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Renormalization of Feynman amplitudes on manifolds by spectral zeta regularization and blow-ups.

Nguyen Viet Dang*
Bin Zhang†

Abstract

Our goal in this paper is to present a generalization of the spectral zeta regularization for general Feynman amplitudes on Riemannian manifolds. Our method uses complex powers of elliptic operators but involves several complex parameters in the spirit of the *analytic renormalization* by Speer, to build mathematical foundations for the renormalization of perturbative interacting quantum field theories. Our main result shows that spectrally regularized Feynman amplitudes admit an analytic continuation as meromorphic germs with linear poles in the sense of the works of Guo–Paycha and the second author. We also give an explicit determination of the affine hyperplanes supporting the poles. Our proof relies on suitable resolution of singularities of products of heat kernels to make them smooth.

As an application of the analytic continuation result, we use a universal projection from meromorphic germs with linear poles on holomorphic germs to construct renormalization maps which subtract singularities of Feynman amplitudes of Euclidean fields. Our renormalization maps are shown to satisfy consistency conditions previously introduced in the work of Nikolov–Todorov–Stora in the case of flat space–times.

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1 Introduction.

Zeta regularization. Let M be a smooth, compact, connected manifold without boundary and P be a symmetric, positive, elliptic pseudodifferential operator on M . Later on, we will specialize to Schrödinger operators of the form $P = -\Delta_g + V$ where $-\Delta_g$ is a Laplace operator and V is a smooth nonnegative potential. But the present discussion applies to any symmetric, positive, elliptic pseudodifferential operator P . Then P admits a discrete spectral resolution [36, Lemma 1.6.3 p. 51] which means there is an increasing sequence of eigenvalues

$$\sigma(P) = \{0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow +\infty\}$$

and corresponding L^2 -basis of eigenfunctions $(e_\lambda)_{\lambda \in \sigma(P)}$ so that $P e_\lambda = \lambda e_\lambda$. In his seminal work [72], Seeley constructed the complex powers $(P^{-s})_{s \in \mathbb{C}}$ of P as a holomorphic family of linear continuous operators acting on suitable scales of Sobolev spaces on the manifold M . In particular for $Re(s) \geq 0$, P^{-s} is bounded in $L^2(M)$. Now let us consider the *spectral zeta function* $\zeta_P(s)$ which is defined as the trace $TR(P^{-s})$ and coincides with the series :

$$\zeta_P(s) = TR(P^{-s}) = \sum_{\lambda \in \sigma(P) \setminus \{0\}} \lambda^{-s}. \tag{1}$$

By Weyl's law on the growth of eigenvalues of P [36, Lemma 1.12.6 p. 113], the operator P^{-s} is trace class and the series $\zeta_P(s) = \sum_{\lambda > 0} \lambda^{-s}$ converges as a holomorphic function in s on the half-plane $Re(s) > \frac{\dim(M)}{\deg(P)}$. Then Seeley showed that $\zeta_P(s)$ admits an **analytic continuation** on the complex plane as a **meromorphic function** [36, Thm 1.12.2 p. 108] with simple poles. In case P is a **differential operator**, $\zeta_P(s)$ is holomorphic at $s = 0$. This result shows one of the first instances of the power of zeta regularization, where we can regularize the divergent series $\sum_{\lambda \in \sigma(P)} 1$ and obtain the value $\zeta_P(0)$ of the spectral zeta function ζ_P at $s = 0$. More importantly, the residues of $\zeta_P(s)$ at its poles can be expressed as multiple of integrals over M of **local invariants** of the operator P [6, p. 299-303] and are intimately related to the heat invariants of P [36, Thm 1.12.2 p. 108].

From zeta regularization to regularized traces. In the same spirit, zeta regularization techniques were also used in global analysis to construct regularized traces for certain algebras of pseudodifferential operators. The above result of Seeley on the analytic continuation of $TR(P^{-s})$ has been generalized to *canonical traces* on pseudodifferential operators by Kontsevich-Vishik [49], to study anomalies of regularized zeta determinants with related works by Lesch [53] among many authors. Then general types of tracial anomalies were discussed in [55, 18, 61], sometimes in relation with quantum field theory, and finally a general notion of trace for *holomorphic families of pseudodifferential operators* appears in the work of Paycha–Scott [62]. An important object underlying all these constructions is the notion of noncommutative residue for any pseudodifferential operator A . This noncommutative residue can be defined by zeta regularization using complex powers of elliptic operators as follows. Choose any symmetric, positive, elliptic differential operator P , then the noncommutative residue of A is defined as the residue at $s = 0$ of the meromorphic continuation of the trace $TR(AP^{-s})$, and is given by a local formula in the symbol of A . In his seminal works, Wodzicki [78, 79] proved that up to some constant factor, this residue is the unique trace on the algebra of pseudodifferential operators. It plays a central role in global analysis and noncommutative geometry. We refer the reader to the monographs [60, 71] for further details on these topics.

Zeta regularization for partition functions. Already in the simple case of spectral zeta functions of the Laplace–Beltrami operator, these regularization methods turn out to be extremely useful to study Euclidean quantum fields on Riemannian manifolds. In the mathematical physics literature, zeta regularization was first applied to quantum field theory on curved spaces by Hawking [43] to give a definition of the partition function of Euclidean QFT. It can also be used to give a mathematical model of the Casimir effect [31]. For topological quantum field theories, following the seminal work of Ray–Singer [66] on analytic torsion, it was soon realized by Schwarz that one can define and calculate the partition function of some abelian BF theories [70] using zeta regularized determinants. Formally, for some flat bundle (E, ∇) over some smooth compact manifold M of dimension d , his formula for the partition function of the BF theory reads

$$\int_{(A,B) \in \Omega^k(M,E) \times \Omega^{n-k-1}(M,E)} \exp\left(-\int_M B \wedge d^\nabla A\right) = \prod_{k=0}^d \det(\Delta^{(k)})^{(-1)^k k/2}$$

where d^∇ is the twisted differential acting on $\Omega^\bullet(M, E)$ and the right hand side is the Ray–Singer analytic torsion of the flat bundle $(E, \nabla) \mapsto M$ which is a topological invariant [56, (10) p. 9]. Then Witten generalized the above work of Schwarz by showing that the perturbative partition function of Chern–Simons theory involved the Ray–Singer analytic torsion and also the eta invariant of Atiyah–Patodi–Singer. Since the formula looks quite complicated, we refer the reader to [56, (12) p. 9]. But the important point is that the formula involves zeta regularized determinants. The main idea underlying the above results is that partition functions are **formally** expressed as functional integrals on some space of fields, these partition functions are then identified with regularized determinants of elliptic operators. For instance, in the case of the Dirichlet action functional $S(\varphi) = \frac{1}{2} \int_M \varphi(-\Delta_g)\varphi dv(x)$ where $-\Delta_g$ is the Laplace–Beltrami operator and dv the Riemannian volume, the partition function Z reads :

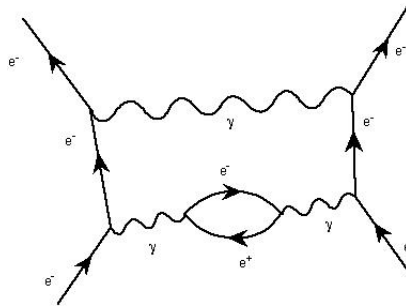
$$Z = \int d\varphi \exp\left(-\frac{1}{2} \int_M \varphi(-\Delta_g)\varphi dv\right) = \det(-\Delta_g)^{-\frac{1}{2}}$$

where $\det(-\Delta_g)$ may be defined as $\exp(-\zeta'(0))$ where ζ is the regularized zeta function of the elliptic operator $(-\Delta_g)$ appearing in the definition of the partition function.

For applications in mathematical physics and in the present work, a particular role will be played by complex powers of generalized Laplacians (more generally elliptic, positive, self-adjoint operators of order 2) and their relation with the heat kernel asymptotics. These methods based on the local asymptotic expansion of the heat kernel are crucial in the local index theory [15] and are also used in the works [8, 4] to give a purely spectral definition of the Einstein–Hilbert action functional following [47, 1, 48].

Another interesting physical property of zeta regularization is its natural covariance which is why it was used in the first place by Hawking. Indeed, for any diffeomorphism $\Phi : M \mapsto M$, the spectrum $\sigma(-\Delta_{\Phi^*g})$ of $-\Delta_{\Phi^*g}$ on (M, Φ^*g) coincides with the spectrum of $-\Delta_g$ on (M, g) and $\sigma(-\Delta_g)$ is thus an **invariant of the Riemannian structure** (M, g) ¹. Therefore zeta regularization is a coordinate independent regularization scheme which depends only on the spectral properties of the Laplacian which in turns is entirely specified by the Riemannian structure (M, g) .

Renormalization in quantum field theory. The present paper is written for analysts and does not require any background in physics or quantum field theory. We present our results in a purely mathematical form. However, we felt that for readers with some interest in QFT, it would be preferable to present some physical motivations and the uninterested reader can skip the present paragraph. QFT is a general framework aimed at describing the fundamental forces and particles. In QFT, we are given some graphs called *Feynman graphs* which pictorially represent complicated interaction process between various particles and we associate to every graph G some number c_G , called *Feynman amplitude*, which is often given by some divergent integral as soon as the graph G contains any loops. The issue is that the above zeta regularization methods can only be used to renormalize one loop graphs as discussed in [17, 1.4 p. 10]. For interacting QFT's, it is not enough to regularize only the partition function and one loop graphs, one must renormalize amplitudes whose corresponding graphs contain an arbitrary number of loops. For instance in quantum electrodynamics (QED) which is the QFT describing the interaction of light and matter, the computation of the probability amplitude of some scattering process for two incoming and two outgoing electrons is represented by the following Feynman diagram :



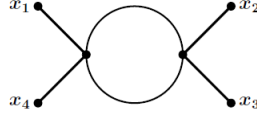
where the electrons are denoted by e^- , positrons by e^+ and photons by γ . The corresponding Feynman amplitude is given by some product of electron propagators, represented by the straight lines, and photon propagators represented by the wiggly lines. These propagators are distributions on $\mathbb{R}^4 \times \mathbb{R}^4$ valued in 4×4 matrices.

For the sake of simplicity, we limit ourselves to scalar theories in the present paper. In these theories, unlike in gauge theories, there is only one scalar valued propagator which is denoted by \mathfrak{G} in the sequel. The topology of the Feynman graphs that we encounter is dictated by the interaction of the theory. For instance in the massless ϕ^4 theory, the only Feynman graphs that we encounter have vertices of degree 4. Our goal is to use spectral zeta regularization to renormalize multiple loop amplitudes for Euclidean QFT on Riemannian manifolds with the aim to relate them to geometric invariants of Riemannian manifolds which is the subject of future work of the authors. Our starting point is the work of Eugene Speer on analytic renormalization in QFT [73, 74, 75] who found an alternative formulation of the usual BPHZ renormalization algorithm, based on analytic regularization with several complex parameters. The analytic structure of the regularized amplitude in these variables encodes rich algebraic structure so that a renormalized amplitude may be defined by the application of a universal projector, independent of the graph in question, to the regularized amplitude. Indeed, we will show that regularized amplitudes are **meromorphic germs with linear poles** and in subsection 6.3,

¹The space of Riemannian structure if the set of pairs (M, g) quotiented by isometries

we will describe a straightforward way of subtracting the divergent part of the regularized amplitudes while keeping only the holomorphic part. Then renormalization will be reformulated in definition 6.4 as the evaluation at some poles of the holomorphic part of the regularized amplitude. This projection is a useful substitute to the BPHZ algorithm and the method pioneered by Connes–Kreimer based on Hopf algebras and Birkhoff factorizations. In our work, a common point with the BPHZ algorithm and Speer’s work, is that we rely on Hepp sectors and resolution of singularities arguments.

Let us show how the idea of *analytic renormalization* works in an example on flat space. On Euclidean space \mathbb{R}^4 , the Green function of the Laplace operator reads $\mathfrak{G}(x, y) = CQ^{-1}(x - y)$ where C is some constant and Q is the quadratic form $Q(v) = \sum_{i=1}^4 v_i^2$. On configuration space $(\mathbb{R}^4)^6$, the Feynman rules assign to the graph



the corresponding amplitude

$$T(x_1, x_2, x_3, x_4) = \int_{(y_1, y_2) \in (\mathbb{R}^4)^2} \mathfrak{G}(x_1, y_1) \mathfrak{G}(x_2, y_1) \mathfrak{G}^2(y_1, y_2) \mathfrak{G}(y_2, x_3) \mathfrak{G}(y_2, x_4) d^4 y_1 d^4 y_2,$$

which is given by some formal product of Green functions. To get rid of the infrared divergence due to the fact that we integrate in some infinite volume $(\mathbb{R}^4)^2$, one may either introduce a sharp cut-off by replacing \mathbb{R}^4 by a finite box, or we may as well insert some smooth compactly supported cut-off function $g \in C_c^\infty(\mathbb{R}^4)$ for each variable $(y_i)_{i \in \{1, 2\}}$ corresponding to the internal vertices of the Feynman graph as follows :

$$T(x_1, x_2, x_3, x_4) = \int_{(\mathbb{R}^4)^2} \mathfrak{G}(x_1, y_1) \mathfrak{G}(x_2, y_1) \mathfrak{G}^2(y_1, y_2) \mathfrak{G}(y_2, x_3) \mathfrak{G}(y_2, x_4) g(y_1) g(y_2) d^4 y_1 d^4 y_2.$$

In fact, it is natural to view the full amplitude $\mathfrak{G}(x_1, y_1) \mathfrak{G}(x_2, y_1) \mathfrak{G}^2(y_1, y_2) \mathfrak{G}(y_2, x_3) \mathfrak{G}(y_2, x_4)$ as a **distribution** in $\mathcal{D}'((\mathbb{R}^4)^6)$, so we may think that we insert some smooth compactly supported cut-off function $g(y_1)g(y_2)$ on $(\mathbb{R}^4)^2$ so that $T(x_1, x_2, x_3, x_4)$ is well-defined as the **pushforward** of the product $\mathfrak{G}(x_1, y_1) \mathfrak{G}(x_2, y_1) \mathfrak{G}^2(y_1, y_2) \mathfrak{G}(y_2, x_3) \mathfrak{G}(y_2, x_4) g(y_1) g(y_2) d^4 y_1 d^4 y_2$ along the fibers of the projection $(\mathbb{R}^4)^6 \mapsto (\mathbb{R}^4)^4$.

In terms of the quadratic function Q , the above amplitude reads :

$$\int_{(y_1, y_2) \in (\mathbb{R}^4)^2} Q^{-1}(x_1, y_1) Q^{-1}(x_2, y_1) Q^{-1}(y_1, y_2) Q^{-1}(y_1, y_2) \\ \times Q^{-1}(y_2, x_3) Q^{-1}(y_2, x_4) g(y_1) g(y_2) d^4 y_1 d^4 y_2.$$

Now for each Q^{-1} in factor in the amplitude, we shall introduce a complex power s as follows :

$$T(\mathbf{s}) = \int_{(y_1, y_2) \in (\mathbb{R}^4)^2} Q^{-s_1}(x_1, y_1) Q^{-s_2}(x_2, y_1) Q^{-s_3}(y_1, y_2) Q^{-s_4}(y_1, y_2) \\ \times Q^{-s_5}(y_2, x_3) Q^{-s_6}(y_2, x_4) g(y_1) g(y_2) d^4 y_1 d^4 y_2$$

where the new amplitude depends on $\mathbf{s} = (s_1, \dots, s_6) \in \mathbb{C}^6$. For $Re(s_i)_{1 \leq i \leq 6}$ large enough, one can easily see that the amplitude defining T is integrable. The main result of Speer is the fact that $T(\mathbf{s})$ admits an analytic continuation in $\mathbf{s} \in \mathbb{C}^6$ as a meromorphic function with linear poles. Then he shows that $T(\mathbf{s})$ decomposes as the sum of a singular part and a holomorphic part at $\mathbf{s} = (1, \dots, 1) \in \mathbb{C}^6$ and renormalization consists in subtracting the singular part and evaluating at $(1, \dots, 1) \in \mathbb{C}^6$.

The main goal of the present paper is to combine the methods from zeta regularization to present a generalization of analytic renormalization to general Riemannian manifolds. Then we will show that the renormalization defined satisfies the consistency axioms of Nikolov–Todorov–Stora in [59] inspired by the seminal works of Epstein–Glaser [30].

2 Main results.

In the present section, we introduce the main objects of study and state the main results of our work. We define first Feynman amplitudes, next we explain how to implement a zeta regularization with several complex parameters then we state the first main analytic continuation Theorem and we finally give a simplified version of our second main theorem concerning applications of the analytic continuation result to renormalization in QFT.

2.1 Feynman amplitudes

We work on a compact, connected Riemannian manifold (M, g) without boundary, the Laplace–Beltrami operator is denoted by Δ_g and $C_{\geq 0}^\infty(M)$ denotes smooth, nonnegative functions on M . For a potential $V \in C_{\geq 0}^\infty(M)$, it is well-known that the Schrödinger operator $P = -\Delta_g + V$ is a second order, symmetric, positive, elliptic differential operator which defines a unique unbounded, self-adjoint operator acting on $L^2(M)$ [76, p. 34-35]. We now generalize the Feynman rules to this case. That is, to every graph we associate a formal product of Green kernels of the operator P . Since on a general manifold, there is no Fourier transform, our Feynman rules are just the Riemannian versions of the Euclidean Feynman rules in **position space** of [19, definition 2.1] (see also [24]).

Definition 2.1 (Feynman rules) *Let $\mathfrak{G}(x, y)$ denote the Green kernel of the operator P , then for a graph G with the set of vertices $V(G)$ and the set of edges $E(G)$, if for any edge $e \in E(G)$, the vertices incident to e are $i(e)$ and $j(e)$ and G has no self-loops, then the Feynman amplitude associated to G is defined as*

$$t_G = \prod_{e \in E(G)} \mathfrak{G}(x_{i(e)}, x_{j(e)}) \quad (2)$$

as a C^∞ function on $M^{V(G)} \setminus \{\text{all diagonals}\}$.

Remark 2.1 *Since the Green kernel \mathfrak{G} is symmetric in its variables, t_G is well-defined. The graphs are not allowed to have self-loops since the Green function \mathfrak{G} is not well-defined on the diagonal hence cannot be evaluated at coinciding points. The above Feynman rules correspond to a perturbative Euclidean QFT where the Lagrangian is already Wick renormalized which explains why self-loops (also called tadpoles in the physics literature) are excluded. In the physics literature, the amplitude reads*

$$t_G = \frac{1}{|\text{Aut}(G)|} \prod_{e \in E(G)} \mathfrak{G}(x_{i(e)}, x_{j(e)})$$

where the combinatorial factor $|\text{Aut}(G)| \in \mathbb{N}$ counts the number of automorphisms of G . We drop this combinatorial factor for simplicity since this does not affect our discussion.

2.2 Multiple spectral zeta regularization.

The operator P^{-s} is defined as a spectral function of the operator P in a very simple way following [36, equation (1.12.13) p. 107] :

Definition 2.2 (Complex powers) For $\operatorname{Re}(s) \geq 0$, for every $u \in L^2(M)$, decompose u in the orthonormal basis $(e_\lambda)_\lambda$ of $L^2(M)$ given by the eigenfunctions of P . Then

$$P^{-s}u = \sum_{\lambda \in \sigma(P) \setminus \{0\}} \lambda^{-s} \langle u, e_\lambda \rangle e_\lambda$$

where the sum on the right hand side converges absolutely in $L^2(M)$ since the eigenvalue $\lambda_i \xrightarrow{i \rightarrow +\infty} +\infty$ hence the sequence $(\lambda_i^{-s})_i$ remains bounded.

The Schwartz kernel of P^{-s} is then by definition :

$$\mathfrak{G}^s(x, y) = \sum_{\lambda \in \sigma(P) \setminus \{0\}} \lambda^{-s} e_\lambda(x) e_\lambda(y) \quad (3)$$

where we abusively denoted by $\mathfrak{G}^s(x, y)$ an actual *distribution* $\mathfrak{G}^s \in \mathcal{D}'(M \times M)$ and the series on the r.h.s. converges in $\mathcal{D}'(M \times M)$. We will later see that \mathfrak{G}^s is actually a function on $M \times M$ for $\operatorname{Re}(s) > \frac{d}{2}$ where $d = \dim(M)$. We shall generalize this regularization to the case $M = \mathbb{R}^d$ with flat Euclidean metric g and $P = -\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$. Our definition of \mathfrak{G}^s in the flat case is similar to the compact case since we define \mathfrak{G}^s with complex powers of the Laplace operator :

Definition 2.3 (Complex powers for flat space.) If $M = \mathbb{R}^d, g$ is a constant quadratic form and $m \in \mathbb{R}_{\geq 0}$ is a mass, then we set :

$$\mathfrak{G}^s(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i\langle (x-y), \xi \rangle}}{(g^{\mu\nu} \xi_\mu \xi_\nu + m^2)^s} d^d \xi = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\langle x-y, x-y \rangle_g}{4t}} e^{-tm^2} t^{s-1} dt.$$

It is immediate from the above formulas that \mathfrak{G}^s is the Schwartz kernel of $(-\Delta_g + m^2)^{-s}$ and that when $s = 1$, we recover the Green function of the operator $(-\Delta_g + m^2)$.

Definition 2.4 (Regularized Feynman rules) Under the above assumptions, we denote by P^{-s} the complex powers of P and by $\mathfrak{G}^s(x, y) \in \mathcal{D}'(M \times M)$ the corresponding Schwartz kernel. Then for a graph G with vertex set $V(G)$ and edge set $E(G)$, the regularized Feynman amplitude reads

$$t_G(s) = \prod_{e \in E(G)} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)}) \quad (4)$$

which is in $C^\infty(M^{V(G)} \setminus \{\text{all diagonals}\})$.

Remark 2.2 We will see later in Lemma 4.1 that \mathfrak{G}^s is actually in $C^k(M \times M)$ for $\operatorname{Re}(s)$ large enough hence it follows that the above Feynman rules also make sense for graphs G with self-loops when $\operatorname{Re}(s)$ is large enough which was not true for $s = 1$ since \mathfrak{G} would be a distribution singular on the diagonal.

Let us state our first main Theorem :

Theorem 2.1 Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d , $dv(x)$ the Riemannian volume and $P = -\Delta_g + V, V \in C_{\geq 0}^\infty(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$. Then for every graph G , on the configuration space $(M^{V(G)}, g_{V(G)})$ endowed with the product metric $g_{V(G)}$, and product volume form $dv_{M^{V(G)}}$, for any test function $\varphi \in C^\infty(M^{V(G)})$,

$$s \mapsto \int_{M^{V(G)}} t_G(s) \varphi dv_{M^{V(G)}} \quad (5)$$

can be analytically continued near $(s_e = 1)_{e \in E(G)}$ as a **meromorphic germ with possible linear poles** on the hyperplanes of equation $\sum_{e \in G'} s_e - |E(G')| = 0$ where G' is a subgraph of G such that $2|E(G')| - b_1(G')d \leq 0$, $|E(G')|$ is the number of edges in G' and $b_1(G')$ the first Betti number of G' .

