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Pascal Boyer

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ABOUT GALOIS REDUCIBILITY OF TORSION COHOMOLOGY CLASSES FOR KHT SHIMURA VARIETIES

by

Boyer Pascal

Abstract. — In [2], we proved that the $\overline{\mathbb{Z}}_l$ -cohomology of KHT Shimura varieties of dimension d which is not prime, whatever is the weight of the coefficients, when the level is large enough at l , always contains non trivial torsion classes. In this work we are interested in the Galois action on the torsion submodules in the cohomology groups of KHT Shimura varieties and we prove, at least when d is prime and $l > 2d - 3$ is such that the minimal prime dividing $l - 1$ is $> d$, then all Galois irreducible subquotients are of dimension strictly less than d .

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1. Introduction

From Matsushima's formula and computation of (\mathfrak{G}, K_∞) -cohomology, we know that tempered automorphic representations contribution in the cohomology of Shimura varieties with complex coefficients, is concentrated in middle degree. In [3], the authors consider the case of $\overline{\mathbb{F}}_l$ -coefficients and prove an analog result under some genericness hypothesis which can be stated as follows. Let \mathfrak{m} be a system of Hecke eigenvalues appearing in $H^{n_0}(\mathrm{Sh}_K, \overline{\mathbb{F}}_l)$ with Sh_K some compact unitary Shimura variety of Kottwitz type associated to some similitude group G . By the main result of [7], one can attach to such \mathfrak{m} , a mod l Galois representation

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_l)$$

where F is the reflex field of Sh_K which, by assumption, contains some imaginary quadratic extension E/\mathbb{Q} . From [3] definition 1.9, we say that \mathfrak{m} is generic (resp. decomposed generic) at some split p in E , if for all places v of F dividing p , the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ satisfy $\lambda_i/\lambda_j \notin \{q_v^{\pm 1}\}$ for all $i \neq j$ (resp. and are pairwise distincts), where q_v is the cardinal of the residue field at v . Then under the hypothesis⁽¹⁾ that there exists such p with \mathfrak{m} generic at p , the integer n_0 above is necessary equals to the relative dimension of Sh_K . In particular the $H^i(\mathrm{Sh}_K, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ are all torsion free.

Another relative question is to understand the Galois action on the torsion submodules of these cohomology groups. For example when the congruence relation is verified, then any Galois irreducible subquotient $\overline{\rho}$ of $H^i(\mathrm{Sh}_K, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ verifies the following property: for almost all places v of F , the minimal polynomial of $\overline{\rho}(\mathrm{Frob}_v)$ is a divisor of the characteristic polynomial $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$. In the following particular cases from [4]:

- when $\overline{\rho}_{\mathfrak{m}}$ is irreducible, induced from a character of $\mathrm{Gal}(\overline{K}/K)$ for K/F a finite galoisian extension;
- or $l \geq d$ and $SL_d(\kappa) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^\times \mathrm{GL}_d(\kappa)$ for a finite extension κ/\mathbb{F}_p ,

then when $\overline{\rho}$ is irreducible of dimension d , necessary it is isomorphic to $\overline{\rho}_{\mathfrak{m}}$. However in the general case, even when the congruence relation is fulfilled, we don't know much about such irreducible subquotients.

⁽¹⁾In their new preprint, Caraiani and Scholze explained that, from an observation of Koshikawa, one can replace decomposed generic by simply generic, in their main statement.

In this work we consider the particular case of Kottwitz-Harris-Taylor Shimura varieties Sh_K of [5] associated to inner forms of GL_d . In [1] we gave a very explicit proof of the fact that if \mathfrak{m} is generic at some split place v then the localized cohomology groups are all torsion free. Moreover we described the shape of the local component at v of any $\overline{\mathbb{Q}}_l$ -automorphic representations raising the multiset of modulo l Satake parameters given by \mathfrak{m} .

