GALOIS REDUCIBILITY AND FREENESS OF THE COHOMOLOGY OF KHT SHIMURA VARIETIES

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by

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Abstract. — In [**9**], we proved that the $\overline{\mathbb{Z}}_l$-cohomology of KHT Shimura varieties of dimension d which is not prime, whatever is the weight of the coefficients, when the level is large enough at l, always contains non trivial torsion classes. In [**5**], and more generally in [**10**] for other compact PEL Shimura varieties, for a generic maximal ideal $\mathfrak{m}$ of the unramified Hecke algebra, the localisation at $\mathfrak{m}$ of these cohomology groups appear to be free. In this work we obtain the same result for $\mathfrak{m}$ such that its associated galoisian $\overline{\mathbb{F}}_l$-representation $\overline{\rho}_{\mathfrak{m}}$ is irreducible.

Contents

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Introduction

From Matsushima's formula and computation of $(\mathfrak{G}, K_\infty)$-cohomology, we know that tempered automorphic representations contribution in the cohomology of Shimura varieties with complex coefficients, is concentrated in middle degree. In [**10**], the authors consider the case of $\overline{\mathbb{F}}_l$-coefficients and prove an analog result under some genericness hypothesis which can be stated as follows. Let $\mathfrak{m}$ be a system of Hecke eigenvalues appearing in $H^{n_0}(\mathrm{Sh}_K, \overline{\mathbb{F}}_l)$ with Sh_K some compact unitary Shimura variety of Kottwitz type associated to some similitude group G. By the main result of [**17**], one can attach to such $\mathfrak{m}$, a mod l Galois representation

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_d(\overline{\mathbb{F}}_l)$$

where F is the reflex field of Sh_K which, by assumption, contains some imaginary quadratic extension $E/\mathbb{Q}$. From [**10**] definition 1.9, we say that $\mathfrak{m}$ is generic (resp. decomposed generic) at some split p in E, if for all places v of F dividing p, the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ satisfy $\lambda_i/\lambda_j \notin \{q_v^{\pm 1}\}$ for all $i \neq j$ (resp. and are pairwise distincts), where q_v is the cardinal of the residue field at v. Then under the hypothesis[(1)] that there exists such p with $\mathfrak{m}$ generic at p, the integer n_0 above is necessary equals to the relative dimension of Sh_K. In particular the $H^i(\mathrm{Sh}_K, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ are all torsion free.

In this work we consider the particular case of Kottwitz-Harris-Taylor Shimura varieties Sh_K of [**14**] associated to inner forms of GL_d. In [**5**] we gave a very explicit proof of the fact that if $\mathfrak{m}$ is generic at some split place v then the localized cohomology groups are all torsion free. Moreover we described the shape of the local component at v of any $\overline{\mathbb{Q}}_l$-automorphic representations raising the multiset of modulo l Satake parameters given by $\mathfrak{m}$. Concerning torsion cohomology classes, in [**9**], we proved, at least when d is not prime, that whatever is the weight ξ,

(1) In their new preprint, Caraiani and Scholze explained that, from an observation of Koshikawa, one can replace decomposed generic by simply generic, in their main statement.

if you increase the level at l, then non trivial torsion cohomology classes always appear. The main result of this paper is the following.

Theorem. — *Let $\mathfrak{m}$ be a system of Hecke eigenvalues such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible, then the localized cohomology groups of Sh_K with coefficients in any $\overline{\mathbb{Z}}_l$-local system V_ξ, are all free.*

Roughly the proof relies on the fact that, cf. [**6**] theorem 3.1.1 which follows easily from [**18**] theorem 5.6, if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, then as a $\mathbb{T}_{S,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,

$$H^{d-1}(X_{U,\bar{\eta}},\mathbb{Z}_l)_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}} \otimes_{\mathbb{T}_{S,\mathfrak{m}}} \rho_{\mathfrak{m}},$$

for some $\mathbb{T}_{S,\mathfrak{m}}$-module $\sigma_{\mathfrak{m}}$ on wich Gal_F acts trivially. The idea is then to prove that if there were torsion then the previous decomposition of the global lattice as a tensorial product could not be possible. To do this we compute the cohomology using two different filtrations of the nearby cycles perverse sheaf through their associated spectral sequence, cf. (3.1.1) and (3.1.2).

The author would like to thanks Koshikawa for helpful conversation about a previous work on the same theme, where he pointed me a major mistake. He then explained me some of his ideas which was very inspiring.

1. Recalls from [5]

1.1. Representations of $GL_d(K)$. — We fix a finite extension $K/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We denote $|-|$ its absolute value.

For a representation π of $GL_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n\,\mathrm{val}\circ\det}.$$

1.1.1. Notations. — *For π_1 and π_2 representations of respectively $GL_{n_1}(K)$ and $GL_{n_2}(K)$, we will denote by*

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1\{\frac{n_2}{2}\} \otimes \pi_2\{-\frac{n_1}{2}\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$GL_{r_1} \times GL_{r_2-r_1} \times \cdots \times GL_{r_k-r_{k-1}}.$$

Recall that a representation ϱ of $GL_d(K)$ is called *cuspidal* (resp. *supercuspidal*) if it's not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zera then these two notions coincides, but this is no more true for $\overline{\mathbb{F}}_l$.

1.1.2. Definition. — *(see* [**22**] *§9 and* [**3**] *§1.4) Let g be a divisor of $d = sg$ and π an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(K)$. The induced representation*

$$\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\}$$

holds a unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi)$ *(resp.* $\mathrm{Speh}_s(\pi)$*); it's a generalized Steinberg (resp. Speh) representation.*

Moreover the induced representation $\mathrm{St}_t(\pi\{\frac{-r}{2}\}) \times \mathrm{Speh}_r(\pi\{\frac{t}{2}\})$ *(resp. of* $\mathrm{St}_{t-1}(\pi\{\frac{-r-1}{2}\}) \times \mathrm{Speh}_{r+1}(\pi\{\frac{t-1}{2}\})$*) owns a unique irreducible subspace (resp. quotient), denoted $LT_\pi(t-1,r)$.*

Remark. These representations $LT_\pi(t-1,r)$ appear in the cohomology of the Lubin-Tate spaces, cf. [**2**].

1.1.3. Proposition. — *(cf.* [**20**] *III.5.10) Let π be an irreducible cuspidal representation of $GL_g(K)$ with a stable $\overline{\mathbb{Z}}$-lattice*[(2]*, then its modulo l reduction is irreducible and cuspidal but not necessary supercuspidal.*

The supercuspidal support of the modulo l reduction of a cuspidal representation, is a segment associated to some irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation ϱ of $GL_{g_{-1}(\varrho)}(F_v)$ with $g = g_{-1}(\varrho)t$ where t is either equal to 1 or of the following shape $t = m(\varrho)l^u$ with $u \geqslant 0$ and where $m(\varrho)$ is some integer associated to ϱ: when ϱ is the trivial representation then $m(1_v)$ is either the order of q modulo l when it is > 1, otherwise $m(1_v) = l$. We say that such π_v is of ϱ-type u with $u \geqslant -1$.

1.1.4. Notation. — *For ϱ an irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation, we denote* Cusp_ϱ *(resp.* $\mathrm{Cusp}_\varrho(u)$ *for some $u \geqslant -1$) the set of*

(2)We say that π is entire.

equivalence classes of irreducible $\overline{\mathbb{Q}}_l$-cuspidal representation whose modulo l reduction has supercuspidal support a segment associated to ϱ (resp. of ϱ-type u).

