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► **To cite this version:**

Sylvain Rousseau, David Helbert. A Theoretic Approach of Classical Operations in Clifford Algebras. [Research Report] Université de Poitiers (France). 2012. hal-02082136

HAL Id: hal-02082136

<https://hal.science/hal-02082136>

Submitted on 28 Mar 2019

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A Theoretic Approach of Classical Operations in Clifford Algebra

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2012

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Chapter 1

Introduction

Clifford algebra concepts have a limited theoretical interest. Their properties are presented in the literature of applied mathematics by considering their existence for granted.

We find in the mathematical literature a rigorous definition of Clifford algebras but that omits to define certain concepts that have a limited theoretical interest. Similarly, works of applied mathematics that present the Clifford algebras make a list of their properties by considering their existence for granted. This report presents a definition of Clifford algebra with other interpretations and demonstrations based on [1], [2], [6] and [3] with practical notions used in [5] and [4].

Chapter 2

Inner Product

2.1 Introduction

We will define the inner product due to Hestenes [5] from an asymmetric product called contraction which is useful in computer science [4].

We first present a definition of Clifford algebra and the universal property and secondly the Clifford algebra dimension. We then present \dagger and α operators to finally propose a definition of the geometric algebra.

2.2 Definition and universal property

In this paper, V is a vector space on a field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , Q is a quadratic form on V and B its polar form:

$$B(x, y) = \frac{Q(x + y) - Q(x) - Q(y)}{2}, \quad \forall x, y \in V.$$

The tensor algebra of V noted $\otimes(V)$ is the unitary graded algebra:

$$\mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

The product is induced by the tensor product and the identity element is noted 1_{\otimes} . The linear subspace of elements of degree h is noted $\otimes^h(V)$.

Definition 1. A Clifford algebra on V with respect to Q , noted $C(V, Q)$ is the quotient of tensor algebra $\otimes(V)$ by the two-sided ideal I_Q spanned by elements of the form:

$$x \otimes x - Q(x), \quad x \in V \subset \otimes(V).$$

The canonical projection of $\otimes(V)$ onto the Clifford algebra $C(V, Q)$ is called i_Q . We set $\bar{x} = i_Q(x)$ for $x \in \otimes(V)$. The quotient $C(V, Q)$ is an associative unitary algebra which contains the identity element $\bar{1}_{\otimes}$. As V generates $\otimes(V)$, $i_Q(V)$ generates $C(V, Q)$. Using the definition, we have the fundamental relationship:

$$\bar{x}^2 = Q(x) \quad \forall x \in V, \tag{2.1}$$

By developing the expression: $(\bar{x} + \bar{y})^2 = Q(x + y)$ with $x, y \in V$:

$$\bar{x}\bar{y} + \bar{y}\bar{x} = 2B(x, y). \tag{2.2}$$

This relationship allows us to characterize orthogonal vectors.

Property 1 (Universal property). Let A be a \mathbb{K} -algebra and $f : V \rightarrow A$ a morphism of \mathbb{K} -vector space such that:

$$\forall x \in V, \quad f(x)^2 = Q(x)1.$$

Then there exists a unique g such that the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ i_Q \downarrow & \nearrow g & \\ C(V, Q) & & \end{array} \quad (2.3)$$

2.3 Dimension of $C(V, Q)$

Lemma 1. If u_1, \dots, u_n span the vector space V , then

$$\overline{1}_{\otimes}, \quad \overline{u_{i_1} u_{i_2} \cdots u_{i_r}}, \quad \text{where } 1 \leq i_1 < i_2 < \cdots < i_r \leq n \text{ and } 1 \leq r \leq n,$$

span the vector space $C(V, Q)$.

Proof. Let u_1, \dots, u_n be vectors generating the vector space V . All elements of $C(V, Q)$ are also linear combination of monomial in the \overline{u}_i , $1 \leq i \leq n$, because $i_Q(V)$ spans $C(V, Q)$. On the other hand,

$$\overline{u}_i \overline{u}_j + \overline{u}_j \overline{u}_i = 2B(u_i, u_j), \quad (2.4)$$

$$(\overline{u}_i)^2 = Q(u_i). \quad (2.5)$$

If we reorder these monomials using the relationship (2.4) then lowering the degree with (2.5), we can assume that all element of $C(V, Q)$ is a linear combination of elements of the form:

$$\overline{1}_{\otimes}, \quad \overline{u_{i_1} u_{i_2} \cdots u_{i_r}}, \quad \text{where } 1 \leq i_1 < i_2 < \cdots < i_r \leq n \text{ and } 1 \leq r \leq n.$$

□

Definition 2. Let F and G be two graded algebras. The tensor product $F \times G$ can be endowed with an associative algebra structure by defining:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{p_2 q_1} a_1 b_1 \otimes a_2 b_2,$$

where a_2 is of degree p_2 and b_1 is of degree q_1 and then extend to $F \times G$ by linearity.

Lemma 1. Let Q and Q' be two quadratic forms on vector spaces of finite dimension V and V' respectively. Then there exists a surjective morphism of algebra of $C(V \oplus V', Q \oplus Q')$ onto $C(V, Q) \otimes C(V', Q')$.

Proof. Let $u : V \oplus V' \longrightarrow C(V, Q) \otimes C(V', Q')$ be defined by

$$u(x, x') = i_Q(x) \otimes 1 + 1 \otimes i_{Q'}(x'),$$

for $x \in V$ and $x' \in V'$. We have:

$$\begin{aligned} (u(x, x'))^2 &= (i_Q(x) \otimes 1 + 1 \otimes i_{Q'}(x'))^2 \\ &= (i_Q(x))^2 \otimes 1 + 1 \otimes (i_{Q'}(x'))^2 + i_Q(x) \otimes i_{Q'}(x') - i_{Q'}(x) \otimes i_Q(x) \\ &= Q(x) + Q'(x') = (Q \oplus Q')(x, x'). \end{aligned}$$

Using the universal property, there exists a unique ψ such that:

$$\begin{array}{ccc} V \oplus V' & \xrightarrow{u} & C(V, Q) \otimes C(V', Q') \\ \downarrow i_{Q \oplus Q'} & \dashrightarrow \psi & \\ C(V \oplus V', Q \oplus Q') & & \end{array}$$

is commutative. On the other hand, as $i_Q(V)$ spans $C(V, Q)$ and $i_{Q'}(V')$ spans $C(V', Q')$, $C(V, Q) \otimes C(V', Q')$ is spanned by elements of form $1 \otimes i_{Q'}(x') = u(0, x')$ and $i_Q(x) \otimes 1 = u(x, 0)$ for $x \in V$ and $x' \in V'$. The map ψ is then surjective. \square

