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# ABOUT GALOIS REDUCIBILITY OF TORSION COHOMOLOGY CLASSES FOR KHT SHIMURA VARIETIES

by

Boyer Pascal

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**Abstract.** — We pursue the original strategy of [1] and we give a new criterion so that the localization of the cohomology of KHT Shimura variety is free. Precisely let  $G$  be a similitude group with signatures  $(1, d - 1), (0, d), \dots, (0, d)$  and  $V_{\xi, \bar{\mathbb{Z}}_l}$  a local system associated to a fixed algebraic representation  $\xi$  of  $G(\mathbb{Q})$ . Consider a system  $\mathfrak{m}$  of Hecke eigenvalues appearing in the free quotient of the cohomology group in middle degree of the Shimura variety  $\text{Sh}_K$  associated to  $G$  and with coefficients in  $V_{\xi, \bar{\mathbb{Z}}_l}$ . Then if the modulo  $l$  galoisian representation  $\bar{\rho}_{\mathfrak{m}}$  is irreducible of dimension  $d$ , and essentially if  $l \geq d + 1$ , the localization at  $\mathfrak{m}$  of every cohomology group of  $\text{Sh}_K$  with coefficients in  $V_{\xi, \bar{\mathbb{Z}}_l}$ , is free.

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## 1. Introduction

From Matsushima's formula and computation of  $(\mathfrak{G}, K_\infty)$ -cohomology, we know that tempered automorphic representations contribution in the cohomology of Shimura varieties with complex coefficients, is concentrated in middle degree. In [5], the authors consider the case of  $\overline{\mathbb{F}}_l$ -coefficients and prove an analog result under some genericness hypothesis which can be stated as follows. Let  $\mathfrak{m}$  be a system of Hecke eigenvalues appearing in  $H^{n_0}(\mathrm{Sh}_K, \overline{\mathbb{F}}_l)$  with  $\mathrm{Sh}_K$  some compact unitary Shimura variety of Kottwitz type associated to some similitude group  $G$ . By the main result of [7], one can attach to such  $\mathfrak{m}$ , a mod  $l$  Galois representation

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_l)$$

where  $F$  is the reflex field of  $\mathrm{Sh}_K$  which, by assumption, contains some imaginary quadratic extension  $E/\mathbb{Q}$ . From [5] definition 1.9, we say that  $\mathfrak{m}$  is decomposed generic at some split  $p$  in  $E$ , if for all places  $v$  of  $F$  dividing  $p$ , the eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$  of  $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$  satisfy  $\lambda_i/\lambda_j \notin \{1, q_v\}$  for all  $i \neq j$ , where  $q_v$  is the cardinal of the residue field at  $v$ . Then under the hypothesis that there exists such  $p$  with  $\mathfrak{m}$  decomposed generic at  $p$ , the integer  $n_0$  above is necessary equals to the relative dimension of  $\mathrm{Sh}_K$ .

In the particular case of Kottwitz-Harris-Taylor Shimura varieties  $\mathrm{Sh}_K$  of [6], in [1] we gave a simpler condition that for one place  $v$  above such  $p$ , for all  $i \neq j$  then  $\lambda_i/\lambda_j \neq q_v^{\pm 1}$ . We also remarked that each system of Hecke eigenvalues  $\mathfrak{m}$  giving rise to a non trivial torsion class, raises in characteristic zero to an irreducible tempered representation so that, in particular, as a Galois module, each torsion class is obtained as a subquotient of the modulo  $l$  reduction of some irreducible  $d$ -dimensional Galois representation.

In [4] and [3], we construct such torsion cohomological classes of weight  $\xi$  an irreducible algebraic representation of  $G(\mathbb{Q})$ , all of them being, as Galois module, irreducible of dimension strictly less than  $d$ . In particular whatever is the weight  $\xi$ , if you increase the level at  $l$ , we proved that non trivial torsion cohomology classes always appear.