Let $u \geqslant 0$, $\pi_{v,u} \in \mathrm{Cusp}_{\varrho}(u)$ and $\tau = \pi_{v,u}[s]_D$. Let then denote ι the image of $\mathrm{Speh}_s(\varrho)$ by the modulo l Jacquet-Langlands correspondance defined at §1.2.4 de [**11**]. Then the modulo l reduction of τ is isomorphic to

$$\iota\{-\frac{m(\tau)-1}{2}\} \oplus \iota\{-\frac{m(\tau)-3}{2}\} \oplus \cdots \oplus \iota\{\frac{m(\tau)-1}{2}\} \tag{1.1.5}$$

where $\iota\{n\} := \iota \otimes q^{-n\,\mathrm{val}\circ\mathrm{nrd}}$.

We want now to recall the notion of level of non degeneracy from [**1**] §4. The mirabolic subgroup $M_d(K)$ of $GL_d(K)$ is the set of matrices with last row $(0, \cdots, 0, 1)$: we denote

$$V_d(K) = \{(m_{i,j} \in P_d(K) : \ m_{i,j} = \delta_{i,j} \text{ for } j < n\}.$$

its unipotent radical. We fix a non trivial character ψ of K and let θ the character of $V_d(K)$ defined by $\theta((m_{i,j})) = \psi(m_{d-1,d})$. For $G = GL_r(K)$ or $M_r(K)$, we denote $\mathrm{Alg}(G)$ the abelian category of algebraic representations of G and ,following [**1**], we introduce

$$\Psi^- : \mathrm{Alg}(M_d(K)) \longrightarrow \mathrm{Alg}(GL_{d-1}(K), \qquad \Phi^- : \mathrm{Alg}(M_d) \longrightarrow \mathrm{Alg}(M_{d-1}(K))$$

defined by $\Psi^- = r_{V_d,1}$ (resp. $\Phi^- = r_{V_d,\theta}$) the functor of V_{d-1} coinvariants (resp. (V_{d-1}, θ)-coinvariants), cf. [**1**] 1.8. We also introduce the normalize compact induced functor

$$\Psi^+ := i_{V,1} : \mathrm{Alg}(GL_{d-1}(K)) \longrightarrow \mathrm{Alg}(M_d(K)),$$
$$\Phi^+ := i_{V,\theta} : \mathrm{Alg}(M_{d-1}(K)) \longrightarrow \mathrm{Alg}(M_d(K)).$$

1.1.6. Proposition. — *([**1**] p451)*

- *The functors Ψ^-, Ψ^+, Φ^- and Φ^+ are exact.*
- $\Phi^- \circ \Psi^+ = \Psi^- \circ \Phi^+ = 0$.
- *Ψ^- (resp. Φ^+) is left adjoint to Ψ^+ (resp. Φ^-) and the following adjunction maps*

$$\mathrm{Id} \longrightarrow \Phi^-\Phi^+, \qquad \Psi^+\Psi^- \longrightarrow \mathrm{Id},$$

are isomorphisms meanwhile

$$0 \to \Phi^+\Phi^- \longrightarrow \mathrm{Id} \longrightarrow \Psi^+\Psi^- \to 0.$$

1.1.7. Definition. — *For $\tau \in \mathrm{Alg}(M_d(K))$, the representation*

$$\tau^{(k)} := \Psi^- \circ (\Phi^-)^{k-1}(\tau)$$

is called the k-th derivative of τ. If $\tau^{(k)} \neq 0$ and $\tau^{(m)} = 0$ for all $m > k$, then $\tau^{(k)}$ is called the highest derivative of τ.

1.1.8. Notation. — *(cf. [**22**] 4.3) Let $\pi \in \mathrm{Alg}(GL_d(K))$ (or $\pi \in \mathrm{Alg}(M_d(K))$). The maximal number k such that $(\pi_{|M_d(K)})^{(k)} \neq (0)$ is called the level of non-degeneracy of π and denoted $\lambda(\pi)$. We can also iterate the construction so that at the end we obtain a partition $\underline{\lambda(\pi)}$ of d.*

1.1.9. Definition. — *A representation π of $GL_d(K)$, over $\bar{\mathbb{Q}}_l$ or $\bar{\mathbb{F}}_l$, is then said generic if its level of non degeneracy $\lambda(\pi)$ is equal to d.*

1.2. Shimura varieties of KHT type. — Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is quadratic imaginary and $F^+/\mathbb{Q}$ totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we will denote

- F_v the completion of F at v,
- $\mathcal{O}_v$ the ring of integers of F_v,
- ϖ_v a uniformizer,
- q_v the cardinal of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F, of dimension d^2 such that at every place x of F, either B_x is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta \in B^{*=-1}$, denote $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and $G/\mathbb{Q}$ the group of similitudes, denoted G_τ in [**14**], defined for every $\mathbb{Q}$-algebra R by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$ split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \tag{1.2.1}$$

where, identifying places of F^+ over x with places of F over y, $x = \prod_i z_i$ in F^+.

Convention: for $x = yy^c$ a place of $\mathbb{Q}$ split in E and z a place of F over y as before, we shall make throughout the text, the following abuse of

notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (1.2.1).

In [**14**], the authors justify the existence of some G like before such that moreover

- if x is a place of $\mathbb{Q}$ non split in E then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [**14**] bottom of page 90, a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

1.2.2. Notation. — *Denote $\mathcal{I}$ the set of open compact subgroup small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $X_{I,\eta} \longrightarrow \operatorname{Spec} F$ the associated Shimura variety of Kottwitz-Harris-Taylor type.*

1.2.3. Definition. — *Define* Spl *the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write* $\mathrm{Spl}(I)$ *the subset of* Spl *of places which doesn't divide the level I.*

In the sequel, v and w will denote places of F in Spl. For such a place v the scheme $X_{I,\eta}$ has a projective model $X_{I,v}$ over $\operatorname{Spec} \mathcal{O}_v$ with special fiber X_{I,s_v}. For I going through $\mathcal{I}$, the projective system $(X_{I,v})_{I\in\mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}^{-1}$ and $\mathrm{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is Artin's isomorphism which sends geometric Frobenius to uniformizers.

1.2.4. Notations. — *For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber $X_{I,\bar{s}_v}$ is denoted*

$$X_{I,\bar{s}_v} =: X_{I,\bar{s}_v}^{\geqslant 1} \supset X_{I,\bar{s}_v}^{\geqslant 2} \supset \cdots \supset X_{I,\bar{s}_v}^{\geqslant d}$$

where $X_{I,\bar{s}_v}^{=h} := X_{I,\bar{s}_v}^{\geqslant h} - X_{I,\bar{s}_v}^{\geqslant h+1}$ is an affine scheme, smooth of pure dimension $d-h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank h. For each $1 \leqslant h < d$, write

$$i_h : X_{I,\bar{s}_v}^{\geqslant h} \hookrightarrow X_{I,\bar{s}_v}^{\geqslant 1}, \quad j^{\geqslant h} : X_{I,\bar{s}_v}^{=h} \hookrightarrow X_{I,\bar{s}_v}^{\geqslant h},$$

and $j^{=h} = i_h \circ j^{\geqslant h}$.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists, [**14**] p.97, then an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leqslant \cdots \leqslant a_{\sigma,d})$. We then denote

$$V_{\xi,\overline{\mathbb{Z}}_l}$$

the associated $\overline{\mathbb{Z}}_l$-local system on $X_{\mathcal{I}}$ Recall that an irreducible automorphic representation Π is said ξ-cohomological if there exists an integer i such that

$$H^i\big((\text{Lie } G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$. Let $d^i_\xi(\Pi_\infty)$ be the dimension of this cohomology group.