Theorem 1. *We have the following structure theorem:*

i) $\dim C(V, Q) = 2^n$ and for all basis (u_1, \dots, u_n) of V , the following elements

$$\overline{1}_{\otimes}, \quad \overline{u}_{i_1} \overline{u}_{i_2} \cdots \overline{u}_{i_r}, \quad \text{where } 1 \leq i_1 < i_2 < \cdots < i_r \leq n \text{ and } 1 \leq r \leq n, \quad (2.6)$$

form a basis for $C(V, Q)$.

ii) The morphism $i_Q : V \longrightarrow C(V, Q)$ is injective.

2.4 † and α operators

2.4.1 α operator

Property 2. *There exists a unique endomorphism of algebra of $C(V, Q)$ denoted α such that:*

$$\alpha(v) = -v, \quad \forall v \in V$$

This endomorphism is an involution called the main involution of the Clifford algebra $C(V, Q)$.

We note the set of even elements $C^+(V, Q) = \{x \in C(V, Q), \alpha(x) = x\}$ and the set of odd elements $C^-(V, Q) = \{x \in C(V, Q), \alpha(x) = -x\}$. We have $C(V, Q) = C^+(V, Q) \oplus C^-(V, Q)$ and $C^+(V, Q)$ is a subalgebra of $C(V, Q)$.

2.4.2 † operator

Property 3. *There exists a unique anti-automorphism of $C(V, Q)$, noted † such that:*

$$v^\dagger = v \quad \forall v \in V$$

This anti-automorphism is an involution.

2.5 Geometric Algebra

2.5.1 Canonical Isomorphism between $\wedge(V)$ and $C(V, Q)$

Vector subspace representation using the exterior product in the Grassmann algebra allows for example to express the belonging relation to a point in a space or the character of a linearly dependent vector subset by simple algebraic relations. We would like to equip a Clifford algebra with such a product. It is sufficient to exhibit an isomorphism between the two spaces and transport the product of an algebra on the other. The question is which isomorphism to choose so that both products perform well with respect to the other. A solution would be to choose a Q -orthogonal base (e_1, \dots, e_n) of V and to send the element $e_1 \wedge \dots \wedge e_n$ on $e_1 \dots e_n$. We can then prove that an isomorphism between vector spaces is defined. We consider here an independent definition of any choice of a base inspired by [1] and [3].

Definition 3. *Let $x \in V$, we define a creation operator for all $a \in \otimes(V)$ by $e_x(a) = x \otimes a$.*

Definition 4. *Let h and j be integers > 0 such that $j \leq h$, (e_1, \dots, e_n) a base of V and φ an element of V^* . We define the contraction operator C_φ^j of $\otimes^h(V)$ in $\otimes^{h-1}(V)$ by:*

$$C_\varphi^j(e_{i_1} \otimes \dots \otimes e_{i_n}) = \varphi(e_{i_j})(e_{i_1} \otimes \dots \otimes \widehat{e_{i_j}} \otimes \dots \otimes e_{i_n}).$$

Property 4. *Let $\varphi \in E^*$, there exists a unique vector space morphism $i_\varphi : \otimes(V) \rightarrow \otimes(V)$ such that:*

$$i_\varphi(1) = 0, \tag{2.7}$$

$$\forall x \in V, \quad i_\varphi \circ e_x + e_x \circ i_\varphi = \varphi(x) \text{Id}. \tag{2.8}$$

Proof. i_φ is already defined on \mathbb{K} . It is defined on $\otimes^i(V)$ by induction. □

Property 5. *We have $i_\varphi = \sum_{j=1}^h (-1)^{j+1} C_\varphi^j$ on $\otimes^h(V)$.*

Proof. The proof can be found by induction on h . By linearity, it is sufficient to prove the equality for a term $x \otimes a$ with $x \in V$ and $a \in \otimes^h(V)$. □

Property 6. *Let $\varphi, \psi \in V^*$, we have the following equalities:*

$$(i_\varphi)^2 = 0 \quad \text{and} \quad i_\varphi \circ i_\psi + i_\psi \circ i_\varphi = 0.$$

Proof. We first remark that $(i_\varphi)^2$ and e_x commute using (2.8). We then prove by induction on h that $(i_\varphi)^2 = 0$ on $\otimes^h(V)$. On the other hand,

$$\begin{aligned} i_\varphi \circ i_\psi + i_\psi \circ i_\varphi &= (i_{\varphi+\psi})^2 - (i_\varphi)^2 - (i_\psi)^2 \\ &= 0 \end{aligned}$$

□

Property 7. Let $a \in \otimes^h(V)$ and $b \in \otimes(V)$, we have the following equality:

$$i_\varphi(a \otimes b) = i_\varphi(a) \otimes b + (-1)^h a \otimes i_\varphi(b). \quad (2.9)$$

Proof. By linearity, we can suppose that $b \in \otimes^k(V)$. We have successively:

$$\begin{aligned} i_\varphi(a \otimes b) &= \sum_{j=1}^{h+k} (-1)^{j-1} C_\varphi^j(a \otimes b) \\ &= \left(\sum_{j=1}^h (-1)^{j-1} C_\varphi^j(a) \right) \otimes b + a \otimes \left(\sum_{j=h+1}^{h+k} (-1)^{j-1} C_\varphi^{j-h}(b) \right) \\ &= i_\varphi(a) \otimes b + (-1)^h a \otimes i_\varphi(b). \end{aligned}$$

□

Property 8. The map i_φ descends to $\wedge(V)$ by:

$$i_\varphi(x_1 \wedge x_2 \wedge \cdots \wedge x_h) = \sum_{j=1}^h (-1)^{j+1} \varphi(x_j) (x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_h).$$

Proof. It is sufficient to prove that ideal I is stable by i_φ . By linearity, it is sufficient to prove that for all $a \in \otimes^h(V)$, $x \in V$ and $b \in \otimes(V)$, we have: $i_\varphi(a \otimes x \otimes x \otimes b) \in I$. By applying (2.9) twice and given that $i_\varphi(x \otimes x) = 0$, we find:

$$i_\varphi(a \otimes x \otimes x \otimes b) = i_\varphi(a) \otimes x \otimes x \otimes b + (-1)^h a \otimes x \otimes x \otimes i_\varphi(b) \in I.$$

□

Lemma 2. Let F be a bilinear form on V , there exists a unique linear endomorphism λ_F on $\otimes(V)$ such that:

$$\lambda_F(1) = 1, \quad (2.10)$$

$$\lambda_F \circ e_x = (e_x + i_x^F) \circ \lambda_F, \quad (2.11)$$

where $i_x^F = i_{x^*}$ and $x^*(y) = F(x, y)$.