Although the previous decomposed genericness hypothesis form a suitable big image assumption, one may ask if it should be possible to replace it, by the condition that  $\bar{\rho}_{\mathfrak{m}}$  is irreducible of dimension  $d$ . It might be too optimistic to expect such a property in full generality but any result

in this direction should be useful at least to deal new cases of Ihara's lemma as in [2]. The main result proved below in §3, is the following.

**Theorem.** — *Let  $\mathfrak{m}$  be a system of Hecke eigenvalues appearing in the free quotient  $H_{free}^{d-1}(\mathrm{Sh}_K \times_F \bar{F}, V_{\xi, \bar{\mathbb{Z}}_l})$  of the middle cohomology of the generic geometric fiber of some KHT Shimura variety  $\mathrm{Sh}_K$ , with coefficients in the local system  $V_{\xi, \bar{\mathbb{Z}}_l}$  associated to an irreducible algebraic representation  $\xi$  of  $G(\mathbb{Q})$ . Suppose that*

- $\bar{\rho}_{\mathfrak{m}}$  is irreducible and
- there exists a place  $v \in \mathrm{Spl}(K)$ , cf. definition 2.1.3, such that the order of  $q_v$  modulo  $l$  is  $\geq d$ .

*Then for all  $i$ , the localized cohomological groups  $H^i(\mathrm{Sh}_K \times_F \bar{F}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  are free.*

*Remark.* Write  $E = \mathbb{Q}[\sqrt{-d}]$  with  $d > 0$  is square free. Then the condition that  $p$  splits in  $E$  is given by a congruence modulo  $4d$ . In particular if  $l \wedge (4d) = 1$ , then by Dirichlet theorem this condition is independent from those concerning the order of  $q_v (= p^r)$  modulo  $l$ . We clearly need  $l \geq d + 1$  and if  $a$  is a generator of  $\mathbb{F}_l^\times$ , we then just need to find  $p \equiv a \pmod{l}$  such that the frobenius at  $p$  is of order prime to  $l - 1$ . Consider the particular case where  $p = q_v$  splits in  $F$ .

- If  $F/\mathbb{Q}$  is a abelian galoisian extension, then this last condition is given in terms of congruences, so if we exclude a finite number of  $l$ , by Dirichlet theorem, this condition is independent from the others so that finally, we only need  $l \geq d + 1$  and  $l \neq 2$  prime to the discriminant of  $E/\mathbb{Q}$  and with some integer associated to the abelian extension  $F/\mathbb{Q}$ .
- If  $F/\mathbb{Q}$  is galoisian but not abelian, then we know that the condition  $p$  splits in  $F$  cannot be described in terms of congruences. In particular as the other conditions are given in terms of congruences, finally we can always find a place  $v$  verifying the conditions of the theorem if  $l \geq d + 1$  and  $l \neq 2$  prime to the discriminant of  $E/\mathbb{Q}$ .

## 2. Recall from [1]

**2.1. Shimura varieties of KHT type.** — Let  $F = F^+E$  be a CM field where  $E/\mathbb{Q}$  is quadratic imaginary and  $F^+/\mathbb{Q}$  totally real with a fixed real embedding  $\tau : F^+ \hookrightarrow \mathbb{R}$ . For a place  $v$  of  $F$ , we will denote

- $F_v$  the completion of  $F$  at  $v$ ,

- $\mathcal{O}_v$  the ring of integers of  $F_v$ ,
- $\varpi_v$  a uniformizer,
- $q_v$  the cardinal of the residual field  $\kappa(v) = \mathcal{O}_v/(\varpi_v)$ .