1.3. Cohomology of Newton strata. —

1.3.1. Notation. — *For $1 \leqslant h \leqslant d$, let $\mathcal{I}_v(h)$ denote the set of open compact subgroups*

$$U_v(\underline{m}, h) := U_v(\underline{m}^v) \times \begin{pmatrix} I_h & 0 \\ 0 & K_v(m_1) \end{pmatrix},$$

where $K_v(m_1) = \text{Ker}\big(GL_{d-h}(\mathcal{O}_v) \longrightarrow GL_{d-h}(\mathcal{O}_v/(\varpi_v^{m_1}))\big)$. We then denote $[H^i(h,\xi)]$ (resp. $[H^i_!(h,\xi)]$) the image of

$$\varinjlim_{I\in\mathcal{I}_v(h)} H^i(X^{\geqslant h}_{I,\bar{s}_v,1}, V_{\xi,\overline{\mathbb{Q}}_l}[d-h]) \qquad \textit{resp.} \quad \varinjlim_{I\in\mathcal{I}_v(h)} H^i(X^{\geqslant h}_{I,\bar{s}_v,1}, j^{\geqslant h}_{1,!} V_{\xi,\bar{\mathbb{Q}}_l}[d-h])$$

inside the Grothendieck $\text{Groth}(v,h)$ *of admissible representations of $G(\mathbb{A}^\infty) \times GL_{d-h}(F_v) \times \mathbb{Z}$.*

Remark. An element $\sigma \in W_v$ acts through $-\deg\sigma \in \mathbb{Z}$ and $\Pi_{p_v,0}(\text{Art}^{-1}(\sigma))$. We moreover consider the action of $GL_h(F_v)$ through $\text{val}\circ\det : GL_h(F_v) \longrightarrow \mathbb{Z}$ and finally $P_{h,d}(F_v)$ through its Levi factor $GL_h(F_v) \times GL_{d-h}(F_v)$, i.e. its unipotent radical acts trivially.

From [**5**] proposition 3.6, for any irreducible tempered automorphic representation Π of $G(\mathbb{A})$ and for every $i \neq 0$, the $\Pi^{\infty,v}$-isotypic component of $[H^i(h,\xi)]$ and $[H^i_!(h,\xi)]$ are zero. About the case $i = 0$, for Π

an irreducible automorphic tempered representation ξ-cohomological, its local component at v looks like

$$\Pi_v \simeq \mathrm{St}_{t_1}(\pi_{v,1}) \times \cdots \times \mathrm{St}_{t_u}(\pi_{v,u}),$$

where for $i = 1, \cdots, u$, $\pi_{v,i}$ is an irreducible cuspidal representation of est une $GL_{g_i}(F_v)$.

1.3.2. Proposition. — *(cf.* [**5**] *proposition 3.9) With the previous notations, we order the $\pi_{v,i}$ such that the first r-ones are unramified characters. Then the $\Pi^{\infty,v}$-isotypic component of $[H^0(h,\xi)]$ is then equals to*

$$\Big(\frac{\sharp\,\mathrm{Ker}^1(\mathbb{Q},G)}{d}\sum_{\Pi'\in\mathcal{U}_G(\Pi^{\infty,v})} m(\Pi')d_\xi(\Pi'_\infty)\Big)\Big(\sum_{1\leqslant k\leqslant r:\ t_k=h}\Pi_v^{(k)}\otimes\chi_{v,k}\Xi^{\frac{d-h}{2}}\Big)$$

where

- $\mathrm{Ker}^1(\mathbb{Q},G)$ *is the subset of elements $H^1(\mathbb{Q},G)$ which become trivial in $H^1(\mathbb{Q}_{p'},G)$ for every prime p';*
- $\Pi_v^{(k)} := \mathrm{St}_{t_1}(\chi_{v,1})\times\cdots\times\mathrm{St}_{t_{k-1}}(\chi_{v,k-1})\times\mathrm{St}_{t_{k+1}}(\chi_{v,k+1})\times\cdots\times\mathrm{St}_{t_u}(\chi_{v,u})$ *and*
- $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ *is defined by $\Xi(\frac{1}{2}) = q_v^{\frac{1}{2}}$.*
- $\mathcal{U}_G(\Pi^{\infty,v})$ *is the set of equivalence classes of irreducible automorphic representations Π' of $G(\mathbb{A})$ such that $(\Pi')^{\infty,v} \simeq \Pi^{\infty,v}$.*

Remark. In particular if $[H^0(h,\xi)]$ has non trivial invariant vectors under some open compact subgroup $I \in \mathcal{I}_v(h)$ which is maximal at v, then the local component of Π at v is of the following shape $\mathrm{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \chi_{v,d-h}$ where the $\chi_{v,i}$ are unramified characters.

1.3.3. Definition. — *For $I \in \mathcal{I}$ a finite level, let $\mathbb{T}_I := \prod_{x\in\mathrm{Unr}(I)}\mathbb{T}_x$ be the unramified Hecke algebra where $\mathrm{Unr}(I)$ is the union of places q where G is unramified and I_x is maximal, and where $\mathbb{T}_x \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.*

Example. For $w \in \mathrm{Spl}(I)$, we have

$$\mathbb{T}_w = \overline{\mathbb{Z}}_l\big[T_{w,i} :\ i = 1, \cdots, d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w,\cdots,\varpi_w}^{i},\overbrace{1,\cdots,1}^{d-i})GL_d(\mathcal{O}_w)\subset GL_d(F_w).$$

The minimal prime ideals of $\mathbb{T}_I$ are the prime ideals above the zero ideal of $\overline{\mathbb{Z}}_l$ and are then in bijection with the prime ideals of $\mathbb{T}_I\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$. To such an ideal, which corresponds to give a collection of Satake parameters, is then associated a unique near equivalence class in the sense of [**19**], denoted $\Pi_{\tilde{\mathfrak{m}}}$ which is the finite set of irreducible automorphic representations cohomological whose multi-set of Satake parameters at each place $x\in\mathrm{Unr}(I)$, is given by $S_{\tilde{\mathfrak{m}}}(x)$ the multi-set of roots of the Hecke polynomial

$$P_{\tilde{\mathfrak{m}},w}(X):=\sum_{i=0}^{d}(-1)^i q_w^{\frac{i(i-1)}{2}}T_{w,i,\tilde{\mathfrak{m}}}X^{d-i}\in\overline{\mathbb{Q}}_l[X]$$

i.e.

$$S_{\tilde{\mathfrak{m}}}(w):=\big\{\lambda\in\mathbb{T}_I\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l/\widetilde{\mathfrak{m}}\simeq\overline{\mathbb{Q}}_l\text{ such that }P_{\tilde{\mathfrak{m}},w}(\lambda)=0\big\}.$$

Thanks to [**14**] and [**19**], we denote

$$\rho_{\tilde{\mathfrak{m}}}:\mathrm{Gal}(\bar{F}/F)\longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

the galoisian representation associated to any $\Pi\in\Pi_{\tilde{\mathfrak{m}}}$. Recall that the modulo l reduction of $\rho_{\tilde{\mathfrak{m}}}$ depends only of $\mathfrak{m}$, and was denoted above $\overline{\rho}_{\mathfrak{m}}$. For every $w\in\mathrm{Spl}(I)$, we also denote $S_{\mathfrak{m}}(w)$ the multi-set of modulo l Satake parameters at w given as the multi-set of roots of

$$P_{\mathfrak{m},w}(X):=\sum_{i=0}^{d}(-1)^i q_w^{\frac{i(i-1)}{2}}\overline{T_{w,i}}X^{d-i}\in\overline{\mathbb{F}}_l[X]$$

i.e.

$$S_{\mathfrak{m}}(w):=\big\{\lambda\in\mathbb{T}_I/\mathfrak{m}\simeq\overline{\mathbb{F}}_l\text{ such that }P_{\mathfrak{m},w}(\lambda)=0\big\}.$$

For ξ an irreducible algebraic representation of $G(\mathbb{Q})$, let $\mathbb{T}_I(\xi)$ be the image of $\mathbb{T}_I$ inside the $\overline{\mathbb{Z}}_l$-endomorphism of the free quotient of $\bigoplus_{i\in\mathbb{N}}H^i(X_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Z}}_l})$.