Proof. λ_F is defined on \mathbb{K} by (2.10). Moreover, for all $x \in V$ and $a \in \otimes^h(V)$, we necessarily have:

$$\lambda_F(x \otimes a) = (e_x + i_x^F(x)) \circ \lambda_F(a) = x \otimes \lambda_F(a) + i_x^F(\lambda_F(a)). \quad (2.12)$$

This defines λ_F on $\otimes^{h+1}(V)$. \square

Lemma 3. *For all $\varphi \in E^*$, λ_F and i_φ commute.*

Proof. By linearity, it is sufficient to prove that λ_F and i_φ commute on $\otimes^h(V) \forall h \geq 0$ and by induction on h . It is obvious for $h = 0$ because $\lambda_F(1) = 1$. Let $h \geq 0$, $a \in \otimes^h(V)$ and $x \in V$, we have:

$$\begin{aligned} \lambda_F(i_\varphi(x \otimes a)) &= \lambda_F(i_\varphi(x) \otimes a - x \otimes i_\varphi(a)) \\ &= \varphi(x)\lambda_F(a) - (e_x + i_x^F)(i_\varphi(\lambda_F(a))) \\ &= i_\varphi(\lambda_F(x \otimes a)). \end{aligned}$$

\square

Lemma 4. *Let F and F' be two bilinear form on E , we have the relationship $\lambda_F \circ \lambda_{F'} = \lambda_{F+F'}$. In particular, λ_F is an isomorphism of $\otimes(V)$.*

Proof. By linearity, it is sufficient to prove that $\lambda_F \circ \lambda_{F'} = \lambda_{F+F'}$ on $\otimes^h(V)$ for all $h \geq 0$. We prove this by induction on h .

The case $h = 0$ is obvious because $\lambda_F(1) = \lambda_{F'}(1) = 1$. Let $h \geq 0$, $a \in \otimes^h(V)$ and $x \in V$, we have:

$$\begin{aligned} \lambda_F(\lambda_{F'}(x \otimes a)) &= \lambda_F(\lambda_{F'}(e_x(a))) \\ &= (e_x + i_x^F)(\lambda_F(\lambda_{F'}(a))) + i_x^{F'}(\lambda_F(\lambda_{F'}(a))) \\ &= \lambda_{F+F'}(x \otimes a), \end{aligned}$$

This proves the relationship on $\otimes^{h+1}(V)$.

In particular $\lambda_F \circ \lambda_{-F} = \lambda_0$ and it is not difficult to show that $\lambda_0 = \text{Id}$. Hence $\lambda_F \circ \lambda_{-F} = \lambda_{-F} \circ \lambda_F = \text{Id}$ and λ_F is an isomorphism of $\otimes(V)$. \square

Property 9. *Let I_Q be the two-sided ideal of $\otimes(V)$ spanned by $x \otimes x - Q(x)$ for $x \in V$. We set $I = I_0$. The λ_B operator sends the ideal I_Q in the ideal I and defines an isomorphism of vector space between $C(V, Q)$ and $\wedge(V)$.*

Proof. It is sufficient to prove that $\lambda_B(I_Q) \subset I$. Indeed the induced morphism will be surjective and $C(V, Q)$ and $\wedge(V)$ have the same dimension. This is done by induction. \square

We obtain by passing to the quotient in (2.11) the relationship $\lambda_B(xu) = x \wedge \lambda_B(u) + i_{x^*}(\lambda_B(u))$. Then by identifying an element of $C(V, Q)$ and its image by λ_B , we find:

$$xu = x \wedge u + i_{x^*}(u). \quad (2.13)$$

An exterior product on $C(V, Q)$ compatible with the geometric product in the sense of (2.13) is then defined.

Remark 1. By applying the \dagger operator to the relationship (2.13) and by assuming that u is a k -vector, we can establish the similar equality:

$$ux = u \wedge x - (-1)^k i_{x^*}(u). \quad (2.14)$$

2.5.2 Definition of an inner product.

We consider the extension of B on the whole Clifford algebra $C(V, Q)$. We first observe that for all $x, y \in V$, $i_{x^*}(y) = B(x, y)$. The map i_{x^*} extends the scalar product to $V \times C(V, Q)$. It exists many possible extensions: the left contraction is an asymmetric product which is useful in computer science[4]. We will define the inner product due to Hestenes [5] from the contraction rather than from property (19).

Definition 5. Let $k > 0$, $x = v_1 \wedge \dots \wedge v_k$ be a decomposable k -vector and $y \in C(V, Q)$. We fix $l_x = i_{v_1^*} \circ \dots \circ i_{v_k^*}$. We call left contraction of x and y and we note it $x \rfloor y$ the element $l_x(y)$. We then extend this contraction by linearity by fixing $\lambda \rfloor y = \lambda y$.

Definition 6. Let x be a r -vector and y a s -vector. We define the inner product of x and y and we note it $x \cdot y$ the element:

$$x \cdot y = \begin{cases} 0 & \text{if } r = 0 \text{ or } s = 0 \\ x \rfloor y & \text{if } r \leq s \\ (-1)^{s(r-1)} y \rfloor x & \text{if } r > s. \end{cases}$$

We then extend this product by linearity.

Remark 2. In fact, if $r = s$ we have $x \rfloor y = y \rfloor x$. The last case in previous definition need not be strict.

Property 10. Let x be a r -vector, y a s -vector, z a t -vector. We have:

- i) $x \cdot y = (-1)^{r(s-1)} y \cdot x$ for $r \leq s$.
- ii) $x \cdot (y \cdot z) = (x \wedge y) \cdot z$ for $r + s \leq t$ and $r, s > 0$.
- iii) $(x \cdot y) \cdot z = x \cdot (y \wedge z)$ for $s + t \leq r$ and $s, t > 0$.
- iv) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for $r + t \leq s$.

