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Pascal Boyer. ABOUT GALOIS REDUCIBILITY OF TORSION COHOMOLOGY CLASSES FOR KHT SHIMURA VARIETIES. 2019. hal-02080476v1

HAL Id: hal-02080476

<https://hal.science/hal-02080476v1>

Preprint submitted on 26 Mar 2019 (v1), last revised 27 Apr 2020 (v4)

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ABOUT GALOIS REDUCIBILITY OF TORSION COHOMOLOGY CLASSES FOR KHT SHIMURA VARIETIES

by

Boyer Pascal

Abstract. — We pursue the original strategy of [1] and we give a new criterion so that the localization of the cohomology of KHT Shimura variety is free. Precisely let G be a similitude group with signatures $(1, d - 1), (0, d), \dots, (0, d)$ and $V_{\xi, \bar{\mathbb{Z}}_l}$ a local system associated to a fixed algebraic representation ξ of $G(\mathbb{Q})$. Consider a system \mathfrak{m} of Hecke eigenvalues appearing in the free quotient of the cohomology group in middle degree of the Shimura variety Sh_K associated to G and with coefficients in $V_{\xi, \bar{\mathbb{Z}}_l}$. Then if the modulo l galoisian representation $\bar{\rho}_{\mathfrak{m}}$ is irreducible of dimension d , and essentially if $l \geq d + 1$, the localization at \mathfrak{m} of every cohomology group of Sh_K with coefficients in $V_{\xi, \bar{\mathbb{Z}}_l}$, is free.

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2010 Mathematics Subject Classification. — 11F70, 11F80, 11F85, 11G18, 20C08.

Key words and phrases. — Shimura varieties, torsion in the cohomology, maximal ideal of the Hecke algebra, localized cohomology, galoisian representation.

1. Introduction

From Matsushima's formula and computation of (\mathfrak{G}, K_∞) -cohomology, we know that tempered automorphic representations contribution in the cohomology of Shimura varieties with complex coefficients, is concentrated in middle degree. In [5], the authors consider the case of $\overline{\mathbb{F}}_l$ -coefficients and prove an analog result under some genericness hypothesis which can be stated as follows. Let \mathfrak{m} be a system of Hecke eigenvalues appearing in $H^{n_0}(\mathrm{Sh}_K, \overline{\mathbb{F}}_l)$ with Sh_K some compact unitary Shimura variety of Kottwitz type associated to some similitude group G . By the main result of [7], one can attach to such \mathfrak{m} , a mod l Galois representation

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_l)$$

where F is the reflex field of Sh_K which, by assumption, contains some imaginary quadratic extension E/\mathbb{Q} . From [5] definition 1.9, we say that \mathfrak{m} is decomposed generic at some split p in E , if for all places v of F dividing p , the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ satisfy $\lambda_i/\lambda_j \notin \{1, q_v\}$ for all $i \neq j$, where q_v is the cardinal of the residue field at v . Then under the hypothesis that there exists such p with \mathfrak{m} decomposed generic at p , the integer n_0 above is necessary equals to the relative dimension of Sh_K .

In the particular case of Kottwitz-Harris-Taylor Shimura varieties Sh_K of [6], in [1] we gave a simpler condition that for one place v above such p , for all $i \neq j$ then $\lambda_i/\lambda_j \neq q_v^{\pm 1}$. We also remarked that each system of Hecke eigenvalues \mathfrak{m} giving rise to a non trivial torsion class, raises in characteristic zero to an irreducible tempered representation so that, in particular, as a Galois module, each torsion class is obtained as a subquotient of the modulo l reduction of some irreducible d -dimensional Galois representation.

In [4] and [3], we construct such torsion cohomological classes of weight ξ an irreducible algebraic representation of $G(\mathbb{Q})$, all of them being, as Galois module, irreducible of dimension strictly less than d . In particular whatever is the weight ξ , if you increase the level at l , we proved that non trivial torsion cohomology classes always appear.

Although the previous decomposed genericness hypothesis form a suitable big image assumption, one may ask if it should be possible to replace it, by the condition that $\bar{\rho}_{\mathfrak{m}}$ is irreducible of dimension d . It might be too optimistic to expect such a property in full generality but any result

in this direction should be useful at least to deal new cases of Ihara's lemma as in [2]. The main result proved below in §3, is the following.

