



**HAL**  
open science

# Galois irreducibility implies cohomology freeness for KHT Shimura varieties

Pascal Boyer

► **To cite this version:**

Pascal Boyer. Galois irreducibility implies cohomology freeness for KHT Shimura varieties. Journal de l'École polytechnique - Mathématiques, 2023, 10, pp.199-232. 10.5802/jep.216 . hal-02080476v4

**HAL Id: hal-02080476**

**<https://hal.science/hal-02080476v4>**

Submitted on 27 Apr 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# GALOIS IRREDUCIBILITY IMPLIES COHOMOLOGY FREENESS FOR KHT SHIMURA VARIETIES

*by*

Boyer Pascal

---

**Abstract.** — Given a KHT Shimura variety provided with an action of its unramified Hecke algebra  $\mathbb{T}$ , we proved in [5], see also [11] for other PEL Shimura varieties, that its localized cohomology groups at a generic maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}$ , appear to be free. In this work, we obtain the same result for  $\mathfrak{m}$  such that its associated galoisian  $\overline{\mathbb{F}}_l$ -representation  $\overline{\rho}_{\mathfrak{m}}$  is irreducible.

## Contents

Introduction.....	2
1. Recalls from [5].....	4
1.1. Representations of $GL_d(K)$ .....	4
1.2. Shimura varieties of KHT type.....	7
1.3. Cohomology of the Newton strata.....	9
2. About the nearby cycle perverse sheaf.....	12
2.1. The case where the level at $v$ is maximal.....	13
2.2. Harris-Taylor perverse sheaves over $\overline{\mathbb{Z}}_l$ .....	14
2.3. Filtrations of the nearby cycles perverse sheaf....	17
2.4. Local behavior of the monodromy over $\overline{\mathbb{F}}_l$ .....	19
3. Irreducibility implies freeness.....	20
3.1. Torsion cohomology classes of Harris-Taylor perverse sheaves.....	21

---

**2010 Mathematics Subject Classification.** — 11F70, 11F80, 11F85, 11G18, 20C08.

**Key words and phrases.** — Shimura varieties, torsion in the cohomology, maximal ideal of the Hecke algebra, localized cohomology, galoisian representation.

3.2. Global torsion and genericity.....	25
3.3. Torsion and modified lattices.....	27
3.4. Global lattices and generic representations.....	32
References.....	38

## Introduction

From Matsushima's formula and computations of  $(\mathfrak{G}, K_\infty)$ -cohomology, we know that tempered automorphic representations contribution in the cohomology of Shimura varieties with complex coefficients, is concentrated in middle degree. If you consider cohomology with coefficients in a very regular local system, then only tempered representations can contribute so that all of the cohomology is concentrated in middle degree.

For  $\overline{\mathbb{Z}}_l$ -coefficients and Shimura varieties of Kottwitz-Harris-Taylor types, we proved in [9], whatever is the weight of the coefficients, when the level is large enough at  $l$ , there are always non trivial torsion cohomology classes, so that the  $\overline{\mathbb{F}}_l$ -cohomology can't be concentrated in middle degree. Thus if you want a  $\overline{\mathbb{F}}_l$ -analog of the previous  $\overline{\mathbb{Q}}_l$ -statement, you must cut some part of the cohomology.

In [5] for KHT Shimura varieties, and more generally in [11] for any PEL proper Shimura variety, we obtain such a result under some genericity hypothesis which can be stated as follows. Let  $(\text{Sh}_K)_{K \subset G(\mathbb{A}^\infty)}$  be a tower, indexed by open compact subgroups  $K$  of  $G(\mathbb{A}^\infty)$ , of compact Shimura varieties of Kottwitz type associated to some similitude group  $G$ : we denote by  $F$  its reflex field. Let then  $\mathfrak{m}$  be a system of Hecke eigenvalues appearing in  $H^{n_0}(\text{Sh}_K \times_F \overline{F}, \overline{\mathbb{F}}_l)$ . By the main result of [17], one can attach to such  $\mathfrak{m}$ , a mod  $l$  Galois representation

$$\overline{\rho}_{\mathfrak{m}} : \text{Gal}(\overline{F}/F) \longrightarrow GL_d(\overline{\mathbb{F}}_l)$$

where  $F$  is the reflex field of  $\text{Sh}_K$  which, by assumption, contains some imaginary quadratic extension  $E/\mathbb{Q}$ . From [11] definition 1.9, we say that  $\mathfrak{m}$  is generic (resp. decomposed generic) at some split  $p$  in  $E$ , if for all places  $v$  of  $F$  dividing  $p$ , the eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$  of  $\overline{\rho}_{\mathfrak{m}}(\text{Frob}_v)$  satisfy  $\lambda_i/\lambda_j \notin \{q_v^{\pm 1}\}$  for all  $i \neq j$  (resp. and are pairwise distincts), where

$q_v$  is the cardinal of the residue field at  $v$ . Then under the hypothesis<sup>(1)</sup> that there exists such  $p$  with  $\mathfrak{m}$  generic at  $p$ , the integer  $n_0$  above is necessary equals to the relative dimension of  $\mathrm{Sh}_K$ . In particular the  $H^i(\mathrm{Sh}_K \times_F \overline{F}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$  are all torsion free.

In this work we consider the particular case of Kottwitz-Harris-Taylor Shimura varieties  $\mathrm{Sh}_K$  of [15] associated to inner forms of  $GL_d$ . Exploiting the fact, which is particular to these Shimura varieties, that the non supersingular Newton strata are geometrically induced, we are then able to prove the following result.

**Theorem** — *We suppose there exists a place  $v$  of  $F$  with residue field of order  $q_v$ , such that the order of  $q_v$  modulo  $l$  is strictly greater than  $d$ . Let then  $\mathfrak{m}$  be a system of Hecke eigenvalues such that  $\overline{\rho}_{\mathfrak{m}}$  is irreducible, then the localized cohomology groups of  $\mathrm{Sh}_K$  with coefficients in any  $\overline{\mathbb{Z}}_l$ -local system  $V_{\xi}$ , are all free.*

*Remark.* The hypothesis about the existence of  $v$  such that  $q_v \in \mathbb{F}_l^{\times}$  is of order  $> d$ , is used three times in the proof:

- first to prove the isomorphism (2.4.1). It is tempting to think that this is still true without this hypothesis.
- Secondly for such a place  $v$  there is no irreducible cuspidal representation  $\pi_v$  of  $GL_g(F_v)$  with  $g > 1$  such that its modulo  $l$  reduction has a supercuspidal support made of characters, cf. the remark after 1.1.4. This simplification is completely harmless and if one want to take care about these cuspidal representations, it suffices to used the proposition 2.4.2 of [7] which is recalled in proposition 2.2.6.
- With this hypothesis we also note that the pro-order of  $GL_d(\mathcal{O}_v)$  is invertible modulo  $l$  so that, concerning torsion cohomology classes, we can easily pass from infinite to maximal level at  $v$ , cf. for example the lemma 3.1.11.

As we can reasonably hope that these three points can be overcome in the future, we write this paper without taking into account simplifications coming from these hypothesis, except when, for the moment, we really need it as explained above.

Note also that if there exists such  $v$  then there exists an infinity of density strictly positive.

---

<sup>(1)</sup>In their new preprint, Caraiani and Scholze explained that, from an observation of Koshikawa, one can replace decomposed generic by simply generic, in their main statement.