2. Computation of torsion cohomology classes

Our strategy to compute the cohomology of the KHT-Shimura variety $X_{I,\bar{\eta}}$ with coefficients in $V_{\xi,\overline{\mathbb{Z}}_l}$, is to realize it as the outcome of the vanishing cycles spectral sequence at some place $v \in \mathrm{Spl}$.

2.1. The case where the level at v is maximal. — As for each $1 \leqslant h \leqslant d-1$, the open Newton stratum $X_{I,\bar{s}_v}^{=h}$ is affine then $H^i(X_{I,\bar{s}_v}^{=h}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h])$ is zero for $i<0$ and free for $i=0$. Using this property and the short exact sequence of free perverse sheaves

$$0 \to i_{h+1,*}V_{\xi,\overline{\mathbb{Z}}_l,|X_{I,\bar{s}_v}^{\geqslant h+1}}[d-h-1] \longrightarrow j_!^{\geqslant h}j^{\geqslant h,*}V_{\xi,\overline{\mathbb{Z}}_l,|X_{I,\bar{s}_v}^{\geqslant h}}[d-h] \longrightarrow V_{\xi,\overline{\mathbb{Z}}_l,|X_{I,\bar{s}_v}^{\geqslant h}}[d-h] \to 0,$$

we then obtain for every $i>0$

$$0 \to H^{-i-1}(X_{I,\bar{s}_v}^{\geqslant h}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(X_{I,\bar{s}_v}^{\geqslant h+1}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1]) \to 0, \tag{2.1.1}$$

and for $i=0$,

$$0 \to H^{-1}(X_{I,\bar{s}_v}^{\geqslant h}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X_{I,\bar{s}_v}^{\geqslant h+1}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1]) \longrightarrow H^0(X_{I,\bar{s}_v}^{\geqslant h}, j_!^{\geqslant h}j^{\geqslant h,*}V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X_{I,\bar{s}_v}^{\geqslant h}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \to \cdots \tag{2.1.2}$$

In [**5**], arguing by induction from $h=d$ to $h=1$, we prove that for a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I(\xi)$ such that $S_{\mathfrak{m}}(v)$ does not contain any subset of the form $\{\alpha, q_v\alpha\}$, all the cohomology groups $H^i(X_{I,\bar{s}_v}^{\geqslant h}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ are free: note that in order to deal with $i \geqslant 0$, one has to use Grothendieck-Verdier duality.

Without this hypothesis, arguing similarly, we conclude that any torsion cohomology class comes from a non strict map

$$H^0_{free}(X_{I,\bar{s}_v}^{\geqslant h+1}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1])_{\mathfrak{m}} \longrightarrow H^0(X_{I,\bar{s}_v}^{\geqslant h}, j_!^{\geqslant h}j^{\geqslant h,*}V_{\xi,\overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}.$$

In particular it raises in characteristic zero to some free subquotient of $H^0(X_{I,\bar{s}_v}^{\geqslant h}, j_!^{\geqslant h}j^{\geqslant h,*}V_{\xi,\overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$.

2.1.3. Proposition. — *Consider as in* [**5**], *h_0 maximal such that there exists i with $H^{d-h_0+i}(X_{I,\bar{s}_v}^{\geqslant h_0}V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m},tor} \neq (0)$, then we have the following properties:*

– $i = 0, 1$;
– *for all* $1 \leqslant h \leqslant h_0$ *and* $i < h_0 - h$ *then* $H^{d-h+i}(X^{\geqslant h}_{I,\bar{s}_v} V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m},tor} = (0)$ *while for* $i = h_0 - h$ *it's non trivial.*

Remark. Note that any system of Hecke eigenvalues $\mathfrak{m}$ of $\mathbb{T}_I$ inside the torsion of some $H^i(X_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l})$ raises in characteristic zero, i.e. is associated to a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I(\xi)$.

2.2. Harris-Taylor perverse sheaves over $\overline{\mathbb{Z}}_l$. — Consider now the ideals $I^v(n) := I^v K_v(n)$ where $K_v(n) := \mathrm{Ker}(GL_d(\mathcal{O}_v) \twoheadrightarrow GL_d(\mathcal{O}_v/\mathcal{M}_v^n))$. Recall then that $X^{=h}_{I^v(n),\bar{s}}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d-h}(\mathcal{O}_v/\mathcal{M}_v^n)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely this means there exists a closed subscheme $X^{=h}_{I^v(n),\bar{s},1}$ stabilized by the Hecke action of $P_{h,d-h}(F_v)$ and such that

$$X^{=h}_{I^v(n),\bar{s}} \simeq X^{=h}_{I^v(n),\bar{s},1} \times_{P_{h,d-h}(\mathcal{O}_v/\mathcal{M}_v^n)} GL_d(\mathcal{O}_v/\mathcal{M}_v^n).$$

Let then denote $\mathfrak{m}^v$ the multiset of Hecke eigenvalues given by $\mathfrak{m}$ but outside v and introduce for Π_h any representation of $GL_h(F_v)$

$$H^i(X^{\geqslant h}_{I^v(\infty),\bar{s}_v,1}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^v} \otimes \Pi_h := \varprojlim_{I^v(n)} H^i(X^{\geqslant h}_{I^v(n),\bar{s}_v,1}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^v} \otimes \Pi_h,$$

as a representation of $GL_h(F_v) \times GL_{d-h}(F_v)$, where $g \in GL_h(F_v)$ acts both on Π_h and on $H^i(X^{\geqslant h}_{I^v(n),\bar{s}_v,1}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$ through the determinant map $\det : GL_h(F_v) \twoheadrightarrow F_v^\times$. Note moreover that the unipotent radical of $P_{h,d}(F_v)$ acts trivially on these cohomology groups and introduce the induced version

$$H^i(X^{\geqslant h}_{I^v(\infty),\bar{s}_v}, \Pi_h \otimes V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^v} \simeq \mathrm{ind}_{P_{h,d}(F_v)}^{GL_d(F_v)} H^i(X^{\geqslant h}_{I^v(\infty),\bar{s}_v,1}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^v} \otimes \Pi_h.$$

More generally, with the notations of [**2**], replace now the trivial representation by an irreducible cuspidal representation π_v of $GL_g(F_v)$ for some $1 \leqslant g \leqslant d$.

2.2.1. Notations. — *Let* $1 \leqslant t \leqslant s := \lfloor d/g \rfloor$ *and* Π_t *any representation of* $GL_{d-tg}(F_v)$. *We then denote*

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_1 \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

the Harris-Taylor local system on the Newton stratum $X^{=tg}_{I,\bar{s}_v,1}$ *where*

– $\mathcal{L}(\pi_v[t]_D)_1$ *is defined thanks to Igusa varieties attached to the representation* $\pi_v[t]_D$ *of the division algebra of dimension* $(tg)^2$ *over* F_v *associated to* $\mathrm{St}_t(\pi_v)$ *by the Jacquet-Langlands correspondence,*
– $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ *defined by* $\Xi(\frac{1}{2}) = q^{1/2}$.

We also introduce the induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \left(\mathcal{L}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}\right) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ *acts trivially and the action of*

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

is given

– *by the action of* g_v^c *on* Π_t *and* $\deg(\sigma_v) \in \mathbb{Z}$ *on* $\Xi^{\frac{tg-d}{2}}$, *and*
– *the action of* $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ *on* $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT(\pi_v, \Pi_t)_1 := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1_h}}[d - tg],$$

and the perverse sheaf

$$P(t, \pi_v)_1 := j_{1,!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v))_1 \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT(\pi_v, \Pi_t)$ *and* $P(t, \pi_v)$, *where*

$$j^{=h} = i^h \circ j^{\geqslant h} : X_{I,\bar{s}}^{=h} \hookrightarrow X_{I,\bar{s}}^{\geqslant h} \hookrightarrow X_{I,\bar{s}}$$

and $\mathbb{L}^\vee$ *is the local Langlands correspondence. Finally we will also use the indice* ξ *in the notations, for example* $HT_\xi(\pi_v, \Pi_t)$, *when we twist the sheaf with* $V_{\xi,\overline{\mathbb{Z}}_l}$.

