show that  $X$  is Stein we must exclude that  $X$  contains compact analytic subsets of positive dimension.

**Lemma 4.2.** *The quotient manifold  $X$  does not contain compact analytic subsets of positive dimension.*

*Proof.* Suppose that  $A \subset X$  is an irreducible compact analytic subset of dimension  $\dim A = k > 0$ .

Applying the Cartan-Leray spectral sequence to the universal covering  $\mathbb{B}_n \rightarrow X$  it follows that  $H_{2k}(X, \mathbb{Z}) = \{0\}$  for all  $k \geq 1$ . Indeed, since we have  $E_2^{p,q} \cong H_p(\Gamma, H_q(\mathbb{B}_n))$  and  $\Gamma$  is free, the only non-zero terms are  $E_2^{0,0}$  and  $E_2^{1,0}$ , which proves the claim.

Consequently,  $[A] \in H_{2k}(X, \mathbb{Z})$  is trivial, hence  $A = \partial M$  for some real submanifold  $M$  of  $X$ .

On the other hand, the Bergman metric of the unit ball is invariant under  $\text{Aut}(\mathbb{B}_n)$  and thus descends to  $X$ . Let  $\omega$  denote the corresponding Kähler form on  $X$ . Then we obtain

$$0 < \int_A \omega^k = \int_M d\omega^k = 0,$$

a contradiction.  $\square$

*Remark.* The argument in the proof of Lemma 4.2 shows that, whenever a free group acts properly on a contractible Stein manifold which carries an invariant Kähler form, the quotient manifold does not contain compact analytic sets of positive dimension.

Combining Lemmas 4.1 and 4.2 we obtain the following

**Theorem 4.3.** *Let  $\Gamma \subset \text{Aut}(\mathbb{B}_n)$  be a Schottky group acting on  $\mathbb{B}_n$ . Then the quotient manifold  $X = \mathbb{B}_n/\Gamma$  is Stein.*

*Remark.* Extending the action of  $\Gamma$  to  $\mathbb{P}_n \supset \mathbb{B}_n$ , we obtain a holomorphic action of the free group  $\Gamma$  of arbitrary rank  $r > 0$  on  $\mathbb{P}_n$  which has domain of discontinuity  $\mathbb{B}_n$ .

*Remark.* Stefan Nemirovski announced Theorem 4.3 during the workshop ‘‘Lie Groups, Invariant Theory & Complex Geometry’’ that took place on June 1st and 2nd 2017 at the University of Duisburg-Essen, Germany. His proof of Lemma 4.2 however uses

a fundamental domain for the action of  $\Gamma$  constructed with the help of bisectors and relies on their complex-analytic properties.

## 5. OPEN PROBLEMS

**5.1. Cohomology groups in small dimensions.** Recall that we had to make the dimension assumption 3.4 in order to calculate the dimensions of several cohomology groups.

**Problem 1.** *Calculate the cohomology groups of the quotient manifolds  $Q_\Gamma$  for which the initial flag manifold  $X$  is isomorphic to the projective spaces  $\mathbb{P}_3$ ,  $\mathbb{P}_5$  and the quadrics  $Q_3$ ,  $Q_5$  and  $Q_6$ .*

**5.2. Generalizations of Blanchard Manifolds.** Following Ma. Kato, a *Blanchard manifold* is a compact complex manifold  $Y$  whose universal covering is isomorphic to the complement of a single projective line in  $\mathbb{P}_3$ . The fundamental group  $\Gamma := \pi_1(Y)$  is isomorphic to  $(\mathbb{Z}^4, +)$ . It is natural to look for generalisations of this situation in higher dimensions. One has first the following negative result.

**Proposition 5.1.** *Let  $C$  be a linearly embedded subspace  $\mathbb{P}_k$  of  $\mathbb{P}_n$ . If  $k < (n - 1)/2$ , then there is no infinite discrete group of automorphisms of  $\Omega := \mathbb{P}_n \setminus C$  acting properly on  $\Omega$ .*

*Proof.* We may suppose that  $C = \{[x] \in \mathbb{P}_n; x_{k+1} = \dots = x_n = 0\}$ . A neighborhood basis of  $C$  is given by the open sets of the form

$$U_\varepsilon := \{[x : y] \in \mathbb{P}_n; \varepsilon\|x\|^2 - (1 - \varepsilon)\|y\|^2 > 0\}, \quad \varepsilon > 0,$$

where  $x = (x_0, \dots, x_k)$  and  $y = (x_{k+1}, \dots, x_n)$ . One verifies directly that the sets  $U_\varepsilon$  are strictly  $k$ -convex. In particular,  $U_\varepsilon$  does not contain any compact analytic subset of dimension bigger than  $k$ .

Suppose now for a moment that there exists an infinite discrete group acting holomorphically and properly on  $\Omega$ . Then for arbitrary  $0 < \varepsilon < 1$  and for all but finitely many  $\gamma \in \Gamma$  we have  $\gamma(C') \subset U_\varepsilon$  where  $C' := \{[x : y] \in \mathbb{P}_n; x = 0\} \cong \mathbb{P}_{n-k-1}$ . It follows from the above discussion that  $n - k - 1 \leq k$ , which is equivalent to  $k \geq (n - 1)/2$ .  $\square$

In order to generalize the notion of Blanchard manifold one may consider the following problem.