Roughly the proof relies on the fact that, cf. [6] theorem 3.1.1 which follows easily from [18] theorem 5.6, if  $\overline{\rho}_m$  is absolutely irreducible, then as a  $\mathbb{T}_{\xi,m}^S[\text{Gal}_{F,S}]$ -module, where  $\text{Gal}_{F,S}$  is the Galois group of the maximal extension of  $F$  unramified outside  $S$ :

$$H_{free}^{d-1}(\text{Sh}_K \times_F \overline{F}, V_{\xi, \overline{\mathbb{Z}}_l})_m \simeq \pi_{\xi,m} \otimes_{\mathbb{T}_{\xi,m}^S} \rho_{\xi,m},$$

where

- $\pi_{\xi,m}$  is a  $\mathbb{T}_{\xi,m}^S$ -module on which  $\text{Gal}_{F,S}$  acts trivially,
- and  $\rho_{\xi,m} : \text{Gal}_{F,S} \rightarrow GL_d(\mathbb{T}_{\xi,m}^S)$  is a Galois representation unramified outside  $S$  such that, cf. [17] V.4.4, for all  $u \notin S$ , then  $\det(1 - X \text{Frob}_u | \rho_{\xi,m})$  is equal to the Hecke polynomial, cf. the end of §1.3.

The idea is then to prove, using the geometric induced structure of Newton strata and the monodromy operator, that if there were non trivial torsion cohomology classes, then the previous decomposition of the global lattice as a tensorial product could not be possible. See the introduction of §3 to have more details about the main steps of the proof.

The author would like to thank Koshikawa for helpful conversations about a previous work on the same theme, where he pointed me a major mistake. He then explained me some of his ideas which was very inspiring.

## 1. Recalls from [5]

**1.1. Representations of  $GL_d(K)$ .** — We fix a finite extension  $K/\mathbb{Q}_p$  with residue field  $\mathbb{F}_q$ . We denote by  $|\cdot|$  its absolute value.

For a representation  $\pi$  of  $GL_d(K)$  and  $n \in \frac{1}{2}\mathbb{Z}$ , set

$$\pi\{n\} := \pi \otimes q^{-n \text{val} \circ \det}.$$

**1.1.1. Notations.** — For  $\pi_1$  and  $\pi_2$  representations of respectively  $GL_{n_1}(K)$  and  $GL_{n_2}(K)$ , we will denote by

$$\pi_1 \times \pi_2 := \text{ind}_{P_{n_1, n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1\left\{\frac{n_2}{2}\right\} \otimes \pi_2\left\{-\frac{n_1}{2}\right\},$$

the normalized parabolic induced representation where for any sequence  $\underline{r} = (0 < r_1 < r_2 < \dots < r_k = d)$ , we write  $P_{\underline{r}}$  for the standard parabolic subgroup of  $GL_d$  with Levi

$$GL_{r_1} \times GL_{r_2-r_1} \times \dots \times GL_{r_k-r_{k-1}}.$$

Recall that a representation  $\varrho$  of  $GL_d(K)$  is called *cuspidal* (resp. *supercuspidal*) if it's not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for  $\overline{\mathbb{F}}_l$ .

**1.1.2. Definition.** — (see [21] §9 and [3] §1.4) Let  $g$  be a divisor of  $d = sg$  and  $\pi$  an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation of  $GL_g(K)$ . The induced representation

$$\pi\left\{\frac{1-s}{2}\right\} \times \pi\left\{\frac{3-s}{2}\right\} \times \cdots \times \pi\left\{\frac{s-1}{2}\right\}$$

holds a unique irreducible quotient (resp. subspace) denoted  $St_s(\pi)$  (resp.  $Speh_s(\pi)$ ); it's a generalized Steinberg (resp. *Speh*) representation.

Moreover the induced representation  $St_t(\pi\{\frac{-r}{2}\}) \times Speh_r(\pi\{\frac{t}{2}\})$  (resp. of  $St_{t-1}(\pi\{\frac{-r-1}{2}\}) \times Speh_{r+1}(\pi\{\frac{t-1}{2}\})$ ) owns a unique irreducible subspace (resp. quotient), denoted  $LT_\pi(t-1, r)$ .

*Remark.* These representations  $LT_\pi(t-1, r)$  appear in the cohomology of the Lubin-Tate spaces, cf. [2].

**1.1.3. Proposition.** — (cf. [20] III.5.10) Let  $\pi$  be an irreducible cuspidal representation of  $GL_g(K)$  with a stable  $\overline{\mathbb{Z}}$ -lattice<sup>(2)</sup>, then its modulo  $l$  reduction is irreducible and cuspidal but not necessary supercuspidal.

The supercuspidal support of the modulo  $l$  reduction of a cuspidal representation, is a segment associated to some irreducible  $\overline{\mathbb{F}}_l$ -supercuspidal representation  $\varrho$  of  $GL_{g_{-1}(\varrho)}(F_v)$  with  $g = g_{-1}(\varrho)t$  where  $t$  is either equal to 1 or of the following shape  $t = m(\varrho)l^u$  with  $u \geq 0$  and where  $m(\varrho)$  is defined as follows.

**1.1.4. Notation.** — We denote by  $m(\varrho)$  the cardinal of the Zelevinsky line of  $\varrho$  if it is not equal to 1, otherwise  $m(\varrho) = 1$ .

*Remark.* When  $\varrho$  is the trivial representation then  $m(1_v)$  is either the order of  $q$  modulo  $l$  when it is  $> 1$ , otherwise  $m(1_v) = 1$ . We say that such  $\pi_v$  is of  $\varrho$ -type  $u$  with  $u \geq -1$ .

<sup>(2)</sup>We say that  $\pi$  is entire.

**1.1.5. Notation.** — For  $\varrho$  an irreducible  $\overline{\mathbb{F}}_l$ -supercuspidal representation, we denote by  $\text{Cusp}_\varrho$  (resp.  $\text{Cusp}_\varrho(u)$  for some  $u \geq -1$ ) the set of equivalence classes of irreducible  $\overline{\mathbb{Q}}_l$ -cuspidal representations whose modulo  $l$  reduction has supercuspidal support a segment associated to  $\varrho$  (resp. of  $\varrho$ -type  $u$ ).

Let  $u \geq 0$ ,  $\pi_{v,u} \in \text{Cusp}_\varrho(u)$  and  $\tau = \pi_{v,u}[s]_D$ . Let then denote by  $\iota$  the image of  $\text{Speh}_s(\varrho)$  by the modulo  $l$  Jacquet-Langlands correspondence defined at §1.2.4 de [13]. Then the modulo  $l$  reduction of  $\tau$  is isomorphic to

$$\iota\left\{-\frac{m(\tau)-1}{2}\right\} \oplus \iota\left\{-\frac{m(\tau)-3}{2}\right\} \oplus \dots \oplus \iota\left\{\frac{m(\tau)-1}{2}\right\} \quad (1.1.6)$$

where  $\iota\{n\} := \iota \otimes q^{-n \text{val} \circ \text{nrđ}}$ .