Theorem. — *Let \mathfrak{m} be a system of Hecke eigenvalues appearing in the free quotient $H_{free}^{d-1}(\mathrm{Sh}_K \times_F \bar{F}, V_{\xi, \bar{\mathbb{Z}}_l})$ of the middle cohomology of the generic geometric fiber of some KHT Shimura variety Sh_K , with coefficients in the local system $V_{\xi, \bar{\mathbb{Z}}_l}$ associated to an irreducible algebraic representation ξ of $G(\mathbb{Q})$. Suppose that*

- $\bar{\rho}_{\mathfrak{m}}$ is irreducible and
- there exists a place $v \in \mathrm{Spl}(K)$, cf. definition 2.1.3, such that the order of q_v modulo l is $\geq d$.

Then for all i , the localized cohomological groups $H^i(\mathrm{Sh}_K \times_F \bar{F}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}}$ are free.

Remark. Write $E = \mathbb{Q}[\sqrt{-d}]$ with $d > 0$ is square free. Then the condition that p splits in E is given by a congruence modulo $4d$. In particular if $l \wedge (4d) = 1$, then by Dirichlet theorem this condition is independent from those concerning the order of $q_v (= p^r)$ modulo l . We clearly need $l \geq d + 1$ and if a is a generator of \mathbb{F}_l^\times , we then just need to find $p \equiv a \pmod{l}$ such that the frobenius at p is of order prime to $l - 1$. Consider the particular case where $p = q_v$ splits in F .

- If F/\mathbb{Q} is a abelian galoisian extension, then this last condition is given in terms of congruences, so if we exclude a finite number of l , by Dirichlet theorem, this condition is independent from the others so that finally, we only need $l \geq d + 1$ and $l \neq 2$ prime to the discriminant of E/\mathbb{Q} and with some integer associated to the abelian extension F/\mathbb{Q} .
- If F/\mathbb{Q} is galoisian but not abelian, then we know that the condition p splits in F cannot be described in terms of congruences. In particular as the other conditions are given in terms of congruences, finally we can always find a place v verifying the conditions of the theorem if $l \geq d + 1$ and $l \neq 2$ prime to the discriminant of E/\mathbb{Q} .

2. Recall from [1]

2.1. Shimura varieties of KHT type. — Let $F = F^+E$ be a CM field where E/\mathbb{Q} is quadratic imaginary and F^+/\mathbb{Q} totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F , we will denote

- F_v the completion of F at v ,

- \mathcal{O}_v the ring of integers of F_v ,
- ϖ_v a uniformizer,
- q_v the cardinal of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F , of dimension d^2 such that at every place x of F , either B_x is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*|_F$ is the complex conjugation. For any $\beta \in B^{*-1}$, denote \sharp_β the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and G/\mathbb{Q} the group of similitudes, denoted G_τ in [6], defined for every \mathbb{Q} -algebra R by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of \mathbb{Q} split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \quad (2.1.1)$$

where, identifying places of F^+ over x with places of F over y , $x = \prod_i z_i$ in F^+ .

Convention: for $x = yy^c$ a place of \mathbb{Q} split in E and z a place of F over y as before, we shall make throughout the text, the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (2.1.1).

In [6], the author justify the existence of some G like before such that moreover

- if x is a place of \mathbb{Q} non split in E then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [6] bottom of page 90, a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

2.1.2. Notation. — Denote \mathcal{I} the set of open compact subgroup small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $X_{I,\eta} \longrightarrow \text{Spec } F$ the associated Shimura variety of Kottwitz-Harris-Taylor type.

2.1.3. Definition. — Define Spl the set of places v of F such that $p_v := v|_{\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write $\text{Spl}(I)$ the subset of Spl of places which doesn't divide the level I .

In the sequel, v will denote a place of F in Spl . For such a place v the scheme $X_{I,\eta}$ has a projective model $X_{I,v}$ over $\text{Spec } \mathcal{O}_v$ with special

fiber X_{I,s_v} . For I going through \mathcal{I} , the projective system $(X_{I,v})_{I \in \mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \text{val} \circ \text{Art}^{-1}$ and $\text{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is Artin's isomorphism which sends geometric Frobenius to uniformizers.

2.1.4. Notations. — For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber X_{I,\bar{s}_v} is denoted

$$X_{I,\bar{s}_v} =: X_{I,\bar{s}_v}^{\geq 1} \supset X_{I,\bar{s}_v}^{\geq 2} \supset \cdots \supset X_{I,\bar{s}_v}^{\geq d}$$

where $X_{I,\bar{s}_v}^{=h} := X_{I,\bar{s}_v}^{\geq h} - X_{I,\bar{s}_v}^{\geq h+1}$ is an affine scheme, smooth of pure dimension $d - h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank h . For each $1 \leq h < d$, write