Let  $B$  be a division algebra with center  $F$ , of dimension  $d^2$  such that at every place  $x$  of  $F$ , either  $B_x$  is split or a local division algebra and suppose  $B$  provided with an involution of second kind  $*$  such that  $*|_F$  is the complexe conjugation. For any  $\beta \in B^{*-1}$ , denote  $\sharp_\beta$  the involution  $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$  and  $G/\mathbb{Q}$  the group of similitudes, denoted  $G_\tau$  in [6], defined for every  $\mathbb{Q}$ -algebra  $R$  by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with  $B^{op} = B \otimes_{F,c} F$ . If  $x$  is a place of  $\mathbb{Q}$  split  $x = yy^c$  in  $E$  then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \quad (2.1.1)$$

where, identifying places of  $F^+$  over  $x$  with places of  $F$  over  $y$ ,  $x = \prod_i z_i$  in  $F^+$ .

**Convention:** for  $x = yy^c$  a place of  $\mathbb{Q}$  split in  $E$  and  $z$  a place of  $F$  over  $y$  as before, we shall make throughout the text, the following abuse of notation by denoting  $G(F_z)$  in place of the factor  $(B_z^{op})^\times$  in the formula (2.1.1).

In [6], the author justify the existence of some  $G$  like before such that moreover

- if  $x$  is a place of  $\mathbb{Q}$  non split in  $E$  then  $G(\mathbb{Q}_x)$  is quasi split;
- the invariants of  $G(\mathbb{R})$  are  $(1, d-1)$  for the embedding  $\tau$  and  $(0, d)$  for the others.

As in [6] bottom of page 90, a compact open subgroup  $U$  of  $G(\mathbb{A}^\infty)$  is said *small enough* if there exists a place  $x$  such that the projection from  $U^v$  to  $G(\mathbb{Q}_x)$  does not contain any element of finite order except identity.

**2.1.2. Notation.** — Denote  $\mathcal{I}$  the set of open compact subgroup small enough of  $G(\mathbb{A}^\infty)$ . For  $I \in \mathcal{I}$ , write  $X_{I,\eta} \longrightarrow \text{Spec } F$  the associated Shimura variety of Kottwitz-Harris-Taylor type.

**2.1.3. Definition.** — Define  $\text{Spl}$  the set of places  $v$  of  $F$  such that  $p_v := v|_{\mathbb{Q}} \neq l$  is split in  $E$  and  $B_v^\times \simeq GL_d(F_v)$ . For each  $I \in \mathcal{I}$ , write  $\text{Spl}(I)$  the subset of  $\text{Spl}$  of places which doesn't divide the level  $I$ .

In the sequel,  $v$  will denote a place of  $F$  in  $\text{Spl}$ . For such a place  $v$  the scheme  $X_{I,\eta}$  has a projective model  $X_{I,v}$  over  $\text{Spec } \mathcal{O}_v$  with special

fiber  $X_{I,s_v}$ . For  $I$  going through  $\mathcal{I}$ , the projective system  $(X_{I,v})_{I \in \mathcal{I}}$  is naturally equipped with an action of  $G(\mathbb{A}^\infty) \times \mathbb{Z}$  such that  $w_v$  in the Weil group  $W_v$  of  $F_v$  acts by  $-\deg(w_v) \in \mathbb{Z}$ , where  $\deg = \text{val} \circ \text{Art}^{-1}$  and  $\text{Art}^{-1} : W_v^{ab} \simeq F_v^\times$  is Artin's isomorphism which sends geometric Frobenius to uniformizers.

**2.1.4. Notations.** — For  $I \in \mathcal{I}$ , the Newton stratification of the geometric special fiber  $X_{I,\bar{s}_v}$  is denoted

$$X_{I,\bar{s}_v} =: X_{I,\bar{s}_v}^{\geq 1} \supset X_{I,\bar{s}_v}^{\geq 2} \supset \cdots \supset X_{I,\bar{s}_v}^{\geq d}$$

where  $X_{I,\bar{s}_v}^{=h} := X_{I,\bar{s}_v}^{\geq h} - X_{I,\bar{s}_v}^{\geq h+1}$  is an affine scheme, smooth of pure dimension  $d - h$  built up by the geometric points whose connected part of its Barsotti-Tate group is of rank  $h$ . For each  $1 \leq h < d$ , write

$$i_h : X_{I,\bar{s}_v}^{\geq h} \hookrightarrow X_{I,\bar{s}_v}^{\geq 1}, \quad j^{\geq h} : X_{I,\bar{s}_v}^{=h} \hookrightarrow X_{I,\bar{s}_v}^{\geq h},$$

and  $j^{=h} = i_h \circ j^{\geq h}$ .