We want now to recall the notion of level of non degeneracy from [1] §4. The mirabolic subgroup  $M_d(K)$  of  $GL_d(K)$  is the set of matrices with last row  $(0, \dots, 0, 1)$ : we denote by

$$V_d(K) = \{(m_{i,j} \in P_d(K) : m_{i,j} = \delta_{i,j} \text{ for } j < n)\}.$$

its unipotent radical. We fix a non trivial character  $\psi$  of  $K$  and let  $\theta$  be the character of  $V_d(K)$  defined by  $\theta((m_{i,j})) = \psi(m_{d-1,d})$ . For  $G = GL_r(K)$  or  $M_r(K)$ , we denote by  $\text{Alg}(G)$  the abelian category of algebraic representations of  $G$  and, following [1], we introduce

$$\Psi^- : \text{Alg}(M_d(K)) \longrightarrow \text{Alg}(GL_{d-1}(K)), \quad \Phi^- : \text{Alg}(M_d) \longrightarrow \text{Alg}(M_{d-1}(K))$$

defined by  $\Psi^- = r_{V_d,1}$  (resp.  $\Phi^- = r_{V_d,\theta}$ ) the functor of  $V_{d-1}$  coinvariants (resp.  $(V_{d-1}, \theta)$ -coinvariants), cf. [1] 1.8. We also introduce the normalized compact induced functor

$$\begin{aligned} \Psi^+ &:= i_{V,1} : \text{Alg}(GL_{d-1}(K)) \longrightarrow \text{Alg}(M_d(K)), \\ \Phi^+ &:= i_{V,\theta} : \text{Alg}(M_{d-1}(K)) \longrightarrow \text{Alg}(M_d(K)). \end{aligned}$$

**1.1.7. Proposition.** — ([1] p451)

- The functors  $\Psi^-$ ,  $\Psi^+$ ,  $\Phi^-$  and  $\Phi^+$  are exact.
- $\Phi^- \circ \Psi^+ = \Psi^- \circ \Phi^+ = 0$ .
- $\Psi^-$  (resp.  $\Phi^+$ ) is left adjoint to  $\Psi^+$  (resp.  $\Phi^-$ ) and the following adjunction maps

$$\text{Id} \longrightarrow \Phi^- \Phi^+, \quad \Psi^+ \Psi^- \longrightarrow \text{Id},$$

are isomorphisms meanwhile

$$0 \rightarrow \Phi^+ \Phi^- \longrightarrow \text{Id} \longrightarrow \Psi^+ \Psi^- \rightarrow 0.$$

**1.1.8. Definition.** — For  $\tau \in \text{Alg}(M_d(K))$ , the representation

$$\tau^{(k)} := \Psi^- \circ (\Phi^-)^{k-1}(\tau)$$

is called the  $k$ -th derivative of  $\tau$ . If  $\tau^{(k)} \neq 0$  and  $\tau^{(m)} = 0$  for all  $m > k$ , then  $\tau^{(k)}$  is called the highest derivative of  $\tau$ .

**1.1.9. Notation.** — (cf. [21] 4.3) Let  $\pi \in \text{Alg}(GL_d(K))$  (or  $\pi \in \text{Alg}(M_d(K))$ ). The maximal number  $k$  such that  $(\pi|_{M_d(K)})^{(k)} \neq (0)$  is called the level of non-degeneracy of  $\pi$  and denoted by  $\lambda(\pi)$ . We can also iterate the construction so that at the end we obtain a partition  $\underline{\lambda}(\pi)$  of  $d$ .

**1.1.10. Definition.** — A representation  $\pi$  of  $GL_d(K)$ , over  $\overline{\mathbb{Q}}_l$  or  $\overline{\mathbb{F}}_l$ , is then said generic if its level of non degeneracy  $\lambda(\pi)$  is equal to  $d$ .

*Remark.* Let  $\pi$  be an essentially square integrable irreducible representation of  $GL_d(K)$  which is entire. Then its modulo  $l$  reduction owns a unique generic irreducible sub-quotient.

**1.2. Shimura varieties of KHT type.** — Let  $F = F^+E$  be a CM field where  $E/\mathbb{Q}$  is quadratic imaginary and  $F^+/\mathbb{Q}$  totally real with a fixed real embedding  $\tau : F^+ \hookrightarrow \mathbb{R}$ . For a place  $v$  of  $F$ , we will denote by

- $F_v$  the completion of  $F$  at  $v$ ,
- $\mathcal{O}_v$  the ring of integers of  $F_v$ ,
- $\varpi_v$  a uniformizer,
- $q_v$  the cardinal of the residual field  $\kappa(v) = \mathcal{O}_v/(\varpi_v)$ .

Let  $B$  be a division algebra with center  $F$ , of dimension  $d^2$  such that at every place  $x$  of  $F$ , either  $B_x$  is split or a local division algebra and suppose  $B$  provided with an involution of second kind  $*$  such that  $*|_F$  is the complex conjugation. For any  $\beta \in B^{*-1}$ , denote by  $\sharp_\beta$  the involution  $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$  and  $G/\mathbb{Q}$  the group of similitudes, denoted  $G_\tau$  in [15], defined for every  $\mathbb{Q}$ -algebra  $R$  by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$



with  $B^{op} = B \otimes_{F,c} F$ . If  $x$  is a place of  $\mathbb{Q}$  split  $x = yy^c$  in  $E$  then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \quad (1.2.1)$$

where, identifying places of  $F^+$  over  $x$  with places of  $F$  over  $y$ ,  $x = \prod_i z_i$  in  $F^+$ .

**Convention:** for  $x = yy^c$  a place of  $\mathbb{Q}$  split in  $E$  and  $z$  a place of  $F$  over  $y$  as before, we shall make throughout the text, the following abuse of notation by denoting  $G(F_z)$  in place of the factor  $(B_z^{op})^\times$  in the formula (1.2.1).

In [15], the authors justify the existence of some  $G$  like before such that moreover

- if  $x$  is a place of  $\mathbb{Q}$  non split in  $E$  then  $G(\mathbb{Q}_x)$  is quasi split;
- the invariants of  $G(\mathbb{R})$  are  $(1, d - 1)$  for the embedding  $\tau$  and  $(0, d)$  for the others.

As in [15] bottom of page 90, a compact open subgroup  $U$  of  $G(\mathbb{A}^\infty)$  is said *small enough* if there exists a place  $x$  such that the projection from  $U^v$  to  $G(\mathbb{Q}_x)$  does not contain any element of finite order except identity.

**1.2.2. Notation.** — Denote by  $\mathcal{I}$  the set of open compact subgroups small enough of  $G(\mathbb{A}^\infty)$ . For  $I \in \mathcal{I}$ , write  $\text{Sh}_{I,\eta} \longrightarrow \text{Spec } F$  for the associated Shimura variety of Kottwitz-Harris-Taylor type.

**1.2.3. Definition.** — Define  $\text{Spl}$  the set of places  $v$  of  $F$  such that  $p_v := v|_{\mathbb{Q}} \neq l$  is split in  $E$  and  $B_v^\times \simeq GL_d(F_v)$ . For each  $I \in \mathcal{I}$ , write  $\text{Spl}(I)$  for the subset of  $\text{Spl}$  of places which does not divide the level  $I$ .

In the sequel,  $v$  and  $w$  will denote places of  $F$  in  $\text{Spl}$ . For such a place  $v$  the scheme  $\text{Sh}_{I,\eta}$  has a projective model  $\text{Sh}_{I,v}$  over  $\text{Spec } \mathcal{O}_v$  with special fiber  $\text{Sh}_{I,s_v}$ . For  $I$  going through  $\mathcal{I}$ , the projective system  $(\text{Sh}_{I,v})_{I \in \mathcal{I}}$  is naturally equipped with an action of  $G(\mathbb{A}^\infty) \times \mathbb{Z}$  such that any  $w_v$  in the Weil group  $W_v$  of  $F_v$  acts by  $-\deg(w_v) \in \mathbb{Z}$ , where  $\deg = \text{val} \circ \text{Art}^{-1}$  and  $\text{Art}^{-1} : W_v^{ab} \simeq F_v^\times$  is Artin's isomorphism which sends geometric Frobenius to uniformizers.

**1.2.4. Notations.** — For  $I \in \mathcal{I}$ , the Newton stratification of the geometric special fiber  $\mathrm{Sh}_{I, \bar{s}_v}$  is denoted by

$$\mathrm{Sh}_{I, \bar{s}_v} =: \mathrm{Sh}_{I, \bar{s}_v}^{\geq 1} \supset \mathrm{Sh}_{I, \bar{s}_v}^{\geq 2} \supset \cdots \supset \mathrm{Sh}_{I, \bar{s}_v}^{\geq d}$$

where  $\mathrm{Sh}_{I, \bar{s}_v}^{=h} := \mathrm{Sh}_{I, \bar{s}_v}^{\geq h} - \mathrm{Sh}_{I, \bar{s}_v}^{\geq h+1}$  is an affine scheme, smooth of pure dimension  $d - h$  built up by the geometric points whose connected part of its Barsotti-Tate group is of rank  $h$ . For each  $1 \leq h < d$ , write

$$i_h : \mathrm{Sh}_{I, \bar{s}_v}^{\geq h} \hookrightarrow \mathrm{Sh}_{I, \bar{s}_v}^{\geq 1}, \quad j^{\geq h} : \mathrm{Sh}_{I, \bar{s}_v}^{=h} \hookrightarrow \mathrm{Sh}_{I, \bar{s}_v}^{\geq h},$$

and  $j^{=h} = i_h \circ j^{\geq h}$ .