With the previous notations, from (1.1.5), we deduce the following equality in the Grothendieck group of Hecke-equivariant local systems

$$m(\varrho) l^u \Big[\mathbb{F}\mathcal{F}_{\xi,\bar{\mathbb{Z}}_l}(t, \pi_{v,u})\Big] = \Big[\mathbb{F}\mathcal{F}_{\xi,\bar{\mathbb{Z}}_l}(tm(\varrho) l^u, \pi_{v,-1})\Big]. \tag{2.2.2}$$

We want now to focus on the perverse Harris-Taylor sheaves. Note first that over $\overline{\mathbb{Z}}_l$, there are two notions of intermediate extension for the two classical t-structures p and $p+$, so that we can define for every $\pi_v \in \mathrm{Cusp}_\varrho$ of $GL_g(F_v)$ and $1 \leqslant t \leqslant d/g$:

$${}^p j_{!*}^{=tg} HT(\pi_{v,}, \Pi_t) \twoheadrightarrow_+ {}^{p+} j_{!*}^{=tg} HT(\pi_v, \Pi_t), \tag{2.2.3}$$

the symbol $\hookrightarrow\!\!\!\!\rightarrow$ meaning bimorphism, ie both a monomorphism and epimorphism, so that the cokernel for the t-structure p (resp. the kernel for $p+$) has support in $X_{I,\bar{s}_v}^{\geqslant tg+1}$. When π_v is a character, i.e. when $g=1$, it's trivial that there are in fact isomorphic but in general there are not.
Remark. One of the main result of [**7**], is to prove that the previous bimorphism is an isomorphism for any $\pi_v \in \mathrm{Cusp}_{\varrho}(-1)$, but as here we'll only deal with ϱ the trivial representation, we don't need the result of [**7**].

We focus now on the case where ϱ is a character and $\pi_v \in \mathrm{Cusp}_{\varrho}(u)$ for $u \geqslant 0$. Let $\mathbb{F}(\bullet) := \bullet \otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ be the modulo l reduction functor.

2.2.4. Proposition. — *(cf. proposition 2.4.2 of* [**7**]*) Let ϱ be a character, i.e. $g_{-1}(\varrho) = 1$, we then have the following equality in the Grothendieck group of Hecke-equivariant perverse sheaves*

$$\mathbb{F}\Big({}^{p}j_{!*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)\Big) = m(\varrho) l^u \sum_{r=0}^{s-tm(\varrho)l^u} {}^{p}j_{!*}^{=tg_u(\varrho)+rg_{-1}(\varrho)} HT\big(\varrho, r_l(\Pi_t) \overrightarrow{\times} V_{\varrho}(r, <\underline{\delta_u})\big) \otimes \Xi^{r\frac{g-1}{2}},$$

where $V_{\varrho}(r, <\underline{\delta_u})\big)$ is defined in loc. cit. as the sum of irreducible constituants of the modulo l reduction of $\mathrm{St}_r(\pi_{v,-1})$ of ϱ-level strictly less than $\underline{\delta_u} := (0, \cdots, 0, 1, 0, \cdots)$, cf. §A.2 of [**7**]

Remark. The proof only uses the fact that $\mathbb{F}$ and ${}^{p}j_!$ commutes and the property that the bimorphism (2.2.3) is a isomorphism for π_v a character. Note moreover that the equality of the previous proposition remains true if we replace ${}^{p}j_{!*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)$ by any perverse sheaf P such that

$${}^{p}j_{!*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t) \hookrightarrow\!\!\!\!\rightarrow_+ P \hookrightarrow\!\!\!\!\rightarrow_+ {}^{p+}j_{!*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t).$$

When we look at $\mathbb{F}P$ for such P, they differ from each of others only in the order of the Jordan-Holder factors: for example they appears in the increasing (resp. decreasing) order of the dimension of their support for ${}^{p}j_{!*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)$ (resp. for ${}^{p+}j_{!*}^{=tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)$).

2.3. The infinite level case at v and w. — Recall that v and w are places of Spl and we focus now on the torsion in the cohomology groups with compact support of $HT(\chi_v, \Pi_t)$ when the level at v and w is infinite.

2.3.1. Notation. — *We will denote $I^{v,w} \in \mathcal{I}$ a finite level outside v and w, and let $\mathfrak{m}^{v,w}$ be the maximal ideal of $\mathbb{T}_{I^{v,x}}(\xi)$ associated to $\mathfrak{m}$.*

Start first from the following resolution of ${}^{p}j_{!*}^{=tg}HT(\chi_v, \Pi_t)$

$$\begin{aligned} 0 \to j_!^{=d}HT(\chi_v, \Pi_t\{\frac{t-s}{2}\}) \times \mathrm{Speh}_{d-t}(\chi_v\{t/2\})) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \cdots \\ \longrightarrow j_!^{=t+1}HT(\chi_v, \Pi_t\{-1/2\} \times \chi_v\{t/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow \\ j_!^{=t}HT(\chi_v, \Pi_t) \longrightarrow {}^{p}j_{!*}^{=t}HT(\chi_v, Pi_t) \to 0. \quad (2.3.2) \end{aligned}$$

Remark. This result is proved in full generality over $\overline{\mathbb{Q}}_l$ for any irreducible cuspidal representation in replacement of χ_v. Over $\overline{\mathbb{Z}}_l$, it's proved in [**7**] for any irreducible representation whose modulo l reduction is supercuspidal. In the case of a character, the argument is trivial as we just have to notice that the strata $X_{I,\bar{s}_v,1}^{\geqslant h}$ are smooth so that the constant sheaf, up to shift, is perverse and so equals to the intermediate extension of the constant sheaf, shifted by $d-h$, on $X_{I,\bar{s}_v,1}^{=h}$. The previous resolution is then just the induced version of this.

By adjunction property, the map

$$\begin{aligned} j_!^{=t+\delta}HT(\chi_v, \Pi_t\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_{\delta}(\chi_v\{t/2\})) \otimes \Xi^{\delta/2} \\ \longrightarrow j_!^{=t+\delta-1}HT(\chi_v, \Pi_t\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \quad (2.3.3) \end{aligned}$$

is given by

$$\begin{aligned} HT(\chi_v, \Pi_t\{\frac{-\delta}{2}\} \times \mathrm{Speh}_{\delta}(\chi_v\{t/2\})) \otimes \Xi^{\delta/2} \longrightarrow \\ {}^{p}i^{t+\delta,!}j_!^{=t+\delta-1}HT(\chi_v, \Pi_t\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \quad (2.3.4) \end{aligned}$$

From [**7**], we have

$$\begin{aligned} {}^{p}i^{t+\delta,!}j_!^{=t+\delta-1}HT(\chi_v, \Pi_t\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \\ \simeq HT\Big(\chi_v, \Pi_t\{\frac{1-\delta}{2}\}) \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{t/2\}\Big) \otimes \Xi^{\delta/2} \\ (2.3.5) \end{aligned}$$

Fact. In particular, up to homothety, the map (2.3.5), and so those of (2.3.4), is unique. Finally as the map of (2.3.2) are strict, the given

maps (2.3.3) are uniquely determined, that is if we forget the infinitesimal parts, these maps are independent of chosen t in (2.3.2).