Concerning torsion cohomology classes, in [2], we proved, at least when d is not prime, that whatever is the weight ξ , if you increase the level at l , then non trivial torsion cohomology classes always appear. For KHT Shimura varieties, the congruence relation is proved by Wedhorn in [9] so that as the torsion classes constructed in [2] corresponds to reducible representations $\overline{\rho}_{\mathfrak{m}}$, one may asks if all irreducible Galois subquotients of the torsion submodules $H_{\text{tor}}^i(\text{Sh}_k, \overline{\mathbb{Z}}_l)$ are of dimension strictly less than the dimension d of Sh_K . The main result in this direction proved below in §3, is the following.

Theorem. — *Suppose that d is an odd prime and $l \geq 2d - 3$ is such that any prime dividing $l - 1$ is $> d$. Then any irreducible $\overline{\mathbb{F}}_l[\text{Gal}(\overline{F}/F)]$ -subquotient of any $H_{\text{tor}}^i(\text{Sh}_K \times_F \overline{F}, V_{\xi, \overline{\mathbb{Z}}_l})[l]$ is of dimension strictly less than d .*

Remark. Another way to formulate the previous statement, is to say that except in middle degree $i = d - 1$, all irreducible Galois subquotients of $H^i(\text{Sh}_K \times_F \overline{F}, V_{\xi, \overline{\mathbb{F}}_l})$ are of dimension $< d$.

The argument rests crucially on the following statement.

Theorem. — (cf. [6] theorem 1.6)

Let G be a finite irreducible subgroup of $GL_d(k)$ with d an odd prime. Assume the characteristic l of k is such that $l > 2d - 3$ (or $l = 0$). Then there exists a semisimple element $x \in G$ with all eigenspaces of dimension at most 1.

2. Recalls from [1]

2.1. Shimura varieties of KHT type. — Let $F = F^+E$ be a CM field where E/\mathbb{Q} is quadratic imaginary and F^+/\mathbb{Q} totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F , we will denote

- F_v the completion of F at v ,
- \mathcal{O}_v the ring of integers of F_v ,

- ϖ_v a uniformizer,
- q_v the cardinal of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F , of dimension d^2 such that at every place x of F , either B_x is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*|_F$ is the complex conjugation. For any $\beta \in B^{*-1}$, denote \sharp_β the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and G/\mathbb{Q} the group of similitudes, denoted G_τ in [5], defined for every \mathbb{Q} -algebra R by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of \mathbb{Q} split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \quad (2.1.1)$$

where, identifying places of F^+ over x with places of F over y , $x = \prod_i z_i$ in F^+ .

Convention: for $x = yy^c$ a place of \mathbb{Q} split in E and z a place of F over y as before, we shall make throughout the text, the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (2.1.1).

In [5], the authors justify the existence of some G like before such that moreover

- if x is a place of \mathbb{Q} non split in E then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [5] bottom of page 90, a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

2.1.2. Notation. — Denote \mathcal{I} the set of open compact subgroup small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $X_{I,\eta} \rightarrow \text{Spec } F$ the associated Shimura variety of Kottwitz-Harris-Taylor type.

2.1.3. Definition. — Define Spl the set of places v of F such that $p_v := v|_{\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write $\text{Spl}(I)$ the subset of Spl of places which doesn't divide the level I .

In the sequel, v will denote a place of F in Spl . For such a place v the scheme $X_{I,\eta}$ has a projective model $X_{I,v}$ over $\text{Spec } \mathcal{O}_v$ with special fiber X_{I,s_v} . For I going through \mathcal{I} , the projective system $(X_{I,v})_{I \in \mathcal{I}}$ is

naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \text{val} \circ \text{Art}^{-1}$ and $\text{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is Artin's isomorphism which sends geometric Frobenius to uniformizers.