To recover renormalized Feynman amplitudes, we follow the strategy of [25, 2.2]. We cannot evaluate $t_G(s)$ at $(s_e = 1)_{e \in E(G)}$ since it might belong to the polar set of t_G . However, applying the machinery from [40] allows us to subtract the polar part of $t_G(s)$ at $(s_e = 1)_{e \in E(G)}$ while keeping a holomorphic part. This is based on an extension of the framework of [40] to distributions valued in meromorphic germs with linear poles constructed in paragraph 6.2. Then to recover the renormalized Feynman amplitude, it suffices to evaluate the holomorphic part at $(s_e = 1)_{e \in E(G)}$. Following Speer [74, section 3], analytic renormalization will be reformulated in definition 6.4 as the evaluation at some poles of the holomorphic part of the regularized amplitude. This idea was recently abstracted in the works [20, 41] in a purely algebraic way where the composition of a projection on the holomorphic part and the evaluation at $(s_e = 1)_{e \in E(G)}$ is called **evaluator** [41, 1.3 p. 6]. The renormalization $\mathcal{R}(t_G)$ of some amplitude t_G is the composition of the operations summarized in the following diagram :

$$t_G \xrightarrow{\text{regularization}} t_G(s) \xrightarrow{\text{projection on holomorphic part}} \pi(t_G(s)) \xrightarrow{\text{evaluation at } s=s_0} \mathbf{ev}|_{s_0}(\pi(t_G(s))) = \mathcal{R}(t_G),$$

where $s_0 = (s_e = 1)_{e \in E(G)}$.

In section 6, we apply the above ideas to the renormalization of quantum field theories on Riemannian manifolds and show the existence of a collection of renormalization maps $(\mathcal{R}_{M^I})_{I \subset \mathbb{N}}$ that roughly assign to each graph G a renormalized amplitude in $\mathcal{D}'(M^{V(G)})$ such that the renormalization maps satisfy the consistency axioms 6.2 which come from the work of Nikolov–Todorov–Stora [59]. Let us state a simplified version of our second main Theorem 6.2 :

Theorem 2.2 *Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d , $dv(x)$ the Riemannian volume and $P = -\Delta_g + V$, $V \in C_{\geq 0}^\infty(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Then for every graph G , on the configuration space $M^{V(G)}$ endowed with the product volume form $dv_{M^{V(G)}}$:*

- *there exists distributions $\pi(t_G(s)), (1 - \pi)(t_G(s))$ such that for any test function $\varphi \in C^\infty(M^{V(G)})$, we have a decomposition*

$$\int_{M^{V(G)}} t_G(s) \varphi dv_{M^{V(G)}} = \underbrace{\int_{M^{V(G)}} (1 - \pi)(t_G(s)) \varphi dv_{M^{V(G)}}}_{\text{meromorphic germ}} + \int_{M^{V(G)}} \pi(t_G(s)) \varphi dv_{M^{V(G)}}$$

where $s \mapsto \int_{M^{V(G)}} \pi(t_G(s)) \varphi dv_{M^{V(G)}}$ is a holomorphic germ at $s_0 = (s_e = 1)_{e \in E(G)} \in \mathbb{C}^{|E(G)|}$.

- *If $\varphi \in C_c^\infty(M^{V(G)} \setminus \text{all diagonals})$ then*

$$\lim_{s \rightarrow s_0} \int_{M^{V(G)}} \pi(t_G(s)) \varphi dv_{M^{V(G)}} = \int_{M^{V(G)}} t_G \varphi dv_{M^{V(G)}} \quad (6)$$

which means $\lim_{s \rightarrow s_0} \pi(t_G(s))$ is a distributional extension of $t_G \in C^\infty(M^{V(G)} \setminus \text{all diagonals})$.

Then the reader is referred to Theorem 6.2 where we prove many important properties enjoyed by the renormalized amplitudes $\lim_{s \rightarrow s_0} \pi(t_G(s))$. The most important being a factorization equation appearing in definition 6.2 which translates in mathematical terms the essential property of locality in Euclidean QFT.

Related works. In recent works of Hairer [42] and Pottel [64, 65], the authors give analytic treatments of the BPHZ algorithm. Like in the present paper, they also start from Feynman amplitudes in **position space** but Hairer works on \mathbb{R}^d with abstract kernels K with specific singularity along

diagonals, whereas we work on Riemannian manifolds but we limit our discussion to Green kernels of Laplace type operators. He also uses Hepp sectors to perform some kind of multiscale analysis to analyze the divergences of the Feynman amplitudes. It would be interesting to compare the renormalization maps defined in the present paper with the valuations in Hairer’s paper [42] and definition 6.2 with the consistency axioms of [42].

Our treatment of renormalization bears a strong inspiration from the seminal work of Epstein–Glaser [30] who were among the first to understand the central role of causality (this is replaced in the current work by locality) in perturbative renormalization. Their work was generalized by Brunetti–Fredenhagen [16] to curved space times while the crucial physical notions of covariance of the renormalization were addressed in the works of Hollands–Wald [45, 46]. A recent investigation of the Epstein–Glaser renormalization using resolution of singularities can be found in the thesis of Berghoff [12, 13] clarifying some previous attempts [10, 11]. Our results seem to be more general since we work in the manifold case and we resolve singularities by hand instead of using the compactifications of configuration space of Fulton–McPherson and de Concini–Procesi.

There is a famous interpretation of the BPHZ renormalization in terms of Hopf algebras pioneered by Connes–Kreimer [21, 22, 23]. This approach using dimensional regularization works essentially in momentum space and does not generalize in a straightforward way to curved spaces. Motivated by problems from number theory, Marcolli–Ceyhan [19] managed to reformulate the Hopf–algebraic approach on configuration space.

Other sources of inspiration for us are the recent works [29, 58], where renormalization is discussed from the point of view of distributional extensions in position à la Epstein–Glaser, using also several complex parameters to perform some analytic continuation of the Feynman amplitude. In particular the paper [58] relates Epstein–Glaser renormalization with the analytic renormalization by Speer.

Renormalization in the Riemannian setting was recently discussed in the book by Costello [24], however it seems his proof of subtraction of counterterms contains some gaps that were fixed by Albert who also extended Costello’s work to manifolds with boundary [2, 3].

Perspectives. A natural extension of our results would be to prove an analytic continuation result for Feynman amplitudes made from Schwartz kernels of holomorphic families of pseudodifferential operators in the sense of Paycha–Scott [62] generalizing the Schwartz kernels of complex powers of Laplace operators. For the sake of simplicity, we limited ourselves to complex powers of Laplace operators because of their explicit relation with heat kernels and leave it to another work for the investigation of the more general case. Another interesting situation is when the manifold M is noncompact with specific asymptotic structure as in scattering theory. Probably in this case, we would need to use resolvents to define complex powers.

It would also be very interesting to test our proof in the Lorentz case with the *Feynman propagator* instead of the Green’s function of the Laplacian. Then a natural question would be what is the substitute in the Lorentz case for the complex powers of the Laplace operators? The first author defined complex regularization of Feynman propagators in some previous work [25] on **analytic Lorentzian space–times** under some very restrictive assumptions of **geodesic convexity**. This was based on the Hadamard parametrix for the Feynman propagator. From our point of view, it would be preferable to define some complex regularization scheme on **smooth Lorentzian space–times** which are **not necessarily geodesically convex**. The scheme should be manifestly covariant as spectral regularization on Riemannian manifolds. Probably, this would be based on the recent results of [9, 32, 33, 35, 26, 27, 77] on the analytic structure of Feynman propagators. It is a reasonable idea to replace the heat kernel asymptotic expansion by the Hadamard parametrix for the Feynman propagator as in [7].

Another interesting direction is to investigate if it is possible to renormalize the amplitudes in Euclidean theory then perform a geometric Wick rotation as in Gérard–Wrochna [34] to build renormalized amplitudes of the corresponding Lorentzian QFT.

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3 Preliminaries.

The goal of the present section is to introduce the language of meromorphic germs with linear poles and give the main definitions since meromorphic germs appear in the formulation of Theorem 2.1. We also introduce their distributional counterpart which we call meromorphic germs of distributions which is the fundamental object needed for the proof of Theorem 2.1. The meromorphic germs of distributions are essentially distributions depending on some parameter $s \in \mathbb{C}^p$, $p \in \mathbb{N}$, such that when they are paired with some test function φ , they give meromorphic germs in s .

3.1 Meromorphic functions with linear poles.

In this paper, all meromorphic functions of several variables $s = (s_1, \dots, s_p) \in \mathbb{C}^p$ have singularities along unions of affine hyperplanes. In fact, we will work with **meromorphic germs with linear poles** in the terminology of [40]. We work in the space \mathbb{R}^p , and with the standard complex structure on $\mathbb{C}^p = \mathbb{R}^p \otimes \mathbb{C}$, let $(\mathbb{R}^p)^*$ be the dual. In the sequel, holomorphic functions on some domain $\Omega \subset \mathbb{C}^p$ and holomorphic germs at $s_0 \in \mathbb{C}^p$ are denoted by $\mathcal{O}(\Omega)$ and $\mathcal{O}_{s_0}(\mathbb{C}^p)$ respectively.

Definition 3.1 (meromorphic germs) *Let $s_0 \in \mathbb{R}^p$, then f is a meromorphic germ with (real) linear poles at s_0 if there are vectors $(L_i)_{1 \leq i \leq m}$ in $(\mathbb{R}^p)^*$, such that*

$$\left(\prod_{i=1}^m L_i(\cdot - s_0) \right) f \in \mathcal{O}_{s_0}(\mathbb{C}^p). \quad (7)$$

Meromorphic germs with linear poles at $s_0 \in \mathbb{C}^p$ are denoted by $\mathcal{M}_{s_0}(\mathbb{C}^p)$.

Geometrically such a meromorphic germ f is singular along some arrangement of affine hyperplanes $\{s \in \mathbb{C}^p \text{ s.t. } L_i(s - s_0) = 0\}_{1 \leq i \leq m}$, intersecting at the point s_0 .

3.2 Meromorphic germs of distributions.

In this paper, we deal with families of distributions $t(s)$ on some smooth second countable manifold X without boundary, depending meromorphically on some parameter s and whose poles are linear. We will also call them distributions valued in meromorphic germs with linear poles and will denote the space of such families by $\mathcal{D}'(X, \mathcal{M})$. We devote this subsection to their proper definition. Our plan is to give the definition gradually starting from holomorphic objects. For a smooth manifold X with given smooth density dv , we will use $\mathcal{D}'(X)$ to denote the space of distributions on X and $\mathcal{D}'(X)$ is defined in the present paper as the topological dual of $C_c^\infty(X) \otimes dv$ which is the space of smooth, compactly supported densities. But in many situations where the density is explicitly given by the geometric problem, we may equivalently think of distributions as the dual of $C_c^\infty(X)$.

Holomorphic families of distributions. Before we discuss meromorphic germs of distributions, let us start smoothly by defining distributions depending holomorphically on some extra parameter.

Definition 3.2 (holomorphic families) *Let $\Omega \subset \mathbb{C}^p$ be a complex domain, and X be a smooth manifold, a holomorphic family of distributions on X parametrized by Ω is a family $(t(s))_{s \in \Omega}$ of distributions on X , such that for every test function $\varphi \in C_c^\infty(X)$, $s \mapsto \langle t(s), \varphi \rangle$ defines a holomorphic function on Ω . Such set of holomorphic families of distributions will be denoted by $\mathcal{D}'(X, \mathcal{O}(\Omega))$.*

Then we next introduce a variant of the above definition involving distributions whose distributional order is bounded by some integer m .

Definition 3.3 (holomorphic families with bounded order) *Let m be an integer and X a smooth manifold, then a distribution t is of order bounded above by m on X if t defines a continuous linear function on $C_c^m(X)$. For a complex domain Ω , a holomorphic family of distributions $(t(s))_{s \in \Omega}$ of order bounded above by m on X , is a family $(t(s))_{s \in \Omega}$ of distributions of order bounded above by m such that for every test function $\varphi \in C_c^m(X)$, $s \mapsto \langle t(s), \varphi \rangle$ defines a holomorphic function on Ω . This set is denoted by $\mathcal{D}'^m(X, \mathcal{O}(\Omega))$.*

Once we defined holomorphic families of distributions where the complex parameter lives on some domain Ω containing some element s_0 , it is natural to give a definition where we want to forget the information about Ω and localize around s_0 . We thus work at the level of holomorphic germs near s_0 .

Definition 3.4 (holomorphic germs) *A holomorphic germ at a point $s_0 \in \mathbb{C}^p$ of distributions on X is an equivalence class of holomorphic families of distributions on X w.r.t. the natural equivalence relation: $(t(s))_{s \in \Omega_1} \sim (u(s))_{s \in \Omega_2}$ if there exists $\Omega_3 \subset \Omega_1 \cap \Omega_2$ such that $s_0 \in \Omega_3$ and $t(s) = u(s)$ for all $s \in \Omega_3$. This set is denoted by $\mathcal{D}'(X, \mathcal{O}_{s_0}(\mathbb{C}^p))$.*

Example 3.1 *The family of distributions $t(s) : \varphi \in C_c^\infty(\mathbb{R}) \mapsto \int_{\mathbb{R}} e^{sx} \varphi(x) dx$ defines a holomorphic germ of distributions at $s = 0$ with real coefficients.*

Meromorphic germs of distributions. Once we have a proper definition for holomorphic families of distributions, we can give a very natural definition of meromorphic families of distributions as follows :

Definition 3.5 (meromorphic family of distributions with linear poles) *For a complex domain $\Omega \subset \mathbb{C}^p$, a meromorphic family of distributions on Ω is a holomorphic family $(t(s))_{s \in \Omega \setminus \{s; L_1 = \dots = L_k(s) = 0\}}$ of distributions, where L_1, \dots, L_k are linear functions on \mathbb{C}^p , such that*

$$t(s) = (L_1(s) \dots L_k(s))^{-1} h(s), \forall s \in \Omega \setminus \{s; L_1 = \dots = L_k(s) = 0\} \quad (8)$$

where $(h(s))_{s \in \Omega} \in \mathcal{D}'(X, \mathcal{O}(\Omega))$.

Now we localize the above definition to germs at $s_0 \in \mathbb{C}^p$:

Definition 3.6 (meromorphic germs of distributions) *A meromorphic germ of distributions at s_0 with linear poles is an equivalence class of meromorphic families of distributions on some neighborhood Ω of s_0 with linear poles under the equivalence relation: $t(s)_{s \in \Omega_1 \setminus Y_1} \sim t'(s)_{s \in \Omega_2 \setminus Y_2}$, where $Y_i = \{s; L_1^i = \dots = L_{k_i}^i(s) = 0\}$, $i = 1, 2$ with linear functions L_j^i , $j = 1, \dots, k_i$, $i = 1, 2$, if there exist a complex domain $\Omega_3 \subset \Omega_1 \cap \Omega_2$, and linear functions $L_1^3, \dots, L_{k_3}^3$ such that $s_0 \in \Omega_3$, $s_0 \in Y_1 \cap Y_2$, $Y_3 = \{s; L_1^3 = \dots = L_{k_3}^3(s) = 0\} \subset Y_1, Y_3 \subset Y_2$, and $t(s)_{s \in \Omega_3 \setminus Y_3} = t'(s)_{s \in \Omega_3 \setminus Y_3}$. The set of meromorphic germs of distribution with real coefficients will be denoted by $\mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$.*

It is simple to show that :

Proposition 3.1 *The set $\mathcal{D}'(X, \mathcal{O}_{s_0}(\mathbb{C}^p))$ is a vector subspace of $\mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$.*

3.3 Power expansions of holomorphic germs

Let us state a convenient proposition about power series expansion of holomorphic families of distributions whose proof is given in the appendix.

Proposition 3.2 *Let X be some smooth manifold, $\Omega \subset \mathbb{C}^p$ and $(t(s))_{s \in \Omega} \in \mathcal{D}'(X, \mathcal{O}(\Omega))$ be some holomorphic family of distributions. Then near every $s_0 \in \Omega$, t_s admits a power series expansion*

$$t(s) = \sum_{\alpha} \frac{(s - s_0)^{\alpha}}{\alpha!} t_{\alpha}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and t_{α} is a distribution in $\mathcal{D}'(X)$ such that for all test functions φ , $\sum_{\alpha} \frac{(s - s_0)^{\alpha}}{\alpha!} t_{\alpha}(\varphi)$ converges as power series near s_0 .

This classical result is just a multivariable version of [37, Theorem 1] which is stated for general locally convex spaces E , we include a proof in the appendix to make our text self-contained.

3.4 From Green functions to the heat kernel.

The fundamental tool we use to investigate the singularities of Feynman amplitudes is the heat kernel. In this section, we recall its main properties and explain how one can express the regularized Green functions and the Feynman amplitudes in terms of the heat kernel.

3.4.1 Heat kernels

The complex powers of $P = -\Delta_g + V$ are related to the heat kernel e^{-tP} in the following way (see also [36, paragraph 1.12.14 p. 112]):

Proposition 3.3 *Let (M, g) be a smooth compact, connected Riemannian manifold without boundary and let $P = -\Delta_g + V$, $V \in C_{\geq 0}^{\infty}(M)$ or $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Set Π to be the spectral projector on $\ker(P)$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 0$ then*

$$P^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} (e^{-tP} - \Pi) t^s \frac{dt}{t} \quad (9)$$

in the sense of bounded operator from $L^2(M, \mathbb{C}) \mapsto L^2(M, \mathbb{C})$ where Γ is the Euler Gamma function. In the sense of Schwartz kernels :

$$\mathfrak{G}^s(x, y) = \frac{1}{\Gamma(s)} \int_0^{\infty} (K_t(x, y) - \Pi(x, y)) t^s \frac{dt}{t}$$

where $K_t \in C^{\infty}((0, \infty) \times M \times M)$ is the heat kernel.

Note that when $0 \notin \sigma(P)$ and M compact or when $M = \mathbb{R}^d$, then we can set $\Pi = 0$.

Proof — The proposition is clear when $M = \mathbb{R}^d$ hence we just discuss the compact case. As a consequence of the compactness of M and the fact that P is an elliptic, positive, self-adjoint operator, P has discrete spectrum denoted by $\sigma(P)$, the eigenfunctions $(e_{\lambda})_{\lambda \in \sigma(P)}$ of P form an orthonormal basis of $L^2(M, \mathbb{C})$, so for any $u \in L^2(M, \mathbb{C})$, $u = \sum \langle u, e_{\lambda} \rangle e_{\lambda}$. By definition, $P^{-s}u = \sum_{\lambda \in \sigma(P), \lambda \neq 0} \lambda^{-s} \langle u, e_{\lambda} \rangle e_{\lambda}$ where the sum on the right hand side converges absolutely in $L^2(M, \mathbb{C})$. And the spectral projector Π on $\ker(P)$ is simply $\Pi(u) = \sum_{\lambda=0} \langle u, e_{\lambda} \rangle e_{\lambda}$.

The heat operator e^{-tP} is a strongly continuous semigroup acting on $L^2(M, \mathbb{C})$. For every $u \in L^2(M, \mathbb{C})$:

$$(e^{-tP} - \Pi)u = \sum_{\lambda \in \sigma(P) \setminus 0} e^{-t\lambda} \langle u, e_{\lambda} \rangle e_{\lambda}$$

where the sum on the r.h.s converges in $L^2(M, \mathbb{C})$.

Therefore for $\lambda > 0$, by a change of variable in the Γ function $\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t\lambda} t^s \frac{dt}{t}$, it follows that the identity $P^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} (e^{-tP} - \Pi) t^s \frac{dt}{t}$ holds true in operator sense where the integral on the r.h.s converges in operator norm. Hence the same identity should hold true for the corresponding Schwartz kernels. ■

3.5 Local asymptotic expansions of heat kernels

We will use the following property of the heat kernel asymptotics [15, Thm 2.30] (see also [69, Thm 7.15]) :

Theorem 3.1 (Minakshisundaram–Pleijel) *Let (M, g) be a compact Riemannian manifold without boundary, ε is the **injectivity radius** of M and $P = -\Delta_g + V$, $V \in C_{\geq 0}^\infty(M)$. Choose some cut-off function $\psi : \mathbb{R}_+ \mapsto [0, 1]$ such that $\psi(s) = 1$ if $s \leq \varepsilon^2/4$ and $\psi(s) = 0$ if $s > 4\varepsilon^2/9$. Let $K_t(x, y) \in C^\infty((0, \infty) \times M \times M)$ denote the heat kernel, then there exist smooth **real valued** functions $a_k(x, y) \in C^\infty(M \times M)$, $k = 0, 1, \dots$, with $a_0(x, y) = 1$, such that for all $(n, p) \in \mathbb{N}^2$, and differential operator $P(x, D_x)$ of degree m , there exists a constant $C > 0$ such that for all $t \in (0, 1]$:*

$$\sup_{(x, y) \in M^2} |P(x, D_x) \partial_t^p \left(K_t(x, y) - \sum_{k=0}^n \psi(\mathbf{d}^2) \frac{e^{-\frac{\mathbf{d}^2(x, y)}{4t}}}{(4\pi t)^{\frac{d}{2}}} a_k(x, y) t^k \right)| \leq C t^{n - \frac{d}{2} - \frac{m}{2} - p} \quad (10)$$

where $\mathbf{d}(\cdot, \cdot) : M \times M \mapsto \mathbb{R}_{\geq 0}$ is the Riemannian distance function.

Note that our statement differs from the statement in [69] from the fact that we use a cut-off function ψ since outside some neighborhood of the diagonal $\Delta \subset M \times M$, K_t vanishes at infinite order in t when $t \rightarrow 0$ (see the proof of [69, Thm 7.15 p. 102]). In case $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$, we have the well known exact formula :

$$K_t(x, y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x-y|_g^2}{4t} - tm^2\right) = \frac{\exp\left(-\frac{|x-y|_g^2}{4t}\right)}{(4\pi t)^{\frac{d}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k t^k m^{2k}}{k!}$$

which already appeared in definition 2.3.

4 Reduction of regularized Feynman amplitudes

Recall our aim was to prove analytic continuation of the regularized amplitude

$$t_G = \prod_{e \in E(G)} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)})$$

in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ where $s_0 = (s_e = 1)_{e \in E(G)}$ and \mathfrak{G}^s is the Schwartz kernel of the complex power P^{-s} . The main goal of this section is to prove a technical Theorem 4.1 which allows us to reduce our main Theorem 2.1 to the proof of an analytic continuation Theorem for simpler analytic objects. These are some kind of Feynman amplitudes introduced in definition 4.3 corresponding to *labelled Feynman graphs* defined in definition 4.2 which are graphs whose edges are decorated by some integer. Intuitively, the amplitude of the labelled graph is obtained from the regularized amplitude $t_G(s)$ where we replace the heat kernels appearing in the formula for the Green function $\mathfrak{G}^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (e^{-tP} - \Pi) dt$, by the heat kernel asymptotic expansion. The integers decorating the edges exactly correspond to the heat coefficients in the heat kernel asymptotic expansion.