Proof. i). It comes directly from the definition.

ii). By inner product bilinearity, we can suppose that x and y are decomposable, i-e $x = x_1 \wedge \dots \wedge x_r$ and $y = y_1 \wedge \dots \wedge y_s$. By using the notations of the definition 5, as $r \leq t$ and $s \leq t$, we have:

$$\begin{aligned} x \cdot (y \cdot z) &= x \rfloor (y \rfloor z) \quad \text{because } r, s > 0 \\ &= l_x(l_y(z)) = l_{x \wedge y}(z) = (x \wedge y) \cdot z \quad \text{because } r + s \leq t. \end{aligned}$$

- iii). It comes from (i) by exchanging the roles of x and z in (ii).
iv). For $r = 0$ or $t = 0$, the relationship is obvious. We thus suppose $r, t > 0$. By using (i) and (ii), we successively have:

$$\begin{aligned}
x \cdot (y \cdot z) &= (-1)^{s(t-1)} x \cdot (z \cdot y) = (-1)^{t(s-1)} (x \wedge z) \cdot y \\
&= (-1)^{t(s-1)} (-1)^{rt} (z \wedge x) \cdot y = (-1)^{t(s-1)} (-1)^{rt} z \cdot (x \cdot y) \\
&= (-1)^{t(s-1)} (-1)^{rt} (-1)^{t(s-r-1)} (x \cdot y) \cdot z = (x \cdot y) \cdot z.
\end{aligned}$$

□

2.5.3 Geometric Algebra

The Clifford algebra is now equipped with an exterior product. We can use concepts introduced in Grassmann algebras. We say that an element of $C(V, Q)$ is *decomposable* if it can be written as the exterior product of any number of element of V . We call the set of k -vectors the set $C^k(V, Q) = \text{Vect} \{x_1 \wedge \cdots \wedge x_k, x_1, \dots, x_k \in V\}$. By convention $C^0(V, Q) = \mathbb{K}$. The $C^k(V, Q)$ spaces are in direct sum $C(V, Q) = \bigoplus_{i=0}^n C^i(V, Q)$ and we have the equality $\dim C^r(V, Q) = \binom{n}{k}$.

We note $\langle \cdot \rangle_i$ the projection of $C(V, Q)$ onto $C^i(V, Q)$, so that:

$$x = \sum_{i=1}^n \langle x \rangle_i. \quad (2.15)$$

We call *degree* of element x of $C(V, Q)$ the integer $\max_{0 \leq i \leq n} \{0\} \cup \{i, \langle x \rangle_i \neq 0\}$. A non-null k -vector is this of degree k .

We now establish two useful lemmas relying vector product and geometric product.

Lemma 2. *If x_1, \dots, x_r is an orthogonal family of element of V , then*

$$x_1 \cdots x_r = x_1 \wedge \cdots \wedge x_r.$$

Proof. By induction on r . The case $r = 1$ is obvious. We suppose $x_1 \cdots x_r = x_1 \wedge \cdots \wedge x_r$ for all $1 \leq r < n$. Let $x_{r+1} \in V$ such that the family (x_1, \dots, x_{r+1}) is orthogonal. We fix $u = x_2 \cdots x_{r+1}$. According to (2.13) we have $x_1 \cdots x_{r+1} = x_1 u = x_1 \wedge u + i_{x^*}(u)$. Yet $i_{x^*}(u) = \sum_{j=2}^{r+1} (-1)^j B(x_1, x_j) (x_2 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{r+1}) = 0$, because $B(x_1, x_j) = 0$ for all $j = 2, \dots, r+1$. We thus have the equality $x_1 \cdots x_r = x_1 \wedge \cdots \wedge x_r$. □

Lemma 3. *Let F be a vector subspace of V , (v_1, \dots, v_r) a base of F and (u_1, \dots, u_r) an orthogonal base of F . If A denotes the transfer matrix of the base (u_1, \dots, u_r) to the base (v_1, \dots, v_r) , we have the equality:*

$$v_1 \wedge \cdots \wedge v_r = \det(A) u_1 \cdots u_r. \quad (2.16)$$

Proof. We have $v_i = \sum_{k=1}^r a_{ki} u_k$, $\forall 1 \leq i \leq r$. We can then write:

$$\begin{aligned} v_1 \wedge \cdots \wedge v_r &= \left(\sum_{i=1}^r a_{i1} u_i \right) \wedge \cdots \wedge \left(\sum_{i=1}^r a_{ir} u_i \right) \\ &= \sum_{\sigma \in \mathcal{S}_r} a_{\sigma(1)1} \cdots a_{\sigma(r)r} u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(r)} \\ &= \det(A) u_1 \wedge \cdots \wedge u_r = \det(A) u_1 \cdots u_r, \end{aligned}$$

□

Property 11. Let v_1, \dots, v_r be vectors, x a k -vector and $y \in C(V, Q)$. Also:

$$i) (v_1 \wedge \cdots \wedge v_r)^\dagger = v_r \wedge \cdots \wedge v_1 = (-1)^{\frac{r(r-1)}{2}} (v_1 \wedge \cdots \wedge v_r).$$

$$ii) x^\dagger = (-1)^{\frac{k(k-1)}{2}} x.$$

$$iii) y^\dagger = \sum_{i=0}^n (-1)^{\frac{i(i-1)}{2}} \langle y \rangle_i.$$

Property 12. Let x be k -vector and $y, z \in C(V, Q)$. We have the following equalities:

$$i) \alpha(x) = (-1)^k x.$$

$$ii) \alpha(y \wedge z) = \alpha(y) \wedge \alpha(z).$$

$$iii) \alpha(y) = \sum_{i=0}^n (-1)^i \langle y \rangle_i.$$

$$iv) C^+(V, Q) = \bigoplus_{\substack{0 \leq i \leq n \\ i \text{ pair}}} C^i(V, Q).$$

$$v) C^-(V, Q) = \bigoplus_{\substack{0 \leq i \leq n \\ i \text{ impair}}} C^i(V, Q).$$

Lemma 5. Let $x \in V$ and $\wedge_x : C(V, Q) \rightarrow C(V, Q)$ such that $\wedge_x(u) = x \wedge u$ for all $u \in C(V, Q)$. Then:

$$\langle \cdot \rangle_{i+1} \circ \wedge_x = \wedge_x \circ \langle \cdot \rangle_i \quad \forall i \text{ such that } 0 \leq i \leq n-1. \quad (2.17)$$

Proof. $\langle \cdot \rangle_i$ and \wedge_x are indeed linear operators, it is sufficient to check (2.17) on each space $C^k(V, Q)$ where $0 \leq k \leq n$. Let $u \in C^k(V, Q)$ and i such as $0 \leq i \leq n-1$.