**Problem 2.** (a) *Is it possible that  $\Omega = \mathbb{P}_n \setminus \mathbb{P}_k$  covers a compact complex manifold if  $n - 2 \geq k \geq (n - 1)/2$ , and  $n > 3$ ?*  
 (b) *Let  $X$  be one of the flag manifolds admitting an action of a Schottky group of arbitrary rank  $r$  and let  $C := C_0$  be a connected component of a movable Schottky pair. Determine all the cases when  $X' := X \setminus C$  is the universal covering of a compact complex manifold, i.e. when  $X'$  admits a free and properly discontinuous co-compact action of of a discrete group of automorphisms.*

**5.3. Generalized Kato theory.** In a series of papers Ma. Kato developed the theory of compact complex 3-folds of class  $L$  using in particular the methods of connected sums and Klein combinations, which allow to produce given two manifolds of class  $L$  a third one of class  $L$ .

Let  $X$  be one of the flag manifolds admitting an action of a Schottky group of arbitrary rank  $r$  and let  $C := C_0 \subset X$  be a connected component of a movable Schottky pair. By imitating Kato's definition, one may define a manifold of class  $C$  as a compact complex manifold containing an open domain which is biholomorphic to an open neighborhood of  $C$  in  $X$ . As in Kato's case it is still possible to produce connected sums and Klein combinations of manifolds of class  $C$ .

**Problem 3.** *Develop a theory of manifolds of class  $C$ .*

## REFERENCES

- [Akh77] D. N. Akhiezer, *Dense orbits with two endpoints*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 2, 308–324, 477.
- [B56] A. Blanchard, *Sur les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. **73** (1956), 157–202.
- [Bow95] B. H. Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Math. J. **77** (1995), no. 1, 229–274.
- [BL02] R. Bremigan and J. Lorch, *Orbit duality for flag manifolds*, Manuscripta Math. **109** (2002), no. 2, 233–261.
- [Can08] A. Cano, *Schottky groups can not act on  $\mathbf{P}_{\mathbf{C}}^{2n}$  as subgroups of  $\mathrm{PSL}_{2n+1}(\mathbf{C})$* , Bull. Braz. Math. Soc. (N.S.) **39** (2008), no. 4, 573–586.
- [CNS13] A. Cano, J. P. Navarrete, and J. Seade, *Complex Kleinian groups*, Progress in Mathematics, vol. 303, Birkhäuser/Springer Basel AG, Basel, 2013.
- [Gol99] W. M. Goldman, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1999, Oxford Science Publications.
- [Gui07] A. Guillot, *Sur les équations d'Halphen et les actions de  $\mathrm{SL}_2(\mathbf{C})$* , Publ. Math. Inst. Hautes Études Sci. No. **105** (2007), 221–294.
- [KLP16] M. Kapovich, B. Leeb and J. Porti, *Some recent results on Anosov representations*, Transform. Groups **21** (2016), no. 4, 1105–1121.
- [Ka82] Ma. Kato, *Examples of Simply Connected Compact Complex 3-folds*, Tokyo J. Math. **5** (1982), no. 2, 341–264.
- [Ka85] Ma. Kato, *On compact complex 3-folds with lines*, Tokyo J. Math. **11** (1985), no. 1, 1–58.
- [Ka89] Ma. Kato, *Factorization of compact complex 3-folds which admit certain projective structures*, Tôhoku Math. J. **41** (1989), 359–397.
- [Ka10] M. Kato, *Compact quotients with positive algebraic dimensions of large domains in a complex projective 3-space*, J. Math. Soc. Japan **62** (2010), no. 4, 1317–1371.
- [Koe10] P. Koebe, *Über die Uniformisierung der algebraischen Kurven. II*, Math. Ann. **69** (1910), no. 1, 1–81.
- [Lár98] F. Lárusson, *Compact quotients of large domains in complex projective space*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 223–246.
- [Li06] G. Link, *Geometry and dynamics of discrete isometry groups of higher rank symmetric spaces*, Geom. Dedicata **122** (2006), 51–75.
- [Ma67] B. Maskit, *A characterization of Schottky groups*, J. Analyse Math. **19** (1967), 227–230.
- [MO16] C. Miebach and K. Oeljeklaus, *Schottky groups acting on homogeneous rational manifolds*, to appear in J. Reine Angew. Math., ahead of print <https://doi.org/10.1515/crelle-2016-0065>
- [Nor86] M. V. Nori, *The Schottky groups in higher dimensions*, The Lefschetz centennial conference, Part I (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1986, pp. 195–197.
- [SV03] J. Seade and A. Verjovsky, *Complex Schottky groups*, Astérisque (2003), no. 287, xx, 251–272, Geometric methods in dynamics. II.

UNIV. LITTORAL CÔTE D'OPALE, EA 2797 – LMPA – LABORATOIRE DE MATHÉMATIQUES  
PURES ET APPLIQUÉES JOSEPH LIOUVILLE, F-62228 CALAIS, FRANCE

*E-mail address:* christian.miebach@univ-littoral.fr

AIX-MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, CMI, 39, RUE F.  
JOLIOT-CURIE, 13453 MARSEILLE CEDEX 13, FRANCE

*E-mail address:* karl.oeljeklaus@univ-amu.fr