Let  $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$  be a fixed embedding and write  $\Phi$  the set of embeddings  $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$  whose restriction to  $E$  equals  $\sigma_0$ . There exists, [6] p.97, then an explicit bijection between irreducible algebraic representations  $\xi$  of  $G$  over  $\overline{\mathbb{Q}}_l$  and  $(d+1)$ -uple  $(a_0, (\bar{a}_\sigma)_{\sigma \in \Phi})$  where  $a_0 \in \mathbb{Z}$  and for all  $\sigma \in \Phi$ , we have  $\bar{a}_\sigma = (a_{\sigma,1} \leq \cdots \leq a_{\sigma,d})$ . We then denote  $V_\xi$  the associated  $\overline{\mathbb{Z}}_l$ -local system on  $X_{\mathcal{I}}$ . Recall that an irreducible automorphic representation  $\Pi$  is said  $\xi$ -cohomological if there exists an integer  $i$  such that

$$H^i((\text{Lie } G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee) \neq (0),$$

where  $U$  is a maximal open compact subgroup modulo the center of  $G(\mathbb{R})$ . Let  $d_\xi^i(\Pi_\infty)$  be the dimension of this cohomology group.

## 2.2. Cohomology of Newton strata. —

**2.2.1. Notation.** — For  $1 \leq h \leq d$ , let  $\mathcal{I}_v(h)$  denote the set of open compact subgroups

$$U_v(\underline{m}, h) := U_v(\underline{m}^v) \times \begin{pmatrix} I_h & 0 \\ 0 & K_v(m_1) \end{pmatrix},$$

where  $K_v(m_1) = \text{Ker}(GL_{d-h}(\mathcal{O}_v) \rightarrow GL_{d-h}(\mathcal{O}_v/(\varpi_v^{m_1}))$ ). We then denote  $[H^i(h, \xi)]$  (resp.  $[H_1^i(h, \xi)]$ ) the image of

$$\varinjlim_{I \in \mathcal{I}_v(h)} H^i(X_{I,\bar{s}_v,1}^{\geq h}, V_{\xi, \overline{\mathbb{Q}}_l}[d-h]) \quad \text{resp.} \quad \varinjlim_{I \in \mathcal{I}_v(h)} H^i(X_{I,\bar{s}_v,1}^{\geq h}, j_{1,!}^{\geq h} V_{\xi, \overline{\mathbb{Q}}_l}[d-h])$$

inside the Grothendieck  $\text{Groth}(v, h)$  of admissible representations of  $G(\mathbb{A}^\infty) \times GL_{d-h}(F_v) \times \mathbb{Z}$ .

*Remark.* An element  $\sigma \in W_v$  acts through  $-\deg \sigma \in \mathbb{Z}$  and  $\Pi_{p_v, 0}(\text{Art}^{-1}(\sigma))$ . We moreover consider the action of  $GL_h(F_v)$  through  $\text{val} \circ \det : GL_h(F_v) \rightarrow \mathbb{Z}$  and finally  $P_{h,d}(F_v)$  through its Levi factor  $GL_h(F_v) \times GL_{d-h}(F_v)$ , i.e. its unipotent radical acts trivially.