4.1 Holomorphicity of Green's function

The next Lemma discusses analytical properties of the regularized Green function of the Schrödinger operator P which is elliptic since its leading part coincides with the Laplace operator, it is therefore automatically self-adjoint by the symmetry assumption [76, p. 35].

Lemma 4.1 *Let (M, g) be a smooth compact, connected Riemannian manifold without boundary and let $P = -\Delta_g + V$, $V \in C_{\geq 0}^\infty(M)$ or $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Denote by K_t the corresponding heat kernel. Then :*

1. For all $k \in \mathbb{N}$, if $\operatorname{Re}(s) > \frac{d}{2} + k$, \mathfrak{G}^s is a C^k function on $M \times M$.
2. For all $k \in \mathbb{N}$, a compact subset $K \subset M \times M \setminus \text{Diagonal}$, the kernel \mathfrak{G}^s is holomorphic in s and valued in $C^k(K)$.
3. If we write

$$\mathfrak{G}^s(x, y) = \mathfrak{G}_{\leq}^s(x, y) + \mathfrak{G}_{\geq}^s(x, y)$$

where

$$\mathfrak{G}_{\geq}^s(x, y) = \frac{1}{\Gamma(s)} \left(\int_0^1 (K_t - \Pi)(x, y) t^{s-1} dt \right), \mathfrak{G}_{\leq}^s(x, y) = \frac{1}{\Gamma(s)} \left(\int_1^\infty (K_t - \Pi)(x, y) t^{s-1} dt \right)$$

then $\mathfrak{G}_{\leq}^s(x, y)$ is holomorphic in s and valued in $C^\infty(M \times M)$ which is denoted by $\mathfrak{G}_{\leq}^s \in \bar{C}^\infty(M \times M, \mathcal{O}(\mathbb{C}^p))$.

The proof of these classical properties, when M is compact, is recalled in the appendix. For $M = \mathbb{R}^d$, they follow from straightforward computations.

4.2 Reduction to local charts and localization near deepest diagonal.

The purpose of the next two Lemmas is to localize the proof of our main Theorem about the analytic continuation of the distribution $t_G(s)$ to neighborhoods of the deepest diagonals in $M^{V(G)}$.

Lemma 4.2 *Let X be a manifold without boundary, $s_0 \in \mathbb{C}^p$ then $t(s) \in \mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$ iff for every $x \in N$, there exists a neighborhood U_x of x such that $t(s)|_{U_x} \in \mathcal{D}'(U_x, \mathcal{M}_{s_0}(\mathbb{C}^p))$.*

If X is non compact, we require that there are linear functions $(L_i)_{i=1}^k$ corresponding to a fixed polar set $Y = \{s; L_1(s) = \dots = L_k(s) = 0\} \subset \mathbb{C}^p$, $s_0 \in Y$ such that $t(s)|_{U_x} \in \mathcal{D}'(U_x, \mathcal{M}_{s_0}(\mathbb{C}^p))$ singular along the polar set Y .

Proof — The direct implication is straightforward. Assume that for every $x \in X$, there exists a neighborhood U_x of x such that $t(s)|_{U_x} \in \mathcal{D}'(U_x, \mathcal{M}_{s_0}(\mathbb{C}^p))$. Then by local compactness, there is a locally finite subcover $(U_i)_i$ of N such that $t(s)|_{U_i} \in \mathcal{D}'(U_i, \mathcal{M}_{s_0}(\mathbb{C}^p))$. Let $(\chi_i)_i$ be a partition of unity where each χ_i is supported in U_i . Then for every test function $\varphi \in C_c^\infty(X)$, $\langle t(s), \varphi \rangle = \sum_i \langle t(s), \chi_i \varphi \rangle$ is a finite sum of meromorphic germs with linear poles at s_0 . In the non compact case, the polar set Y is fixed. Therefore the sum is a meromorphic germ with linear poles at s_0 . This is a finite sum, hence $s \mapsto \langle t(s), \varphi \rangle$ is meromorphic with linear poles at s_0 with linear poles. ■

The next Lemma is inspired by the seminal work of Popineau and Stora [63] and it states that it is enough to work our analytic continuation problem for the distributions $t_G \in \mathcal{D}'(M^{V(G)})$ near the deepest diagonals :

Lemma 4.3 (Popineau–Stora Lemma) *If for any graph G , and any $x \in M$, there is some neighborhood U_x of x , such that $t_G|_{U_x^{V(G)}} \in \mathcal{D}'(U_x^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^p))$ where $s_0 = (s_e = 1)_{e \in E(G)}$, then $t_G \in \mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^p))$, $s_0 = (s_e = 1)_{e \in E(G)}$ for every graph G .*

Proof — We prove it by induction on the number of vertices of the graph G . For $|V(G)| = 2$, $t_G = \prod_{e=1}^E \mathfrak{G}^{s_e}(x, y)$. For a point $(x, y) \in M^2 = M^{V(G)}$, if $x \neq y$, consider neighborhoods V_x of x in M and V_y of y in M , such that $V_x \cap V_y = \emptyset$, then $\{V_x \times V_y\}_{(x,y) \in M^2, x \neq y} \cup \{U_x \times U_x\}_{x \in M}$ form an open cover of M^2 , it has a locally finite subcover $\{W_i\}_i$ with $W_i = V_{x_i} \times V_{y_i}$ or $U_{x_i} \times U_{x_i}$. Let $\{\chi_i\}_i$ be a partition of unity supported on $\{W_i\}_i$, then $t_G = \sum_i t_G \chi_i$, where each $t_G \chi_i$ is holomorphic at s_0 if the support of χ_i does not intersect the diagonal by Lemma 4.1 or has meromorphic continuation at s_0 by assumption. Now the claim follows from Lemma 4.2 applied to the manifold $X = M^{V(G)}$.

Now $|V(G)| = n > 2$ and assume the result holds for all graphs whose number of vertices are strictly less than n . Denote by $d_n = \{x_1 = \dots = x_n\} \subset M^n$, the deepest diagonal in the configuration space M^n . For $(x_1, \dots, x_n) \in M^n \setminus d_n$, let $I = \{i \mid x_i = x_1\}$, and $I^c = \{1, \dots, n\} \setminus I$, then $I \neq \emptyset$, $I^c \neq \emptyset$, and for any $j \in I^c$ there are neighborhoods U_j of x_1 and V_j of x_j such that $U_j \cap V_j = \emptyset$. Let $\mathcal{V} = (\bigcap_{j \in I^c} U_j)^{|I|} \times \prod_{j \in I^c} V_j$, then $\mathcal{V} \subset M^n \setminus d_n$, and $x_i \neq x_j, \forall (i, j) \in I \times I^c$ for all $(x_1, \dots, x_n) \in \mathcal{V}$. Then we partition the set of edges $E(G)$ of the graph G in three parts : $E(G) = E_I \cup E_{I^c} \cup E_{II^c}$, where E_I (resp E_{I^c}) is the set of edges of G whose incident vertices are in I (resp I^c) i.e. every edge $e \in E_I$ (resp $e \in E_{I^c}$) is bounded by vertices in I (resp I^c). The remaining subset of edges is denoted by E_{II^c} and is made of all edges $e \in E(G)$ which are neither in E_I nor in E_{I^c} . By that, we mean the edges of E_{II^c} only connect some vertex in I with another vertex of I^c . So we write $(x_1, \dots, x_n) = (x_i, x_j)_{i \in I, j \in I^c} \in M^I \times M^{I^c}$. Similarly the complex variables $(s_e)_{e \in E(G)}$ attached to the edges of G will be divided in three groups corresponding to the edges E_I, E_{I^c} and E_{II^c} respectively. Then we decompose the amplitude t_G as a product of three factors :

$$\begin{aligned} t_G((s_e)_{e \in E(G)}; (x_i, x_j)_{i \in I, j \in I^c}) &= t_I((s_e)_{e \in E_I}; (x_i)_{i \in I}) t_{I^c}(((s_e)_{e \in E_{I^c}}; (x_j)_{j \in I^c}) \\ &\times t_{II^c}((s_e)_{e \in E_{II^c}}; (x_i, x_j)_{i \in I, j \in I^c}) \end{aligned}$$

where $t_I = \prod_{e \in E_I} \mathfrak{G}^{s_e}, t_{I^c} = \prod_{e \in E_{I^c}} \mathfrak{G}^{s_e}, t_{II^c} = \prod_{e \in E_{II^c}} \mathfrak{G}^{s_e}$.

By the induction assumption, both t_I and t_{I^c} are distributions in $\mathcal{D}'(M^I, \mathcal{M}_{s_{0I}}(\mathbb{C}^{E_I})), s_{0I} = (s_e = 1)_{e \in E_I}$ and $\mathcal{D}'(M^{I^c}, \mathcal{M}_{s_{0I^c}}(\mathbb{C}^{E_{I^c}})), s_{0I^c} = (s_e = 1)_{e \in E_{I^c}}$ respectively. Then by Lemma 7.1, the exterior product $t_I((s_e)_{e \in E_I}; (x_i)_{i \in I}) t_{I^c}(((s_e)_{e \in E_{I^c}}; (x_j)_{j \in I^c})$ of distributions depending on different variables is an element in $\mathcal{D}'(M^n, \mathcal{M}_{s_{0II^c}}(\mathbb{C}^{E_I \cup E_{I^c}})), s_{0II^c} = (s_e = 1)_{e \in E_I \cup E_{I^c}}$. Now the factor t_{II^c} contains only product of propagators $\mathfrak{G}^{s_e}(x_i, x_j)$ where $x_i \neq x_j$, so in the open subset $\mathcal{V} \subset M^n$, $\mathfrak{G}^{s_e}(x_i, x_j) \in C^\infty(M \times M \setminus \text{Diagonal}, \mathcal{O}(\mathbb{C}))$ in the variables (x_i, x_j) by Lemma 4.1. Thus on \mathcal{V} , $t_{II^c} \in C^\infty(\mathcal{V}, \mathcal{O}(\mathbb{C}^{E_{II^c}}))$ which means t_{II^c} is holomorphic in the parameter $(s_e)_{e \in E_{II^c}} \in \mathbb{C}^{E_{II^c}}$. We conclude that near any element of M^n , there is some open neighborhood $U \subset M^n$ such that $t_G \in \mathcal{D}'(U, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)})), s_0 = (s_e = 1)_{e \in E(G)}$. Then by Lemma 4.2, $t_G \in \mathcal{D}'(M^n, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)})), s_0 = (s_e = 1)_{e \in E(G)}$. \blacksquare

4.3 Reductions to integrals on cubes.

In the representation of the Green function as integral of the heat kernel over $(0, +\infty)$, we would like to get rid of the low energy part which is $\mathfrak{G}_{\leq}^s = \int_1^\infty dt (K_t - \Pi) t^{s-1}$ which is smooth and holomorphic in s so it does not contribute to the singularities of t_G . We thus reduce the study of t_G , to the study of some formula P_G which contains only integrals over cubes which are easier to handle and contain all the singularities of t_G .

Definition 4.1 For a graph G , and $E \subset E(G)$, the subgraph induced by E is the subgraph H of G , such that $E(H) = E, V(H) = \{v \in V(G) \mid v \text{ is a vertex incident to some } e \in E\}$.

Proposition 4.1 If for every graph G , the product

$$P_G(s) = \prod_{e \in E(G)} \frac{1}{\Gamma(s_e)} \left(\int_0^1 (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e-1} d\ell_e \right)$$

extends to $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ at $s_0 = (s_e = 1)_{e \in E(G)}$, then $t_G(s)$ extends to $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ at $s_0 = (s_e = 1)_{e \in E(G)}$.

Proof — For all $\text{Re}(s_e) > \frac{d}{2}$, since $\mathfrak{G}^s, \mathfrak{G}_{\geq}^s, \mathfrak{G}_{\leq}^s$ all belong to $C^0(M \times M)$ by Lemma 4.1, the following product makes perfect sense :

$$t_G(s) = \prod_{e \in E(G)} \left(\mathfrak{G}_{\leq}^{s_e} + \mathfrak{G}_{\geq}^{s_e} \right) = \prod_{e \in E(G)} \mathfrak{G}_{\geq}^{s_e} + \prod_{e \in E(G)} \mathfrak{G}_{\leq}^{s_e} + \sum_{E_1 \cup E_2 = E(G)} \left(\prod_{e \in E_1} \mathfrak{G}_{\geq}^{s_e} \right) \left(\prod_{e \in E_2} \mathfrak{G}_{\leq}^{s_e} \right)$$

where the sum runs over all partitions $E(G) = E_1 \cup E_2, E_1 \neq \emptyset, E_2 \neq \emptyset$. Therefore :

$$t_G(s) = P_G(s) + \underbrace{\prod_{e \in E(G)} \mathfrak{G}_{\leq}^{s_e}} + \sum_{E_1 \cup E_2 = E(G)} P_{G(E_1)}(s) \left(\underbrace{\prod_{e \in E_2} \mathfrak{G}_{\leq}^{s_e}} \right)$$

where $G(E_1)$ is the induced subgraph of G by the subset E_1 . The terms underbraced are in C^∞ functions depending holomorphically on the parameters $s \in \mathbb{C}^{E(G)}$ near $(s_e = 1)_{e \in E(G)}$ since each $\mathfrak{G}_{\leq}^s \in C^\infty(M \times M, \mathcal{O}(\mathbb{C}))$. By assumption, for all induced subgraph $G(E_1) \subset G$, $P_{G(E_1)}$ extends to $\mathcal{D}'(M^{V(G(E_1))}, \mathcal{M}_{s_0}(\mathbb{C}^{E_1}))$, $s_0 = (s_e = 1)_{e \in E_1}$. Therefore by Lemma 7.3, each product $P_{G(E_1)}(s) \left(\prod_{e \in E_2} \mathfrak{G}_{\leq}^{s_e} \right)$ has analytic continuation in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e \in E(G)}$. ■

Therefore it is sufficient to study :

$$P_G(s) = \prod_{e \in E(G)} \frac{1}{\Gamma(s_e)} \left(\int_0^1 (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e - 1} d\ell_e \right) \quad (11)$$

Lemma 4.4 *Let G be a graph with E edges and*

$$P_G(s) = \prod_{e \in E(G)} \frac{1}{\Gamma(s_e)} \left(\int_{[0,1]^{E(G)}} \prod_{e \in E(G)} (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e - 1} d\ell_e \right). \quad (12)$$

If for all $e \in \{1, \dots, E\}$, $\operatorname{Re}(s_e) > \frac{d}{2}$, then the integral defining $P_G(s)$ converges absolutely in $[0, 1]^E$ uniformly in $(x_1, \dots, x_{|V(G)|}) \in M^{V(G)}$.

Proof — First when $M = \mathbb{R}^d$ or if M is compact and $0 \notin \ker(P)$ then $\Pi = 0$. Otherwise, if $0 \in \ker(P)$, then the Schwartz kernel of Π must a constant function (see appendix 7.3). Therefore, it is sufficient that $\operatorname{Re}(s) > 0$ so that $\int_0^1 \Pi(x, y) \ell^{s-1} d\ell$ is Riemann integrable. Now by Theorem 3.1, there exists a constant $C_0 > 0$ such that for $\ell \in (0, 1]$ and for all $(x, y) \in M^2$:

$$\left| \left(K_\ell(x, y) - \frac{1}{(4\pi\ell)^{\frac{d}{2}}} \psi(\mathbf{d}^2(x, y)) e^{-\frac{\mathbf{d}^2(x, y)}{4\ell}} \sum_{0 \leq k \leq \frac{d}{2} + 1} a_k(x, y) \ell^k \right) \right| \leq C_0.$$

So by the triangular inequality and by positivity of the heat kernel, we have the bound

$$0 \leq K_\ell(x, y) \leq \frac{1}{(4\pi\ell)^{\frac{d}{2}}} \psi(\mathbf{d}^2(x, y)) e^{-\frac{\mathbf{d}^2(x, y)}{4\ell}} \sum_{0 \leq k \leq \frac{d}{2} + 1} |a_k(x, y)| \ell^k + C_0.$$

From which we can bound the integral :

$$\begin{aligned} \int_0^1 K_\ell(x, y) |\ell^{s-1}| d\ell &\leq \int_0^1 \frac{1}{(4\pi)^{\frac{d}{2}}} \psi(\mathbf{d}^2(x, y)) e^{-\frac{\mathbf{d}^2(x, y)}{4\ell}} \sum_{0 \leq k \leq \frac{d}{2} + 1} |a_k(x, y) \ell^{k+s-\frac{d}{2}-1}| + C_0 |\ell^{s-1}| d\ell \\ &\leq \int_0^1 \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{0 \leq k \leq \frac{d}{2} + 1} |a_k(x, y) \ell^{k+s-\frac{d}{2}-1}| + C_0 |\ell^{s-1}| d\ell \end{aligned}$$

since $\psi e^{-\frac{\mathbf{d}^2}{4\ell}} \leq 1$ and the right hand side is absolutely integrable when $\operatorname{Re}(s) > \frac{d}{2}$. Therefore

$$\left(\int_{[0,1]^{E(G)}} \prod_{e \in E(G)} (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e - 1} d\ell_e \right) = \prod_{e \in E(G)} \int_0^1 (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e - 1} d\ell_e$$

is a product of convergent Riemann integrals, the above integral inversions make sense by Fubini which yields the claim of our Lemma. ■

Now we set :

$$I_G(s) = \prod_{e \in E(G)} \frac{1}{\Gamma(s_e)} \int_0^1 K_{\ell_e}(x_{i(e)}, x_{j(e)}) \ell_e^{s_e-1} d\ell_e \quad (13)$$

which is well-defined as soon as $Re(s_e) > \frac{d}{2}, \forall e \in E(G)$ by the above arguments. Then

$$P_G(s) = \sum_{E \subset E(G)} I_{G(E)}(s) \prod_{e \in E(G) \setminus E} \frac{\Pi(x_{i(e)}, x_{j(e)})}{s_e} \quad (14)$$

where $G(E)$ is the induced subgraph by the subset of edges $E \subset E(G)$. By the fact that $\int_0^1 \Pi \ell^{s-1} d\ell = \frac{\Pi}{s} \in C^\infty(M \times M, \mathcal{O}_1(\mathbb{C}))$, which is holomorphic near $s = 1$, the products of spectral projectors do not contribute to the poles. So we can further reduce our study to the analytic continuation of $I_G(s)$.

4.4 Distributional order

In this step, we introduce a further reduction by replacing each K_ℓ in the integral formula of $I_G(s)$ by its heat asymptotic expansion and try to control the remainders.

$$\frac{1}{\Gamma(s)} \int_0^1 K_\ell \ell^{s-1} d\ell = \frac{1}{\Gamma(s)} \int_0^1 \frac{e^{-\frac{d^2(x,y)}{4\ell}}}{(4\pi\ell)^{\frac{d}{2}}} \left(\sum_{k=0}^p a_k(x,y) \psi(\mathbf{d}^2(x,y)) \ell^k \right) + h_p(\ell, x, y) \ell^{s-1} d\ell$$

where $h_p(\ell, x, y)$ is the remainder in the heat asymptotics which satisfies the estimate $\|h_p\|_m \leq C \ell^{p-\frac{d}{2}-\frac{m}{2}}$ by Theorem 3.1 and ψ is the cut-off function from Theorem 3.1.

We first introduce some refinement of Feynman graphs to keep track of the information on the heat coefficients for every edge. So these are basically Feynman graphs whose edges are decorated by integers which correspond to heat coefficients.

Definition 4.2 (labelled graph) For a set S , an S -labelled graph is a pair (G, \vec{k}) where \vec{k} is a map $E(G) \rightarrow S$. If S is \mathbb{N} , we call it shortly **labelled graph**, and for $e \in E(G)$, we use the short notation k_e to denote the element $\vec{k}(e) \in \mathbb{N}$. If $S = \mathbb{R}_{>0}$, then the map $E(G) \mapsto \mathbb{R}_{>0}$, called the length function, is denoted by ℓ and we call such pair (G, ℓ) a **metric graph**. If ℓ is injective, then (G, ℓ) is called **strict metric graph**.

We next define Feynman amplitudes attached to labelled graphs.

Definition 4.3 For every labelled graph (G, \vec{k}) , we define the corresponding amplitude $I_{G, \vec{k}}(s)$ as follows :

$$I_{G, \vec{k}}(s) = \prod_{e \in E(G)} \frac{1}{\Gamma(s_e)} \int_{[0,1]^E} \prod_{e \in E(G)} \left(\frac{e^{-\frac{d^2}{4\ell_e}}}{(4\pi)^{\frac{d}{2}}} a_{k_e} \psi(\mathbf{d}^2) \right) (x_{i(e)}, x_{j(e)}) \ell_e^{k_e - \frac{d}{2} + s_e - 1} d\ell_e \quad (15)$$

which is well-defined and holomorphic in $(s_e)_{e \in E(G)} \in \mathbb{C}^{E(G)}$ on the domain $s_e > \frac{d}{2}, e \in E(G)$ by exactly the same proof as in Lemma 4.4.

Proposition 4.2 If for every graph G , there is $m \in \mathbb{N}$ depending on G , such that for all labels $\vec{k} \in \mathbb{N}^{E(G)}$, for all $x \in M$, there is an open neighborhood $U_x \subset M$ of x such that $I_{G, \vec{k}}(s)$ has analytic continuation in $\mathcal{D}'^m(U_x^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e \in E(G)}$, then for all G , $t_G(s)$ extends in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e \in E(G)}$.