Let  $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$  be a fixed embedding and write  $\Phi$  for the set of embeddings  $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$  whose restriction to  $E$  equals  $\sigma_0$ . There exists then, cf. [15] p.97, an explicit bijection between irreducible algebraic representations  $\xi$  of  $G$  over  $\overline{\mathbb{Q}}_l$  and  $(d+1)$ -uple  $(a_0, (\overline{a}_\sigma)_{\sigma \in \Phi})$  where  $a_0 \in \mathbb{Z}$  and for all  $\sigma \in \Phi$ , we have  $\overline{a}_\sigma = (a_{\sigma, 1} \leq \cdots \leq a_{\sigma, d})$ . We then denote by

$$V_{\xi, \overline{\mathbb{Z}}_l}$$

the associated  $\overline{\mathbb{Z}}_l$ -local system on  $\mathrm{Sh}_{\mathcal{I}}$ . Recall that an irreducible automorphic representation  $\Pi$  is said  $\xi$ -cohomological if there exists an integer  $i$  such that

$$H^i((\mathrm{Lie} G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee) \neq (0),$$

where  $U$  is a maximal open compact subgroup modulo the center of  $G(\mathbb{R})$ . Let  $d_\xi^i(\Pi_\infty)$  be the dimension of this cohomology group.

### 1.3. Cohomology of the Newton strata. —

**1.3.1. Notation.** — For  $1 \leq h \leq d$ , let  $\mathcal{I}_v(h)$  be the set of open compact subgroups

$$U_v(\underline{m}, h) := U_v(\underline{m}^v) \times \begin{pmatrix} I_h & 0 \\ 0 & K_v(m_1) \end{pmatrix},$$

where  $K_v(m_1) = \mathrm{Ker}(GL_{d-h}(\mathcal{O}_v) \rightarrow GL_{d-h}(\mathcal{O}_v/(\varpi_v^{m_1})))$ . We then denote by  $[H^i(h, \xi)]$  (resp.  $[H_i^i(h, \xi)]$ ) the image of

$$\varinjlim_{I \in \mathcal{I}_v(h)} H^i(\mathrm{Sh}_{I, \bar{s}_v, 1}^{\geq h}, V_{\xi, \overline{\mathbb{Q}}_l}[d-h]) \quad \text{resp.} \quad \varinjlim_{I \in \mathcal{I}_v(h)} H^i(\mathrm{Sh}_{I, \bar{s}_v, 1}^{\geq h}, j_{1,!}^{\geq h} V_{\xi, \overline{\mathbb{Q}}_l}[d-h])$$

inside the Grothendieck  $\mathrm{Groth}(v, h)$  of admissible representations of  $G(\mathbb{A}^\infty) \times GL_{d-h}(F_v) \times \mathbb{Z}$ .

*Remark.* An element  $\sigma \in W_v$  acts through  $-\deg \sigma \in \mathbb{Z}$  and  $\Pi_{p_v,0}(\text{Art}^{-1}(\sigma))$ . We moreover consider the action of  $GL_h(F_v)$  through  $\text{val} \circ \det : GL_h(F_v) \rightarrow \mathbb{Z}$  and finally  $P_{h,d}(F_v)$  through its Levi factor  $GL_h(F_v) \times GL_{d-h}(F_v)$ , i.e. its unipotent radical acts trivially.

From [5] proposition 3.6, for any irreducible tempered automorphic representation  $\Pi$  of  $G(\mathbb{A})$  and for every  $i \neq 0$ , the  $\Pi^{\infty,v}$ -isotypic component of  $[H^i(h, \xi)]$  and  $[H_!^i(h, \xi)]$  are zero. About the case  $i = 0$ , for  $\Pi$  an irreducible automorphic tempered representation  $\xi$ -cohomological, its local component at  $v$  looks like

$$\Pi_v \simeq \text{St}_{t_1}(\pi_{v,1}) \times \cdots \times \text{St}_{t_u}(\pi_{v,u}),$$

where for  $i = 1, \dots, u$ ,  $\pi_{v,i}$  is an irreducible cuspidal representation of est une  $GL_{g_i}(F_v)$ .

**1.3.2. Proposition.** — (cf. [5] proposition 3.9) *With the previous notations, we order the  $\pi_{v,i}$  such that the first  $r$ -ones are unramified characters. Then the  $\Pi^{\infty,v}$ -isotypic component of  $[H^0(h, \xi)]$  is then equals to*

$$\left( \frac{\#\text{Ker}^1(\mathbb{Q}, G)}{d} \sum_{\Pi' \in \mathcal{U}_G(\Pi^{\infty,v})} m(\Pi') d_\xi(\Pi'_\infty) \right) \left( \sum_{1 \leq k \leq r: t_k=h} \Pi_v^{(k)} \otimes \chi_{v,k} \chi^{\frac{d-h}{2}} \right)$$

where

- $\text{Ker}^1(\mathbb{Q}, G)$  is the subset of elements  $H^1(\mathbb{Q}, G)$  which become trivial in  $H^1(\mathbb{Q}_{p'}, G)$  for every prime  $p'$ ;
- $\Pi_v^{(k)} := \text{St}_{t_1}(\chi_{v,1}) \times \cdots \times \text{St}_{t_{k-1}}(\chi_{v,k-1}) \times \text{St}_{t_{k+1}}(\chi_{v,k+1}) \times \cdots \times \text{St}_{t_u}(\chi_{v,u})$  and
- $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_l^\times$  is defined by  $\Xi(\frac{1}{2}) = q_v^{\frac{1}{2}}$ .
- $\mathcal{U}_G(\Pi^{\infty,v})$  is the set of equivalence classes of irreducible automorphic representations  $\Pi'$  of  $G(\mathbb{A})$  such that  $(\Pi')^{\infty,v} \simeq \Pi^{\infty,v}$ .

*Remark.* In particular if  $[H^0(h, \xi)]$  has non trivial invariant vectors under some open compact subgroup  $I \in \mathcal{I}_v(h)$  which is maximal at  $v$ , then the local component of  $\Pi$  at  $v$  is of the following shape  $\text{St}_h(\chi_{v,1}) \times \chi_{v,2} \times \cdots \times \chi_{v,d-h}$  where the  $\chi_{v,i}$  are unramified characters.

**1.3.3. Definition.** — *For a finite set  $S$  of places of  $\mathbb{Q}$  containing the places where  $G$  is ramified, denote by  $\mathbb{T}_{abs}^S := \prod_{x \notin S} \mathbb{T}_{x,abs}$  the abstract*

unramified where  $\mathbb{T}_{x,abs} \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$  for  $T_x$  a split torus,  $W_x$  the spherical Weyl group and  $X^{un}(T_x)$  the set of  $\overline{\mathbb{Z}}_l$ -unramified characters of  $T_x$ .