2.1.4. Notations. — For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber X_{I, \bar{s}_v} is denoted

$$X_{I, \bar{s}_v} =: X_{I, \bar{s}_v}^{\geq 1} \supset X_{I, \bar{s}_v}^{\geq 2} \supset \cdots \supset X_{I, \bar{s}_v}^{\geq d}$$

where $X_{I, \bar{s}_v}^{=h} := X_{I, \bar{s}_v}^{\geq h} - X_{I, \bar{s}_v}^{\geq h+1}$ is an affine scheme, smooth of pure dimension $d - h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank h . For each $1 \leq h < d$, write

$$i_h : X_{I, \bar{s}_v}^{\geq h} \hookrightarrow X_{I, \bar{s}_v}^{\geq 1}, \quad j^{\geq h} : X_{I, \bar{s}_v}^{=h} \hookrightarrow X_{I, \bar{s}_v}^{\geq h},$$

and $j^{=h} = i_h \circ j^{\geq h}$.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0 . There exists, [5] p.97, then an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d + 1)$ -uple $(a_0, (\vec{a}_\sigma)_{\sigma \in \Phi})$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\vec{a}_\sigma = (a_{\sigma, 1} \leq \cdots \leq a_{\sigma, d})$. We then denote V_ξ the associated $\overline{\mathbb{Z}}_l$ -local system on $X_{\mathcal{I}}$. Recall that an irreducible automorphic representation Π is said ξ -cohomological if there exists an integer i such that

$$H^i((\text{Lie } G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$. Let $d_\xi^i(\Pi_\infty)$ be the dimension of this cohomology group.

2.2. Cohomology of Newton strata. —

2.2.1. Notation. — For $1 \leq h \leq d$, let $\mathcal{I}_v(h)$ denote the set of open compact subgroups

$$U_v(\underline{m}, h) := U_v(\underline{m}^v) \times \begin{pmatrix} I_h & 0 \\ 0 & K_v(m_1) \end{pmatrix},$$

where $K_v(m_1) = \text{Ker}(GL_{d-h}(\mathcal{O}_v) \rightarrow GL_{d-h}(\mathcal{O}_v/(\varpi_v^{m_1}))$). We then denote $[H^i(h, \xi)]$ (resp. $[H^i_i(h, \xi)]$) the image of

$$\varinjlim_{I \in \mathcal{I}_v(h)} H^i(X_{I, \bar{s}_v, 1}^{\geq h}, V_{\xi, \overline{\mathbb{Q}}_l}[d-h]) \quad \text{resp.} \quad \varinjlim_{I \in \mathcal{I}_v(h)} H^i(X_{I, \bar{s}_v, 1}^{\geq h}, j_{1, !}^{\geq h} V_{\xi, \overline{\mathbb{Q}}_l}[d-h])$$

inside the Grothendieck $\text{Groth}(v, h)$ of admissible representations of $G(\mathbb{A}^\infty) \times GL_{d-h}(F_v) \times \mathbb{Z}$.

Remark. An element $\sigma \in W_v$ acts through $-\deg \sigma \in \mathbb{Z}$ and $\Pi_{p_v, 0}(\text{Art}^{-1}(\sigma))$. We moreover consider the action of $GL_h(F_v)$ through $\text{val} \circ \det : GL_h(F_v) \rightarrow \mathbb{Z}$ and finally $P_{h,d}(F_v)$ through its Levi factor $GL_h(F_v) \times GL_{d-h}(F_v)$, i.e. its unipotent radical acts trivially.

From [1] proposition 3.6, for any irreducible tempered automorphic representation Π of $G(\mathbb{A})$ and for every $i \neq 0$, the $\Pi^{\infty, v}$ -isotypic component of $[H^i(h, \xi)]$ and $[H_i^i(h, \xi)]$ are zero. About the case $i = 0$, for Π an irreducible automorphic tempered representation ξ -cohomological, its local component at v looks like

$$\Pi_v \simeq \text{St}_{t_1}(\pi_{v,1}) \times \cdots \times \text{St}_{t_u}(\pi_{v,u}),$$

where for $i = 1, \dots, u$, $\pi_{v,i}$ is an irreducible cuspidal representation of est une $GL_{g_i}(F_v)$.