We want now to copy the arguments of §2.1. For every $1 \leqslant h \leqslant d$, let denote $i(h)$ the smaller index i such that $H^i(X_{I^{v,w},\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v, \Pi_h))_{\mathfrak{m}}$ has non trivial torsion: if it doesn't exists then set $i(h) = +\infty$. By duality, as ${}^p j_{!*}^{=h} = {}^{p+} j_{!*}^{=h}$ for Harris-Taylor local systems associated to character, note that when $i(h)$ is finite then $i(h) \leqslant 0$. Suppose there exists h with $i(h)$ finite and denote h_0 the bigger such h.

2.3.6. Lemma. — *For $1 \leqslant h \leqslant t_0$ then $i(h) = h - h_0$.*

Proof. — Note first that for every $h_0 \leqslant h \leqslant s$, then the cohomology groups of $j_!^{=h} HT_\xi(\chi_v, \Pi_h)$ are torsion free. Indeed there exists a filtration

$$(0) = \mathrm{Fil}^0(\chi_v, h) \hookrightarrow \mathrm{Fil}^{-d}(\chi_v, h) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^{-h}(\chi_v, h) = j_!^{=h} HT(\chi_v, \Pi_h)$$

with graduates

$$\mathrm{gr}^{-k}(\chi_v, h) \simeq {}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}).$$

The ξ-associated spectral sequence localized at $\mathfrak{m}^{v,w}$ is then concentrated in middle degree and torsion free. Then the spectral sequence associated to (2.3.2) has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the maps

$$\begin{aligned} H^0\big(X_{I^{v,w},\bar{s},v}, j_!^{=h+\delta} HT_\xi(\chi_v, \Pi_h\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{t/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}^{v,w}} \longrightarrow \\ H^0\big(X_{I^{v,w},\bar{s},v}, j_!^{=h+\delta-1} HT_\xi(\chi_v, \Pi_h\{\frac{1-\delta}{2}\}) \\ \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}^{v,w}} \quad (2.3.7) \end{aligned}$$

are strict. Then from the previous fact stressed after (2.3.5), this property remains true when we consider the associated spectral sequence for $1 \leqslant h' \leqslant h_0$.

Consider now $h = h_0$ where we know the torsion to be non trivial. From what was observed above we then deduce that the map

$$H^0\big(X_{I^{v,w},\bar{s},v}, j_!^{=h_0+1} HT_\xi(\chi_v, \Pi_{h_0}\{\frac{-1}{2}\}) \times \chi_v\{h_0/2\})) \otimes \Xi^{1/2}\big)_{\mathfrak{m}^{v,w}} \longrightarrow H^0\big(X_{I^{v,w},\bar{s},v}, j_!^{=h_0} HT_\xi(\chi_v, \Pi_{h_0})\big)_{\mathfrak{m}^{v,w}} \quad (2.3.8)$$

has a non trivial torsion cokernel so that $i(h_0) = 0$.

Finally for any $1 \leqslant h \leqslant h_0$, the map like (2.3.8) for $h + \delta - 1 < h_0$ are strict so that the $H^i(X_{I^{v,w},\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v, \Pi_h))_{\mathfrak{m}^{v,w}}$ are zero for $i < h - h_0$ while when $h + \delta - 1 = h_0$ its cokernel has non trivial torsion which gives then a non trivial torsion class in $H^{h-h_0}(X_{I^{v,w},\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v, \Pi_h))_{\mathfrak{m}^{v,w}}$. □

From the previous proof we also deduce that all cohomology classes of any of the $H^i(X_{I^{v,w},\bar{s}_v}, \mathcal{P}_\xi(t, \chi_v))_{\mathfrak{m}^{v,w}}$ comes from non strictness of some of the map (2.3.8) where $\Pi_v := \mathrm{St}_t(\chi_v)$. In the following we will focus on $H^i(X_{I^{v,w},\bar{s}_v}, \mathcal{P}_\xi(t, \chi_v))_{\mathfrak{m}^{v,w}}[l]$ as a $\overline{\mathbb{F}}_l$-representation of $GL_d(F_v) \times GL_d(F_w)$. More precisely we are interested in irreducible such sub-quotients which have maximal non-degeneracy level either at v or w. Let first fix such non degeneracy level $\underline{\lambda}$ for $GL_d(F_v)$ in the sense of notation 1.1.8, which is maximal for torsion classes in $H^0(X_{I^{v,w},\bar{s}_v}, \mathcal{P}_\xi(t, \chi_v))_{\mathfrak{m}^{v,w}}[l]$ for various $1 \leqslant t \leqslant d$.

2.3.9. Lemma. — *Let ϱ be a $\overline{\mathbb{F}}_l$-character of $F_v^\times$ and $\pi_v \in \mathrm{Cusp}_\varrho$. For $i \neq 0, 1$ all $\overline{\mathbb{F}}_l[GL_d(F_v)]$-irreducible sub-quotients of $H^i(X_{I^{v,w},\bar{s}_v}, \mathcal{P}_\xi(t, \pi_v))_{\mathfrak{m}^{v,w}}[l]$ have a level of non degeneracy strictly less than $\underline{\lambda}$.*

Remark. Recall that we only know, a priori, that the $\mathcal{P}(t, \pi_v)$ only verify

$${}^p j_{!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v)) \hookrightarrow\!\!\!\!\rightarrow_+ \mathcal{P}(t, \pi_v) \hookrightarrow\!\!\!\!\rightarrow_+ {}^{p+} j_{!*}^{=tg} HT(\pi_v, \mathrm{St}_t(\pi_v)).$$

Proof. — If $\pi_v \in \mathrm{Cusp}_\varrho(-1)$ is a character then the result follows from above. In particular the result remains true for ${}^p j_{!*}^{=t} HT_{\xi,\overline{\mathbb{F}}_l}(\varrho, \Pi_t)[d-t]$, i.e. if $H^I(X_{I^{v,w},\bar{s}_v}, {}^p j_{!*}^{=t} HT_{\xi,\overline{\mathbb{F}}_l}(\varrho, \Pi_t)[d-t])_{\mathfrak{m}^{v,w}}$ has an irreducible subquotient with level of non-deneracy greater or equal to $\underline{\lambda}$ then $i = 0, 1$.[(3)] We

[(3)] Recall that the $\overline{\mathbb{Q}}_l$-cohomology groups localized at $\mathfrak{m}$ are concentrated in degree $i = 0$.

then conclude using the equality of proposition 2.2.4 and its following remark. □

3. Proof of the theorem

3.1. Filtrations of the nearby cycles perverse sheaf. — Let denote

$$\Psi_{I,v} := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])(\frac{d-1}{2})$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber $X_{I,\bar{s}_v}$ of X_I. We also denote $\Psi_{I,\xi,v} := \Psi_{I,v} \otimes V_{\xi,\overline{\mathbb{Z}}_l}$.

Using the Newton stratification and following the constructions of [**8**], we can define a $\overline{\mathbb{Z}}_l$-filtration $\mathrm{Fil}^\bullet(\Psi_{I,v})$ whose graduates are free, isomorphic to some free perverse Harris-Taylor sheaf. Moreover in [**7**] proposition 3.1.3, we prove the following splitting

$$\Psi_{I,v} \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_{I,\varrho}$$

where $\mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ is the set of inertial equivalence classes of irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation of $GL_g(F_v)$, with the property the irreducible constituant of $\Psi_{I,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are exactly the perverse Harris-Taylor sheaves, of level I, associated to a irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations of some $GL_g(F_v)$ such that the supercuspidal support of the modulo l reduction of π_v is a segment associated to the inertial class ϱ.