*Example.* For  $w \in \text{Spl}$ , we have

$$\mathbb{T}_{w,abs} = \overline{\mathbb{Z}}_l[T_{w,i} : i = 1, \dots, d],$$

where  $T_{w,i}$  is the characteristic function of

$$GL_d(\mathcal{O}_w) \text{diag}(\overbrace{\varpi_w, \dots, \varpi_w}^i, \overbrace{1, \dots, 1}^{d-i}) GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

**1.3.4. Notation.** — Let  $\mathbb{T}_\xi^S$  be the image of  $\mathbb{T}_{abs}^S$  inside

$$\bigoplus_{i=0}^{2d-2} \varinjlim_I H^i(\text{Sh}_{I,\bar{\eta}}, V_{\xi, \overline{\mathbb{Q}}_l})$$

where the limit concerned the ideals  $I$  which are maximal at each places outside  $S$ .

*Remark.* In [5], we proved that if  $\mathfrak{m}$  is a maximal ideal of  $\mathbb{T}^S$  such that there exists  $i$  with  $H^i(\text{Sh}_{I,\bar{\eta}}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}} \neq (0)$  for  $I$  maximal at each places outside  $S$ , then  $(\mathbb{T}_\xi^S)_{\mathfrak{m}} \neq (0)$ , i.e. torsion cohomology classes raise in characteristic zero.

The minimal prime ideals of  $\mathbb{T}_\xi^S$  are the prime ideals above the zero ideal of  $\overline{\mathbb{Z}}_l$  and are then in bijection with the prime ideals of  $\mathbb{T}_\xi^S \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ . To such an ideal, which corresponds to give a collection of Satake parameters, is then associated a unique near equivalence class in the sense of [19], denoted by  $\Pi_{\tilde{\mathfrak{m}}}$ , which is the finite set of irreducible automorphic cohomological representations whose multi-set of Satake parameters at each place  $x \in \text{Unr}(I)$ , is given by  $S_{\tilde{\mathfrak{m}}}(x)$  the multi-set of roots of the Hecke polynomial

$$P_{\tilde{\mathfrak{m}},w}(X) := \sum_{i=0}^d (-1)^i q_w^{\frac{i(i-1)}{2}} T_{w,i,\tilde{\mathfrak{m}}} X^{d-i} \in \overline{\mathbb{Q}}_l[X]$$

i.e.

$$S_{\tilde{\mathfrak{m}}}(w) := \{ \lambda \in \mathbb{T}_\xi^S \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l / \tilde{\mathfrak{m}} \simeq \overline{\mathbb{Q}}_l \text{ such that } P_{\tilde{\mathfrak{m}},w}(\lambda) = 0 \}.$$

Thanks to [15] and [19], we denote by

$$\rho_{\tilde{\mathfrak{m}}} : \text{Gal}(\overline{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

the galoisian representation associated to any  $\Pi \in \Pi_{\tilde{\mathfrak{m}}}$ . Recall that the modulo  $l$  reduction of  $\rho_{\tilde{\mathfrak{m}}}$  depends only of  $\mathfrak{m}$ , and was denoted above  $\overline{\rho}_{\mathfrak{m}}$ . For every  $w \in \text{Spl}(I)$ , we also denote by  $S_{\mathfrak{m}}(w)$  the multi-set of modulo  $l$  Satake parameters at  $w$  given as the multi-set of roots of

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^d (-1)^i q_w^{\frac{i(i-1)}{2}} \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_l[X]$$

i.e.

$$S_{\mathfrak{m}}(w) := \{ \lambda \in \mathbb{T}_{\xi}^S / \mathfrak{m} \simeq \overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda) = 0 \}.$$

Using the arguments of [12] and following [17] V.4.4, we then deduce the existence, for  $\text{Gal}_{F,S}$  the Galois group of the maximal extension of  $F$  unramified outside  $S$ , of

$$\rho_{\xi,\mathfrak{m}} : \text{Gal}_{F,S} \longrightarrow GL_d((\mathbb{T}_{\xi}^S)_{\mathfrak{m}})$$

interpolating the  $\rho_{\tilde{\mathfrak{m}}}$ , so that in particular for all  $u \notin S$ ,  $\det(1 - X \text{Frob}_u | \rho_{\mathfrak{m}})$  is equal to the Hecke polynomial.

*Remark.* In [17], the author constructs  $\rho_{\mathfrak{m}} : \text{Gal}_{F,S} \longrightarrow GL_d((\mathbb{T}_{\xi}^S)_{\mathfrak{m}}/J)$  where  $J$  is a nilpotent ideal but it seems from incoming work that one can arrange  $J$  to be zero.

## 2. About the nearby cycle perverse sheaf

Our strategy to compute the cohomology of the KHT-Shimura variety  $\text{Sh}_{I,\overline{\eta}}$  with coefficients in  $V_{\xi,\overline{\mathbb{Z}}_l}$ , is to realize it as the outcome of the nearby cycles spectral sequence at some place  $v \in \text{Spl}$ .

Note that the role of the local system  $V_{\xi,\overline{\mathbb{Z}}_l}$  associated to  $\xi$  is completely harmless when dealing with sheaves: one just have to add a tensor product with it to all the statements without the index  $\xi$ . In the following we will sometimes not mention the index  $\xi$  in the statements to make formulas more readable. Of course when looking at the cohomology groups, the role of  $V_{\xi,\overline{\mathbb{Z}}_l}$  is crucial as it selects the automorphic representations which contribute to the cohomology.

**2.1. The case where the level at  $v$  is maximal.** — By the smooth base change theorem, we have  $H^i(\mathrm{Sh}_{I, \bar{\eta}_v}, V_\xi) \simeq H^i(\mathrm{Sh}_{I, \bar{s}_v}, V_\xi)$ . As for each  $1 \leq h \leq d-1$ , the open Newton stratum  $\mathrm{Sh}_{I, \bar{s}_v}^{\leq h}$  is affine then  $H_c^i(\mathrm{Sh}_{I, \bar{s}_v}^{\leq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h])$  is zero for  $i < 0$  and free for  $i = 0$ . Using this property and the short exact sequence of free perverse sheaves

$$\begin{aligned} 0 \rightarrow i_{h+1,*} V_{\xi, \bar{\mathbb{Z}}_l, |\mathrm{Sh}_{I, \bar{s}_v}^{\geq h+1}}[d-h-1] &\longrightarrow j_!^{\geq h} j^{\geq h,*} V_{\xi, \bar{\mathbb{Z}}_l, |\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}}[d-h] \\ &\longrightarrow V_{\xi, \bar{\mathbb{Z}}_l, |\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}}[d-h] \rightarrow 0, \end{aligned}$$

we then obtain for every  $i > 0$

$$0 \rightarrow H^{-i-1}(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1]) \rightarrow 0, \quad (2.1.1)$$

and for  $i = 0$ ,

$$\begin{aligned} 0 \rightarrow H^{-1}(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) &\longrightarrow H^0(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1]) \longrightarrow \\ H^0(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h,*} V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) &\longrightarrow H^0(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h]) \rightarrow \dots \end{aligned} \quad (2.1.2)$$

In [5], arguing by induction from  $h = d$  to  $h = 1$ , we prove that for a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_\xi^S$  such that  $S_{\mathfrak{m}}(v)$  does not contain any subset of the form  $\{\alpha, q_v \alpha\}$ , all the cohomology groups  $H^i(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  are free: note that in order to deal with  $i \geq 0$ , one has to use the Grothendieck-Verdier duality.

Without this hypothesis, arguing similarly, we conclude that any torsion cohomology class comes from a non strict map

$$H_{free}^0(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h+1}, V_{\xi, \bar{\mathbb{Z}}_l}[d-h-1])_{\mathfrak{m}} \longrightarrow H^0(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h,*} V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}. \quad (2.1.3)$$

In particular it raises in characteristic zero to some free subquotient of  $H^0(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h}, j_!^{\geq h} j^{\geq h,*} V_{\xi, \bar{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$ .