2.2.2. Proposition. — (cf. [1] proposition 3.9) *With the previous notations, we order the $\pi_{v,i}$ such that the first r -ones are unramified characters. Then the $\Pi^{\infty, v}$ -isotypic component of $[H^0(h, \xi)]$ is then equals to*

$$\left(\frac{\#\text{Ker}^1(\mathbb{Q}, G)}{d} \sum_{\Pi' \in \mathcal{U}_G(\Pi^{\infty, v})} m(\Pi') d_\xi(\Pi'_\infty) \right) \left(\sum_{1 \leq k \leq r: t_k = h} \Pi_v^{(k)} \otimes \chi_{v,k} \Xi^{\frac{d-h}{2}} \right)$$

where

- $\text{Ker}^1(\mathbb{Q}, G)$ is the subset of elements $H^1(\mathbb{Q}, G)$ which become trivial in $H^1(\mathbb{Q}_{p'}, G)$ for every prime p' ;
- $\Pi_v^{(k)} := \text{St}_{t_1}(\chi_{v,1}) \times \cdots \times \text{St}_{t_{k-1}}(\chi_{v,k-1}) \times \text{St}_{t_{k+1}}(\chi_{v,k+1}) \times \cdots \times \text{St}_{t_u}(\chi_{v,u})$
and
- $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_l^\times$ is defined by $\Xi(\frac{1}{2}) = q_v^{\frac{1}{2}}$.
- $\mathcal{U}_G(\Pi^{\infty, v})$ is the set of equivalence classes of irreducible automorphic representations Π' of $G(\mathbb{A})$ such that $(\Pi')^{\infty, v} \simeq \Pi^{\infty, v}$.

Remark. In particular if $[H^0(h, \xi)]$ has non trivial invariant vectors under some open compact subgroup $I \in \mathcal{I}_v(h)$ which is maximal at v , then the local component of Π at v is of the following shape $\text{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h}$ where the $\chi_{v,i}$ are unramified characters.

2.3. Torsion cohomology classes. — For $I \in \mathcal{I}$ a finite level, let $\mathbb{T}_I := \prod_{x \in \text{Unr}(I)} \mathbb{T}_x$ be the unramified Hecke algebra where $\text{Unr}(I)$ is the union of places q where G is unramified and I_x is maximal, and where $\mathbb{T}_x \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ the set of $\overline{\mathbb{Z}}_l$ -unramified characters of T_x .

Example. For $w \in \text{Spl}(I)$, we have

$$\mathbb{T}_w = \overline{\mathbb{Z}}_l[T_{w,i} : i = 1, \dots, d],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w) \text{diag}(\overbrace{\varpi_w, \dots, \varpi_w}^i, \overbrace{1, \dots, 1}^{d-i}) GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

For every $w \in \text{Spl}(I)$, the multi-set of Satake parameters at w corresponds to the roots of the Hecke polynomial

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^d (-1)^i q_w^{\frac{i(i-1)}{2}} \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_l[X]$$

i.e.

$$S_{\mathfrak{m}}(w) := \{\lambda \in \mathbb{T}_I/\mathfrak{m} \simeq \overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda) = 0\}.$$

For ξ an irreducible algebraic representation of $G(\mathbb{Q})$, let $\mathbb{T}_I(\xi)$ be the image of \mathbb{T}_I inside the $\overline{\mathbb{Z}}_l$ -endomorphism of the free quotient of $\bigoplus_{i \in \mathbb{N}} H^i(X_{I,\bar{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})$.

Remark. Note that any system of Hecke eigenvalues \mathfrak{m} of \mathbb{T}_I inside the torsion of some $H^i(X_{I,\bar{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})$ raise in characteristic zero, i.e. is associated to a maximal ideal \mathfrak{m} of $\mathbb{T}_I(\xi)$.