From [1] proposition 3.6, for any irreducible tempered automorphic representation  $\Pi$  of  $G(\mathbb{A})$  and for every  $i \neq 0$ , the  $\Pi^{\infty, v}$ -isotypic component of  $[H^i(h, \xi)]$  and  $[H_i^i(h, \xi)]$  are zero. About the case  $i = 0$ , for  $\Pi$  an irreducible automorphic tempered representation  $\xi$ -cohomological, its local component at  $v$  looks like

$$\Pi_v \simeq \text{St}_{t_1}(\pi_{v,1}) \times \cdots \times \text{St}_{t_u}(\pi_{v,u}),$$

where for  $i = 1, \dots, u$ ,  $\pi_{v,i}$  is an irreducible cuspidal representation of est une  $GL_{g_i}(F_v)$ .

**2.2.1. Proposition.** — (cf. [1] proposition 3.9) *With the previous notations, we order the  $\pi_{v,i}$  such that the first  $r$ -ones are unramified characters. Then the  $\Pi^{\infty, v}$ -isotypic component of  $[H^0(h, \xi)]$  is then equals to*

$$\left( \frac{\#\text{Ker}^1(\mathbb{Q}, G)}{d} \sum_{\Pi' \in \mathcal{U}_G(\Pi^{\infty, v})} m(\Pi') d_\xi(\Pi'_\infty) \right) \left( \sum_{1 \leq k \leq r: t_k = h} \Pi_v^{(k)} \otimes \chi_{v,k} \Xi^{\frac{d-h}{2}} \right)$$

where

- $\text{Ker}^1(\mathbb{Q}, G)$  is the subset of elements  $H^1(\mathbb{Q}, G)$  which become trivial in  $H^1(\mathbb{Q}_{p'}, G)$  for every prime  $p'$ ;
- $\Pi_v^{(k)} := \text{St}_{t_1}(\chi_{v,1}) \times \cdots \times \text{St}_{t_{k-1}}(\chi_{v,k-1}) \times \text{St}_{t_{k+1}}(\chi_{v,k+1}) \times \cdots \times \text{St}_{t_u}(\chi_{v,u})$   
and
- $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_l^\times$  is defined by  $\Xi(\frac{1}{2}) = q_v^{\frac{1}{2}}$ .
- $\mathcal{U}_G(\Pi^{\infty, v})$  is the set of equivalence classes of irreducible automorphic representations  $\Pi'$  of  $G(\mathbb{A})$  such that  $(\Pi')^{\infty, v} \simeq \Pi^{\infty, v}$ .

*Remark.* In particular if  $[H^0(h, \xi)]$  has non trivial invariant vectors under some open compact subgroup  $I \in \mathcal{I}_v(h)$  which is maximal at  $v$ , then the local component of  $\Pi$  at  $v$  is of the following shape  $\text{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h}$  where the  $\chi_{v,i}$  are unramified characters.

**2.3. Torsion cohomology classes.** — For  $I \in \mathcal{I}$  a finite level, let  $\mathbb{T}_I := \prod_{x \in \text{Unr}(I)} \mathbb{T}_x$  be the unramified Hecke algebra where  $\text{Unr}(I)$  is the union of places  $q$  where  $G$  is unramified and  $I_x$  is maximal, and where  $\mathbb{T}_x \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$  for  $T_x$  a split torus,  $W_x$  the spherical Weyl group and  $X^{un}(T_x)$  the set of  $\overline{\mathbb{Z}}_l$ -unramified characters of  $T_x$ . For  $\xi$  an irreducible algebraic representation of  $G(\mathbb{Q})$ , let  $\mathbb{T}_I(\xi)$  be the image of  $\mathbb{T}_I$  inside the  $\overline{\mathbb{Z}}_l$ -endomorphism of the free quotient of  $H^{d-1}(X_{I, \bar{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})$ .

*Remark.* Note that any system of Hecke eigenvalues  $\mathfrak{m}$  of  $\mathbb{T}_I$  inside the torsion of some  $H^i(X_{I, \bar{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})$  raise in characteristic zero, i.e. is associated to a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_I(\xi)$ . Moreover, as we will be only interested in systems of Hecke eigenvalues given rise to a  $\overline{\mathbb{F}}_l$ -representation of  $\text{Gal}(\bar{F}/F)$  which is irreducible of dimension  $d$ , it suffices to consider the middle degree cohomology of  $X_I$ .