Remark. In [**7**], we proved that if you always use the adjunction maps $j_!^{=h} j^{=h,*} \to \mathrm{Id}$ then all the previous graduates of $\Psi_{I,\varrho}$ are isomorphic to p-intermerdiate extensions. In the following we will only consider the case where ϱ is a character in which case the p and $p+$ intermediate extension associated to character $\chi_{v,-1} \in \mathrm{Cusp}_{\varrho}(-1)$, coincides. Note that in the following we will not use the results of [**7**].

We then have a spectral sequence

$$E_1^{p,q} = H^{p+q}(X_{I,\bar{s}_v}, \mathrm{gr}^{-p}(\Psi_{I,\xi,v})) \Rightarrow H^{p+q}(X_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l}). \tag{3.1.1}$$

Let consider now the filtration of stratification of $\Psi_{I,\varrho}$ constructed using the adjunction morphisms $j_!^{=t} j^{=t,*}$ as in [**4**]

$$\mathrm{Fil}_!^0(\Psi_{I,\varrho}) \hookleftarrow \mathrm{Fill}_!^1(\Psi_{I,\varrho}) \hookleftarrow \mathrm{Fill}_!^2(\Psi_{I,\varrho}) \hookleftarrow \cdots \hookleftarrow \mathrm{Fill}_!^d(\Psi_{I,\varrho})$$

where $\mathrm{Fill}^t_!(\Psi_{I,\varrho})$ is the saturated image of $j_!^{=t}j^{=g,*}\Psi_{I,\varrho} \longrightarrow \Psi_{I,\varrho}$. We then denote $\mathrm{gr}^k_{I,\varrho,!}$ the graduates and

$$E^{p,q}_{!,\varrho,1} = H^{p+q}(X_{I,\bar{s}_v}, \mathrm{gr}^{-p}_{!,I,\xi,\varrho}) \Rightarrow H^{p+q}(X_{I,\bar{\eta}_v}, \Psi_{I,\xi,\varrho}). \tag{3.1.2}$$

3.2. About torsion cohomology classes. — Recall we have fixed two places $v, w \in \mathrm{Spl}$ as well as a finite level $I^{v,w}$ outside v, w. We then simply denote Ψ and Ψ_ξ, the inductive system of perverse sheaves indexed by the finite level $I^{v,w}I_vI_w \in \mathcal{I}$.

Fix some character χ_v of type ϱ and let denote $\mathrm{Fil}^g_{!,\chi_v}(\Psi) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Fil}^g_!(\Psi_\varrho)$ such that $\mathrm{Fil}^g_{!,\chi_v}(\Psi) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathrm{Fil}^g_!(\Psi_{\chi_v})$ where Ψ_{χ_v} is the direct factor of $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ associated to χ_v, cf. [**4**]. From the main result of [**7**], which can also be deduce easily from [**16**] and the comparison theorem of Faltings-Fargues cf. [**13**], we have then the following resolution of $\mathrm{Fil}^g_{!,\chi_v}(\Psi)$

$$\begin{aligned} 0 \to j_!^{=d}HT(\chi_v, \mathrm{Speh}_d(\chi_v)) \otimes \mathbb{L}(\chi_v(\frac{d-1}{2})) \longrightarrow \\ j_!^{=d-1}HT(\chi_v, \mathrm{Speh}_{d-1}(\chi_v)) \otimes \mathbb{L}(\chi_v(\frac{d-2}{2})) \longrightarrow \\ \cdots \longrightarrow j_!^{=1}HT(\chi_v, \chi_v) \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{Fil}^1_{!,\chi_v}(\Psi) \to 0 \end{aligned} \tag{3.2.1}$$

We can then apply the arguments of the previous section so that $H^i(X_{I^{v,w},\bar{s}_v}, \mathrm{Fil}^1_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$ has non trivial torsion for $i = 1 - t_0$ and with free quotient zero for $i \neq 0$. Clearly we can also repeat the same arguments for the other $\mathrm{gr}^t_{!,\chi_v} \hookrightarrow\!\!\!\!\rightarrow \mathrm{gr}^t_!(\Psi_\varrho)$ with

$$\begin{aligned} 0 \to j_!^{=d}HT(\chi_v, LT_{\chi_v}(t-1, d-t)) \otimes \mathbb{L}(\chi_v(\frac{d-2t+1}{2})) \longrightarrow \\ j_!^{=d-1}HT(\chi_v, LT_{\chi_v}(t-1, d-t-1)) \otimes \mathbb{L}(\chi_v(\frac{d-2t}{2})) \longrightarrow \\ \cdots \longrightarrow j_!^{=t}HT(\chi_v, \mathrm{St}_t(\chi_v)) \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{Fil}^t_{!,\chi_v}(\Psi) \to 0. \end{aligned} \tag{3.2.2}$$

Finally all the torsion cohomology classes of the $H^i(X_{I,\bar{s}_v}, \mathrm{gr}^t_{!,\chi_v})_{\mathfrak{m}^v}$ come from non strictness of the maps

$$H^{-1}(X_{I,\bar{s}_v}, j_!^{=h+1}HT(\chi_v, \Pi_{h+1}))_{\mathfrak{m}^v} \longrightarrow H^0(X_{I,\bar{s}_v}, j_!^{=h}HT(\chi_v, \Pi_h))_{\mathfrak{m}^v} \tag{3.2.3}$$

where (Π_h, Π_{h+1}) is of the shape $\Big(LT_{\chi_v}(t-1, h-t), LT_{\chi_v}(t-1, h+1-t)\Big)$.

3.2.4. Lemma. — *As a $\overline{\mathbb{F}}_l[GL_d(F_v)]$-module, for every i, the l-torsion of $H^i(X_{I,\bar{s}_v},\overline{\mathbb{Z}}_l)_{\mathfrak{m}^v}$ does not have an irreducible generic sub-quotient.*

Proof. — Recall first that, as by hypothesis $\overline{\rho}_{\mathfrak{m}}$ is irreducible, the $\overline{\mathbb{Q}}_l$-version of the spectral sequence (3.1.2) degenerates at E_1 so that in particular all the torsion cohomology classes appear in the E_1 terms. The result then follows from the previous maps (3.2.3) and the fact that for any $r>0$, the modulo l reduction of $LT_{\chi_v}(t-1,r)$ does not admit any irreducible generic sub-quotient. □

Recall we argue by absurdity so there should exist non trivial cohomology classes, so there should exist such h so that the cokernel of the map (3.2.3) has torsion. We then denote h_0 the greatest such h. Then as the $\overline{\mathbb{Q}}_l$-cohomology groups localized at $\mathfrak{m}^{v,w}$ of the graduates of $\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi)$ are concentrated in degree 0, we then have a filtration of the free quotient of $H^0(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$ whose graduate are lattices $\Gamma_{!,\chi_v}(h_0,i)$ of the free quotients of $H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(h_0+i,\chi_v)(\frac{h_0-1-i}{2}))_{\mathfrak{m}^{v,w}}$.