Proof — Let $n = |V(G)|$ and $U = U_x$. By Lemma 4.3 which allows us to localize our analytic continuation proof near the deepest diagonal of M^n , we only need to prove that $t_G(s)$ extends as a meromorphic germ of distributions at $(s_e = 1)_{e \in E(G)}$ on U^n . For a test function φ ,

$$\begin{aligned}
& \langle I_G(s), \varphi \rangle \\
&= \int_{U^n} \left(\prod_{e \in E(G)} \frac{1}{\Gamma(s_e)} \int_0^1 K_{\ell_e} \ell_e^{s_e-1} d\ell_e \right) \varphi dv(x_1) \dots dv(x_n) \\
&= \int_{U^n} \prod_{e \in E(G)} \left(\sum_{k_e=0}^p \frac{1}{\Gamma(s_e)} \int_0^1 \frac{e^{-\frac{d^2}{4\ell_e}}}{(4\pi)^{\frac{d}{2}}} a_{k_e} \psi(\mathbf{d}^2) \ell_e^{k_e+s_e-\frac{d}{2}-1} d\ell_e + \frac{1}{\Gamma(s_e)} \int_0^1 h_p \ell_e^{s_e-1} d\ell_e \right) \varphi dv(x_1) \dots dv(x_n) \\
&= \int_{U^n} \left(\sum_{E_1 \cup E_2 = E(G)} \prod_{e \in E_1} \left(\sum_{k_e=0}^p \frac{1}{\Gamma(s_e)} \int_0^1 \frac{e^{-\frac{d^2}{4\ell_e}}}{(4\pi)^{\frac{d}{2}}} a_{k_e} \psi(\mathbf{d}^2) \ell_e^{k_e+s_e-\frac{d}{2}-1} d\ell_e \right) \prod_{e \in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p \ell_e^{s_e-1} d\ell_e \right) \right) \\
&\times \varphi dv(x_1) \dots dv(x_n)
\end{aligned}$$

where the sum runs over partitions $E_1 \cup E_2 = E(G)$. Therefore we obtain

$$\langle I_G(s), \varphi \rangle = \int_{U^n} \sum_{E_1 \cup E_2 = E(G)} \sum_{\vec{k} \in \{0, \dots, p\}^{E_1}} I_{G(E_1), \vec{k}}(s) \underbrace{\prod_{e \in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x_{i(e)}, x_{j(e)}) \ell_e^{s_e-1} d\ell_e \right)}_{\varphi dv(x_1) \dots dv(x_n)}$$

where the summation is over all $\vec{k} \in \{0, 1, \dots, p\}^{E(G)}$. By Theorem 3.1, we have the two estimates

$$|D_x^m h_p(\ell_e, x, y) \ell_e^{s_e-1}| \leq C \ell_e^{p-\frac{d}{2}-\frac{m}{2}+s_e-1},$$

and

$$|D_x^m h_p(\ell_e, x, y) \ell_e^{s_e-1} \log \ell_e| \leq C \ell_e^{p-\frac{d}{2}-\frac{m}{2}+s_e-1+\varepsilon},$$

for some $\varepsilon > 0$. So when $p > \frac{d+m}{2} - 1$, for every $e \in E_2$, there is a small neighborhood of $s_e = 1$ such that the integral $\int_0^1 h_p(\ell_e, x_{i(e)}, x_{j(e)}) \ell_e^{s_e-1} d\ell_e$ is absolutely convergent and depends holomorphically on s_e . Hence the term underbraced belongs to $C^m(U^n, \mathcal{O}_{s_0}(\mathbb{C}^{E_2}))$, $s_0 = (s_e = 1)_{e \in E_2}$, where $G(E_2)$ is the graph induced by E_2 . Now we conclude the proof by noticing that the product of $I_{G(E_1), \vec{k}}(s) \in \mathcal{D}'^m(U^n, \mathcal{M}_{s_0}(\mathbb{C}^{E_1}))$, $s_0 = (s_e = 1)_{e \in E_1}$ with some element in $C^m(U^n, \mathcal{O}_{s_0}(\mathbb{C}^{E_2}))$, $s_0 = (s_e = 1)_{e \in E_2}$ yields an element of $\mathcal{D}'^m(U^n, \mathcal{M}_{s_0}(\mathbb{C}^{E_1 \cup E_2}))$, $s_0 = (s_e = 1)_{e \in E_1 \cup E_2}$ by Lemma 7.3 proved in the appendix. \blacksquare

The next Theorem is the main result from the present section and summarizes all reduction steps performed above :

Theorem 4.1 (Reduction Theorem) *Assume that for every graph G , there is an integer $m(G)$, such that for any $x \in M$, there is a chart U_x of M around x such that for all \vec{k} , $I_{G, \vec{k}}(s)|_{U_x^n}$ admits an analytic continuation in $\mathcal{D}'^m(G)$ ($U_x^n, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)})$), $s_0 = (s_e = 1)_{e \in E(G)}$. Then for a given graph G , let $m = \sup_{G' \subset G} m(G')$, for any $p > \frac{d+m}{2} - 1$, we have a decomposition*

$$t_G(s) = \sum_{G' \subset G} \mathbf{m}_{G'}(s) h_{G \setminus G'}(s) \tag{16}$$

where the sum runs over induced subgraphs G' of G , $\mathbf{m}_{G'}(s) = \sum_{\vec{k} \in \{0, \dots, p\}^{E(G')}} I_{G', \vec{k}}(s) \in \mathcal{D}'^m(M^{V(G')}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G')}))$ and $h_{G \setminus G'}(s) \in C^m(M^{V(G \setminus G')}, \mathcal{O}_{s_0}(\mathbb{C}^{E(G) \setminus E(G')}))$. In particular, $t_G(s)$ extends as an element in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e \in E(G)}$.

So the above Theorem allows to reduce the proof of Theorem 2.1 to the analytic continuation of the simpler objects $I_{G, \vec{k}}(s)$ if we can control the distributional order of $I_{G, \vec{k}}(s)$ independently of the label $\vec{k} \in \mathbb{N}^{E(G)}$.

Proof — We use the following decomposition formula which summarizes the above three reduction steps, namely the reduction on cubes, the elimination of the spectral projector and the extraction of labelled graphs :

$$\begin{aligned} t_G(s)|_{U_x^n} &= \sum_{E_1 \cup E_2 \cup E_3 \cup E_4 = E(G)} \left(\sum_{\vec{k} \in \{0, \dots, p\}^{E_1}} \underbrace{I_{G(E_1), \vec{k}}(s)} \right) \prod_{e \in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x, y) \ell_e^{s_e-1} d\ell_e \right) \\ &\times \prod_{e \in E_3} \frac{\Pi(x_{i(e)}, y_{j(e)})}{s_e} \prod_{e \in E_4} \mathfrak{G}_{\leq}^{s_e} \end{aligned}$$

where the sum runs over partitions $E_1 \cup E_2 \cup E_3 \cup E_4 = E(G)$. Then consider $m = \sup_{E_1 \subset E(G)} m(G(E_1))$ which is the supremum of distributional orders $m(G(E_1))$ for $E_1 \subset E(G)$, m is finite by assumption and bounds the distributional order of all the terms $I_{G(E_1), \vec{k}}(s)$ underbraced. Moreover, we saw in the proof of Proposition 4.2 that if we choose $p > \frac{d+m}{2} - 1$, then the product

$$\prod_{e \in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x, y) \ell_e^{s_e-1} d\ell_e \right) \prod_{e \in E_3} \frac{\Pi(x_{i(e)}, y_{j(e)})}{s_e} \prod_{e \in E_4} \mathfrak{G}_{\leq}^{s_e}$$

belongs to $C^m(M^{V(G(E_2 \cup E_3 \cup E_4))}, \mathcal{O}_{s_0}(\mathbb{C}^{E_2 \cup E_3 \cup E_4}))$, $s_0 = (s_e = 1)_{e \in E_2 \cup E_3 \cup E_4}$. Therefore the whole product $t_G(s) \in \mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $(s_e = 1)_{e \in E(G)}$. The above complicated formula can be written very concisely as

$$t_G(s) = \sum_{G' \subset G} \mathbf{m}_{G'}(s) h_{G \setminus G'}(s)$$

where the sum runs over induced subgraphs G' of G , $\mathbf{m}_{G'}(s) = \sum_{\vec{k} \in \{0, \dots, p\}^{E(G')}} I_{G', \vec{k}}(s)$ and $h_{G \setminus G'} = \prod_{e \in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x, y) \ell_e^{s_e-1} d\ell_e \right) \prod_{e \in E_3} \frac{\Pi(x_{i(e)}, y_{j(e)})}{s_e} \prod_{e \in E_4} \mathfrak{G}_{\leq}^{s_e}$ where $E_2 \cup E_3 \cup E_4$ forms a partition of $E(G) \setminus E(G')$. \blacksquare

5 Desingularization of parameter space.

Now that we reduced the proof of Theorem 2.1 to the proof of Theorem 4.1, we start by studying in local coordinates the amplitudes $I_{G, \vec{k}}(s) \in \mathcal{D}'(M^{V(G)})$ corresponding to labelled graphs (G, \vec{k}) .

Fixing charts. For any $x \in M$, take a coordinate chart U around x such that $U \cong \mathbb{R}^d$ and $\bar{U} \subset M$ is compact and $\mathbf{d}(y_1, y_2) < \frac{\varepsilon}{2}$ for any $y_1, y_2 \in U$. Since the volume form dv on a Riemannian manifold reads $|\det(g)| d^d x$ in local coordinate chart, we may absorb the smooth function $|\det(g)|$ in the test function φ and forget about the determinant of the metric in the local coordinate system. We number the vertices of G by $\{1, \dots, n\}$, and the edges by integers $\{1, \dots, E\}$. For every edge e , we associate vertices $i(e), j(e)$ which are incident to e . Then for a test function φ with $\text{supp}(\varphi) \subset U^n$,

$$\begin{aligned} &\langle I_{G, \vec{k}}(s), \varphi \rangle \\ &= \frac{1}{(4\pi)^{\frac{dE}{2}}} \prod_{e=1}^E \frac{1}{\Gamma(s_e)} \int_{[0,1]^E} d\ell_1 \dots d\ell_E \left(\int_{U^n} \prod_{e=1}^E \exp\left(-\frac{\mathbf{d}^2}{4\ell_e}\right) \ell_e^{s_e+k_e-\frac{d}{2}-1} a_{k_e} \psi(\mathbf{d}^2) \tilde{\varphi} d^d x_1 \dots d^d x_n \right) \end{aligned}$$

where $\tilde{\varphi} = |\det(g)|\varphi$. This formula is well-defined when $Re(s_e) > \frac{d}{2}, \forall e \in \{1, \dots, E\}$ since the integration on $[0, 1]^E$ is absolutely convergent, the integral on U^n converges absolutely by compactness of the support of φ , hence we can integrate in order by Fubini Theorem. Furthermore, arguing as in the proof of Proposition 4.2 show that $\langle I_{G, \vec{k}}(s), \varphi \rangle$ is holomorphic in $s \in \mathbb{C}^E$ when $Re(s_e) > \frac{d}{2}$.

By our choice of U , \mathbf{d}^2 is smooth on $U \times U$, it is enough to prove that :

$$\frac{1}{(4\pi)^{\frac{dE}{2}}} \int_{[0,1]^E} d\ell_1 \dots d\ell_E \left(\int_{\mathbb{R}^{dn}} \prod_{e=1}^E \exp\left(-\frac{\mathbf{d}^2}{4\ell_e}\right) \ell_e^{s_e+k_e-\frac{d}{2}-1} a_{k_e} \psi(\mathbf{d}^2) \tilde{\varphi} d^d x_1 \dots d^d x_n \right)$$

extends to a meromorphic germ of distribution at $(s_e = 1)$. Note that this argument also applies to the case where $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$.

5.1 Smoothness problems and the need to resolve singularities.

Assume we work on flat space \mathbb{R}^d , then to study the analytic continuation of $I_{G, \vec{k}}$, we need to study integrals of the form :

$$\int_{[0,1]^E} d\ell_1 \dots d\ell_E \left(\int_{\mathbb{R}^{dn}} \prod_{e=1}^E \exp\left(-\frac{|x_{i(e)} - x_{j(e)}|^2}{4\ell_e}\right) \ell_e^{s_e+k_e-\frac{d}{2}-1} a_{k_e} \psi(\mathbf{d}^2) \tilde{\varphi} d^d x_1 \dots d^d x_n \right).$$

The analytic continuation would come from integration by parts on the cube $[0, 1]^E$ w.r.t. the variables (ℓ_1, \dots, ℓ_E) . However, we see immediately that $e^{-|x-y|^2/4\ell}$ is not a smooth function of $(\ell, x, y) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$. The problem occurs at the set $X = \{\ell = 0, x - y = 0\} \subset [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$. A solution in global analysis is to consider the following smooth map :

$$\pi : (t, x, h) \in [0, 1] \times (\mathbb{R}^d)^2 \mapsto (t, x, x + \sqrt{th}) \in [0, 1] \times (\mathbb{R}^d)^2.$$

Note that after pull-back by π , we find that $e^{-|x-y|^2/4\ell} \circ \pi(x, t, h) = e^{-|h|^2/4}$ which is now a smooth function near the preimage $\pi^{-1}(X) = \{t = 0\}$ in $[0, 1] \times (\mathbb{R}^d)^2$. We say that we resolved the singularities of $e^{-|x-y|^2/4\ell}$. For a discussion of why one needs to use blow-ups to study heat kernels and applications to index theory, the reader is referred to [54, p. 253]. Similarly, the product of exponentials $\prod_{e=1}^E \exp\left(-\frac{|x_{i(e)} - x_{j(e)}|^2}{4\ell_e}\right)$ appearing in Feynman amplitudes is also not smooth on the whole domain of integration $(\ell_1, \dots, \ell_E, x_1, \dots, x_n) \in [0, 1]^E \times \mathbb{R}^{dn}$ and integration by parts cannot be done. It follows that we must resolve the products $\prod_{e=1}^E \exp\left(-\frac{|x_{i(e)} - x_{j(e)}|^2}{4\ell_e}\right)$ to make them smooth which is discussed in paragraph 5.4. Such resolution of singularities were studied by Speer on flat space building on the work of Hepp. Also, when (M, g) is an analytic Riemannian manifold or when $M = \mathbb{R}^d$ with constant Euclidean metric, one can use Hironaka's resolution of singularities as in [5] or Bernstein-Sato polynomials to regularize Feynman amplitudes [25, 44]. However, on a Riemannian manifold (M, g) , if for all $m \in M$, there is an open subset U containing m and a local coordinate system $(x_1, \dots, x_d) : U \subset M \mapsto \mathbb{R}^d$ such that for every $(m_1, m_2) \in U^2$, $\mathbf{d}(m_1, m_2) = \sqrt{\sum_{i=1}^d (x_i(m_1) - x_i(m_2))^2}$ then (M, g) is flat. Otherwise for generic Riemannian manifolds (M, g) , it is not possible to find good coordinates to make the distance function locally quadratic because of curvature. This makes our resolution of singularities more difficult to handle than the one appearing in the work of Speer and the fact that we work in the C^∞ case and not in the analytic or algebraic category prevents us from using directly Hironaka's resolution of singularities or Bernstein-Sato polynomials. Following the tradition in QFT [38, 68], our strategy is essentially combinatorial and our blow-ups are encoded by spanning trees of Feynman graphs whose definition is recalled in the next paragraph.

5.2 Spanning trees of metric graphs.

Let us first collect some definitions and classical results on graphs which are close to [50, paragraph 2.1]. Recall that for all graph we consider in the present paper, since we assume the graph has no self-loop, every edge e is adjacent to two different vertices.

Definition 5.1 For a graph G ,

- a path from vertex u to w in a graph G is a sequence $(u = v_1, e_1, v_2, \dots, v_n, e_n, v_{n+1} = w)$, where $v_i \in V(G)$, $e_j \in E(G)$ such that the vertices bounding e_i are v_i and v_{i+1} , u is the initial vertex and w is the terminal vertex, n is called the length. A path is simple if all the edges are distinct. If $u = w$, it is called a **cycle**.
- The set of subgraphs is ordered as follows, we say $G_1 \subset G_2$ for two subgraphs G_1, G_2 of G if $E(G_1) \subset E(G_2)$. A forest T is a graph without any simple cycle and a tree is a connected forest.
- A **spanning forest** T of a graph G is a subgraph of G which is a forest and is maximal for the inclusion relation among subgraphs which are forests. If T is a tree, it is called **spanning tree**. For any graph G , we will often use the following equivalent characterizations of spanning forests which is a classical result in graph theory [52, p. 40-41]. A subgraph $T \subset G$ is a spanning forest if and only if T is a forest whose complement contains $b_1(G)$ edges :

$$b_1(G) = |E(G)| - |E(T)| \quad (17)$$

where $b_1(G)$ is the first Betti number of G .

- To a metric graph (G, ℓ) , a metric filtration of G is an increasing family of subgraphs $G_1 \subset G_2 \subset \dots \subset G_E$ where G_i is **induced by the shortest i edges** where $E = |E(G)|$ is the number of edges in G . For a strict metric graph, the metric filtration is unique.
- For every forest $T \subset G$ and every subgraph G_i of G , we define $T|_{G_i}$ as the subgraph of G_i **induced by the subset of edges $E(T) \cap E(G_i) \subset E(G_i)$** . We will call $T|_{G_i}$ **the trace** of T in G_i .
- If T is a subgraph of G induced by $E(T)$, and $e \in E(G) \setminus E(T)$ then we define $T \cup e$ as the subgraph of G induced by $E(T) \cup e$. For every edge $e \in E(G)$, the graph $G \setminus e$ is the subgraph induced by $E(G) \setminus e$.

Definition 5.2 For any permutation $\sigma \in S_E$ of $\{1, \dots, E\}$, the simplex Δ_σ :

$$\Delta_\sigma = \{\ell_{\sigma(1)} < \dots < \ell_{\sigma(E)}\}. \quad (18)$$

is called a **sector** of $[0, 1]^E$.

Before we proceed, let us remark that for a graph G , an element $\ell \in [0, 1]^{E(G)}$, which is a map $\ell : E(G) \rightarrow [0, 1]$, naturally defines a metric graph (G, ℓ) . To a strict metric graph G , the metric induces a natural strict ordering of edges by the length which defines an element $\ell \in [0, 1]^{E(G)}$ in a unique sector. The next Theorem, due to Kruskal, aims to show how from a strict connected metric graph (G, ℓ) , one can produce some algorithm which extracts a unique spanning tree T in G .

Theorem 5.1 (Kruskal) For a connected strict metric graph (G, ℓ) , let $G_1 \subset \dots \subset G_E = G$ be the unique metric filtration of G . We denote by N_i the first Betti number $b_1(G_i)$ of G_i . Then there exists a unique spanning tree T of G such that for all $i \in \{1, \dots, E\}$, its trace $T|_{G_i}$ is a spanning forest of G_i .

Proof — Let $\ell : E(G) \rightarrow (0, \infty)$ be the length function. We shall assume that the edges $E(G)$ are numbered as $\{e_1, \dots, e_E\}$ in such a way that $i < j \implies \ell(e_i) < \ell(e_j)$. We construct the tree by the Kruskal algorithm [51] as described in [38, p. 107]. Notice that the requirement that T is a tree

implies that T together with all traces $T|_{G_i}$ contain no simple cycles. So $T|_{G_i}$ is a forest and therefore its complement in G_i contains at least N_i edges of G_i . Also notice that for any graph G , for every $e \in E(G)$, we have the inequality $0 \leq b_1(G) - b_1(G \setminus e) \leq 1$. This implies that the sequence N_1, N_2, \dots is increasing.

Now we can construct the desired spanning tree T : start from G_1 which has only one edge $\{e_1\}$ hence contains no simple cycle, $N_1 = 0$. Let us denote by i_1 the first integer such that $b_1(G_{i_1}) = 1$, similarly define $\{i_2, \dots, i_{N_E}\} \subset \{1, \dots, E\}$ such that $b_1(G_{i_2}) = 2, \dots, b_1(G_{i_{N_E}}) = N_E = b_1(G)$ and every i_j is the smallest integer so that $b_1(G_{i_j}) = j$ for any $j = 1, \dots, N_E$. Set $i_0 = 1$, then we have an increasing family of subgraphs $G_{i_0} \subset G_{i_1} \subset \dots \subset G_{i_{N_E}} \subset G$. Let $G_i = G_{i-1} \cup e_i$ and we set $T = G \setminus \bigcup_{j=1}^{N_E} e_{i_j}$. We prove that the subgraph T constructed above has the property that its trace $T|_{G_i}$ to every subgraph G_i is a spanning forest in G_i by induction for $j = 1, \dots, E$. First, we initialize the induction for $j = 1$, G_1 contains just one edge hence $T|_{G_1} = G_1$ is a spanning tree in G_1 . Assume that $T|_{G_k}$ is a spanning forest in G_k , then there are two cases:

Case 1 : $b_1(G_k) = b_1(G_{k+1})$ i.e. $N_k = N_{k+1}$ so $e_{k+1} \in T$, and $T|_{G_{k+1}} = T|_{G_k} \cup e_{k+1}$, and let us prove that $T|_{G_k} \cup e_{k+1}$ is a spanning forest in G_{k+1} . First $T|_{G_k} \cup e_{k+1}$ contains no simple cycle. Since if it contained a simple cycle γ , then e_{k+1} would belong to γ and therefore $T|_{G_k}$ would be a spanning forest in G_{k+1} , so $b_1(G_{k+1}) = b_1(G_k) + 1$ by equation (17) which contradicts $b_1(G_k) = b_1(G_{k+1})$. $T|_{G_k} \cup e_{k+1}$ is thus a forest. $T|_{G_k}$ is spanning in G_k hence $T|_{G_k} \cup e_{k+1}$ meets all vertices of G_{k+1} and is spanning in G_{k+1} .

Case 2 : $b_1(G_k) + 1 = b_1(G_{k+1})$ and by definition $T|_{G_{k+1}} = T|_{G_k}$. $T|_{G_k}$ is obviously a forest in G_{k+1} , its complement in G_{k+1} contains $b_1(G_k) + 1 = b_1(G_{k+1})$ edges by construction which implies it is spanning by equation 17.

Now we use induction to prove the uniqueness of T , in fact, we prove $T|_{G_k}$ is unique for any k . The initial step is trivial, and in general there are two cases. Either $b_1(G_k) = b_1(G_{k+1})$, then $T|_{G_{k+1}} = T|_{G_k} \cup e_{k+1}$ or $b_1(G_k) + 1 = b_1(G_{k+1})$ then $T|_{G_{k+1}} = T|_{G_k}$, so our algorithm produces a unique spanning tree. ■

Corollary 5.1 *Let (G, ℓ) be a strict metric graph and T be the unique spanning forest in T from Theorem 5.1. Then for every edge $e \in E(G) \setminus E(T)$, there is a unique simple cycle γ_e in $T \cup e$, such that for any edge $e' \in \gamma_e \setminus \{e\}$, $\ell(e) > \ell(e')$.*

Proof — Since T is a spanning tree, $e \in E(G) \setminus E(T)$, so there is a unique simple cycle γ_e in $T \cup e$. By our construction, if $e = e_{i_j}$, then $T|_{G_{i_{j-1}}} = T|_{G_{i_j}}$, and $T|_{G_{i_{j-1}}} \cup e_{i_j}$ contains only one simple cycle γ_e , so for any edge $e' \in \gamma_e$, $e' \neq e$, $\ell(e) > \ell(e')$. ■

5.3 Approximation of the Riemannian distance in normal coordinates.

For a smooth Riemannian manifold (M, g) , and any $x \in M$, $g(x)$ is an inner product in $T_x M$ which induces an isomorphism $g(x) : T_x M \rightarrow T_x^* M$ and thus an inner product $g^{-1}(x)$ on $T_x^* M$ by $g^{-1}(x)(w_1, w_2) = g(x)(g^{-1}(x)(w_1), g^{-1}(x)(w_2))$. This defines a smooth metric g^{-1} on $T^* M$.