We argue by absurdity and we suppose there exists  $I \in \mathcal{I}$  such that there exists non trivial torsion cohomology classes in the  $\mathfrak{m}$ -localized cohomology of  $\mathrm{Sh}_{I, \bar{\eta}_v}$  with coefficients in  $V_{\xi, \bar{\mathbb{Z}}_l}$ . Fix such finite level  $I$ .

**2.1.4. Proposition.** — *Consider as in [5],  $h_0(I)$  maximal such that there exists  $i \in \mathbb{Z}$  with  $H^{d-h_0(I)+i}(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h_0(I)}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}, tor} \neq (0)$ . Then we have the following properties:*

- $i = 0, 1$ ;
- for all  $1 \leq h \leq h_0(I)$  and  $i < h - h_0(I)$ ,

$$H^{d-h+i}(\mathrm{Sh}_{I, \bar{s}_v}^{\geq h} V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}, \mathrm{tor}} = (0)$$

while for  $i = h - h_0(I)$  it's non trivial.

*Remark.* Note that any system of Hecke eigenvalues  $\mathfrak{m}$  of  $\mathbb{T}_{\xi}^S$  inside the torsion of some  $H^i(\mathrm{Sh}_{I, \bar{\eta}_v}, V_{\xi, \bar{\mathbb{Z}}_l})$  raises in characteristic zero, i.e. is associated to a minimal prime ideal  $\tilde{\mathfrak{m}}$  of  $\mathbb{T}_{\xi}^{S \cup \{v\}}$ . More precisely, using the remark following the proposition 1.3.2, there exists  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  such that the local component at  $v$  of  $\pi_{\tilde{\mathfrak{m}}} \simeq \mathrm{St}_{h_0(I)+1}(\chi_v) \times \chi_{v,1} \times \cdots \times \chi_{v,d-h_0(I)-1}$  where  $\chi_v, \chi_{v,1}, \dots, \chi_{v,d-h_0(I)-1}$  are characters of  $F_v^{\times}$ .

**2.2. Harris-Taylor perverse sheaves over  $\bar{\mathbb{Z}}_l$ .** — Consider now the ideals  $I^v(n) := I^v K_v(n)$  where  $K_v(n) := \mathrm{Ker}(GL_d(\mathcal{O}_v) \rightarrow GL_d(\mathcal{O}_v/\mathcal{M}_v^n))$ . Recall then that  $\mathrm{Sh}_{I^v(n), \bar{s}}^{\leq h}$  is geometrically induced under the action of the parabolic subgroup  $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ , defined as the stabilizer of the first  $h$  vectors of the canonical basis of  $F_v^d$ . Concretely this means there exists a closed subscheme  $\mathrm{Sh}_{I^v(n), \bar{s}_v, \bar{\Gamma}_h}^{\leq h}$  stabilized by the Hecke action of  $P_{h,d}(F_v)$  and such that

$$\mathrm{Sh}_{I^v(n), \bar{s}_v}^{\leq h} \simeq \mathrm{Sh}_{I^v(n), \bar{s}_v, \bar{\Gamma}_h}^{\leq h} \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} GL_d(\mathcal{O}_v/\mathcal{M}_v^n).$$

**2.2.1. Notation.** — For any  $g \in GL_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ , we denote by  $\mathrm{Sh}_{I^v(n), \bar{s}_v, g}^{\leq h}$  the pure Newton stratum defined as the image of  $\mathrm{Sh}_{I^v(n), \bar{s}_v, \bar{\Gamma}_h}^{\leq h}$  by  $g$ .

Let then denote by  $\mathfrak{m}^v$  the multiset of Hecke eigenvalues given by  $\mathfrak{m}$  but outside  $v$  and introduce for  $\Pi_h$  any representation of  $GL_h(F_v)$

$$H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v, \bar{\Gamma}_h}^{\geq h} V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}^v} \otimes \Pi_h := \varinjlim_n H^i(\mathrm{Sh}_{I^v(n), \bar{s}_v, \bar{\Gamma}_h}^{\geq h} V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}^v} \otimes \Pi_h,$$

as a representation of  $GL_h(F_v) \times GL_{d-h}(F_v)$ , where  $g \in GL_h(F_v)$  acts both on  $\Pi_h$  and on  $H^i(\mathrm{Sh}_{I^v(n), \bar{s}_v, \bar{\Gamma}_h}^{\geq h} V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}^v}$  through the determinant map  $\det : GL_h(F_v) \rightarrow F_v^{\times}$ . Note moreover that the unipotent radical of  $P_{h,d}(F_v)$  acts trivially on these cohomology groups and introduce the induced version

$$H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}^{\geq h}, \Pi_h \otimes V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}^v} \simeq \mathrm{ind}_{P_{h,d}(F_v)}^{GL_d(F_v)} H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v, \bar{\Gamma}_h}^{\geq h} V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}^v} \otimes \Pi_h.$$

More generally, with the notations of [2], replace now the trivial representation by an irreducible cuspidal representation  $\pi_v$  of  $GL_g(F_v)$  for some  $1 \leq g \leq d$ .

**2.2.2. Notations.** — Let  $1 \leq t \leq s := \lfloor d/g \rfloor$  and  $\Pi_t$  any representation of  $GL_{d-tg}(F_v)$ . We then denote by

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

the Harris-Taylor local system on the Newton stratum  $\mathrm{Sh}_{I, \overline{s}, \overline{1_{tg}}}^{\overline{tg}}$  where

- $\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}}$  is defined thanks to Igusa varieties attached to the representation  $\pi_v[t]_D$  of the division algebra of dimension  $(tg)^2$  over  $F_v$  associated to  $\mathrm{St}_t(\pi_v)$  by the Jacquet-Langlands correspondence,
- $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$  defined by  $\Xi(\frac{1}{2}) = q^{1/2}$ .

We also introduce the induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \left( \mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \right) \times_{P_{tg,d}(F_v)} GL_d(F_v),$$

where the unipotent radical of  $P_{tg,d}(F_v)$  acts trivially and the action of

$$(g^{\infty, v}, \left( \begin{array}{cc} g_v^c & * \\ 0 & g_v^{et} \end{array} \right), \sigma_v) \in G(\mathbb{A}^{\infty, v}) \times P_{tg,d}(F_v) \times W_v$$

is given

- by the action of  $g_v^c$  on  $\Pi_t$  and  $\deg(\sigma_v) \in \mathbb{Z}$  on  $\Xi^{\frac{tg-d}{2}}$ , and
- the action of  $(g^{\infty, v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty, v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$  on  $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Xi^{\frac{tg-d}{2}}$ .

We also introduce

$$HT(\pi_v, \Pi_t)_{\overline{1_{tg}}} := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1_{tg}}}[d - tg],$$

and the perverse sheaf

$$P(t, \pi_v)_{\overline{1_{tg}}} := j_{1, !}^{\overline{tg}} HT(\pi_v, \mathrm{St}_t(\pi_v))_{\overline{1_{tg}}} \otimes \mathbb{L}(\pi_v),$$

and their induced version,  $HT(\pi_v, \Pi_t)$  and  $P(t, \pi_v)$ , where

$$j^{\overline{h}} = i^{\overline{h}} \circ j^{\geq \overline{h}} : \mathrm{Sh}_{I, \overline{s}}^{\overline{h}} \hookrightarrow \mathrm{Sh}_{I, \overline{s}}^{\geq \overline{h}} \hookrightarrow \mathrm{Sh}_{I, \overline{s}}$$

and  $\mathbb{L}^\vee$ , the dual of  $\mathbb{L}$ , is the local Langlands correspondence. Finally we will also use the indice  $\xi$  in the notations, for example  $HT_\xi(\pi_v, \Pi_t)$ , when we twist the sheaf with  $V_{\xi, \overline{\mathbb{Z}}_l}$ .