$$i_h : X_{I,\bar{s}_v}^{\geq h} \hookrightarrow X_{I,\bar{s}_v}^{\geq 1}, \quad j^{\geq h} : X_{I,\bar{s}_v}^{=h} \hookrightarrow X_{I,\bar{s}_v}^{\geq h},$$

and $j^{=h} = i_h \circ j^{\geq h}$.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0 . There exists, [6] p.97, then an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$ -uple $(a_0, (\bar{a}_\sigma)_{\sigma \in \Phi})$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\bar{a}_\sigma = (a_{\sigma,1} \leq \cdots \leq a_{\sigma,d})$. We then denote V_ξ the associated $\overline{\mathbb{Z}}_l$ -local system on $X_{\mathcal{I}}$. Recall that an irreducible automorphic representation Π is said ξ -cohomological if there exists an integer i such that

$$H^i((\text{Lie } G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$. Let $d_\xi^i(\Pi_\infty)$ be the dimension of this cohomology group.

2.2. Cohomology of Newton strata. —

2.2.1. Notation. — For $1 \leq h \leq d$, let $\mathcal{I}_v(h)$ denote the set of open compact subgroups

$$U_v(\underline{m}, h) := U_v(\underline{m}^v) \times \begin{pmatrix} I_h & 0 \\ 0 & K_v(m_1) \end{pmatrix},$$

where $K_v(m_1) = \text{Ker}(GL_{d-h}(\mathcal{O}_v) \rightarrow GL_{d-h}(\mathcal{O}_v/(\varpi_v^{m_1}))$). We then denote $[H^i(h, \xi)]$ (resp. $[H_1^i(h, \xi)]$) the image of

$$\varinjlim_{I \in \mathcal{I}_v(h)} H^i(X_{I,\bar{s}_v,1}^{\geq h}, V_{\xi, \overline{\mathbb{Q}}_l}[d-h]) \quad \text{resp.} \quad \varinjlim_{I \in \mathcal{I}_v(h)} H^i(X_{I,\bar{s}_v,1}^{\geq h}, j_{1,!}^{\geq h} V_{\xi, \overline{\mathbb{Q}}_l}[d-h])$$

inside the Grothendieck $\text{Groth}(v, h)$ of admissible representations of $G(\mathbb{A}^\infty) \times GL_{d-h}(F_v) \times \mathbb{Z}$.

Remark. An element $\sigma \in W_v$ acts through $-\deg \sigma \in \mathbb{Z}$ and $\Pi_{p_v, 0}(\text{Art}^{-1}(\sigma))$. We moreover consider the action of $GL_h(F_v)$ through $\text{val} \circ \det : GL_h(F_v) \rightarrow \mathbb{Z}$ and finally $P_{h,d}(F_v)$ through its Levi factor $GL_h(F_v) \times GL_{d-h}(F_v)$, i.e. its unipotent radical acts trivially.

From [1] proposition 3.6, for any irreducible tempered automorphic representation Π of $G(\mathbb{A})$ and for every $i \neq 0$, the $\Pi^{\infty, v}$ -isotypic component of $[H^i(h, \xi)]$ and $[H_i^i(h, \xi)]$ are zero. About the case $i = 0$, for Π an irreducible automorphic tempered representation ξ -cohomological, its local component at v looks like

$$\Pi_v \simeq \text{St}_{t_1}(\pi_{v,1}) \times \cdots \times \text{St}_{t_u}(\pi_{v,u}),$$

where for $i = 1, \dots, u$, $\pi_{v,i}$ is an irreducible cuspidal representation of est une $GL_{g_i}(F_v)$.

2.2.1. Proposition. — (cf. [1] proposition 3.9) *With the previous notations, we order the $\pi_{v,i}$ such that the first r -ones are unramified characters. Then the $\Pi^{\infty, v}$ -isotypic component of $[H^0(h, \xi)]$ is then equals to*

$$\left(\frac{\#\text{Ker}^1(\mathbb{Q}, G)}{d} \sum_{\Pi' \in \mathcal{U}_G(\Pi^{\infty, v})} m(\Pi') d_\xi(\Pi'_\infty) \right) \left(\sum_{1 \leq k \leq r: t_k = h} \Pi_v^{(k)} \otimes \chi_{v,k} \Xi^{\frac{d-h}{2}} \right)$$

where

- $\text{Ker}^1(\mathbb{Q}, G)$ is the subset of elements $H^1(\mathbb{Q}, G)$ which become trivial in $H^1(\mathbb{Q}_{p'}, G)$ for every prime p' ;
- $\Pi_v^{(k)} := \text{St}_{t_1}(\chi_{v,1}) \times \cdots \times \text{St}_{t_{k-1}}(\chi_{v,k-1}) \times \text{St}_{t_{k+1}}(\chi_{v,k+1}) \times \cdots \times \text{St}_{t_u}(\chi_{v,u})$ and
- $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_l^\times$ is defined by $\Xi(\frac{1}{2}) = q_v^{\frac{1}{2}}$.
- $\mathcal{U}_G(\Pi^{\infty, v})$ is the set of equivalence classes of irreducible automorphic representations Π' of $G(\mathbb{A})$ such that $(\Pi')^{\infty, v} \simeq \Pi^{\infty, v}$.