For a maximal ideal \mathfrak{m} of $\mathbb{T}_I(\xi)$ and $v \in \text{Spl}$, the multiset $S_{\mathfrak{m}}(v)$ of modulo l Satake parameters at the place v , corresponds to the multi-set of eigenvalues of $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_v)$ where $\bar{\rho}_{\mathfrak{m}}$ is the modulo l galoisian representation associated to \mathfrak{m} .

As for each $1 \leq h \leq d-1$, the open Newton stratum $X_{I,\bar{s}_v}^{=h}$ is affine then $H^i(X_{I,\bar{s}_v}^{=h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])$ is zero for $i < 0$ and free for $i = 0$. Using this property and the short exact sequence of free perverse sheaves

$$\begin{aligned} 0 \rightarrow i_{h+1,*} V_{\xi, \overline{\mathbb{Z}}_l, |X_{I,\bar{s}_v}^{\geq h+1}}[d-h-1] &\longrightarrow j_!^{\geq h} j^{\geq h,*} V_{\xi, \overline{\mathbb{Z}}_l, |X_{I,\bar{s}_v}^{\geq h}}[d-h] \\ &\longrightarrow V_{\xi, \overline{\mathbb{Z}}_l, |X_{I,\bar{s}_v}^{\geq h}}[d-h] \rightarrow 0, \end{aligned}$$

we then obtain for every $i > 0$

$$0 \rightarrow H^{-i-1}(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1]) \rightarrow 0, \quad (2.3.1)$$

and for $i = 0$,

$$\begin{aligned} 0 \rightarrow H^{-1}(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) &\longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1]) \longrightarrow \\ &H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h, *} V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) \rightarrow \dots \end{aligned} \quad (2.3.2)$$

In [1], arguing by induction from $h = d$ to $h = 1$, we prove that, under the hypothesis that $S_{\mathfrak{m}}(v)$ does not contain any subset of the form $\{\alpha, q_v \alpha\}$, all the cohomology groups $H^i(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}}$ are free: note that in order to deal with $i \geq 0$, one has to use Grothendieck-Verdier duality.

Without this hypothesis, arguing similarly, we conclude that any torsion cohomology class comes from a non strict map

$$H_{free}^0(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1])_{\mathfrak{m}} \longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h, *} V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}.$$

In particular it raises in characteristic zero to some free subquotient of $H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h, *} V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$.

3. Proof of the theorem

The minimal prime ideals of $\mathbb{T}_I(\xi)$ are the prime ideals above the zero ideal of $\bar{\mathbb{Z}}_l$ and are then in bijection with the prime ideals of $\mathbb{T}_I(\xi) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$. To such an ideal, which corresponds to give a collection of Satake parameters, is then associated an unique near equivalence class in the sense of [8], denoted $\Pi_{\tilde{\mathfrak{m}}}$ which is the finite set of irreducible automorphic representations ξ -cohomological whose multi-set of Satake parameters at each place $x \in \text{Unr}(I)$, is given by $S_{\tilde{\mathfrak{m}}}(x)$. Thanks to [5] and [8], we denote

$$\rho_{\tilde{\mathfrak{m}}} : \text{Gal}(\bar{F}/F) \longrightarrow GL_d(\bar{\mathbb{Q}}_l)$$

the galoisian representation associated to any $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$. Recall that the modulo l reduction of $\rho_{\tilde{\mathfrak{m}}}$ depends only of \mathfrak{m} , and was denoted above $\bar{\rho}_{\mathfrak{m}}$.