If $k \neq i$, $\langle \cdot \rangle_{i+1} \circ \wedge_x(u) = \langle x \wedge u \rangle_{i+1} = 0$ and $\wedge_x \circ \langle \cdot \rangle_i(u) = \wedge_x(\langle u \rangle_i) = 0$.

If $k = r$, $\langle \cdot \rangle_{i+1} \circ \wedge_x(u) = \langle x \wedge u \rangle_{i+1} = x \wedge u$ and $\wedge_x \circ \langle \cdot \rangle_i(u) = \wedge_x(\langle u \rangle_i) = x \wedge u$. □

Property 13. Let x be a r -vector and y a s -vector. If $r \neq 0$ or $s \neq 0$, we have $x \wedge y = \langle xy \rangle_{r+s}$.

Proof. If $r = 0$ or $s = 0$, the equality is obvious. We now suppose $r, s > 0$. We proceed by induction on $n = r + s$. The case $r = s = 1$ for $n = 2$ is obvious. We suppose $n > 2$ and $r + s = n$. By linearity, we can suppose that x is decomposable, we thus have $x = x_1 \wedge x'$ and then:

$$\begin{aligned}
x \wedge y &= x_1 \wedge x' \wedge y \\
&= x_1 \wedge \langle x' y \rangle_{r+s-1}, \quad \text{by recurrence hypothesis,} \\
&= \wedge_{x_1} \circ \langle \cdot \rangle_{r+s-1}(x' y) = \langle \cdot \rangle_{r+s} \circ \wedge_{x_1}(x' y), \quad \text{according lemma 5,} \\
&= \langle x_1 \wedge (x' y) \rangle_{r+s} = \langle x_1 x' y - i_{x_1^*}(x' y) \rangle_{r+s} = \langle x_1 x' y \rangle_{r+s} \\
&= \langle (x_1 \wedge x') y + i_{x_1^*}(x' y) \rangle_{r+s} = \langle x y \rangle_{r+s},
\end{aligned}$$

Hence the relationship at rank n . □

Property 14. *Let x be a r -vector and y a s -vector. If $r \neq 0$ and $s \neq 0$, we have $x \cdot y = \langle xy \rangle_{|r-s|}$.*

Proof. By linearity, we can suppose that x and y are decomposable. We use induction on r .

1. If $r = 1$, by using (2.13) we have:

$$\begin{aligned}
x \cdot y &= i_{x^*}(y) = i_{x^*}(y) + \langle x \wedge y \rangle_{|s-1|} = \langle i_{x^*}(y) \rangle_{s-1} + \langle x \wedge y \rangle_{|s-1|} \\
&= \langle i_{x^*}(y) + x \wedge y \rangle_{|s-1|} = \langle xy \rangle_{|s-1|}.
\end{aligned}$$

2. We suppose $r > 1$, there are two cases:

- a) If $r \leq s$, we fix $x = x_1 \wedge \cdots \wedge x_r$ with orthogonal vectors x_1, \dots, x_r . By using (2.13), we have:

$$\begin{aligned}
x \cdot y &= (x_1 \wedge \cdots \wedge x_r) \cdot y = (x_1 \wedge \cdots \wedge x_{r-1}) \cdot (x_r \cdot y) \\
&= (x_1 \cdots x_{r-1}) \cdot (x_r \cdot y) = \langle (x_1 \cdots x_{r-1})(x_r \cdot y) \rangle_{|r-1-(s-1)|} \\
&= \langle (x_1 \cdots x_{r-1})(x_r y - x_r \wedge y) \rangle_{|r-s|} \\
&= \langle x_1 \cdots x_r y \rangle_{|r-s|} - \langle (x_1 \wedge \cdots \wedge x_r \wedge y) \rangle_{|r-s|} = \langle xy \rangle_{|r-s|}.
\end{aligned}$$

- b) If $r \geq s$, we fix $y = y_1 \wedge \cdots \wedge y_s$ with orthogonal vectors y_1, \dots, y_s . By using (2.14), we have:

$$\begin{aligned}
x \cdot y &= x \cdot (y_1 \wedge \cdots \wedge y_s) = (x \cdot y_1) \cdot (y_2 \wedge \cdots \wedge y_s) \\
&= \langle (x \cdot y_1) y_2 \cdots y_s \rangle_{|r-1-(s-1)|} = \langle (x y_1 - x \wedge y_1) y_2 \cdots y_s \rangle_{|r-s|} \\
&= \langle x y_1 \cdots y_s \rangle_{|r-s|} - \langle x \wedge y_1 \wedge \cdots \wedge y_s \rangle_{|r-s|} = \langle xy \rangle_{|r-s|},
\end{aligned}$$

Hence the relationship at rank r . □

2.6 Conclusion

We have presented a definition of the Clifford Algebra with a theoretical viewpoint. Defined products will then allow us to write linear transformations of the space V . We had to be interested in the case where $\mathbb{K} = \mathbb{R}$.

Chapter 3

Geometric Transformations

3.1 Introduction

We are interested in the case where $\mathbb{K} = \mathbb{R}$. We assume that the quadratic form Q is non-degenerated and has the signature (p, q) . For simplicity we also note $\mathbb{R}_{p,q}$ the Clifford algebra $C(V, Q)$ and $\mathbb{R}_{p,q}^k$ the k -vectors of this algebra. 0-vectors, 1-vectors, 2-vectors and 3-vectors are also called respectively scalars, vectors, bivectors and trivectors. We will introduce the concept of pseudo-vectors and the operation of duality that allows to algebraize the transition to the orthogonal of a subspace of V . Products defined in the part I will then allow us to write linear transformations of the space V .

In a first time we define the invertibility then the duality. We secondly present the orthogonal projection onto a subspace, the orthogonal symmetry and the rotation.