Remark. In particular if $[H^0(h, \xi)]$ has non trivial invariant vectors under some open compact subgroup $I \in \mathcal{I}_v(h)$ which is maximal at v , then the local component of Π at v is of the following shape $\text{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h}$ where the $\chi_{v,i}$ are unramified characters.

2.3. Torsion cohomology classes. — For $I \in \mathcal{I}$ a finite level, let $\mathbb{T}_I := \prod_{x \in \text{Unr}(I)} \mathbb{T}_x$ be the unramified Hecke algebra where $\text{Unr}(I)$ is the union of places q where G is unramified and I_x is maximal, and where $\mathbb{T}_x \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ the set of $\overline{\mathbb{Z}}_l$ -unramified characters of T_x . For ξ an irreducible algebraic representation of $G(\mathbb{Q})$, let $\mathbb{T}_I(\xi)$ be the image of \mathbb{T}_I inside the $\overline{\mathbb{Z}}_l$ -endomorphism of the free quotient of $H^{d-1}(X_{I, \bar{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})$.

Remark. Note that any system of Hecke eigenvalues \mathfrak{m} of \mathbb{T}_I inside the torsion of some $H^i(X_{I, \bar{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})$ raise in characteristic zero, i.e. is associated to a maximal ideal \mathfrak{m} of $\mathbb{T}_I(\xi)$. Moreover, as we will be only interested in systems of Hecke eigenvalues given rise to a $\overline{\mathbb{F}}_l$ -representation of $\text{Gal}(\bar{F}/F)$ which is irreducible of dimension d , it suffices to consider the middle degree cohomology of X_I .

For a maximal ideal \mathfrak{m} of $\mathbb{T}_I(\xi)$ and $v \in \text{Spl}$, we denote $S_{\mathfrak{m}}(v)$ the multi-set of modulo l Satake parameters at the place v . Recall that it corresponds to the multi-set of eigenvalues of $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_v)$ where $\bar{\rho}_{\mathfrak{m}}$ is the modulo l galoisian representation associated to \mathfrak{m} . In the following we will suppose that $\bar{\rho}_{\mathfrak{m}}$ is irreducible of dimension d , so that for all $i \neq 0$, the localization $H_{free}^i(X_{I, \bar{s}_v}^{-h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$ and $H_{free}^i(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$ are zero.

As for each $1 \leq h \leq d-1$, the open Newton stratum X_{I, \bar{s}_v}^{-h} is affine then $H^i(X_{I, \bar{s}_v}^{-h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])$ is zero for $i < 0$ and free for $i = 0$. Using this property and the short exact sequence of free perverse sheaves

$$\begin{aligned} 0 \rightarrow i_{h+1, *} V_{\xi, \overline{\mathbb{Z}}_l, |X_{I, \bar{s}_v}^{\geq h+1}}[d-h-1] &\longrightarrow j_!^{\geq h} j^{\geq h, *} V_{\xi, \overline{\mathbb{Z}}_l, |X_{I, \bar{s}_v}^{\geq h}}[d-h] \\ &\longrightarrow V_{\xi, \overline{\mathbb{Z}}_l, |X_{I, \bar{s}_v}^{\geq h}}[d-h] \rightarrow 0, \end{aligned}$$

we then obtain for every $i > 0$

$$0 \rightarrow H^{-i-1}(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h-1]) \rightarrow 0, \quad (2.3.1)$$

and for $i = 0$,

$$\begin{aligned} 0 \rightarrow H^{-1}(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) &\longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h-1]) \longrightarrow \\ &H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h, *} V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) \rightarrow \dots \end{aligned} \quad (2.3.2)$$

In [1], arguing by induction from $h = d$ to $h = 1$, we prove that, under the hypothesis that $S_{\mathfrak{m}}(v)$ does not contain any subset of the form $\{\alpha, q_v \alpha\}$,

all the cohomology groups $H^i(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}}$ are free: note that in order to deal with $i \geq 0$, one has to use Grothendieck-Verdier duality.