Let $G_I = \prod_{x \in \text{Spl}(I)} \text{Gal}(\bar{F}_x/F_x)$. By Chebotarev theorem, recall that a representation of $\text{Gal}(\bar{F}/F)$ is irreducible if and only if its restriction to G_I is irreducible. The idea is, thanks to the theorem of [6] recalled in the introduction, to resume the proof of theorem 4.18 of [1]. More precisely, we argue by absurdity by considering an irreducible $\bar{\mathbb{F}}_l[G_I]$ -subquotient

of some $H_{\text{tor}}^i(X_{I, \bar{s}_v}, V_{\xi, \bar{\mathbb{Z}}_l})[l]$ with dimension d and we want to prove, using the previous theorem of [6], that $l - 1$ has a divisor $\leq d$.

Consider an irreducible subquotient $\bar{\rho}$ of $H^i(X_{I, \bar{\eta}}, V_{\xi, \bar{\mathbb{Z}}_l}[d - 1])_{\mathfrak{m}}[l]$ associated to some maximal ideal \mathfrak{m} of \mathbb{T}_I . From the results of [1] recalled in the previous section, cf. proposition 2.2.2, there exists then a prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ such that for any $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$ its local component at v is of the following shape

$$\text{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h},$$

where $\chi_{v,1}, \dots, \chi_{v,d-h}$ are unramified characters of F_v^\times and $h \geq 2$. Moreover $\bar{\chi}_{v,1}(\frac{1-h}{2})$ is a subquotient of the restriction $\bar{\rho}_v$ of $\bar{\rho}$ to the decomposition group at v . Recall the Eichler-Shimura relation proved by Wedhorn [9] in our case.

3.1. Theorem. — (cf. [4] 3.3.1) *For all $w \in \text{Spl}(I)$, the action of*

$$\sum_{i=0}^d (-1)^i q_w^{\frac{i(i-1)}{2}} T_{w,i} \text{Frob}_w^{d-i}$$

on every $H^j(X_{I, \bar{\eta}}, V_{\xi, \bar{\mathbb{Z}}_l})$ is zero.

In particular, $\bar{\rho}_v$ can be written as

$$\bar{\rho}_v \simeq \bigoplus_{i=1}^h m_{1,i} \bar{\chi}_{v,1} \left(\frac{1-h}{2} + i \right) \oplus \bigoplus_{j=2}^{d-h} m_j \bar{\chi}_{v,j}$$

where $m_{1,i}$ and m_j belongs to \mathbb{N} and $\sum_{i=1}^h m_{1,i} + \sum_{j=2}^{d-h} m_j = \dim \bar{\rho} = d$.

We are now considering a place v such that the eigenvalues of $\bar{\rho}(\text{Frob}_v)$ are simple. Note necessary we need $\bar{\chi}_{v,1}, \dots, \bar{\chi}_{v,d-h}$ to be pairwise distinct, the order of q modulo l to be $\geq h$ and then $\bar{\rho}_v$ to be equal to $\bar{\rho}_{\mathfrak{m},v}$.

We can now resume the argument of the proof of theorem 4.18 of [1] which is the following. Note that each time we produce as above a torsion class in the cohomology, it gives an Frob_v -eigenvalue $\lambda := \bar{\chi}_{v,1}(\frac{1-h}{2})(q_v^{-1})$ so that $q_v \lambda$ is also an eigenvalue of $\bar{\rho}_{\mathfrak{m},v}(\text{Frob}_v)$ and so of $\bar{\rho}_v(\text{Frob}_v)$ associated to another torsion class. As $\bar{\rho}_v$ is finite dimensional and multiplicity free, we must have a cycle $\lambda, \lambda q_v, \dots, \lambda q_v^{r-1}$ with $r \leq d$ and $q_v^r = 1 \pmod{l}$, so that in particular $q_v \in \mathbb{F}_l^\times$ is of order $\leq d$ which is in contradiction with our hypothesis on $l - 1$ which could not be divisible by $r \leq d$.

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BOYER PASCAL • *E-mail* : boyer@math.univ-paris13.fr, Université Paris 13,
Sorbonne Paris Cité, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse (France),
PerCoLaTor: ANR-14-CE25