3.2 Inversibility

An interest of Clifford algebras is that you can take the inverse of a non-isotropic vector. Indeed, let v be a non-isotropic vector. We have $v^2 = Q(v)$ then $v \frac{v}{Q(v)} = \frac{v}{Q(v)} v = 1$ and $v^{-1} = \frac{v}{Q(v)}$.

You can also notice that every k -decomposable nonzero vector u being a regular vector subspace is invertible. Such a vector can be written $\lambda v_1 \cdots v_k$ where the v_i anticommute, $Q(v_i) = \epsilon_i$, $\epsilon_i = \pm 1$ and $\lambda \in \mathbb{R}^*$. We have:

$$uu^\dagger = u^\dagger u = \lambda^2 Q(v_1) \cdots Q(v_k) = \lambda^2 \epsilon,$$

with $\epsilon = \epsilon_1 \cdots \epsilon_k$ and thus

$$u^{-1} = \frac{\epsilon}{\lambda^2} u^\dagger.$$

3.3 Duality

We choose an orthonormal base (e_1, \dots, e_n) of V . We call pseudo-vector relative to the base (e_1, \dots, e_n) the element:

$$I = e_1 \cdots e_n.$$

We have the following relationships:

$$I^{-1} = I^\dagger = (-1)^{\frac{n(n-1)}{2}} I,$$

and if u is k -vector,

$$uI = (-1)^{k(n-1)}Iu.$$

Definition 7. We call dual of an element $u \in C(V, Q)$ the element \tilde{u} such that:

$$\tilde{u} = uI^{-1}.$$

Remark 3. We remark that we have $\tilde{u} = u \cdot I^{-1}$ for all non scalar u . In fact the pseudo-vector can be defined with the inner product which in this case, is a left contraction.

Lemma 1. Let E be a subspace of V and E_0 its kernel. Let (e_1, \dots, e_p) an orthogonal base of E_0 and U a space such that $U \perp E_0 = E$. There exists f_1, \dots, f_p such that $P_i = \mathbb{R}e_i + \mathbb{R}f_i$ is an hyperbolic plane and such that:

$$F' = P_1 \perp \dots \perp P_p \perp U,$$

is an orthogonal sum in E .

Proof. By induction on p . The case $p = 0$ is obvious. We suppose the lemma true at rank $p - 1$, $p \geq 1$. We fix $E_1 = U + \text{Vect}(e_1, \dots, e_{p-1})$. We have $E_1 \subsetneq E$. Let $f_p \in E_1^\perp / E^\perp$, we then have $f_p \cdot e_p \neq 0$ and $e_1 \cdot f_p = \dots = e_{p-1} \cdot f_p = 0$. By replacing f_p by $f_p + \lambda e_p$ with a well chosen λ we can suppose that f_p is isotropic and that $e_p \cdot f_p = 1$. The plane $P_p = \text{Vect}(e_p, f_p)$ is then hyperbolic. We apply the induction hypothesis to the subspace $E_1 \perp P_p$ whose kernel is $\text{Vect}(e_1, \dots, e_{p-1})$. There exists then $p - 1$ element f_1, \dots, f_{p-1} such that:

$$F' = P_1 \perp \dots \perp P_{p-1} \perp P_p \perp U,$$

hence the relationship at rank p . □

Property 15. Let E be a non zero subspace of V and E^\perp its orthogonal. If u represents E then \tilde{u} represents E^\perp .

Proof. i) We suppose E regular. Let e_{n-k+1}, \dots, e_n an orthogonal base of E which is completed in an orthogonal base e_1, \dots, e_n of V . The subspace E is represented by $u = e_{n-k+1} \wedge \dots \wedge e_n = e_{n-k+1} \cdots e_n$. We have then:

$$\begin{aligned} \tilde{u} &= e_{n-k+1} \cdots e_n I^{-1} \\ &= e_{n-k+1} \cdots e_n e_n \cdots e_1 \\ &= e_{n-k} \cdots e_1 \\ &= e_{n-k} \wedge \dots \wedge e_1, \end{aligned}$$

which represents the orthogonal of E .

ii) If E is not regular. Let $E_0 = E \cap E^\perp$, A such that $E = E_0 \oplus A$, (a_1, \dots, a_q) an orthogonal base of A and (e_1, \dots, e_p) an orthogonal base of E_0 . By applying lemma 1, let f_1, \dots, f_p be elements such that $P_i = \text{Vect}(e_i, f_i)$ is an hyperbolic

plane and $E' = P_1 \perp \dots \perp P_p \perp U$ is regular. Let G such that $G \perp E'$ and (g_1, \dots, g_k) a base of G . We remark that the family:

$$g_1, \dots, g_k, \frac{e_1 - f_1}{\sqrt{2}}, \dots, \frac{e_p - f_p}{\sqrt{2}}, \frac{e_1 + f_1}{\sqrt{2}}, \dots, \frac{e_p + f_p}{\sqrt{2}}, a_1, \dots, a_q,$$

is an orthonormal base of V . A pseudo-scalar is thus written:

$$I = g_1 \cdots g_k \cdot \frac{e_1 - f_1}{\sqrt{2}} \cdots \frac{e_p - f_p}{\sqrt{2}} \cdot \frac{e_1 + f_1}{\sqrt{2}} \cdots \frac{e_p + f_p}{\sqrt{2}} \cdot a_1 \cdots a_q,$$

From which:

$$\begin{aligned} I^{-1} &= a_q \cdots a_1 \cdot \frac{e_p + f_p}{\sqrt{2}} \cdots \frac{e_1 + f_1}{\sqrt{2}} \cdot \frac{e_p - f_p}{\sqrt{2}} \cdots \frac{e_1 - f_1}{\sqrt{2}} \cdot g_k \cdots g_1 \\ &= a_q \cdots a_1 \cdot \frac{f_p e_p - e_p f_p}{2} \cdots \frac{f_1 e_1 - e_1 f_1}{2} \cdot g_k \cdots g_1. \end{aligned}$$