Without this hypothesis, arguing similarly, we conclude that any torsion cohomology class comes from a non strict map

$$H_{free}^0(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1]) \longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h, *} V_{\xi, \bar{\mathbb{Z}}_l}[d-h]).$$

In particular it raises in characteristic zero to some free subquotient of $H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h, *} V_{\xi, \bar{\mathbb{Z}}_l}[d-h])$.

3. Proof of the theorem

The minimal prime ideals of $\mathbb{T}_I(\xi)$ are the prime ideals above the zero ideal of $\bar{\mathbb{Z}}_l$ and are then in bijection with the prime ideals of $\mathbb{T}_I(\xi) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$. To such an ideal, which corresponds to give a collection of Satake parameters, is then associated an unique near equivalence class in the sense of [8], denoted $\Pi_{\tilde{\mathfrak{m}}}$ which is the finite set of irreducible automorphic representations ξ -cohomological whose multi-set of Satake parameters at each place $x \in \text{Unr}(I)$, is given by $S_{\tilde{\mathfrak{m}}}(x)$. Thanks to [6] and [8], we denote

$$\rho_{\tilde{\mathfrak{m}}} : \text{Gal}(\bar{F}/F) \longrightarrow GL_d(\bar{\mathbb{Q}}_l)$$

the galoisian representation associated to any $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$. Recall that the modulo l reduction of $\rho_{\tilde{\mathfrak{m}}}$ depends only of \mathfrak{m} , and was denoted above $\bar{\rho}_{\mathfrak{m}}$.

By hypothesis there exists a place $v \in \text{Spl}(I)$ such that the order of q_v modulo l is $\geq d$, where q_v is the order of the residue field at v . We argue now by absurdity, supposing there exists a non trivial torsion class of some $H_{tor}^i(X_{I, \bar{s}_v}^{\geq 1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}[l]$.

Consider then h maximal such that there exists i for which $H_{tor}^i(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}[l]$ is non trivial. Note then using (2.3.1) and (2.3.2), we must have $i = 0$. Consider then an irreducible sub- $\bar{\mathbb{F}}_l[\mathbb{T}_I(\xi) \times \text{Gal}(\bar{F}_v/F_v)]$ -module of $H_{tor}^0(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}[l]$. From proposition 2.2.1,

- there exists then a prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ such that for any $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$ its local component at v is of the following shape

$$\text{St}_{h+1}(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h-1},$$

where $\chi_{v,1}, \dots, \chi_{v,d-h-1}$ are unramified characters of F_v^\times . In particular

$$\bar{\chi}_{v,1}\left(-\frac{h}{2}\right), \quad \bar{\chi}_{v,1}\left(-\frac{h-2}{2}\right), \quad \cdots, \quad \bar{\chi}_{v,1}\left(\frac{h}{2}\right)$$

are irreducible subquotients of the restriction $\bar{\rho}_{m,v}$ of $\bar{\rho}_m$ to the decomposition group at v .

– Moreover the Galois action is isotypic for $\bar{\chi}_{v,1}(\frac{-h}{2})$.

Using repeatedly (2.3.1), we then have the same property concerning $H_{tor}^{1-h}(X_{I,\bar{s}_v}^{\geq 1}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}} \simeq H_{tor}^{1-h}(X_{I,\bar{\eta}_v}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}$. Now this last space has a full action of $\text{Gal}(\bar{F}/F)$ and each torsion class is obtain as before, i.e. comes from a quotient of $H_c^0(X_{I,\bar{s}_v}^{\geq h}, V_{\xi,\bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$.

Note then that each $\bar{\mathbb{F}}_l[\text{Gal}(\bar{F}_v/F_v)]$ -irreducible torsion class of $H_{tor}^{1-h}(X_{I,\bar{\eta}_v}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}[l]$ verifies the following property: it is an irreducible subquotient $\bar{\chi}_v$ of $\bar{\rho}_{m,v}$ such that $\bar{\chi}_v(1)$ is also an irreducible subquotient of $\bar{\rho}_{m,v}$. Now as $\bar{\rho}_m$ is supposed to be irreducible and as $H_{tor}^{1-h}(X_{I,\bar{\eta}_v}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}$ is a $\text{Gal}(\bar{F}/F)$ -module, we conclude that for each irreducible subquotient $\bar{\chi}_v$ of $\bar{\rho}_{m,v}$ then $\bar{\chi}_v(1)$ is also such a subquotient, so that all the Zelevinski line of $\bar{\chi}_v$ is contained in the supercuspidal support of $\bar{\rho}_m$ so that in particular the order of q_v modulo l is necessary $\leq d$ which is not the case by hypothesis.

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