For every $x \in M$, we will use normal coordinates (U, ϕ, x^μ) around x , without loss of generality U is assumed to be geodesically convex. The use of normal coordinates will be crucial since it allows us to approximate the squared distance $\mathbf{d}^2(x, y)$ by $|x - y|^2$ in local coordinates in Lemma 5.2. In some other coordinate chart, this approximation might not be as good. On (U, ϕ, x^μ) , there are two metrics: the Riemannian metric g and the Euclidean metric h :

$$h\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = h_{\mu\nu} = \delta_{\mu\nu}.$$

For this Euclidean metric, we will use $|x - y|$ to denote the induced distance. We recall that at the origin, we have the identity $g_{\mu\nu}(0) = h_{\mu\nu} = \delta_{\mu\nu}$. The following Lemma which dates back to Hadamard can be found in [28, Lemma 8.3 p. 90], [57, (A.3) p. 31], [67, (38) p. 171] :

Lemma 5.1 (Hadamard) *We denote by \mathbf{d} the Riemannian distance and $\phi = \mathbf{d}^2$. Then there exists a neighborhood U of the diagonal $\subset M \times M$, such that $\phi \in C^\infty(U)$ and is symmetric, that is $\phi(x, y) = \phi(y, x)$, ϕ vanishes along the diagonal at order 2 and ϕ satisfies the Hamilton–Jacobi equation :*

$$\boxed{g^{-1}(d_x\phi(x, y), d_x\phi(x, y)) = 4\phi(x, y).} \quad (19)$$

Next we state an important Lemma which gives informations on the jets of the function $\phi = \mathbf{d}^2$ along the diagonal in $M \times M$.

Lemma 5.2 *For $x_0 \in M$, if (U, ϕ) is a normal coordinate system around x_0 such that $\phi(x_0) = 0$ and the square of Riemannian distance ϕ is smooth on $U \times U$, then on $U \times U$,*

$$\boxed{\phi(x, y) - g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu),} \quad (20)$$

vanishes along the diagonal at order 3.

The proof can be found in appendix 7.5.

5.4 Resolving singularities using spanning trees.

Let (G, ℓ) be a connected strict metric graph with edge set $E(G)$ identified with the set of integers $\{1, \dots, E\}$ in such a way that $0 \leq \ell_1 < \dots < \ell_E \leq 1$. This means that the metric graph (G, ℓ) lies in a fixed sector $\Delta = \{0 < \ell_1 < \dots < \ell_E < 1\} \subset [0, 1]^E$, $\overline{\Delta}$ denotes its closure $\{0 \leq \ell_1 \leq \dots \leq \ell_E \leq 1\}$. It is associated with a unique spanning tree T by Theorem 5.1 and the vertices of both graphs G and T are numbered by $\{1, \dots, n\}$. For any $(i, j) \in \{1, \dots, n\}^2$, we denote by \overrightarrow{ij} the unique simple path in T from i to j .

The product $\prod_{e \in E(G)} e^{-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}} \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))$, where ψ is the cut-off function from Theorem 3.1, is not smooth near the algebraic set $X = \cup_{e \in E(G)} \{\ell_e(x_{i(e)} - x_{j(e)}) = 0\} \subset \mathbb{R}^{dn} \times \overline{\Delta}$. What we do next is give a recipe to resolve the singularities of such products by some explicit map π which is defined as follows :

Definition 5.3 *In the above notation, the map from $\mathbb{R}^d \times (\mathbb{R}^d)^{E(T)} \times [0, 1]^E$ to $\mathbb{R}^{dn} \times \overline{\Delta}$ is :*

$$\boxed{\pi : \underbrace{(x, (h_e)_{e \in E(T)}, (t_k)_{k=1}^E)}_{\in \mathbb{R}^d \times (\mathbb{R}^d)^{E(T)} \times [0, 1]^E} \mapsto \underbrace{\left(x + \sum_{e \in \overrightarrow{1i}} \left(\prod_{j \geq e} t_j \right) h_e \right)_{i=1}^n, \left(\prod_{k \geq e} t_k^2 \right)_{e=1}^E}_{\in \mathbb{R}^{dn} \times \overline{\Delta}}} \quad (21)$$

where the sum runs over all edges e in the path $\overrightarrow{1i}$. The map π depends on spanning tree T hence on the strict ordering of $E(G)$ induced by the metric ℓ .

In the sequel, we shall denote elements of the target space $\mathbb{R}^{dn} \times \overline{\Delta}$ by $(x_i, \ell_e)_{1 \leq i \leq n, 1 \leq e \leq E}$. We check that the map $\pi : \mathbb{R}^d \times (\mathbb{R}^d)^{E(T)} \times [0, 1]^E \mapsto \mathbb{R}^{dn} \times \overline{\Delta}$ is a diffeomorphism outside some subset of measure zero.

Proposition 5.1 *The map π is a smooth diffeomorphism from $\mathbb{R}^d \times \mathbb{R}^{d(n-1)} \times (0, 1)^E$ to $\mathbb{R}^{dn} \times \Delta$.*

Proof — It is one to one since we can explicitly invert π as $t_E = \ell_E^{\frac{1}{2}}$, $t_e = \left(\frac{\ell_{e+1}}{\ell_e} \right)^{\frac{1}{2}}$ for $e \leq E-1$ and the linear map: $(x, (h_e)_{e \in E(T)}) \in \mathbb{R}^d \times \mathbb{R}^{d(n-1)} \mapsto \left(x_i = x + \sum_{e \in \overrightarrow{1i}} \left(\prod_{j \geq e} t_j \right) h_e \right)_{i=1}^n \in \mathbb{R}^{dn}$ is **invertible** when $(t_j)_j \in (0, 1)^E$. Then the diffeomorphism property follows from an explicit calculation of the differential of π whose determinant does not vanish when $t \in (0, 1)^E$. ■

Finally we may state the main Theorem from this section :

Theorem 5.2 (Resolution of singularities.) *Let g be a Riemannian metric on \mathbb{R}^d and $\mathbf{d} : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}_{\geq 0}$ be the Riemannian distance whose injectivity radius is ε . Let π be the map defined by equation (21). For any $\psi(t) \in C_c^\infty(\mathbb{R})$ such that $\psi(t) = 1$ when $t \leq \frac{\varepsilon^2}{4}$ and $\psi(t) = 0$ when $t > \frac{4\varepsilon^2}{9}$, for every edge $e \in E(G)$ bounded by the vertices $(i(e), j(e))$, the pull-back :*

$$\pi^* \left(\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{\ell_e} \right) \quad (22)$$

defines a smooth function in $\mathbb{R}^d \times \mathbb{R}^{d(n-1)} \times [0, 1]^E$.

Proof — For $e \in E(T)$, set $x_{j(e)} - x_{i(e)} = \pm h_e$. Recall that h_e has in fact d components $(h_e^\mu)_{\mu=1}^d$. Then by Lemma 5.2 :

$$\begin{aligned} \pi^* \left(\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{\ell_e} \right) &= \frac{g_{\mu\nu}(x_{i(e)}) h_e^\mu h_e^\nu (\prod_{i \geq e} t_i)^2 + R(\pi^* x_{i(e)}, \pi^* x_{i(e)} \pm (\prod_{i \geq e} t_i) h_e)}{(\prod_{i \geq e} t_i)^2} \\ &= g_{\mu\nu}(x_{i(e)}) h_e^\mu h_e^\nu + r_e(t, x, h) \end{aligned}$$

where

$$r_e(t, x, h) = \frac{R(\pi^* x_{i(e)}, \pi^* x_{i(e)} + (\prod_{i \geq e} t_i) h_e)}{(\prod_{i \geq e} t_i)^2} = \frac{O((\prod_{i \geq e} t_i)^3 \|h_e\|^3)}{(\prod_{i \geq e} t_i)^2} = O((\prod_{i \geq e} t_i) \|h_e\|^3)$$

vanishes at order 3 in $(h_e)_{e \in T}$ and at order 1 in $(t_e)_{e=1}^E$ by Lemma 5.2 and r_e is smooth.

If $e \notin E(T)$, then

$$\pi^* \left(\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{\ell_e} \right) = \pi^* \left(\frac{g_{\mu\nu}(x_{i(e)}) (x_{i(e)}^\mu - x_{j(e)}^\mu) (x_{i(e)}^\nu - x_{j(e)}^\nu)}{\ell_e} \right) + \pi^* \left(\frac{R(x_{i(e)}, x_{j(e)})}{\ell_e} \right).$$

where

$$\begin{aligned} &\pi^* \left(\frac{g_{\mu\nu}(x_{i(e)}) (x_{i(e)}^\mu - x_{j(e)}^\mu) (x_{i(e)}^\nu - x_{j(e)}^\nu)}{\ell_e} \right) \\ &= \frac{g_{\mu\nu}(x_{i(e)}) (\sum_{e' \in \gamma_e \setminus e} \varepsilon(e') \left(\prod_{j \geq e'} t_j \right) h_{e'}^\mu) (\sum_{e' \in \gamma_e \setminus e} \varepsilon(e') \left(\prod_{j \geq e'} t_j \right) h_{e'}^\nu)}{(\prod_{i \geq e} t_i)^2} \end{aligned}$$

where $\varepsilon(e') = \pm 1$ and γ_e is the unique simple cycle in $T \cup e$ in Corollary 5.1. The important fact is that for all edge e' in the path $\gamma_e \setminus \{e\}$, $e' < e$. It follows that :

$$\begin{aligned} &\pi^* \left(\frac{g_{\mu\nu}(x_{i(e)}) (x_{i(e)}^\mu - x_{j(e)}^\mu) (x_{i(e)}^\nu - x_{j(e)}^\nu)}{\ell_e} \right) \\ &= g_{\mu\nu}(x_{i(e)}) \left(\sum_{e' \in \gamma_e \setminus e} \varepsilon(e') \left(\prod_{e' \leq j < e} t_j \right) h_{e'}^\mu \right) \left(\sum_{e' \in \gamma_e \setminus e} \varepsilon(e') \left(\prod_{e' \leq j < e} t_j \right) h_{e'}^\nu \right) \end{aligned}$$

which is smooth since the product $(\prod_{i \geq e} t_i)^2$ on the denominator cancel out with the same powers appearing on the numerator. The same argument applies to the remainder term $\pi^* \left(\frac{R(x_{i(e)}, x_{j(e)})}{\ell_e} \right)$. ■

5.5 Change of variables

For a test function φ supported in \mathbb{R}^{dn} , since the map π is a smooth diffeomorphism from $\mathbb{R}^d \times \mathbb{R}^{d(n-1)} \times (0,1)^E$ to $\mathbb{R}^{dn} \times \Delta$, we can take it as a change of variables for integration:

$$\begin{aligned} \langle I_{G, \vec{k}}(s), \varphi \rangle &= \frac{1}{(4\pi)^{\frac{dE}{2}}} \left(\prod_{e=1}^E \frac{1}{\Gamma(s_e)} \right) \int_{[0,1]^E} dl_1 \dots dl_E \int_{\mathbb{R}^{dn}} \prod_{e=1}^E \exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \\ &\quad \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \ell_e^{s_e+k_e-\frac{d}{2}-1} a_{k_e}(x_{i(e)}, x_{j(e)}) \tilde{\varphi} d^d x_1 \dots d^d x_n \\ &= \frac{1}{(4\pi)^{\frac{dE}{2}}} \left(\prod_{e=1}^E \frac{1}{\Gamma(s_e)} \right) \sum_{\sigma \in S_E} \int_{\Delta_\sigma} dl_1 \dots dl_E \int_{\mathbb{R}^{dn}} \prod_{e=1}^E \exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \\ &\quad \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \ell_e^{s_e+k_e-\frac{d}{2}-1} a_{k_e}(x_{i(e)}, x_{j(e)}) \tilde{\varphi} d^d x_1 \dots d^d x_n \end{aligned}$$

where σ runs over the group S_E of permutations of $\{1, \dots, E\}$. The open simplices Δ_σ do not cover the unit cube $[0,1]^E$. However the complement of $\cup_\sigma \Delta_\sigma$ in $[0,1]^E$ has zero Lebesgue measure. Since for $Re(s_e)_{e=1, \dots, E}$ large enough, the integral $\int_{[0,1]^E} \prod_{e \in E(G)} \frac{e^{-\frac{d^2}{4\ell_e}}}{(4\pi)^{\frac{d}{2}}} a_{k_e} \psi(\mathbf{d}^2) \ell_e^{k_e-\frac{d}{2}+s_e-1} d\ell_e$ is absolutely convergent and depends holomorphically in $s \in \mathbb{C}^E$, we have the equality of integrals

$$\int_{[0,1]^E} \prod_{e \in E(G)} \frac{e^{-\frac{d^2}{4\ell_e}}}{(4\pi)^{\frac{d}{2}}} a_{k_e} \psi(\mathbf{d}^2) \ell_e^{k_e-\frac{d}{2}+s_e-1} d\ell_e = \sum_{\sigma \in S(E)} \int_{\Delta_\sigma} \prod_{e \in E(G)} \frac{e^{-\frac{d^2}{4\ell_e}}}{(4\pi)^{\frac{d}{2}}} a_{k_e} \psi(\mathbf{d}^2) \ell_e^{k_e-\frac{d}{2}+s_e-1} d\ell_e,$$

where both sides depend holomorphically on $(s_e)_e$ for $Re(s_e)_{e=1, \dots, E}$ large enough.

Now we can carry the change of variables in the fixed sector $\Delta = \{0 < \ell_1 < \dots < \ell_E < 1\}$ (the other terms will be obtained by permutation) which yields an expression of the form :

$$\begin{aligned} &\int_{\Delta} \prod_{e=1}^E \frac{d\ell_e}{\ell_e} \left(\int_{(\mathbb{R}^d)^n} \prod_{e=1}^E \exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \ell_e^{s_e+k_e-\frac{d}{2}} a_{k_e}(x_{i(e)}, x_{j(e)}) \tilde{\varphi} d^d x_1 \dots d^d x_n \right) \\ &= 2^E \int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} \int_{(\mathbb{R}^d)^n} \pi^* \left(\prod_{e=1}^E \exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \right) \\ &\times \pi^* \left(\tilde{\varphi} \prod_{e \in E(G)} a_{k_e} \right) \left(\prod_{e \in E(G)} \ell_e^{(s_e+k_e)-\frac{d}{2}} \right) \left(\prod_{e \in E(T)} \ell_e^{\frac{d}{2}} \right) d^d x \prod_{e \in E(T)} d^d h_e. \end{aligned}$$

We can further simplify the product $\prod_{e \in E(G)} \ell_e(t)^{(s_e+k_e)-\frac{d}{2}} \prod_{e \in E(T)} \ell_e(t)^{\frac{d}{2}}$ as :

$$\begin{aligned} &\prod_{e \in E(G)} \left(\prod_{i \geq e} t_i \right)^{2(s_e+k_e)-d} \prod_{e \in E(T)} \left(\prod_{i \geq e} t_i \right)^d = \prod_{e \in E(T)} \left(\prod_{i \geq e} t_i \right)^{2s_e+2k_e} \prod_{e \notin E(T)} \left(\prod_{i \geq e} t_i \right)^{2s_e-d+2k_e} \\ &= \prod_{e \in E(G)} \left(\prod_{i \geq e} t_i \right)^{2s_e+2k_e} \prod_{e \notin E(T)} \left(\prod_{i \geq e} t_i \right)^{-d} \\ &= t_E^{2s_E+2k_E} (t_E t_{E-1})^{2s_{E-1}+2k_{E-1}} \dots (t_E \dots t_1)^{2s_1+2k_1} (t_E \dots t_{i_k})^{-d} \dots (t_E \dots t_{i_1})^{-d} \end{aligned}$$

where $(i_1 < \dots < i_k) \subset \{1, \dots, E\}$ are the numbers decorating the edges in the complement of $E(T)$ and $k = b_1(G)$, hence :

$$\prod_{e \in E(T)} \ell_e(t)^{2s_e+2k_e} \prod_{e \notin E(T)} \ell_e(t)^{2s_e-d+2k_e} = \prod_{e=1}^E t_e^{\sum_{i \leq e} 2s_i+2k_i-d b_1(G_e)} \quad (23)$$

where G_e denotes the graph induced by the first e edges $\{1, \dots, e\}$. This in turns implies that we obtain the simplified form :

$$\begin{aligned} & \int_{\Delta} \prod_{e=1}^E \frac{d\ell_e}{\ell_e} \left(\int_{(\mathbb{R}^d)^n} \prod_{e=1}^E \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \ell_e^{s_e+k_e-\frac{d}{2}} a_{k_e}(x_{i(e)}, x_{j(e)}) \tilde{\varphi} d^d x_1 \dots d^d x_n \right) \\ &= \int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} t_e^{(\sum_{i \leq e} 2s_i + 2k_i) - db_1(G_e)} \int_{(\mathbb{R}^d)^n} A(t_e, x, h_e) d^d x \prod_{e \in E(T)} d^d h_e \end{aligned} \quad (24)$$

where

$$A((t_e)_{e=1}^E, x, (h_e)_{e \in E(T)}) = 2^E \pi^* \left(\left(\prod_{e=1}^E \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) a_{k_e} \right) \tilde{\varphi} \right).$$

The coordinates $((t_e)_{e=1}^E, x, (h_e)_{e \in E(T)})$ on $[0, 1]^E \times \mathbb{R}^d \times (\mathbb{R}^d)^{E(T)}$ will be shortly denoted by (t, x, h) for simplicity. We now prove the smoothness in $t \in [0, 1]^E$ of the partial integral $\int_{(\mathbb{R}^d)^n} A(t, x, h) d^d x \prod_{e \in E(T)} d^d h_e$ which is needed to ensure analytic continuation.

Lemma 5.3 *The map $t \mapsto \int_{(\mathbb{R}^d)^n} A(t, x, h) d^d x \prod_{e \in E(T)} d^d h_e$ belongs to $C^\infty([0, 1]^E)$.*

Proof — The smoothness of A is a direct consequence of Theorem 5.2. Now start from the definition of A :

$$\begin{aligned} A(t, x, h) &= 2^E \pi^* \left(\prod_{e \in E(T)} \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) a_{k_e} \right) \\ &\times \pi^* \left(\prod_{e \notin E(T)} \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) a_{k_e} \right) \\ &\times \tilde{\varphi}(x + \sum_{e \in (r,1)} (\prod_{j \geq e} t_j) h_e, \dots, x + \sum_{e \in (r,n)} (\prod_{j \geq e} t_j) h_e). \end{aligned}$$

Then use the key fact that since the open neighborhood $(\cong \mathbb{R}^d)$ is chosen small enough and has compact closure, there exists a fixed constant $\delta > 0$ such that for all $x, y \in \mathbb{R}^d$, we have the following bound on the Riemannian distance :

$$\delta |x - y| \leq \mathbf{d}(x, y) = |x - y| + o(|x - y|) \leq \delta^{-1} |x - y|$$

which follows from Lemma 5.2 since $\mathbf{d}^2(x, y) - \sum g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu)$ vanishes along the diagonal at order 3, and locally $\sum g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu)$ is bounded by some multiple of $|x - y|$ by compactness of the neighborhood. It follows that

$$\frac{\delta^2 |x - y|^2}{4\ell_e} \leq \frac{\mathbf{d}(x, y)^2}{4\ell_e} \leq \frac{\delta^{-2} |x - y|^2}{4\ell_e}$$

which implies that for all edges $e \in E(T)$, we have the bound :

$$\pi^* \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \leq \pi^* \exp\left(-\frac{\delta^2 |x - y|^2}{4\ell_e}\right) = \exp\left(-\frac{\delta^2 |h_e|^2}{4}\right).$$

This allows us to use the product $\pi^* \left(\prod_{e \in E(T)} \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \right)$ to control the exponential decay of A :

$$\pi^* \left(\prod_{e \in E(T)} \exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right) \right) \leq \prod_{e \in E(T)} \exp\left(-\frac{\delta^2 |h_e|^2}{4}\right).$$

By smoothness of the heat coefficients a_k , by compactness of the support of $\varphi \in C_c^\infty(U^n)$ thus of $\tilde{\varphi}$, and by the definition of π , for every multi-index α , we find that there exists some constant $C_\alpha > 0$ s.t.

$$|\partial_t^\alpha \pi^* \left(\tilde{\varphi} \prod_{e \notin E(T)} \exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \prod_{e=1}^E a_{k_e} \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) \right)| \leq C_\alpha \left(1 + \sum_{e \in E(T)} |h_e| \right)^{|\alpha|},$$

the partial derivatives in t contributes the powers of h .

Therefore we have the bound for all (t, x, h) :

$$|\partial_t^\alpha A(t_e, x, h_e)| \leq C_\alpha \prod_{e \in E(T)} \exp \left(-\frac{\delta^2 |h_e|^2}{4} \right) \left(1 + \sum_{e \in E(T)} |h_e| \right)^{|\alpha|}$$

with A compactly supported in x . Then smoothness of $t \mapsto \int_{(\mathbb{R}^d)^n} A(t, x, h) d^d x \prod_{e \in E(T)} d^d h_e$ follows from smoothness of the integrand which is the function $A \in C^\infty([0, 1]^E \times (\mathbb{R}^d)^n)$, and any derivative $\partial_t^\alpha A$ has fast decrease in h when $|h| \rightarrow +\infty$ and compact support in the variable $x \in \mathbb{R}^d$. Therefore all derivatives in $\partial_t^\alpha A$ are integrable, uniformly in $t \in [0, 1]^E$ and the conclusion follows from classical results on integrals depending smoothly on parameters. \blacksquare

Lemma 5.4 [*Jet Lemma*] *Fix the sector Δ corresponding to the system of inequalities $\{0 \leq \ell_1 \leq \dots \leq \ell_E\}$. Let*

$$A((t_e)_{e=1}^E, x, (h_e)_{e \in T}) = \pi^* \left(\prod_{e=1}^E \left(\exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) a_{k_e} \right) \tilde{\varphi} \right).$$

Then the k -jet of

$$\chi(t_1, \dots, t_E) = \int_{(\mathbb{R}^d)^n} A((t_e)_{e=1}^E, x, (h_e)_{e \in E(T)}) d^d x \prod_{e \in E(T)} d^d h_e$$

depends continuously on the k -jet of $(a_{k_i}, i = 1, \dots, E, \varphi, \mathbf{d}^2, g)$.

Proof — The claim follows from the formulas defining A and the change of variables π and repeated application of the chain rule to $\pi^* \left(\prod_{e=1}^E \left(\exp \left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e} \right) \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)})) a_{k_e} \right) \varphi \right)$. \blacksquare

Recall that from Theorem 4.1, the main Theorem 2.1 reduces to an analytic continuation result for the amplitudes $I_{G, \vec{k}}(s)$ corresponding to the labelled Feynman graphs (G, \vec{k}) . The problem was that the integral formula for $I_{G, \vec{k}}(s)$ involved some product of heat kernels which required blow-ups which were performed in sectors. The following Proposition shows how the integral expression $I_{G, \vec{k}}(s)$ simplifies after blow-up :

Proposition 5.2 *Let us consider $I_{G, \vec{k}}(s)$ as in Equation (15), for any $e \in \{1, \dots, E\} \simeq E(G)$ and any permutation $\sigma \in S_E$, $G_{\sigma(e)}$ is the subgraph of G induced by the collection of edges $\{\sigma(1), \dots, \sigma(e)\} \subset E(G)$. Then for every test function $\varphi \in C_c^\infty(U^n)$, there exists a family $\chi_\sigma \in C^\infty([0, 1]^E)$ indexed by $\sigma \in S_E$ such that the identity :*

$$\int_{\mathbb{R}^{dn}} I_{G, \vec{k}}(s) \varphi d^{nd} x = \prod_{e=1}^E \frac{1}{\Gamma(s_e)} \sum_{\sigma \in S_E} \int_{[0, 1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} t_e^{\sum_{i \leq e} (2s_{\sigma(i)} + 2k_{\sigma(i)}) - db_1(G_{\sigma(e)})} \chi_\sigma(t) \quad (25)$$

holds true for all $Re(s_e), e \in \{1, \dots, E\}$ large enough and both sides are holomorphic in $s \in \mathbb{C}^E$.