With the previous notations, from (1.1.6), we deduce the following equality in the Grothendieck group of Hecke-equivariant local systems

$$m(\varrho)l^u \left[ \mathbb{F}\mathcal{L}_{\xi, \bar{\mathbb{Z}}_l}(\pi_{v,u}[t]_D) \right] = \left[ \mathbb{F}\mathcal{L}_{\xi, \bar{\mathbb{Z}}_l}(\pi_{v,-1}[tm(\varrho)l^u]_D) \right]. \quad (2.2.3)$$

We want now to focus on the perverse Harris-Taylor sheaves. Note first that over  $\bar{\mathbb{Z}}_l$ , there are two notions of intermediate extension associated to the two classical  $t$ -structures  $p$  and  $p+$ . So for every  $\pi_v \in \text{Cusp}_\varrho$  of  $GL_g(F_v)$  and  $1 \leq t \leq d/g$ , we can define:

$${}^p j_{!*}^{-tg} HT(\pi_v, \Pi_t) \hookrightarrow_+ {}^{p+} j_{!*}^{-tg} HT(\pi_v, \Pi_t), \quad (2.2.4)$$

the symbol  $\hookrightarrow_+$  meaning bimorphism, i.e. both a monomorphism and epimorphism, so that the cokernel for the  $t$ -structure  $p$  (resp. the kernel for  $p+$ ) has support in  $\text{Sh}_{I, \bar{s}_v}^{\geq tg+1}$ . When  $\pi_v$  is a character, i.e. when  $g = 1$ , the associated bimorphisms are isomorphisms, as explained in the following lemma, but in general there are not.

**2.2.5. Lemma.** — *With the previous notations, we have an isomorphism*

$${}^p j_{I_h, !*}^{\geq h} HT(\chi_v, \Pi_h) \simeq {}^{p+} j_{I_h, !*}^{\geq h} HT(\chi_v, \Pi_h).$$

*Proof.* — Recall that  $\text{Sh}_{I, \bar{s}_v, \bar{1}_h}^{\geq h}$  is smooth over  $\text{Spec } \bar{\mathbb{F}}_p$ . As, up to a modification of the action of the fundamental group through the character  $\chi_v$ , we have

$$HT(\chi_v, \Pi_h)_{\bar{1}_h}[h-d] = (\bar{\mathbb{Z}}_l)_{|\text{Sh}_{I, \bar{s}_v, \bar{1}_h}^{\geq h}} \otimes \Pi_h.$$

Then  $HT(\chi_v, \Pi_h)_{\bar{1}_h}$  is perverse for the two  $t$ -structures with

$$i_{\bar{1}_h}^{h \leq +1, *} HT(\chi_v, \Pi_h)_{\bar{1}_h} \in {}^p \mathcal{D}^{<0} \quad \text{and} \quad i_{\bar{1}_h}^{h \leq +1, !} HT(\chi_v, \Pi_h)_{\bar{1}_h} \in {}^{p+} \mathcal{D}^{\geq 1}.$$

□

*Remark.* One of the main result of [7], is to prove that the previous lemma holds for any  $\pi_v \in \text{Cusp}_\varrho(-1)$ .

As explained in the introduction, with the hypothesis on the order of  $q_v$  modulo  $l$  which is supposed to be strictly greater than  $d$ , for  $\varrho$  the trivial representation, we do not need to bother about the  $\pi_v \in \text{Cusp}_\varrho(u)$  for  $u \geq 0$ , cf. the remark after 1.1.4. However the next proposition tells us that you can easily express the cohomology of the Harris-Taylor perverse sheaves associated to these  $\pi_v \in \text{Cusp}_\varrho(u)$  for  $u \geq 0$ , in terms of

those of associated to characters. So the hypothesis on the order of  $q_v$  is not really crucial to overcome this difficulty, see for example the lemma 3.1.13.

**2.2.6. Proposition.** — (cf. proposition 2.4.2 of [7]) Let  $\mathbb{F}(\bullet) := \bullet \otimes_{\mathbb{Z}_l}^{\mathbb{L}}$   $\overline{\mathbb{F}}_l$  be the modulo  $l$  reduction functor. We then have the following equality in the Grothendieck group of Hecke-equivariant perverse sheaves

$$\mathbb{F}\left({}^p j_{1*}^{:tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)\right) = m(\varrho) l^u \sum_{r=0}^{s-tm(\varrho)l^u} {}^p j_{1*}^{:tg_u(\varrho)+rg_{-1}(\varrho)} HT(\varrho, r_l(\Pi_t) \overrightarrow{\times} V_\varrho(r, < \underline{\delta}_u)) \otimes \Xi^{r \frac{g-1}{2}},$$

where  $V_\varrho(r, < \underline{\delta}_u)$  is defined in loc. cit. as the sum of irreducible constituents of the modulo  $l$  reduction of  $\text{St}_r(\pi_{v,-1})$  of  $\varrho$ -level strictly less than  $\underline{\delta}_u := (0, \dots, 0, 1, 0, \dots)$ , cf. §A.2 of [7]

*Remark.* The proof only uses the fact that  $\mathbb{F}$  and  ${}^p j_!$  commutes and the property that the bimorphism (2.2.4) is an isomorphism for  $\pi_v$  a character. Note moreover that the equality of the previous proposition remains true if we replace  ${}^p j_{1*}^{:tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)$  by any perverse sheaf  $P$  such that

$${}^p j_{1*}^{:tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t) \hookrightarrow_+ P \hookrightarrow_+ {}^{p+} j_{1*}^{:tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t).$$

When we look at  $\mathbb{F}P$  for such  $P$ , they differ from each of others only in the order of the Jordan-Holder factors: for example they appear in the increasing (resp. decreasing) order of the dimension of their support for  ${}^p j_{1*}^{:tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)$  (resp. for  ${}^{p+} j_{1*}^{:tg_u(\varrho)} HT(\pi_{v,u}, \Pi_t)$ ).

**2.3. Filtrations of the nearby cycles perverse sheaf.** — Let denote by

$$\Psi_{I,v} := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])\left(\frac{d-1}{2}\right)$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber  $\text{Sh}_{I,\bar{s}_v}$  of  $\text{Sh}_I$ . We also denote by  $\Psi_{I,\xi,v} := \Psi_{I,v} \otimes V_{\xi,\overline{\mathbb{Z}}_l}$ .

Using the Newton stratification and following the constructions of [8], we can define a  $\overline{\mathbb{Z}}_l$ -filtration  $\text{Fill}^\bullet(\Psi_{I,v})$  whose graded parts are free, isomorphic to some free perverse Harris-Taylor sheaf. Moreover in [7] proposition 3.1.3, we proved the following splitting

$$\Psi_{I,v} \simeq \bigoplus_{g=1}^d \bigoplus_{\varrho \in \text{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_{I,\varrho},$$

where  $\text{Scusp}_{\overline{\mathbb{F}}_l}(g)$  is the set of inertial equivalence classes of irreducible  $\overline{\mathbb{F}}_l$ -supercuspidal representations of  $GL_g(F_v)$ , with the property that, with the notations of [2], the irreducible sub-quotients of

$$\Psi_{I,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \text{Cusp}_\varrho} \Psi_{I,\pi_v}$$

are exactly the perverse Harris-Taylor sheaves, of level  $I$ , associated to an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representations of some  $GL_g(F_v)$  such that the supercuspidal support of the modulo  $l$  reduction of  $\pi_v$  is a segment associated to the inertial class  $\varrho$ .

*Remark.* In [7], we proved that if you always use the adjunction maps  $j_1^{=h} j^{=h,*} \rightarrow \text{Id}$  then all the previous graded parts of  $\Psi_{I,\varrho}$  are isomorphic to  $p$ -intermediate extensions. In the following we will only consider the case where  $\varrho$  is a character in which case, cf. lemma 2.2.5, the  $p$  and  $p+$  intermediate extensions associated to character  $\chi_{v,-1} \in \text{Cusp}_\varrho(-1)$ , coincides. Note that in the following we will not use the results of [7].