For a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_I(\xi)$  and  $v \in \text{Spl}$ , we denote  $S_{\mathfrak{m}}(v)$  the multi-set of modulo  $l$  Satake parameters at the place  $v$ . Recall that it corresponds to the multi-set of eigenvalues of  $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_v)$  where  $\bar{\rho}_{\mathfrak{m}}$  is the modulo  $l$  galoisian representation associated to  $\mathfrak{m}$ . In the following we will suppose that  $\bar{\rho}_{\mathfrak{m}}$  is irreducible of dimension  $d$ , so that for all  $i \neq 0$ , the localization  $H_{free}^i(X_{I, \bar{s}_v}^{-h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$  and  $H_{free}^i(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$  are zero.

As for each  $1 \leq h \leq d-1$ , the open Newton stratum  $X_{I, \bar{s}_v}^{-h}$  is affine then  $H^i(X_{I, \bar{s}_v}^{-h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h])$  is zero for  $i < 0$  and free for  $i = 0$ . Using this property and the short exact sequence of free perverse sheaves

$$\begin{aligned} 0 \rightarrow i_{h+1,*} V_{\xi, \overline{\mathbb{Z}}_l, |X_{I, \bar{s}_v}^{\geq h+1}}[d-h-1] &\longrightarrow j_!^{\geq h} j^{\geq h,*} V_{\xi, \overline{\mathbb{Z}}_l, |X_{I, \bar{s}_v}^{\geq h}}[d-h] \\ &\longrightarrow V_{\xi, \overline{\mathbb{Z}}_l, |X_{I, \bar{s}_v}^{\geq h}}[d-h] \rightarrow 0, \end{aligned}$$

we then obtain for every  $i > 0$

$$0 \rightarrow H^{-i-1}(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h-1]) \rightarrow 0, \quad (2.3.1)$$

and for  $i = 0$ ,

$$\begin{aligned} 0 \rightarrow H^{-1}(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) &\longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h-1]) \longrightarrow \\ H^0(X_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h,*} V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) &\longrightarrow H^0(X_{I, \bar{s}_v}^{\geq h}, V_{\xi, \overline{\mathbb{Z}}_l}[d-h]) \rightarrow \dots \end{aligned} \quad (2.3.2)$$

In [1], arguing by induction from  $h = d$  to  $h = 1$ , we prove that, under the hypothesis that  $S_{\mathfrak{m}}(v)$  does not contain any subset of the form  $\{\alpha, q_v \alpha\}$ ,



all the cohomology groups  $H^i(X_{I,\bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  are free: note that in order to deal with  $i \geq 0$ , one has to use Grothendieck-Verdier duality.

Without this hypothesis, arguing similarly, we conclude that any torsion cohomology class comes from a non strict map

$$H_{free}^0(X_{I,\bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1]) \longrightarrow H^0(X_{I,\bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h,*} V_{\xi, \bar{\mathbb{Z}}_l}[d-h]).$$

In particular it raises in characteristic zero to some free subquotient of  $H^0(X_{I,\bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h,*} V_{\xi, \bar{\mathbb{Z}}_l}[d-h])$ .

### 3. Proof of the theorem

The minimal prime ideals of  $\mathbb{T}_I(\xi)$  are the prime ideals above the zero ideal of  $\bar{\mathbb{Z}}_l$  and are then in bijection with the prime ideals of  $\mathbb{T}_I(\xi) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$ . To such an ideal, which corresponds to give a collection of Satake parameters, is then associated an unique near equivalence class in the sense of [8], denoted  $\Pi_{\tilde{\mathfrak{m}}}$  which is the finite set of irreducible automorphic representations  $\xi$ -cohomological whose multi-set of Satake parameters at each place  $x \in \text{Unr}(I)$ , is given by  $S_{\tilde{\mathfrak{m}}}(x)$ . Thanks to [6] and [8], we denote

$$\rho_{\tilde{\mathfrak{m}}} : \text{Gal}(\bar{F}/F) \longrightarrow GL_d(\bar{\mathbb{Q}}_l)$$

the galoisian representation associated to any  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$ . Recall that the modulo  $l$  reduction of  $\rho_{\tilde{\mathfrak{m}}}$  depends only of  $\mathfrak{m}$ , and was denoted above  $\bar{\rho}_{\mathfrak{m}}$ .