In the next paragraph, we will proceed to the meromorphic continuation of the r.h.s of equation (25) as meromorphic function with linear poles in s and we will also bound the distributional order of $I_{G, \vec{k}}(s)$ independently of the label \vec{k} .

5.6 Integration by parts, bounding orders and pole decomposition.

Now the proof of Theorem 2.1 on the analytic continuation of $t_G(s)$ is reduced to the meromorphic continuation in $s \in \mathbb{C}^E$ of the right hand side of Equation (25). The meromorphic continuation comes from integration by parts as shown in the next Lemma and its corollary.

Lemma 5.5 *Let E be a positive integer, for any smooth function ψ on $[0, 1]^E$,*

$$I_s(\psi) = \int_{[0,1]^E} t_1^{s_1} \dots t_E^{s_E} \psi(t_1, \dots, t_E) d^E t$$

can be analytically extended to a meromorphic germ at $(s_e = p_e)_{e=1}^E \in \mathbb{Z}^E$, more precisely, let $I = \{i \mid p_i < 0\}$

$$\left(\prod_{i \in I} (s_i - p_i) \right) I_s(\psi) \tag{26}$$

extends to a holomorphic germ at $(s_e = p_e)_e$ and $I_s \in \mathcal{D}'^m([0, 1]^E, \mathcal{M}_{s_0}(\mathbb{C}^E))$, $s_0 = (p_1, \dots, p_E)$, $m = \sum_{i \in I} |p_i|$.

The proof of this Lemma, given in the appendix, follows from integration by parts. One consequence of this Lemma is

Corollary 5.2 *Denote by $1_{[0,1]^E}$ the indicator function of the unit cube $[0, 1]^E \subset \mathbb{R}^E$. Let (L_1, \dots, L_E) be linear functions in $s \in \mathbb{C}^E$ with real coefficients $L_i \in (\mathbb{R}^E)^*$, $1 \leq i \leq E$, and $(a_1, \dots, a_E) \in \mathbb{Z}^E$. Set $I = \{i \mid a_i < 0\}$, then*

$$\left(\prod_{i \in I} L_i(s) \right) \int_{[0,1]^E} t_1^{L_1(s)+a_1} \dots t_E^{L_E(s)+a_E} \psi(t_1, \dots, t_E) d^E t$$

is a holomorphic germ at $s = 0 \in \mathbb{C}^E$, and

$$1_{[0,1]^E} t_1^{L_1(s)+a_1} \dots t_E^{L_E(s)+a_E}$$

extends to an element in $\mathcal{D}'^m(\mathbb{R}^E, \mathcal{M}_0(\mathbb{C}^E))$ where $m = \sum_{i \in I} |a_i|$ and the polar set is contained in $\{\prod_{i \in I} L_i = 0\}$.

Applying Corollary 5.2 to the r.h.s. of Equation (25) shows that :

Lemma 5.6 *Let S_σ be the set of all subgraphs $H \in \{G_{\sigma(1)} \subset \dots \subset G_{\sigma(E)} = G\}$ such that $b_1(H) \geq 1$, then*

$$\prod_{H \in S_\sigma} \left(\sum_{e \in E(H)} s_e - |E(H)| \right) \left(\int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} t_e^{\sum_{i \leq e} (2s_{\sigma(i)} + 2k_{\sigma(i)}) - db_1(G_{\sigma(e)})} \chi_\sigma(t) \right)$$

is a holomorphic germ at $(s_e = 1)_e$.

Proof — By Corollary 5.2 and a change of variables $s'_e = s_e - 1$, $e \in \{1, \dots, E\}$, we need to consider the following set of indices :

$$I = \{e \mid e + \sum_{i \leq e} k_{\sigma(i)} - \frac{d}{2} b_1(G_{\sigma(e)}) \leq 0\} \subset \{1, \dots, E\}$$

which is contained in $\{e \mid b_1(G_{\sigma(e)}) \geq 1\}$, which yields the conclusion. ■

Let us comment on the above bound on the location of the pole. First the bound seems suboptimal since the set of indices $I = \{e \mid e + \sum_{i \leq e} k_{\sigma(i)} - \frac{d}{2} b_1(G_{\sigma(e)}) \leq 0\} \subset \{1, \dots, E\}$ is only a subset of $\{e \mid b_1(G_{\sigma(e)}) \geq 1\} \subset \{1, \dots, E\}$. However, it is important for us that we can give a bound on the location of the poles which does not depend on the multi-index \vec{k} since poles from the original Feynman amplitude $t_G(s)$ do not depend on \vec{k} . The formula of Theorem 4.1 expresses t_G as a sum of $I_{G, \vec{k}}$ for some \vec{k} . Hence the poles of $t_G(s)$ come from contributions from the poles of $I_{G, \vec{k}}$. Therefore it is convenient to have some \vec{k} independent bound for poles of $I_{G, \vec{k}}$. Finally, we bound the distributional order of $I_{G, \vec{k}}$ and also give precise location on the affine planes supporting the poles of $I_{G, \vec{k}}$ in the following :

Proposition 5.3 (Poles of $I_{G, \vec{k}}$ and distributional order) *Let G be a graph whose set of edges is in bijection with $\{1, \dots, E\}$. For any $e \in \{1, \dots, E\} \simeq E(G)$ and any permutation $\sigma \in S_E$, $G_{\sigma(e)}$ is the subgraph of G induced by the collection of edges $\{\sigma(1), \dots, \sigma(e)\} \subset E(G)$. For every permutation $\sigma \in S_E$, we associate the filtration $\{G_{\sigma(1)} \subset \dots \subset G_{\sigma(E)} = G\}$, and we consider the set S_σ of all subgraphs $H \in \{G_{\sigma(1)} \subset \dots \subset G_{\sigma(E)} = G\}$ such that $b_1(H) \geq 1$. For every $\vec{k} \in \mathbb{N}^{E(G)}$, the distribution $I_{G, \vec{k}}(s)$ defined in definition 15 can be analytically continued to $\mathcal{D}'^m(U^n, \mathcal{M}_{s_0}(\mathbb{C}^E))$, $s_0 = (s_e = 1)_{e=1}^E$ where*

$$m = \sum_{H \subset G; 2|E(H)| - db_1(H) - 1 < 0} db_1(H) - 2|E(H)| + 1. \quad (27)$$

For every test function φ ,

$$\int_{\mathbb{R}^{dn}} I_{G, \vec{k}}(s) \varphi d^{nd}x = \sum_{\sigma \in S_E} \prod_{H \in S_\sigma} \frac{1}{(\sum_{i \in H} s_i - E(H))} f_\sigma(s)$$

where f_σ is holomorphic germ at $(s_e = 1)_e$.

Remark 5.1 *The bound on the distributional order depends only on the topology of the graph G and the dimension d and not on the element $\vec{k} \in \mathbb{N}^E$. The bound on the distributional order is also not sharp since we should only sum over subgraphs $G' \in \{G_{\sigma(1)}, \dots, G_{\sigma(E)}\}$ such that $2|E(G')| - db_1(G') - 1 < 0$ then take the supremum over all permutations σ .*

Proof — We proved in Proposition 5.2 that for every labelled graph (G, \vec{k}) , and every test function $\varphi \in C_c^\infty(U^n)$, $n = |V(G)|$, there exists a family $\chi_\sigma(t)$ of smooth functions on the cube $[0, 1]^E$ indexed by permutations $\sigma \in S_E$ such that:

$$\int_{\mathbb{R}^{dn}} I_{G, \vec{k}}(s) \varphi d^{nd}x = \prod_{e=1}^E \frac{1}{\Gamma(s_e)} \sum_{\sigma \in S_E} \int_{[0, 1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} t_e^{\sum_{i \leq e} (2s_{\sigma(i)} + 2k_{\sigma(i)}) - db_1(G_{\sigma(e)})} \chi_\sigma(t).$$

By applying Lemma 5.6 to

$$\sum_{\sigma \in S_E} \int_{[0, 1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} t_e^{\sum_{i \leq e} (2s_{\sigma(i)} + 2k_{\sigma(i)}) - db_1(G_{\sigma(e)})} \chi_\sigma(t),$$

we obtain the meromorphic continuation of $s \mapsto \int_{\mathbb{R}^{dn}} I_{G, \vec{k}}(s) \varphi d^{nd}x$ with the bound on the location of poles. To show that $I_{G, \vec{k}}(s)$ is actually an element of $\mathcal{D}'^m(U^n, \mathcal{M}_{s_0}(\mathbb{C}^E))$, $s_0 = (s_e = 1)_{e=1}^E$, we need to show that

$$\varphi \in C_c^\infty(U^n) \mapsto \int_{\mathbb{R}^{dn}} I_{G, \vec{k}}(\cdot) \varphi d^{nd}x \in \mathcal{M}_{s_0}(\mathbb{C}^E)$$

depends linearly on the m -jet of φ for some m . By Corollary 5.2, the integral

$$\int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} t_e^{\sum_{i \leq e} (2s_{\sigma(i)} + 2k_{\sigma(i)}) - db_1(G_{\sigma(e)})} \chi_{\sigma}(t)$$

depends linearly on the m -jet of χ_{σ} for

$$m = \sum_{G' \subset G; 2|E(G')| - db_1(G') - 1 < 0} db_1(G') - 2|E(G')| + 1.$$

Then by Lemma 5.4, the m -jet of χ_{σ} depends continuously on the m -jet of φ which yields the result. ■

Now let us restate the first main Theorem from our paper and conclude its proof :

Theorem 5.3 *Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d , $dv(x)$ the Riemannian volume and $P = -\Delta_g + V$, $V \in C_{\geq 0}^{\infty}(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + \lambda^2$, $\lambda \in \mathbb{R}_{\geq 0}$. Then for every graph G ,*

$$t_G(s) = \prod_{e \in E(G)} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)}) \quad (28)$$

can be analytically continued as an element of $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ where $s_0 = (s_e = 1)_{e \in E(G)}$, with linear poles supported on the union of affine hyperplanes

$$\bigcup_{G'} \left\{ \sum_{e \in G'} s_e - |E(G')| = 0 \right\}$$

where the union runs over subgraphs G' of G such that $2|E(G')| - b_1(G')d \leq 0$ and $|E(G')|$ is the number of edges in G' .

Proof — From Theorem 4.1, one has a decomposition :

$$t_G(s)|_{U_x^n} = \sum_{G' \subset G} \left(\sum_{\vec{k} \in \{0, \dots, p\}^{E_1}} \underbrace{I_{G', \vec{k}}(s)} \right) \times h_{G \setminus G'}(s)$$

where the sum runs over subgraphs $G' \subset G$ and $h_{G \setminus G'}(s) \in C^m(M^{V(G)}, \mathcal{O}_{s_0}(\mathbb{C}^{E(G) \setminus E(G')}))$, $s_0 = (s_e = 1)_{e \in E(G) \setminus E(G')}$ as soon as $p > \frac{d+m}{2} - 1$ for $m = \sum_{G' \subset G; 2|E(G')| - db_1(G') - 1 < 0} db_1(G') - 2|E(G')| + 1$. Therefore, the analytic continuation of $t_G(s)$ should follow from the analytic continuation of $I_{G', \vec{k}}$ for all subgraphs G' of G and the fact that the distributional order of $I_{G', \vec{k}}$ is bounded from above by some integer $m(G')$ independent of $\vec{k} \in \mathbb{N}^{E(G')}$. But Proposition 5.3 precisely gives us that the distributional order of every $I_{G', \vec{k}}$ is bounded from above by some integer which depends only on the topology of G' . Now following the notations from Proposition 5.3, for every subgraph $G' \subset G$, we denote by $S_{E(G')}$ the permutations of the edges $E(G') = \{1, \dots, E'\}$. For every permutation $\sigma \in S_{E(G')}$ corresponds a canonical filtration $\{G'_{\sigma(1)} \subset \dots \subset G'_{\sigma(E')}\}$ of G' and S_{σ} denotes the set of all subgraphs $H \in \{G'_{\sigma(1)} \subset \dots \subset G'_{\sigma(E')}\}$ such that $b_1(H) \geq 1$. Finally doing all the bookkeeping, we find that

$$t_G(s) = \sum_{G' \subset G} \sum_{\sigma \in S_{E(G')}} \left(\prod_{H \in S_{\sigma}} \frac{1}{(\sum_{i \in H} s_i) - E(H)} \right) h_{\sigma}(s)$$

where $h_{\sigma}(s) \in \mathcal{D}'(M^{V(G)}, \mathcal{O}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e \in E(G)}$. ■

6 Renormalization of Feynman amplitudes

In this second part of our paper, we shall apply the analytic continuation results derived in the previous part to the renormalization of Feynman amplitudes on Riemannian manifolds.

6.1 Renormalization maps.

For a smooth manifold (M, g) and for every finite subset $I \subset \mathbb{N}$, we denote by M^I the configuration space of points labelled by I . For $J \subset I$ with $|J| \geq 2$, D_J is the subset $\{(x_i)_{i \in I} \text{ s.t. } x_j = x_k \text{ for } j, k \in J\}$ of M^I called J -diagonal. Let $\Delta_I = \cup_{J \subset I, |J| \geq 2} D_J$ be the maximal diagonal.

Definition 6.1 (Labelling vertices) *Let $I \subset \mathbb{N}, |I| < +\infty$ be some finite subset of integers. A graph with vertices labelled by I is a pair (G, ι) , where G is a graph, ι is an injective map from $V(G)$ to I .*

For a graph with vertices labelled by I , (G, ι) , defines an element

$$t_G = \prod_{e \in E(G)} \mathfrak{G}(x_{i(e)}, x_{j(e)}),$$

where $(i(e), j(e)) \in I^2$ and t_G is a smooth function on $M^I \setminus \Delta_I$. For a finite subset I of \mathbb{N} , let us denote by $\mathcal{F}(M^I)$ the linear span of t_G of all (G, ι) with $\iota(V(G)) \subset I$ as smooth functions on $M^I \setminus \Delta_I$.

For a linear map $\mathcal{R} : E \mapsto \mathcal{D}'(M)$ where E is a vector space and M is a smooth manifold, and any open subset $U \subset M$, let $i_U : U \hookrightarrow M$ denote the inclusion map. Then $\mathcal{R}|_U = i_U^* \mathcal{R} : E \mapsto \mathcal{D}'(U)$ is the pull-back of \mathcal{R} by i_U . Following recent work by Nikolov–Stora–Todorov [59], we can give a definition of renormalization as follows :

Definition 6.2 *A renormalization is a sequence of (not necessarily continuous) linear maps $\mathcal{R}_I : \mathcal{F}(M^I) \mapsto \mathcal{D}'(M^I)$ indexed by finite subsets I of \mathbb{N} , which satisfies the following system of **functional equations** :*

- For $I \subset J$, and $t \in \mathcal{F}(M^I)$,

$$\boxed{\mathcal{R}_J(t) = \mathcal{R}_I(t).} \quad (29)$$

*This is the **compatibility** condition for the family of linear maps.*

- $\forall t \in \mathcal{F}(M^I), \varphi \in C_c^\infty(M^I \setminus \Delta_I)$,

$$\boxed{\langle \mathcal{R}_I(t), \varphi \rangle = \langle t, \varphi \rangle.} \quad (30)$$

*This means that $\mathcal{R}_I(t)$ is a **distributional extension** of $t \in C^\infty(M^I \setminus \Delta_I)$.*

- For a graph (G, ι) with vertices labelled by $J \subset \mathbb{N}$, and $I \subset J = \iota(V(G))$, set $I^c = J \setminus I$, let $E_I = \{e \in E(G); i(e), j(e) \in I^2\}$, $E_{I^c} = \{e \in E(G); i(e), j(e) \in I^{c2}\}$, $E_{II^c} = E(G) \setminus (E_I \cup E_{I^c})$, and we denote by G_I, G_{I^c}, G_{II^c} the corresponding induced subgraphs of G . For open subsets U, V of M with $\text{dist}(U, V) > 0$, denote by $U^I \times V^{I^c}$ the subset $\{(x_j)_{j \in J} \in M^J \text{ s.t. } x_i \in U, \forall i \in I, x_i \in V, \forall i \in I^c\} \subset M^J$. Then :

$$\boxed{\mathcal{R}_J|_{U^I \times V^{I^c}}(t_G) = (\mathcal{R}_J|_{U^I}(t_{G_I}) \boxtimes \mathcal{R}_J|_{V^{I^c}}(t_{G_{I^c}})) t_{G_{II^c}}}$$

*as distributions in $\mathcal{D}'(U^I \times V^{I^c})$. This means that renormalization must preserve **locality**.*

- Let $\Phi : M \mapsto M$ be an orientation preserving diffeomorphism and we denote by $\Phi_I : M^I \mapsto M^I$ the induced diffeomorphism on configuration space M^I . We assume that the renormalization

maps depend on the Riemannian metric g and denote it by $\mathcal{R}[g] = (\mathcal{R}[g]_I)_I$ to stress this dependence. Then the covariance equation on renormalization maps reads for all graph (G, ι) with vertices labelled by I :

$$\boxed{\mathcal{R}[\Phi^*g]_I(\Phi_I^*t_G) = \Phi_I^*(\mathcal{R}[g]_I(t_G))}. \quad (31)$$

This axiom of functorial nature ensures the renormalization is **covariant**.

The following property follows from the locality condition : for a graph (G, ι) with vertices labelled by I , if G is the disjoint union of G_1 and G_2 , $\iota(V(G_1)) \subset I_1$, $\iota(V(G_2)) \subset I_2$, $I_1 \cap I_2 = \emptyset$ then :

$$\boxed{\mathcal{R}_{I_1 \cup I_2}(t_G) = \mathcal{R}_{I_1}(t_{G_1}) \boxtimes \mathcal{R}_{I_2}(t_{G_2})}$$

as distributions in $\mathcal{D}'(M^{I_1 \cup I_2})$.

6.2 Decompositions of meromorphic germs of distributions

Our goal in this paragraph is to extend the decomposition of [40] (see also [14, Appendix]) of the space \mathcal{M}_{s_0} of meromorphic germs with linear poles at $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$ to their distributional counterpart $\mathcal{D}'(\cdot, \mathcal{M}_{s_0})$ defined in paragraph 3.2. This decomposition plays an essential role in our definition of renormalization maps by projections. Recall we denoted by \mathcal{O}_{s_0} the space of holomorphic germs at s_0 .

Let us fix a nondegenerate bilinear form

$$Q(\cdot, \cdot) : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R},$$

which induces a nondegenerate bilinear form

$$Q^*(\cdot, \cdot) : (\mathbb{R}^p)^* \times (\mathbb{R}^p)^* \rightarrow \mathbb{R}.$$

We can now define the concept of polar germ [40, definition 2.3], a polar germ at s_0 is a meromorphic germ of the form $\frac{1}{L_1^{n_1}(s-s_0) \cdots L_k^{n_k}(s-s_0)} h(\ell(s-s_0))$, where L_1, \dots, L_k are linearly independent linear functions in $(\mathbb{R}^p)^*$, $(n_1, \dots, n_k) \in \mathbb{N}_{>0}^k$, $\ell = (\ell_1, \dots, \ell_n) : \mathbb{R}^p \rightarrow \mathbb{R}^n$ defined by linear functions ℓ_1, \dots, ℓ_n , and h is a holomorphic germ at $0 \in \mathbb{C}^n$, such that $Q^*(L_i, \ell_j) = 0$, $1 \leq i \leq k$, $1 \leq j \leq n$. Let \mathcal{P}_{s_0} be the linear span of polar germs at s_0 in \mathcal{M}_{s_0} .

Notice the polar set is defined by real linear functions, by similar proof as in [40] using some geometry of cones, we have the following :

Proposition 6.1 *There is a decomposition:*

$$\mathcal{M}_{s_0} = \mathcal{O}_{s_0} \oplus \mathcal{P}_{s_0}.$$

Proof — We assume w.l.o.g that $s_0 = 0$. Our setting is slightly more general than that in [40] since we deal with complex coefficients whereas [40] deals with real coefficients. However the proofs can be readily adapted in a straightforward way as we shall indicate below. Just like Lemma 2.9 and Theorem 2.10 in [40], we have $\mathcal{M}_0 = \mathcal{O}_0 + \mathcal{P}_0$, where the proof is the same for all coefficients since it relies on the decomposition of the meromorphic germ $\frac{1}{L_1^{n_1} \cdots L_k^{n_k}}$ as a sum of polar germs and Taylor expansion of the numerator.

To prove that the sum is direct, we need to show $\mathcal{O}_0 \cap \mathcal{P}_0 = \{0\}$, where we used two properties: a *projectively properly positioned* family of simplicial real fractions is linearly independent over \mathbb{C} . This is an analog of Proposition 3.5 in [40]. When taking real and imaginary parts, this conclusion follows from Proposition 3.5 in [40].

A projectively properly positioned family of polar germs at zero in our sense is non-holomorphic, as in Theorem 3.6 in [40], we can show that if a linear combination of projectively properly positioned family of polar germs is holomorphic then it is zero.

We may conclude as in [40, Thm 4.15], take any decomposition of $f \in \mathcal{P}_0 \cap \mathcal{O}_0$ as $f = \sum g_i$ where $(g_i)_i$ is a finite set of polar germs. By [40, Lemmas 4.10 and 4.11] there is a choice of the family of the supporting cones of the polar germs such that the union of the cones does not contain any nonzero linear subspace and this family of supporting cones has a properly positioned subdivision. Then we transform the finite sum of polar germs $\sum h_i$ into a sum $\sum \tilde{h}_j$ of *projectively properly positioned* polar germs $(\tilde{h}_j)_j$ using the subdivision operator from [40, p. 18]. Finally, we get $\sum \tilde{h}_j = f \in \mathcal{O}_0$ where the r.h.s is holomorphic hence $\sum \tilde{h}_j = 0$ by the non-holomorphicity [40, Thm 3.6]. Finally, this yields $\mathcal{O}_0 \cap \mathcal{P}_0 = \{0\}$. \blacksquare

Now we extend the concept of polar germs to distributions valued in polar germs.