If $u = e_1 \cdots e_p \cdot a_1 \cdots a_q$ represents E , we also have:

$$\begin{aligned} \tilde{u} &= u I^{-1} \\ &= e_1 \cdots e_p \cdot a_1 \cdots a_q \cdot a_q \cdots a_1 \cdot \frac{f_p e_p - e_p f_p}{2} \cdots \frac{f_1 e_1 - e_1 f_1}{2} \cdot g_k \cdots g_1 \\ &= e_1 \cdots e_p \cdot \frac{f_p e_p - e_p f_p}{2} \cdots \frac{f_1 e_1 - e_1 f_1}{2} \cdot g_k \cdots g_1 \\ &= e_1 \cdots e_p \cdot \frac{f_p e_p}{2} \cdots \frac{f_1 e_1}{2} \cdot g_k \cdots g_1 \\ &= \frac{e_p f_p e_p}{2} \cdots \frac{e_1 f_1 e_1}{2} \cdot g_k \cdots g_1 \\ &= e_p \cdots e_1 \cdot g_k \cdots g_1, \end{aligned}$$

which represents $E^\perp = E_0 \oplus G$. □

Lemma 6. *Let x be an element of $C(V, Q)$ and d an integer such as $0 \leq d \leq n$. We also have:*

$$\widetilde{\langle x \rangle_d} = \langle \tilde{x} \rangle_{n-d}$$

Proof. By linearity, it is sufficient to check that the duality operator sends a k -vector on a $(n - k)$ -vector. We can convince oneself by taking an orthogonal base (e_1, \dots, e_n) of V . □

Property 16. *Let a be a vector and M a multivector. We have the following relationship of duality:*

$$\widetilde{a \wedge M} = a \cdot \widetilde{M}, \tag{3.1}$$

and

$$\widetilde{a \cdot M} = a \wedge \widetilde{M}. \tag{3.2}$$

Proof. By linearity, we can suppose that M is a k -vector. if $k = n$, the equality is obvious. Otherwise we have successively with lemma 6 and properties 18 and 19 (appendices 3.7).

$$\begin{aligned}\widetilde{a \wedge M} &= \langle \widetilde{aM} \rangle_{k+1} \\ &= \langle \widetilde{aM} \rangle_{n-(k+1)} \\ &= \langle \widetilde{aM} \rangle_{n-k-1} \\ &= a \cdot \widetilde{M}.\end{aligned}$$

The relationship (3.2) is deduced from (3.1) by replacing M by \widetilde{M} and by taking the dual of obtained relationship. \square

3.4 Orthogonal projection onto a subspace

Property 17. Let E be a regular subspace of V of dimension k such as $1 \leq k \leq n$ and let p_E be the orthogonal projection onto E . Let u be a decomposable k -vector which represents E . We have the following relationship:

$$p_E(v) = (v \cdot u)u^{-1}. \quad (3.3)$$

Proof. As all elements representing a given subspace are the same up to a positive factor, it is sufficient to prove (3.3) for example for $u = e_1 \wedge \dots \wedge e_k$ with (e_1, \dots, e_k) an orthonormal base of E because E is regular. We complete with an orthonormal base $\mathcal{B} = (e_1, \dots, e_n)$ of V . Let $v \in V$ and $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ its coordinates in the base \mathcal{B} . We have:

$$v = \sum_{i=1}^n \lambda_i e_i,$$

and

$$p_E(v) = \sum_{i=1}^k \lambda_i e_i.$$

We remark that for $1 \leq i \leq n$:

$$i_{e_i^*}(u) = \begin{cases} e_1 \wedge \dots \wedge \check{e}_i \wedge \dots \wedge e_k & \text{if } 1 \leq i \leq k \\ 0 & \text{if } k < i \leq n \end{cases}$$

By linearity, we thus find:

$$i_{v^*}(u) = \sum_{i=1}^k (-1)^{i-1} \lambda_i e_1 \wedge \dots \wedge \check{e}_i \wedge \dots \wedge e_k.$$

It is sufficient to remark that:

$$e_1 \wedge \dots \wedge \check{e}_i \wedge \dots \wedge e_k = e_1 \cdots \check{e}_i \cdots e_k,$$

and

$$(e_1 \cdots \check{e}_i \cdots e_n)(e_1 \cdots e_n)^{-1} = (-1)^{i-1} e_i,$$

To conclude:

$$p_E(v) = (v \cdot u)u^{-1}.$$

□

3.5 Orthogonal symmetry

Let E be a regular subspace of V and let s_E be the orthogonal symmetry with respect to E . It is connected to the orthogonal projection p_E by the formula:

$$s_E = 2p_E - \text{Id}.$$

We thus find, for all $v \in V$:

$$\begin{aligned} s_E(v) &= 2p_E(v) - v \\ &= 2(v \cdot u)u^{-1} - v \\ &= (2v \cdot u - vu)u^{-1} \\ &= (v \cdot u - v \wedge u)u^{-1}. \end{aligned}$$

Where u is a vector, we have $u \wedge v = -v \wedge u$, we thus find:

$$s_E(v) = uvu^{-1}.$$

Where u represents a hyperplan, we have $u = \tilde{n}$ with n a non-isotropic vector. We also have:

$$\begin{aligned} s_E(v) &= (v \cdot \tilde{n} - v \wedge \tilde{n})\tilde{n}^{-1} \\ &= (\widetilde{v \wedge n} - \widetilde{v \cdot n})\tilde{n}^{-1} \\ &= -\widetilde{nv}\tilde{n}^{-1} \\ &= -nvI^{-1}In^{-1} \\ &= -nvn^{-1}. \end{aligned}$$

In this two cases and up to the sign we obtain a rotation of angle π around an axis and a symmetry with respect to a hyperplane by conjugating the vector v by an invertible element of the Clifford algebra. This motivates the definition of the Clifford group in the next section.

3.6 Rotation

3.6.1 Clifford group

Let G be the set of invertible elements $g \in C(V, Q)$ such that:

$$\varphi_g(x) = gxg^{-1} \in V, \quad \forall x \in V.$$

Then G is a group and $\varphi_g \in \text{GL}(V)$.

Definition 8. The multiplicative group G of invertible elements $g \in C(V, Q)$ such that:

$$g x g^{-1} \in V, \quad \forall x \in V,$$

is called the Clifford group.

The intersection $G \cap C^+(V, Q)$ is called the even Clifford group noted G^+ .

Theorem 2. i) φ is a representation of G in $O(Q)$ whose kernel is the set of invertible elements of the center of $C(V, Q)$.

ii) $V \cap G$ is the set of non-isotropic vectors of V and if $g \in V \cap G$, $-\varphi_g$ is the orthogonal symmetry with respect to the hyperplane g^\perp .

iii) If $\dim V = 2r$, $\varphi(G) = O(Q)$, $\varphi(G^+) = SO(Q)$.

iv) If $\dim V = 2r + 1$, $\varphi(G) = \varphi(G^+) = SO(Q)$.