By hypothesis there exists a place  $v \in \text{Spl}(I)$  such that the order of  $q_v$  modulo  $l$  is  $\geq d$ , where  $q_v$  is the order of the residue field at  $v$ . We argue now by absurdity, supposing there exists a non trivial torsion class of some  $H_{tor}^i(X_{I,\bar{s}_v}^{\geq 1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}[l]$ .

Consider then  $h$  maximal such that there exists  $i$  for which  $H_{tor}^i(X_{I,\bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}[l]$  is non trivial. Note then using (2.3.1) and (2.3.2), we must have  $i = 0$ . Consider then an irreducible sub- $\bar{\mathbb{F}}_l[\mathbb{T}_I(\xi) \times \text{Gal}(\bar{F}_v/F_v)]$ -module of  $H_{tor}^0(X_{I,\bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}[l]$ . From proposition 2.2.1,

- there exists then a prime ideal  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  such that for any  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$  its local component at  $v$  is of the following shape

$$\text{St}_{h+1}(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h-1},$$

where  $\chi_{v,1}, \dots, \chi_{v,d-h-1}$  are unramified characters of  $F_v^\times$ . In particular

$$\bar{\chi}_{v,1}\left(-\frac{h}{2}\right), \quad \bar{\chi}_{v,1}\left(-\frac{h-2}{2}\right), \quad \cdots, \quad \bar{\chi}_{v,1}\left(\frac{h}{2}\right)$$

are irreducible subquotients of the restriction  $\bar{\rho}_{m,v}$  of  $\bar{\rho}_m$  to the decomposition group at  $v$ .

– Moreover the Galois action is isotypic for  $\bar{\chi}_{v,1}(\frac{-h}{2})$ .

Using repeatedly (2.3.1), we then have the same property concerning  $H_{tor}^{1-h}(X_{I,\bar{s}_v}^{\geq 1}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}} \simeq H_{tor}^{1-h}(X_{I,\bar{\eta}_v}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}$ . Now this last space has a full action of  $\text{Gal}(\bar{F}/F)$  and each torsion class is obtain as before, i.e. comes from a quotient of  $H_c^0(X_{I,\bar{s}_v}^{\geq h}, V_{\xi,\bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$ .

Note then that each  $\bar{\mathbb{F}}_l[\text{Gal}(\bar{F}_v/F_v)]$ -irreducible torsion class of  $H^{1-h}(X_{I,\bar{\eta}_v}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}[l]$  verifies the following property: it is an irreducible subquotient  $\bar{\chi}_v$  of  $\bar{\rho}_{m,v}$  such that  $\bar{\chi}_v(1)$  is also an irreducible subquotient of  $\bar{\rho}_{m,v}$ . Now as  $\bar{\rho}_m$  is supposed to be irreducible and as  $H_{tor}^{1-h}(X_{I,\bar{\eta}_v}, V_{\xi,\bar{\mathbb{Z}}_l}[d-1])_{\mathfrak{m}}$  is a  $\text{Gal}(\bar{F}/F)$ -module, we conclude that for each irreducible subquotient  $\bar{\chi}_v$  of  $\bar{\rho}_{m,v}$  then  $\bar{\chi}_v(1)$  is also such a subquotient, so that all the Zelevinski line of  $\bar{\chi}_v$  is contained in the supercuspidal support of  $\bar{\rho}_m$  so that in particular the order of  $q_v$  modulo  $l$  is necessary  $\leq d$  which is not the case by hypothesis.

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