Definition 6.3 *Let M be a smooth manifold and $p \in \mathbb{N}$. A polar germ of distributions at $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$ is some element of $\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p))$ of the form $\frac{1}{L_1^{n_1}(s-s_0) \cdots L_k^{n_k}(s-s_0)} h(\ell(s-s_0))$, where L_1, \dots, L_k are linearly independent linear functions in $(\mathbb{R}^p)^*$, $(n_1, \dots, n_k) \in \mathbb{N}_{>0}^k$, $\ell = (\ell_1, \dots, \ell_n) : \mathbb{R}^p \rightarrow \mathbb{R}^n$ defined by linear functions ℓ_1, \dots, ℓ_n , and $h \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$ such that $Q^*(L_i, \ell_j) = 0$, $1 \leq i \leq k$, $1 \leq j \leq n$. We denote by $\mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$ the linear span of polar germs of distributions at s_0 in $\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p))$.*

Lemma 6.1 *If $\frac{1}{L_1^{n_1}(s-s_0) \cdots L_k^{n_k}(s-s_0)} h(\ell(s-s_0))$ and $\frac{1}{M_1^{m_1}(s-s_0) \cdots M_p^{m_p}(s-s_0)} g(\ell'(s-s_0))$ represent the same nonzero meromorphic germ of distributions, then $k = p$, and $M_1, \dots, M_m, L_1, \dots, L_k$ can be rearranged in such a way that L_i is a multiple of M_i and $n_i = m_i$ for $1 \leq i \leq k$.*

Proof — Since this meromorphic germ is not zero, we can take a test function φ such that $h(\ell(s-s_0))(\varphi)$ is not identically zero, then $\frac{1}{L_1^{n_1}(s-s_0) \cdots L_k^{n_k}(s-s_0)} h(\ell(s-s_0))(\varphi)$ and $\frac{1}{M_1^{m_1}(s-s_0) \cdots M_p^{m_p}(s-s_0)} g(\ell'(s-s_0))(\varphi)$ represent the same polar germ, by the same proof as in [40, Lemma 2.8], we have the conclusion. \blacksquare

We then prove the promised decomposition Theorem for $\mathcal{D}'(M, \mathcal{M}_{s_0})$ which generalizes the result in [40].

Theorem 6.1 *Let M be some smooth manifold and $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$. We have the direct sum decomposition :*

$$\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) = \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p)) \oplus \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p)).$$

Proof — Without loss of generality, we can assume that $s_0 = 0$. For $t \in \mathcal{D}'(M, \mathcal{M}_0(\mathbb{C}^p))$, by definition, there exist $L_1, \dots, L_k \in (\mathbb{R}^p)^*$ such that $L_1 \cdots L_k t \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$. By partial fractions decompositions as in the proof of [40, Lemma 2.9 property a)], we may assume there is $(n_1, \dots, n_k) \in \mathbb{N}_{>0}^k$ such that $L_1^{n_1} \cdots L_k^{n_k} t \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$ with L_1, \dots, L_k linearly independent and $(n_1, \dots, n_k) \in \mathbb{N}_{>0}^k$.

Now let us expand L_1, \dots, L_k to a basis (e_1, \dots, e_p) of $(\mathbb{R}^p)^*$ with $e_i = L_i$, $1 \leq i \leq k$ and $Q(e_i, e_j) = 0$ for $1 \leq i \leq k$, $k+1 \leq j \leq p$. Then by Proposition 3.2, we have the power series expansion for :

$$z_1^{n_1} \cdots z_k^{n_k} t = \sum_{\alpha \in \mathbb{N}^p} \frac{z^\alpha}{\alpha!} t_\alpha,$$

where $z = \sum z_i e_i^* \in (\mathbb{C}^p)^*$. So when we apply $z_1^{n_1} \cdots z_k^{n_k} t$ against the test function φ , we obtain $z_1^{n_1} \cdots z_k^{n_k} t(\varphi) = \sum_{\alpha \in \mathbb{N}^p} \frac{z^\alpha}{\alpha!} t_\alpha(\varphi)$, which is absolutely convergent in a small neighborhood of $0 \in \mathbb{C}^p$.

Let $S = \{d = (d_1, \dots, d_p) \in \mathbb{N}^p \mid d \neq (0, \dots, 0), 0 \leq d_i \leq n_i\}$. For $d \in S$, let $I(d) = \{i \mid d_i \neq 0\} \subset \{1, \dots, p\}$, and set

$$N_d = \{\alpha \in \mathbb{N}^p \mid \alpha_i = n_i - d_i \text{ if } i \in I(d), \alpha_i \geq n_i \text{ if } i \in \{1, \dots, k\} \setminus I(d), \alpha_i \in \mathbb{N} \text{ if } i \in \{k+1, \dots, p\}\}.$$

Then we note that $N_d \cap N_e = \emptyset$ if $d \neq e \in S$ and most importantly, we have the partition $\mathbb{N}^p = \bigcup_{d \in S} N_d$.

Now for $z_i \neq 0$, $1 \leq i \leq k$,

$$t(\varphi) = \sum_{\alpha \in \mathbb{N}^p} \frac{z^{\alpha - \alpha_0}}{\alpha!} t_\alpha(\varphi) = \sum_{d \in S} \sum_{\alpha \in N_d} \frac{z^{\alpha - \alpha_0}}{\alpha!} t_\alpha(\varphi),$$

where $\alpha_0 = (n_1, \dots, n_k, 0, \dots, 0)$. And we have

$$\sum_{\alpha \in N_d} \frac{z^{\alpha - \alpha_0}}{\alpha!} t_\alpha(\varphi) = \frac{1}{z_{I(d)}^d} \sum_{\alpha \in N_d} \frac{z_{[p] \setminus I(d)}^{\alpha - \alpha_0}}{\alpha!} t_\alpha(\varphi),$$

where $z_{I(d)}^d = \prod_{i \in I(d)} z_i^{d_i}$, $z_{[p] \setminus I(d)}^{\alpha - \alpha_0} = \prod_{i \in \{1, \dots, k\} \setminus I(d)} z_i^{\alpha_i - s_i} \prod_{i \in \{k+1, \dots, p\}} z_i^{\alpha_i}$.

Let

$$h_d = \sum_{\alpha \in N_d} \frac{z_{[p] \setminus I(d)}^{\alpha - \alpha_0}}{\alpha!} t_\alpha.$$

By Proposition 3.2, for every compact $K \subset U$ there exists $C > 0$ and some continuous seminorm P for the Fréchet topology of $C_K^\infty(U)$ such that $|\forall \varphi \in C_K^\infty(U), |t_\alpha(\varphi)| \leq \frac{\alpha!}{r^{|\alpha|}} CP(\varphi)$ for all multi-indices $\alpha \in \mathbb{N}^p$. This implies that for every $0 < R < r$, for all $|z| < R$,

$$\left| \sum_{\alpha \in \mathbb{N}^p} \frac{z^\alpha}{\alpha!} t_\alpha(\varphi) \right| \leq \sum_{\alpha \in \mathbb{N}^p} \left| \frac{z^\alpha}{\alpha!} t_\alpha(\varphi) \right| \leq \sum_{\alpha \in \mathbb{N}^p} \frac{R^{|\alpha|}}{\alpha!} \frac{\alpha!}{r^{|\alpha|}} CP(\varphi) = \left(1 - \frac{R}{r}\right)^{-p} CP(\varphi).$$

Therefore, $|(\prod_{i \in [k] \setminus I(d)} z_i^{s_i}) h_d(\varphi)| \leq \sum_{\alpha \in \mathbb{N}^p} \left| \frac{z^\alpha}{\alpha!} t_\alpha(\varphi) \right| \leq \left(1 - \frac{R}{r}\right)^{-p} CP(\varphi)$, so $(\prod_{i \in [k] \setminus I(d)} z_i^{s_i}) h_d \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$, so $h_d \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$.

Now by definition, $\frac{1}{z_{I(d)}^d} h_d$ is a polar germ of distributions if $d \neq (0, \dots, 0)$, $h_{(0, \dots, 0)} \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$, and

$$\boxed{t = \sum_{d \in S} \frac{1}{z_{I(d)}^d} h_d \in \mathcal{D}'(M, \mathcal{P}_0(\mathbb{C}^p)) + \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))} \quad (32)$$

where the singular part reads as a finite sum of polar germs as a corollary of the above argument. So we have $\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) = \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p)) + \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$. To show it is a direct sum, if $t \in \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p)) \cap \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$, then for any test function ϕ , $t(\phi) \in \mathcal{P}_{s_0} \cap \mathcal{O}_{s_0}$, so $t(\phi) = 0$ by Proposition 6.1, which implies $t = 0$. \blacksquare

A consequence of the decomposition Theorem is the following

Proposition 6.2 *Let M be a smooth manifold, $p \in \mathbb{N}$ and $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$. There exists a projection*

$$\boxed{\pi_p : \mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) \mapsto \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p))}$$

which sends distribution valued in meromorphic germs at s_0 to distribution valued in holomorphic germs at s_0 such that $\ker(\pi_p) = \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$.

Remark 6.1 *Note that π_p is uniquely determined by the vector subspace of polar germs which are in turn uniquely determined by the choice of the canonical quadratic form $Q : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}$ that we fixed at the beginning of the present section.*

In appendix 7.2, we show some useful Lemmas on the functorial properties of the projection π_p for $p \in \mathbb{N}$. As a consequence of [40], we have a similar projection, still denoted a bit abusively by π_p , at the germ level : $\pi_p : \mathcal{M}_{s_0} \mapsto \mathcal{O}_{s_0}$. It follows that the two projectors are related by the following equation :

Corollary 6.1 *Let X be a smooth manifold and $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$. For all $t(s) \in \mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$ and for all test function $\varphi \in C_c^\infty(X)$,*

$$\boxed{(\pi_p t(s))(\varphi) = \pi_p(t(s)(\varphi)).}$$

6.3 A renormalization map by projections

From now on, for any integer p , we fix the canonical quadratic form Q on \mathbb{R}^p : $Q(x) = \sum_{i=1}^p |x_i|^2$ and we study germs at $s_0 = (1, \dots, 1) \in \mathbb{R}^p$. We denote by $\mathbf{ev}|_{s_0}$ the evaluation of some holomorphic germ at s_0 . The properties of the family of projections $(\pi_p)_{p \in \mathbb{N}}$ allow to us to give a definition of renormalization maps as follows :

Definition 6.4 (Renormalization maps by projections) For $I \subset \mathbb{N}$, we define the renormalization map \mathcal{R}_I as follows: for a graph (G, ι) with vertices labelled by I ,

$$\boxed{\mathcal{R}_I(t_G) = \mathbf{ev}|_{s_0} (\pi_{|E(G)|} (t_G(s)))}$$

where \mathfrak{G}^s is the Schwartz kernel of $(-\Delta)^{-s}$.

Theorem 6.2 (Renormalization Theorem) Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d , $dv(x)$ the Riemannian volume and $P = -\Delta_g + V$, $V \in C_{\geq 0}^\infty(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$. For every finite subset $I \subset \mathbb{N}$, for every graph (G, ι) with vertices labelled by $I \subset \mathbb{N}$, we define $\mathcal{R}_I(t_G) \in \mathcal{D}'(M^I)$ as in definition 6.4 and extend it by linearity to the vector space $\mathcal{F}(M^I)$.

Then the collection of renormalization maps $(\mathcal{R}_I)_{I \subset \mathbb{N}, |I| < +\infty}$ satisfies the functional equations of definition 6.2.

Proof — The compatibility condition is encoded in the family of projections. For simplicity of notations, we choose to drop the subindex s_0 for the space of germs so we will write \mathcal{M}, \mathcal{O} instead of $\mathcal{M}_{s_0}, \mathcal{O}_{s_0}$ and it will always be understood from the context that we consider holomorphic and meromorphic germs localized at $s_0 = (1, \dots, 1) \in \mathbb{C}^p$ for some $p \in \mathbb{N}$. Furthermore, we also write π for the projections instead of $\pi_{|E(G)|}$ where it will be understood that for every graph G , $\pi(t_G(s))$ means $\pi_{|E(G)|}(t_G(s))$.

We now prove that $\mathcal{R}_I(t_G)$ is a distributional extension of t_G . By Lemma 4.1, on $M^I \setminus \Delta_I$, for every $e \in E(G)$, every Green function $\mathfrak{G}^{s_e} \in C^\infty(M^I \setminus \Delta_I, \mathcal{O})$ are in fact smooth and depend holomorphically on s_e . We also have the convergence $\mathfrak{G}^{s_e} \rightarrow \mathfrak{G}$ in $C^\infty(M^2 \setminus \Delta_2)$ when $s_e \rightarrow 1$. Therefore, for any test function $\varphi \in C_c^\infty(M^I \setminus \Delta_I)$, by Corollary 6.1, $\pi(t_G(s))(\varphi) = \pi(t_G(s)(\varphi)) = t_G(s)(\varphi)$ since $t_G(s)(\varphi)$ is holomorphic at $s_0 = (s_e = 1)_{e \in E(G)} \in \mathbb{C}^{E(G)}$, and

$$\mathcal{R}_I(t_G)(\varphi) = \mathbf{ev}|_{s_0} \pi(t_G(s))(\varphi) = \mathbf{ev}|_{s_0} t_G(s)(\varphi) = t_G(\varphi).$$

Now let us prove the locality. For a graph (G, ι) with vertices labelled by $J \subset \mathbb{N}$, and $I \subset J = \iota(V(G))$, set $I^c = J \setminus I$, let $E_I = \{e \in E(G); i(e), j(e) \in I\}$, $E_{I^c} = \{e \in E(G); i(e), j(e) \in I^c\}$, $E_{II^c} = E(G) \setminus (E_I \cup E_{I^c})$, and we denote by (G_I, G_{I^c}, G_{II^c}) the corresponding induced subgraphs of G . Start from $t_G(s) = t_{G_I}(s_I) t_{G_{I^c}}(s_{I^c}) t_{G_{II^c}}(s_{II^c})$ where $s = (s_e)_{e \in E(G)}$, $s_I = (s_e)_{e \in E_I}$, $s_{I^c} = (s_e)_{e \in E_{I^c}}$, $s_{II^c} = (s_e)_{e \in E_{II^c}}$. For a pair of disjoint open subsets (U, V) such that $\text{dist}(U, V) > 0$, denote by $\{(x_j)_{j \in J} \in M^J \text{ s.t. } x_i \in U, \forall i \in I, x_i \in V, \forall i \in I^c\}$ the open subset of configuration space M^J . Then note that the product $t_{G_{II^c}}(s_{II^c}) = \prod_{e \in E_{II^c}} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)}) \in C^\infty(U^I \times V^{I^c}, \mathcal{O}(\mathbb{C}^{E_{II^c}}))$. It follows by Lemma 7.3 proved in the appendix that

$$\pi(t_G(s)) = \pi(t_{G_I}(s_I) t_{G_{I^c}}(s_{I^c}) t_{G_{II^c}}(s_{II^c})) = \pi(t_{G_I}(s_I) t_{G_{I^c}}(s_{I^c})) t_{G_{II^c}}(s_{II^c}).$$

Now the distributions $t_{G_I}(s_I) \in \mathcal{D}'(U^I, \mathcal{M}(\mathbb{C}^{E_I}))$ and $t_{G_{I^c}}(s_{I^c}) \in \mathcal{D}'(V^{I^c}, \mathcal{M}(\mathbb{C}^{E_{I^c}}))$ depend on different variables, therefore by Lemma 7.2, $\pi(t_{G_I}(s_I) \boxtimes t_{G_{I^c}}(s_{I^c})) = \pi(t_{G_I}(s_I)) \boxtimes \pi(t_{G_{I^c}}(s_{I^c}))$. Then as distributions on $U^I \times V^{I^c}$:

$$\begin{aligned} \mathcal{R}_J(t_G) &= \mathbf{ev}|_{(s_e=1)_{e \in E(G)}} (\pi(t_{G_I}(s_I)) \boxtimes \pi(t_{G_{I^c}}(s_{I^c})) \times t_{G_{II^c}}(s_{II^c})) \\ &= \mathbf{ev}|_{(s_e=1)_{e \in E_I}} \pi(t_{G_I}(s_I)) \mathbf{ev}|_{(s_e=1)_{e \in E_{I^c}}} \pi(t_{G_{I^c}}(s_{I^c})) \mathbf{ev}|_{(s_e=1)_{e \in E_{II^c}}} t_{G_{II^c}}(s_{II^c}) \\ &= \mathcal{R}_I(t_{G_I}) \mathcal{R}_{I^c}(t_{G_{I^c}}) t_{G_{II^c}}|_{U^I \times V^{I^c}} \end{aligned}$$

where $t_{G_{II^c}}$ is smooth on $U^I \times V^{I^c}$ which yields the desired equation. \blacksquare

7 Appendix: technical details.

7.1 Proof of Proposition 3.2.

Proof — Without loss of generality assume that $z = 0$. By definition and the multidimensional Cauchy's formula [39, p. 3], for any polydisc $D_1 \times \cdots \times D_p$ around $z = 0$, for any test function $\varphi \in C_c^\infty(M)$ and any λ in the polydisc :

$$t(\lambda)(\varphi) = \frac{1}{(2\pi i)^p} \int_{\partial D_1} \cdots \int_{\partial D_p} \frac{t(z)(\varphi) dz_1 \dots dz_p}{(z_1 - \lambda_1) \dots (z_p - \lambda_p)} = \frac{1}{(2\pi i)^p} \int_{\partial D_1} \cdots \int_{\partial D_p} \sum_{\alpha} \lambda^\alpha \frac{t(z)(\varphi) dz_1 \dots dz_p}{z_1^{\alpha_1+1} \dots z_p^{\alpha_p+1}}.$$

So for any multi-index α and any test function φ , we define the functional t_α by

$$t_\alpha(\varphi) = \frac{\alpha!}{(2\pi i)^p} \int_{\partial D_1} \cdots \int_{\partial D_p} \frac{t(z)(\varphi) dz_1 \dots dz_p}{z_1^{\alpha_1+1} \dots z_p^{\alpha_p+1}},$$

then the series $\sum_{|\alpha| \geq 0} \frac{s^\alpha}{\alpha!} t_\alpha(\varphi)$ converges absolutely to $t(s)(\varphi)$.

First note that the functional t_α is linear, it remains to prove that t_α is continuous. For that, it suffices to show that for every compact $K \subset M$, the restriction of t_α to the Fréchet space $C_K^\infty(M)$ of test functions supported in K is continuous. For fixed s , $t(s)$ is linear continuous on $C_K^\infty(M)$ therefore there exists some constant $C(s)$ and continuous seminorm P of $C_K^\infty(M)$ such that $|t(s)(\varphi)| \leq C(s)P(\varphi), \forall \varphi \in C_K^\infty(M)$. Conversely for fixed φ , $s \in D_1 \times \cdots \times D_p \mapsto t(s)(\varphi)$ is bounded by holomorphicity. By an application of the uniform boundedness principle since $C_K^\infty(M)$ is Fréchet, for every compact $K \subset M$, there exists $C > 0$ and some continuous seminorm P for the Fréchet topology of $C_K^\infty(M)$ such that :

$$\forall \varphi \in C_K^\infty(M), \sup_{s \in \partial D_1 \times \cdots \times \partial D_p} |t(s)(\varphi)| \leq CP(\varphi).$$

Assuming that all discs ∂D_i have radius r , it immediately follows that t_α satisfies a distributional version of Cauchy's bound:

$$\forall \varphi \in C_K^\infty(M), |t_\alpha(\varphi)| \leq \frac{\alpha!}{r^{|\alpha|}} CP(\varphi). \quad (33)$$

This also implies that for all $\varphi \in C_K^\infty(M)$, the power series $\sum_{\alpha} \frac{s^\alpha}{\alpha!} t_\alpha(\varphi)$ converges for $|\lambda| < r$ i.e. the convergence radius equals r . \blacksquare

7.2 Products of meromorphic germ of distributions in different variables.

In this part, we prove some useful Lemmas on products of meromorphic germ of distributions in different variables.

Lemma 7.1 *Let (X_1, X_2) be smooth manifolds, $\mu_1 \in \mathbb{R}^{p_1} \subset \mathbb{C}^{p_1}$ and $\mu_2 \in \mathbb{R}^{p_2} \subset \mathbb{C}^{p_2}$. If $t_1(s_1) \in \mathcal{D}'(X_1, \mathcal{M}_{\mu_1}(\mathbb{C}^{p_1}))$ and $t_2(s_2) \in \mathcal{D}'(X_2, \mathcal{M}_{\mu_2}(\mathbb{C}^{p_2}))$ then the external tensor product $t_1(s_1) \boxtimes t_2(s_2)$ is a well-defined element in $\mathcal{D}'(X_1 \times X_2, \mathcal{M}_{(\mu_1, \mu_2)}(\mathbb{C}^{p_1+p_2}))$.*

Proof — Denote by dv_1, dv_2 some smooth densities on X_1, X_2 respectively. Since every compact subset $K \subset X_1 \times X_2$, can be covered by some finite number of products of compacts of the form $K_1 \times K_2$, by Lemma 4.2 it suffices to show that for all compacts $K_1 \subset X_1, K_2 \subset X_2$, the element $t(s_1; x)t(s_2; y)|_{K_1 \times K_2}$ is a well-defined meromorphic family of distribution in $\mathcal{D}'(K_1 \times K_2)$ at $(\mu_1, \mu_2) \in \mathbb{C}^{p_1} \times \mathbb{C}^{p_2}$ with linear poles. Hence we can assume, without loss of generality, that we work over some product $K_1 \times K_2 \subset X_1 \times X_2$ of compact subsets and we assume without loss of generality that we work around $(\mu_1, \mu_2) = (0, 0)$. There exists monomials $P(s_1) = L_1(s_1) \dots L_k(s_1)$ and $Q(s_2) = M_1(s_2) \dots M_l(s_2)$, where $(L_i)_{i=1}^k, (M_i)_{i=1}^l$ are linear functions, such that $P(s_1)t_1(s_1)$ and $Q(s_2)t_2(s_2)$ are holomorphic germs of distributions at $s_1 = \mu_1, s_2 = \mu_2$ respectively. Therefore

by Proposition 3.2, we know that $P(s_1)t_1(s_1)$ and $Q(s_2)t_2(s_2)$ both admit Laurent series expansions $P(s_1)t_1(s_1) = \sum_{\alpha_1} s_1^{\alpha_1} u_{\alpha_1}$, $Q(s_2)t_2(s_2) = \sum_{\alpha_2} s_2^{\alpha_2} v_{\alpha_2}$ where there exists two integers (m_1, m_2) corresponding to the distributional orders of $(t_1|_{K_1}, t_2|_{K_2})$ and two positive real numbers (r_1, r_2) such that for all multi-index $(\alpha_1, \alpha_2) \in \mathbb{N}^{p_1+p_2}$, we have the bounds :

$$\|u_{\alpha_1}\|_{(C^{m_1})'} \leq C_1 r_1^{|\alpha_1|}, \quad \|v_{\alpha_2}\|_{(C^{m_2})'} \leq C_2 r_2^{|\alpha_2|}. \quad (34)$$