Proof. (i). The application $\varphi_g = \varphi(g)$ is an isometry because:

$$\begin{aligned} Q(\varphi_g(x)) &= Q(g x g^{-1}) \\ &= (g x g^{-1})^2 \\ &= g x^2 g^{-1} \\ &= Q(x). \end{aligned}$$

The application φ is thus a morphism of G in $O(Q)$. Then let $g \in \ker \varphi$, g is invertible and we have $g x g^{-1} = x$ for all $x \in V$ i.e. $g \in Z \cap G$ because V spans $C(V, Q)$. Reciprocally, let g an invertible element of the center of $C(V, Q)$, we have $\varphi_g = \text{Id}_V$ thus $g \in \ker \varphi$.

(ii). Let $g \in V \cap G$, g be invertible, thus $g^2 = Q(v) \neq 0$. Reciprocally let g be a non-isotropic vector of V . Then g has for invertible $g(Q(v))^{-1}$ and we have:

$$\begin{aligned} \varphi_g(x) &= \frac{g x g}{Q(v)} \\ &= \frac{a(2B(x, a) - a x)}{Q(a)} \\ &= -x + \frac{2B(x, a)}{Q(a)} a. \end{aligned}$$

Thus $g \in V \cap G$ and $-\varphi_g$ is the orthogonal symmetry of hyperplane g^\perp .

(iii). Let $u \in O(Q)$. By the theorem of Cartan-Dieudonné, u is written as reflection product i.e. $u = u_1 \circ \dots \circ u_h$ with $u_i = -\varphi(x_i)$ and x_i is a non-isotropic vector, $1 \leq i \leq h$. Finally $u = (-1)^h \varphi(g)$ with $g = x_1 \cdots x_h$. We deduce $SO(Q) \subset \varphi(G^+)$.

In other hand, as n is even we remark that $\varphi(I) = -\text{Id}_V$ thus if h is odd $u = \varphi(Ig)$. We have thus:

$$\varphi(G) = O(Q).$$

It remains to prove that $\varphi(G^+) \subset SO(Q)$. Suppose by contradiction that $\varphi(t) = u$ with $t \in G^+$ et $u \in O(Q) \setminus SO(Q)$. According to the above, we can also write $u = \varphi(Ig)$.

We have thus $Igt^{-1} \in \ker \varphi$ then $Igt^{-1} \in \mathbb{K}$ because n is even which is a contradiction because $\alpha(Igt^{-1}) = -1$.

(iv). If n is odd, using the same reasoning we find:

$$\mathrm{SO}(Q) \subset \varphi(G^+).$$

Suppose by contradiction that $-\mathrm{Id}_V \in \mathrm{O}(Q) \setminus \mathrm{SO}(Q)$ be an element of $\varphi(G)$. Let $t \in G$ such as $txt^{-1} = -t$ for all $x \in V$. We have $tIt^{-1} = -I$ but $I \in Z$ because n is odd which is a contradiction. We have also $\varphi(G) \subset \mathrm{SO}(Q)$ because $-\mathrm{Id}_V$ and $\mathrm{SO}(Q)$ span $\mathrm{O}(Q)$. Thus we have the following inclusions:

$$\mathrm{SO}(Q) \subset \varphi(G^+) \subset \varphi(G) \subset \mathrm{SO}(Q),$$

From which we conclude:

$$\varphi(G) = \varphi(G^+) = \mathrm{SO}(Q).$$

□

3.6.2 Reduced Clifford group

According to the previous theorem in all cases we have $\varphi(G^+) = \mathrm{SO}(Q)$. We search to reduce G^+ while maintaining the surjectivity of φ .

Lemma 7. *For all $g \in G^+$, $g^\dagger g \in \mathbb{R}^*$ and $N : g \mapsto g^\dagger g$ is a homomorphism of G^+ in \mathbb{R}^* .*

Proof. We prove that G is stable by \dagger operator.

Let $g \in G$ and $x \in V$. Fix $x' = \varphi_g(x)$. We have:

$$x' = gxg^{-1},$$

From which, by applying \dagger operator:

$$x' = (g^\dagger)^{-1}xg^\dagger. \tag{3.4}$$

Also $\varphi_{(g^\dagger)^{-1}}(x) \in V$ thus $(g^\dagger)^{-1} \in G$ from which $g^\dagger \in G$. And moreover:

$$\begin{aligned} \varphi_{g^\dagger g}(x) &= g^\dagger gx(g^\dagger g)^{-1} \\ &= g^\dagger x'(g^\dagger)^{-1} \\ &= x, \quad \text{according to (3.4).} \end{aligned}$$

Also $g^\dagger g$ belongs to the center Z of the Clifford algebra.

If moreover $g \in G^+$ we have $g^\dagger g \in G^+ \cap Z = \mathbb{R}^*$.

It remains to prove that N is an homomorphism. Let $g, h \in G^+$ we have:

$$\begin{aligned} N(gh) &= (gh)^\dagger gh \\ &= h^\dagger N(g)h \\ &= N(g)N(h). \end{aligned}$$

□

Definition 9. *The subgroup of elements $g \in G^+$ such that $|N(g)| = 1$ is called spinor group of Clifford algebra $\mathbb{R}_{p,q}$ and it is noted $\text{Spin}(Q)$.*

Thus we remark that transformations induced by a metric Q on a vector space V can be express as simple algebraic operations in Clifford algebra of V associated to Q .

3.7 Conclusion

We have presented Clifford Algebra with a theoretical viewpoint and their properties and certain concepts that have a theoretical interest for their applications in physics and computer science.

Appendice

The proofs of the property of the wedge product and the one of the inner product can be found in the part I "A Theoretic Approach of Classical Operations in Clifford Algebra (I): the Inner Product".

Property 18. *Let x be a r -vector and y a s -vector. If $r \neq 0$ or $s \neq 0$, we have $x \wedge y = \langle xy \rangle_{r+s}$.*

Property 19. *Let x be a r -vector and y a s -vector. If $r \neq 0$ and $s \neq 0$, we have $x \cdot y = \langle xy \rangle_{|r-s|}$.*

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