3.2.5. Lemma. — *Let denote $\Gamma(\chi_v,t)$ the lattice of the free quotient of $H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(t,\chi_v))_{\mathfrak{m}^{v,w}}$. With the previous notations, there exists $1\leqslant i\leqslant d-h_0$ and a short exact sequence*

$$0\to\Gamma_{!,\chi_v}(h_0,i)\longrightarrow\Gamma(\chi_v,h_0+i)\longrightarrow T\to 0$$

where T is a sub-quotient of the torsion submodule of the cokernel of

$$H^0(X_{I^{v,w},\bar{s}_v},j_!^{=h_0+1}HT_\xi(\chi_v,\mathrm{St}_{h_0+1}(\chi_v)))_{\mathfrak{m}^{v,w}}\longrightarrow H^0(X_{I^{v,w},\bar{s}_v},j_!^{=h_0}HT_\xi(\chi_v,\mathrm{St}_{h_0}(\chi_v)))_{\mathfrak{m}^{v,w}}.\quad(3.2.6)$$

Proof. — By maximality of h_0, note that for $h_0<t\leqslant d$, the cohomology groups of $\mathcal{P}_\xi(t,\chi_v)$ and $j_!^{=t}HT_\xi(\pi_v,\Pi_t)$, are all free after localization by $\mathfrak{m}^{v,w}$. Using the same argument as before through the spectral sequence associated to (3.2.2), we then deduce that $H^i(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$ are all free for $i\neq 0,1$ while for $i=0$ the torsion is non trivial given by the non strictness of

$$H^0(X_{I^{v,x},\bar{s}_v},j_!^{=h_0+1}HT_\xi(\chi_v,LT_{\chi_v}(h_0-1,1)))_{\mathfrak{m}^{v,w}}\longrightarrow H^0(X_{I^{v,w},\bar{s}_v},j_!^{=h_0}HT_\xi(\chi_v,\mathrm{St}_{h_0}(\chi_v)))_{\mathfrak{m}^{v,w}}.\quad(3.2.7)$$

Concerning $H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(\chi_v,h_0))_{\mathfrak{m}^{v,w}}$, its torsion submodule is parabolic induced, so that beside those coming from the non strictness of (3.2.7), there is also the contribution given by the non strictness of (3.2.6).

In particular, when we compute the cohomology of $\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi)$ through its filtration of stratification with graduate the $\mathcal{P}_\xi(h_0+k,\chi_v)$ with $0\leqslant k\leqslant d-h_0$, this extra torsion part which appears as a quotient of $H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(\chi_v,h_0))_{\mathfrak{m}^{v,w}}$, must be used to modify the lattice $\Gamma(\chi_v,h_0+i)$ in another one $\Gamma_{!,\chi_v}(h_0,i)$ well adapted to the free quotient of $H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(\chi_v,h_0))_{\mathfrak{m}^{v,w}}$.

□

Arguing as in the proof of lemma 2.3.6, using (3.2.3), we have the following result.

3.2.8. Lemma. — *For every $1\leqslant t$, let $j(t)$ the minimal integer j such that the torsion of $H^j(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^t_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$ is non trivial. Then*

$$j(t)=\begin{cases}+\infty & \text{if } t\geqslant h_0+1,\\ t-h_0 & \text{for } 1\leqslant t\leqslant h_0.\end{cases}$$

Moreover in case $1\leqslant t\leqslant h_0$, for the action of $GL_d(F_w)$, up to multiplicities, we recover the same irreducible sub-quotients in the l-torsion of $H^0(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$.

Fact: through the spectral sequence (3.1.2), we then deduce that the l-torsion of $H^{1-h_0}(X_{I^{v,w},\bar{s}_v},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^{v,w}}$ is non trivial and, for the action of $GL_d(F_w)$, up to multiplicities, we recover the same irreducible sub-quotients in the l-torsion of $H^0(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$.

3.3. Global lattices. — In lemma 3.2.4, we proved that the l-torsion of any $H^i(X_{I^{v,w},\bar{s}_v},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^{v,w}}$, as a $\overline{\mathbb{F}}_l[GL_d(F_v)]$, never contains any generic sub-quotient. In this section we want to prove the opposite, i.e. there should exist generic sub-quotient which gives us our contradiction so that our hypothesis there should exist non trivial torsion classes, is in fact absurd. By symmetry between the places v and w, it suffices to do it for the action of $GL_d(F_w)$.

3.3.1. Lemma. — *With the notations of the previous section, the lattices* $\Gamma_{!,\chi_v}(h_0,i)$ *and* $\Gamma(\chi_v,h_0+i)$ *are isomorphic.*

Proof. — Recall that $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible. Following [**18**] §5 and especially theorem 5.6, see also [**6**] theorem 3.1.1, we know that as a $\mathbb{T}_{S,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,

$$H^{d-1}(X_{I^{v,w},\bar{\eta}},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^{v,w}}\simeq\sigma_{\mathfrak{m}^{v,w}}\otimes_{\mathbb{T}_{S,\mathfrak{m}^{v,w}}}\rho_{\mathfrak{m}^{v,w}},$$

for some $\mathbb{T}_{S,\mathfrak{m}^{v,w}}$-module $\sigma_{\mathfrak{m}^{v,w}}$ on wich Gal_F acts trivially. In particular for $\Pi^{\infty,I^{v,w}}\otimes\sigma(\Pi)$ a direct factor of $H^{d-1}(X_{I^{v,w},\bar{\eta}_{v_1}},V_{\xi,\overline{\mathbb{Q}}_l})_{\mathfrak{m}^{v,w}}$, and its lattice given by the $\overline{\mathbb{Z}}_l$-cohomology is a tensorial product $\Gamma_G\otimes\Gamma_W$ of a stable lattice Γ_G (resp. Γ_W) of $\Pi^{\infty,I^{v,w}}$ (resp. of $L_d(\Pi^{\vee}_{v_1})$).

With the notations of lemme 3.2.5, for any automorphic Π which appears in the $\mathfrak{m}$-cohomology of $\mathcal{P}(h+i,\chi_v)$, we denote $\Gamma_{!,\chi_v}(h_0,i,\Pi)$ (resp. $\Gamma(\chi_v,h_0+i,\Pi)$) the lattice $\Gamma_{!,\chi_v}(h_0,i)\cap\Pi^{\infty,I}\otimes\sigma$ (resp. $\Gamma(\chi_v,h_0+i)\cap\Pi^{\infty,I}\otimes\sigma$) induces by $\Pi^{\infty,I}\otimes\sigma\hookrightarrow H^0(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^{h_0}_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$ (resp. $\Pi^{\infty,I}\otimes\sigma\hookrightarrow H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(h+i,\chi_v)(\frac{h_0-1-i}{2})_{\mathfrak{m}^{v,w}})$. We then consider such Π so that the map short exact sequence of lemma 3.2.5 induces a non trivial map $\Gamma_{!,\chi_v}(h_0,i,\Pi)\hookrightarrow\Gamma(\chi_v,h_0+i,\Pi)$: as in loc. cit., the cokernel T is non trivial, we know such Π should exists.

Through the spectral sequence associated to the filtration $\mathrm{Fill}^\bullet_!(\Psi_{\xi,\varrho})$, we then obtain a filtration of the free quotient of $H^0(X_{I^{v,w},\bar{s}_v},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^{v,w}}$ and so of $\Pi^{\infty,I}\otimes\sigma(\Pi)$ as a submodule. Recall that for every $h\geqslant h_0$, the E_1 terms of the spectral sequence computing $\mathrm{Fil}^h_{!,\chi_v}(\Psi_\xi)$ are torsion free, so that the lattices of the $\Pi^{\infty,I}\otimes\sigma$ parts of $H^0(X_{I^{v,w},\bar{s}_v},\mathrm{Fil}^h_{!,\chi_v}(\Psi_\xi))_{\mathfrak{m}^{v,w}}$, are given by the associated lattices of $H^0(X_{I^{v,w},\bar{s}_v},\mathcal{P}_\xi(h+\delta))_{\mathfrak{m}^{v,w}}$.

Now as the global lattice is a tensor product, we then conclude that $\Gamma_{!,\chi_v}(h_0,i,\Pi)$ and $\Gamma(\chi_v,h_0+i,\Pi)$ have to be isomorphic. □

In particular considering only the action of $GL_d(F_w)$, the l-torsion of the cokernel T in lemma 3.2.5 should contains a generic representation. So using the last fact of the previous section, we then deduce that as a $\overline{\mathbb{F}}_l[GL_d(F_w)]$-module, the l-torsion of $H^{1-h_0}(X_{I^{v,w},\bar{s}_v},V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}^{v,w}}$ has a generic sub-quotient.

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