We define the series $P(s_1)t_1(s_1) \boxtimes Q(s_2)t_2(s_2) = \sum_{\alpha_1, \alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} u_{\alpha_1} \boxtimes v_{\alpha_2}$ and we shall prove that the above series converges for $|s_1| + |s_2|$ small enough in the sense that for every test function $\varphi(x_1, x_2)$ supported in $K_1 \times K_2$, the series

$$\sum_{\alpha_1, \alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} u_{\alpha_1} \boxtimes v_{\alpha_2}(\varphi) = \sum_{\alpha_1, \alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} \int_{X_1 \times X_2} u_{\alpha_1}(x_1) v_{\alpha_2}(x_2) \varphi(x_1, x_2) dv_1(x_1) dv_2(x_2)$$

converges absolutely. We first prove it for some element $\varphi = \varphi_1 \boxtimes \varphi_2 \in C_{K_1}^\infty(X_1) \boxtimes C_{K_2}^\infty(X_2) \subset C_{K_1 \times K_2}^\infty(X_1 \times X_2)$ which is a tensor product of two elements. For $s \in \mathbb{C}^p$, we shall use the notation $\|s\| = \sup_{j \in \{1, \dots, p\}} |s_j|$ and for some multi-index $\alpha \in \mathbb{N}^p$, $|\alpha| = \sum_{j=1}^p \alpha_j$. Then the series converges thanks to the bound :

$$\begin{aligned} & \left| \sum_{\alpha_1, \alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} \int_{X_1 \times X_2} u_{\alpha_1}(x_1) v_{\alpha_2}(x_2) \varphi(x_1, x_2) dv_1(x_1) dv_2(x_2) \right| \\ & \leq \sum_{\alpha_1, \alpha_2} \|s_1^{\alpha_1} s_2^{\alpha_2}\| \left| \int_{X_1} u_{\alpha_1}(x_1) \varphi_1(x_1) dv_1(x_1) \int_{X_2} v_{\alpha_2}(x_2) \varphi_2(x_2) dv_2(x_2) \right| \\ & \leq \sum_{\alpha_1, \alpha_2} \|s_1\|^{|\alpha_1|} C_1 r_1^{|\alpha_1|} \|\varphi_1\|_{C^{m_1}(X_1)} \|s_2\|^{|\alpha_2|} C_2 r_2^{|\alpha_2|} \|\varphi_2\|_{C^{m_2}(X_2)} \\ & \leq \sum_{\alpha_1, \alpha_2} C_1 (\|s_1\| r_1)^{|\alpha_1|} C_2 (\|s_2\| r_2)^{|\alpha_2|} \|\varphi\|_{C^m(X_1 \times X_2)} \end{aligned}$$

for any $m \geq \sup(m_1, m_2)$ where the r.h.s is absolutely convergent for s_1, s_2 small enough. Then we conclude by using the fact that the completed tensor product $C_{K_1}^\infty(X_1) \widehat{\boxtimes} C_{K_2}^\infty(X_2)$ coincides with $C_{K_1 \times K_2}^\infty(X_1 \times X_2)$ where the topology for which we do the completion does not matter since $C_{K_i}^\infty(X_i)$ are Fréchet nuclear spaces. Therefore the algebraic tensor product $C_{K_1}^\infty(X_1) \boxtimes C_{K_2}^\infty(X_2)$ is dense in $C_{K_1 \times K_2}^\infty(X_1 \times X_2)$ and the inequality

$$\left| \sum_{\alpha_1, \alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} \int_{X_1 \times X_2} u_{\alpha_1}(x_1) v_{\alpha_2}(x_2) \varphi(x_1, x_2) dv_1(x_1) dv_2(x_2) \right| \leq \sum_{\alpha_1, \alpha_2} C_1 (\|s_1\| r_1)^{|\alpha_1|} C_2 (\|s_2\| r_2)^{|\alpha_2|} \|\varphi\|_{C^m(X_1 \times X_2)}$$

holds true for all $\varphi \in C_{K_1 \times K_2}^\infty(X_1 \times X_2)$. ■

For every $p \in \mathbb{C}^p$, $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$, let $\pi_p : \mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) \mapsto \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p))$ be the projection from Proposition 6.2. Then :

Lemma 7.2 *Under the assumptions of the previous Lemma, the following equation holds true :*

$$\boxed{\pi_{p_1+p_2}(t_1 \boxtimes t_2) = \pi_{p_1}(t_1) \boxtimes \pi_{p_2}(t_2)}. \quad (35)$$

Proof — The proof of equation 35 goes as follows, we decompose t_1 and t_2 as $t_1 = \pi_{p_1}(t_1) + (1 - \pi_1)(t_1)$ and $t_2 = \pi_{p_2}(t_2) + (1 - \pi_2)(t_2)$ where $(\pi_{p_1}(t_1), \pi_{p_2}(t_2)) \in \mathcal{D}'(X_1, \mathcal{O}_{\mu_1}(\mathbb{C}^{p_1})) \times \mathcal{D}'(X_2, \mathcal{O}_{\mu_2}(\mathbb{C}^{p_2}))$ and $((1 - \pi_1)(t_1), (1 - \pi_2)(t_2)) \in \mathcal{D}'(X_1, \mathcal{P}_{\mu_1}(\mathbb{C}^{p_1})) \times \mathcal{D}'(X_2, \mathcal{P}_{\mu_2}(\mathbb{C}^{p_2}))$. Then note that

$$\begin{aligned} t_1 \boxtimes t_2 &= \underbrace{\pi_{p_1}(t_1) \boxtimes \pi_{p_2}(t_2)}_{\in \mathcal{D}'(X_1 \times X_2, \mathcal{O}_{(\mu_1, \mu_2)}(\mathbb{C}^{p_1+p_2}))} \\ &+ \underbrace{(1 - \pi_1)(t_1) \boxtimes \pi_2(t_2) + \pi_1(t_1) \boxtimes (1 - \pi_2)(t_2) + (1 - \pi_1)(t_1) \boxtimes (1 - \pi_2)(t_2)}_{\in \mathcal{D}'(X_1 \times X_2, \mathcal{P}_{(\mu_1, \mu_2)}(\mathbb{C}^{p_1+p_2}))} \end{aligned}$$

where the term $(1 - \pi_1)(t_1) \boxtimes \pi_2(t_2) + \pi_1(t_1) \boxtimes (1 - \pi_2)(t_2) + (1 - \pi_1)(t_1) \boxtimes (1 - \pi_2)(t_2)$ is a finite sum of polar germs by equation (32). It follows that $\pi_{p_1+p_2}(t_1 \boxtimes t_2) = \pi_{p_1}(t_1) \boxtimes \pi_{p_2}(t_2)$ by the uniqueness of the decomposition which follows from Theorem 6.1. ■

By a similar proof as in the above Lemma, we also have :

Lemma 7.3 *Let X be a smooth manifold, $U \subset X$ an open subset and $m \in \mathbb{N}$. Set $(p_1, p_2) \in \mathbb{N}^2$ to be an arbitrary pair of integers and $(\mu_1, \mu_2) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \subset \mathbb{C}^{p_1+p_2}$. Let $t(s_1) \in \mathcal{D}'^m(U, \mathcal{M}_{\mu_1}(\mathbb{C}^{p_1}))$ and $h(s_2) \in C^m(U, \mathcal{O}_{\mu_2}(\mathbb{C}^{p_2}))$. Then the product $t(s_1)h(s_2)$ is an element of $\mathcal{D}'^m(U, \mathcal{M}_{(\mu_1, \mu_2)}(\mathbb{C}^{p_1+p_2}))$ which satisfies the equation :*

$$\boxed{\pi_{p_1+p_2}(t(s_1)h(s_2)) = \pi_{p_1}(t(s_1))h(s_2)}. \quad (36)$$

Proof — Without loss of generality, we can work locally since all local results can be glued together by partition of unity thanks to Lemma 4.2. For every test function $\varphi \in C_c^\infty(U)$, $\langle t(s_1)h(s_2), \varphi \rangle = \langle t(s_1), \underbrace{h(s_2)\varphi}_{\in C_c^m(U)} \rangle$ hence the product $t(s_1)h(s_2)$ is well defined in $\mathcal{D}'^m(U)$ as soon as both $t(s_1), h(s_2)$

exist. We now explain the meromorphicity of $(s_1, s_2) \mapsto \langle t(s_1)h(s_2), \varphi \rangle$ at $(\mu_1, \mu_2) \in \mathbb{C}^{p_1+p_2}$. Since $t(s) \in \mathcal{D}'^m(U, \mathcal{M}_{\mu_1}(\mathbb{C}^{p_1}))$, there exists $u(s) \in \mathcal{D}'^m(U, \mathcal{O}_{\mu_1}(\mathbb{C}^{p_1}))$ and linear functions (L_1, \dots, L_k) such that $(L_1(s) \cdots L_k(s))t(s) = u(s)$. Therefore the product $t(s_1)h(s_2)$ also reads $\frac{1}{L_1(s_1) \cdots L_k(s_1)}u(s_1)h(s_2)$. Then using power expansions in $s_1 - \mu_1$ for $u(s_1)$ as in Theorem 3.2 and expanding $h(s_2)$ in powers of $s_2 - \mu_2$ where coefficients are in $C^m(U)$, we easily show that $u(s_1)h(s_2) \in \mathcal{D}'^m(U, \mathcal{O}_{(\mu_1, \mu_2)}(\mathbb{C}^{p_1+p_2}))$ for $(s_1, s_2) \in \mathbb{C}^{p_1+p_2}$ close enough to $(\mu_1, \mu_2) \in \mathbb{C}^{p_1+p_2}$ which proves $t(s_1)h(s_2) \in \mathcal{D}'^m(U, \mathcal{M}_{(\mu_1, \mu_2)}(\mathbb{C}^{p_1+p_2}))$.

The equation $\pi_{p_1+p_2}(th) = h\pi_{p_1}(t)$ immediately follows from the fact that $\pi_{p_2}(h(s_2)) = h(s_2)$ since h is holomorphic and $h(s_2)(1 - \pi_{p_1})(t(s_1))$ is valued in polar germs. ■

7.3 Proof of Lemma 4.1.

Proof — Since our Riemannian manifold (M, g) is connected, $\ker(P)$ contains only constant functions. Indeed $Pu = 0$ implies that $u \in C^\infty$ by elliptic regularity and $0 = \langle u, -\Delta_g u \rangle + \langle u, Vu \rangle \implies \langle \nabla u, \nabla u \rangle = 0 \implies \nabla u = 0$ thus u is constant on connected components. Let us determine explicitly the spectral projector Π , it should satisfy for all u :

$$0 = \langle 1, u - \Pi(u) \rangle = \int_M (u - \Pi(u)) = \int_M u dx - \Pi(u) \text{Vol}(M) \implies \Pi(u) = \frac{\int_M u dx}{\text{Vol}(M)}.$$

The Schwartz kernel of the spectral projector Π is therefore the constant function $\Pi(x, y) = \text{Vol}(M)^{-1}$.

The first two claims about the Schwartz kernel $\mathfrak{G}^s(x, y)$ follow from [72, Theorem 4 p. 302] in the celebrated work of Seeley, by applying his Theorem to $A = P - \Pi$ which is a well defined elliptic pseudodifferential operator of order 2.

For the third claim, we start from the formula $\mathfrak{G}^s = \int_0^\infty (e^{-tP} - \Pi)(x, y)t^{s-1} dt$ and our proof exactly follows the proof of [8, Proposition 1] where we replace the heat semigroup $e^{t\Delta_g}$ in their proof by the semigroup $(e^{-tP} - \Pi)$ whose Schwartz kernel is $K_t - \Pi$ and is denoted by p_t . Start from the formula $p_t(x, y) = \langle \delta_x, (e^{-tP} - \Pi)\delta_y \rangle_{L^2(M)} = \langle (e^{-\frac{t}{2}P} - \Pi)\delta_x, (e^{-\frac{t}{2}P} - \Pi)\delta_y \rangle_{L^2(M)}$. For any integers (k, l, m) , $|\partial_t^m P_x^k P_y^l p_t(x, y)| = |P_x^{k+m} P_y^l p_t(x, y)|$ since $\partial_t^m (e^{-tP} - \Pi) = P^m (e^{-tP} - \Pi)$. Hence,

$$|\partial_t^m P_x^k P_y^l p_t(x, y)| \leq \| (e^{-(t-\varepsilon)P} - \Pi) \|_{B(L^2(M))} \| P_x^{k+m} (e^{-\frac{\varepsilon}{2}P} - \Pi) \delta_x \|_{L^2(M)} \| P_y^l (e^{-\frac{\varepsilon}{2}P} - \Pi) \delta_y \|_{L^2(M)}.$$

Therefore taking the supremum over $(x, y) \in M \times M$ yields :

$$\|\partial_t^m P_x^k P_y^l p_t\|_{C^0(M \times M)} \leq \| (e^{-(t-\varepsilon)P} - \Pi) \|_{B(L^2(M))} \| P_x^{k+m} (e^{-\frac{\varepsilon}{2}P} - \Pi) \delta_x \|_{L^2(M)} \| P_y^l (e^{-\frac{\varepsilon}{2}P} - \Pi) \delta_y \|_{L^2(M)}$$

where both $\|P_x^{k+m}(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_x\|_{L^2(M)}$ and $\|P_y^l(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_y\|_{L^2(M)}$ are finite since both $(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_x$ and $(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_y$ are smooth functions because the semigroup $(e^{tP} - \Pi)_{t \in \mathbb{R}_{\geq 0}}$ is smoothing. Furthermore, the term $\|(e^{-(t-\varepsilon)P} - \Pi)\|_{B(L^2(M))}$ has exponential decay when $t \rightarrow +\infty$ since $(e^{-(t-\varepsilon)P} - \Pi)$ is a smoothing operator which has a gap in the spectrum, indeed by spectral theory $e^{-tP}u = \sum_{\lambda \in \sigma(P)} e^{-t\lambda}\Pi_\lambda(u)$ where Π_λ is the spectral projector on the eigenspace of eigenvalue λ and the r.h.s. converges absolutely in all Sobolev spaces $H^s(M)$, $s \geq 0$ when $t > 0$. More generally, we obtain decay estimates of the form

$$\begin{aligned} \|\partial_t^m p_t\|_{C^k(M \times M)} &\leq \sum_{l_1, l_2 \leq \frac{k}{2} + 1} \|\partial_t^m P_x^{k_1} P_y^{k_2} p_t\|_{C^0(M \times M)} \\ &\leq C_{k,m} \|(e^{-(t-\varepsilon)P} - \Pi)\|_{B(L^2(M))} \leq C_{k,m} e^{-(t-\varepsilon)\lambda_1} \end{aligned}$$

where $\lambda_1 > 0$ is the smallest non zero eigenvalue of P which exists since $\sigma(P)$ is a discrete subset of $[0, +\infty)$. It follows that the integral $\int_1^\infty t^{s-1} p_t dt$ converges absolutely for all $s \in \mathbb{C}$ and is valued in all Banach spaces $C^k(M \times M)$, $k \in \mathbb{N}$ since :

$$\left\| \int_1^\infty t^{s-1} p_t dt \right\|_{C^k(M \times M)} \leq \int_1^\infty t^{Re(s)-1} \|p_t\|_{C^k(M \times M)} dt \leq C_k \int_1^\infty t^{Re(s)-1} e^{-(t-\varepsilon)\lambda_1} dt.$$

The integral $\int_1^\infty t^{s-1} p_t dt$ depends holomorphically in s since

$$\left\| \int_1^\infty \left(\frac{d}{ds}\right)^l t^{s-1} p_t dt \right\|_{C^k(M \times M)} \leq C_k \int_1^\infty t^{Re(s)-1} \log(t)^l e^{-(t-\varepsilon)\lambda_1} dt \quad (37)$$

where the r.h.s. is absolutely convergent and we can conclude by dominated convergence arguments. ■

7.4 Proof of Lemma 5.5.

Proof — First notice that when $Re(s_i) > -1$, $i = 1, \dots, E$, this integral is absolutely convergent and holomorphic in s .

Now If $E = 1$, then by integration by parts, for $Re(s) > -1$,

$$\int_{[0,1]} t^s \psi(t) dt = \sum_{i=0}^{k-1} (-1)^i \frac{1}{l_i(s)} \psi^{(i)}(1) + (-1)^k \frac{1}{l_{k-1}(s)} \int_{[0,1]} t^{s+k} \psi^{(k)}(t) dt,$$

where $l_i(s) = (s+1)\dots(s+i+1)$, the l.h.s is a meromorphic function when $Re(s) > -k-1$ with possible poles at $s = -1, \dots, -k$, so it extends to a meromorphic function on $Re(s) > -k-1$.

In general, for $Re(s_i) > -1$, $i = 1, \dots, E$, and $k_1, \dots, k_E \in \mathbb{Z}_{>0}$,

$$\begin{aligned} I_s(\psi) &= \int_{[0,1]^E} t_1^{s_1} \dots t_E^{s_E} \psi(t_1, \dots, t_E) d^E t \\ &= \sum_{\{j_1, \dots, j_m\} \subset \{1, \dots, E\}} \sum_{\substack{j \neq j_1, \dots, j_m \\ i_j = 0, \dots, k_j - 1}} \frac{(-1)^{i_j}}{l_{i_j}(s_j)} \prod_{i=1}^m \frac{(-1)^{k_{j_i}}}{l_{k_{j_i}-1}(s_{j_i})} \\ &\quad \int_{[0,1]^m} \prod_{j=j_1, \dots, j_m} t^{s_j + k_j} \left(\prod_{j \neq j_1, \dots, j_m} \partial_{t_j}^{i_j} \right) \partial_{t_{j_1}}^{k_{j_1}} \dots \partial_{t_{j_m}}^{k_{j_m}} \psi|_{t_j=1, j \neq j_1, \dots, j_m} dt_{j_1} \dots dt_{j_m} \quad (38) \end{aligned}$$

the r.h.s is a meromorphic function when $Re(s_i) > -k_i - 1$. So $I_s(\psi)$ extends to a meromorphic germ at any point in \mathbb{Z}^E .

Now at a given point $(p_e)_e \in \mathbb{Z}^E$, $\frac{1}{s_e - a_e}$ is holomorphic except $a_e = p_e$, therefore

$$\left(\prod_{i \in I} (s_i - p_i) \right) I_s(\psi)$$

is a holomorphic germ at $(p_e)_e$. The distribution order of $I_s(\psi)$ at the point (p_e) can be read from Equation (38) easily. \blacksquare

7.5 Proof of Lemma 5.2.

Proof — In the chart $(U \times U, (x^\mu, y^\nu))$, let us consider the Taylor expansion of $\phi(x, y)$, $\phi(x, y) = \sum_{k \geq 0} \phi_{[k]}(x, y)$, where $\phi_{[k]}(x, y) = \sum_{|\alpha|+|\beta|=k} \frac{x^\alpha y^\beta}{\alpha! \beta!} \partial_x^\alpha \partial_y^\beta \phi(0, 0)$. Obviously $\phi_{[0]}(x, y) = 0$. By symmetry and $\phi(x, x) = 0$, we know that

$$\phi_{[1]}(x, y) = 0.$$

By symmetry and $\phi(x, x) \equiv 0$, we know that $\phi_{[2]}(x, y) = \sum_\mu a_\mu (x^\mu - y^\mu)^2$, now by the fact $\phi(0, y) = \|y\|^2$, we find that :

$$\phi_{[2]}(x, y) = \sum (x^\mu - y^\mu)^2. \quad (39)$$

In fact, let us take $x = y$ in Equation (19),

$$g^{-1}(x) \left(\frac{\partial \phi}{\partial x^\mu}(x, x) dx^\mu, \frac{\partial \phi}{\partial x^\nu}(x, x) dx^\nu \right) = 0$$

which means

$$\frac{\partial \phi}{\partial x^\mu}(x, x) dx^\mu \equiv 0 \Rightarrow \frac{\partial \phi}{\partial x^\mu}(x, x) \equiv 0.$$

By symmetry

$$\frac{\partial \phi}{\partial y^\mu}(x, x) \equiv 0. \quad (40)$$

Now let us make a change of variables on $V \times W \rightarrow U \times U$ given by

$$(v, h) \mapsto (v, v + h),$$

we can take V, W small enough such that $V \times W$ is a coordinate chart around (x_0, x_0) . Let $\tilde{\phi}(v, h) = \phi(v, v + h)$. Take a partial Taylor expansion in h for $\tilde{\phi}$,

$$\tilde{\phi}(v, h) = \tilde{\phi}(v, 0) + \frac{\partial \tilde{\phi}}{\partial h^\mu}(v, 0) h^\mu + \frac{1}{2} \frac{\partial^2 \tilde{\phi}}{\partial h^\mu \partial h^\nu}(v, 0) h^\mu h^\nu + \varepsilon_3$$

where ε_3 vanishes at order 3 in h .

We know

$$\tilde{\phi}(v, 0) = \phi(v, v) = 0,$$

by Equation (40),

$$\frac{\partial \tilde{\phi}}{\partial h^\mu}(v, 0) = \frac{\partial \phi}{\partial y^\mu}(v, v) = 0.$$

By chain rule,

$$\frac{\partial^2 \tilde{\phi}}{\partial h^\mu \partial h^\nu}(v, 0) = \frac{\partial^2 \phi}{\partial y^\mu \partial y^\nu}(v, v)$$

Equation (19) shows :

$$\frac{\partial\phi}{\partial x^\mu}(x, y)g^{\mu\nu}(x)\frac{\partial\phi}{\partial x^\nu}(x, y) = 4\phi(x, y).$$

Taking $\frac{\partial^2}{\partial x^{\mu_1}\partial x^{\nu_1}}$ on both sides and let $x = y = v$, we have

$$\frac{\partial^2\phi}{\partial x^\mu\partial x^{\mu_1}}(v, v)g^{\mu\nu}(v)\frac{\partial^2\phi}{\partial x^\nu\partial x^{\nu_1}}(v, v) + \frac{\partial^2\phi}{\partial x^\mu\partial x^{\nu_1}}(v, v)g^{\mu\nu}(v)\frac{\partial^2\phi}{\partial x^\nu\partial x^{\mu_1}}(v, v) = 4\frac{\partial^2\phi}{\partial x^{\mu_1}\partial x^{\nu_1}}(v, v).$$

that is

$$\frac{\partial^2\phi}{\partial x^\mu\partial x^{\mu_1}}(v, v)g^{\mu\nu}(v)\frac{\partial^2\phi}{\partial x^\nu\partial x^{\nu_1}}(v, v) = 2\frac{\partial^2\phi}{\partial x^{\mu_1}\partial x^{\nu_1}}(v, v).$$

Notice that $\frac{\partial^2\phi}{\partial x^\mu\partial x^\nu}(v, v)$ is invertible since $\frac{\partial^2\phi}{\partial x^\mu\partial x^\nu}(0, 0) = \delta_{\mu\nu}$ by (39) and if U is chosen small enough. Then we get that

$$\frac{\partial^2\phi}{\partial x^\mu\partial x^\nu}(v, v) = 2g_{\mu\nu}(v).$$

Since ϕ is symmetry, we know

$$\frac{\partial^2\phi}{\partial y^\mu\partial y^\nu}(v, v) = \frac{\partial^2\phi}{\partial x^\mu\partial x^\nu}(v, v) = 2g_{\mu\nu}(v).$$

So

$$\tilde{\phi}(v, h) = g_{\mu\nu}(v)h^\mu h^\nu + \varepsilon_3$$

which concludes the proof. ■

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