Denoting by  $\text{grr}^k(\Psi_{I,\xi,v}) := \text{Fill}^k(\Psi_{I,\xi,v}) / \text{Fill}^{k-1}(\Psi_{I,\xi,v})$ , we then have a spectral sequence

$$E_1^{p,q} = H^{p+q}(\text{Sh}_{I,\bar{s}_v}, \text{grr}^{-p}(\Psi_{I,\xi,v})) \Rightarrow H^{p+q}(\text{Sh}_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l}), \quad (2.3.1)$$

where we recall that

$$\begin{aligned} {}^p j_{!*}^{=tg} HT_\xi(\pi_v, \text{St}_t(\pi_v)) \left( \frac{1-t+2i}{2} \right) \hookrightarrow_+ \text{grr}^k(\Psi_{I,\xi,v}) \hookrightarrow_+ \\ {}^{p+} j_{!*}^{=tg} HT_\xi(\pi_v, \text{St}_t(\pi_v)) \left( \frac{1-t+2i}{2} \right), \end{aligned} \quad (2.3.1)$$

for some irreducible cuspidal representation  $\pi_v$  of  $GL_g(F_v)$  with  $1 \leq t \leq d/g$  and  $0 \leq i \leq \lfloor d/g \rfloor - 1$ .

Let consider now the filtration of stratification of  $\Psi_{I,\xi,\varrho}$  constructed using the adjunction morphisms  $j_!^{-t}j^{=t,*}$  as in [4]

$$\mathrm{Fil}_!^0(\Psi_{I,\xi,\varrho}) \hookrightarrow \mathrm{Fil}_!^1(\Psi_{I,\xi,\varrho}) \hookrightarrow \mathrm{Fil}_!^2(\Psi_{I,\xi,\varrho}) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}_!^d(\Psi_{I,\xi,\varrho})$$

where  $\mathrm{Fil}_!^t(\Psi_{I,\xi,\varrho})$  is the saturated image of  $j_!^{-t}j^{=t,*}\Psi_{I,\xi,\varrho} \longrightarrow \Psi_{I,\xi,\varrho}$ . We then denote by  $\mathrm{gr}_{I,\xi,\varrho,!}^k$  the graded parts and

$$E_{!,\varrho,1}^{p,q} = H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{gr}_{I,\xi,\varrho,!}^{-p}) \Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar{\eta}_v}, \Psi_{I,\xi,\varrho}). \quad (2.3.2)$$

**2.4. Local behavior of the monodromy over  $\bar{\mathbb{F}}_l$ .** — We suppose now  $l \geq d$  so that<sup>(3)</sup> the nilpotent monodromy operator  $N_v$  at  $v$  is defined over  $\bar{\mathbb{Z}}_l$

We moreover suppose that the order of  $q_v$  modulo  $l$  is strictly greater than  $d$  so that for any  $\bar{\mathbb{F}}_l$ -character  $\varrho$  of  $F_v^\times$ , the irreducible sub-quotients of  $\Psi_{I,\varrho} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$  are Harris-Taylor perverse sheaves associated to a character  $\chi_v \in \mathrm{Cusp}_\varrho(-1)$ . In Iwahori level, i.e. when  $I_v$  is upper triangular modulo  $\varpi_v$ , from the Rapoport-Zink description of the nearby cycles, cf. [10] lemma 3.1.7,  $N_v$  induces isomorphisms

$$N_v : \mathcal{P}_{I,\bar{\mathbb{Z}}_l}(t, \chi_v)\left(\frac{-r}{2}\right) \longrightarrow \mathcal{P}_{I,\bar{\mathbb{Z}}_l}(t, \chi_v)\left(\frac{2-r}{2}\right) \quad (2.4.1)$$

for all characters  $\chi_v \in \mathrm{Cusp}_\varrho(-1)$ , all  $1 \leq t \leq d$  and all  $3-t \leq r \leq t-1$  with  $r \equiv t-1 \pmod{2}$ .

Note that with our hypothesis that the order of  $q_v$  is  $> d$ , then as for any  $1 \leq t \leq d$ , the modulo  $l$  reduction of  $\mathrm{St}_t(\chi_v)$  is irreducible, then the Harris-Taylor local systems  $HT(t, \mathrm{St}_t(\chi_v))$  have, up to isomorphism, only one stable  $\bar{\mathbb{Z}}_l$ -lattice. In particular whatever is the level  $I$ , then  $N_v$  in (2.4.1), induces a homothety which should have to be an isomorphism as it is for Iwahori level at  $v$ .

*Remark.* It is tempting to conjecture that (2.4.1) is always an isomorphism so that it should be possible to remove the hypothesis on  $l$  in the main theorem of this paper.

<sup>(3)</sup>Note that in the Taylor serie of  $\ln(1-x)$  applied to a nilpotent operator  $x$  with  $x^d = 0$ , only the first  $d$  terms contribute where all the denominator are inversible in  $\bar{\mathbb{Z}}_l$ .

### 3. Irreducibility implies freeness

Recall that we argue by absurdity by assuming that there exists non trivial cohomology classes in some of  $H^i(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\bar{\mathbb{Z}}_l})_{\mathfrak{m}}$ . The strategy is then to choose a place  $v \in \mathrm{Spl}_I$  such that the order of  $q_v$  modulo  $l$  is strictly greater than  $d$ , and to compute the middle cohomology group  $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  through the spectral sequence of vanishing cycles. This spectral sequence gives us in particular a filtration of the free quotient of  $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  where we can easily read the action of the monodromy operator  $N_v$  at  $v$ . We will in fact consider different sorts of level  $I$  relatively to the place  $v$  and another one denoted  $w$  verifying the same hypothesis as  $v$ :

- either with infinite level at  $v$  or with  $I_v$  some particular Iwahori subgroup;
- either maximal or infinite at  $w$ .

By hypothesis, there exists non trivial torsion cohomology classes in level  $I$  with  $I_v$  and  $I_w$  maximal. As moreover we supposed the order of both  $q_v$  and  $q_w$  to be strictly greater than  $d$ , then the functors of invariants by any open compact subgroups either at  $v$  or  $w$ , is exact. Thus this allows us to argue similarly with all the mentioned level, cf. lemma 3.1.11. Let now explain the main steps of the following sections.

- We first analyse, following the arguments of the previous section, torsion cohomology classes of Harris-Taylor perverse sheaves and we deduce, cf. lemma 3.1.13, that, as  $\overline{\mathbb{F}}_l$ -representations, irreducible sub-quotients of  $l$ -torsion of their cohomology with higher non degeneracy level, appears in degree 0, 1.
- In §3.2, we analyse, following the previous strategy, the torsion cohomology classes of the graded parts  $\mathrm{gr}_!^t(\Psi_\rho)$  of the filtration of stratification constructed using the adjunction property  $j_!^{=t} j^{=t,*} \rightarrow \mathrm{Id}$ . We then deduce, cf. lemma 3.2.5, that the  $l$ -torsion of  $H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, V_{\xi,\bar{\mathbb{Z}}_l})_{\mathfrak{m}^v}$  does not have any irreducible generic subquotient whose supercuspidal support is made of characters.
- In section 3.3, we obtain two fundamental results.
  - First, cf. lemma 3.3.2, under the hypothesis that there exists non trivial torsion cohomology classes, we show that

the graded pieces  $\Gamma_k$  of the filtration of the free quotient of  $H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, V_{\xi, \bar{\mathbb{Z}}_l})_{\mathfrak{m}^v}$  are not always given by the lattice of  $H^0(\mathrm{Sh}_{I, \bar{s}_v}, \mathcal{P}(t, \chi_v)^{\frac{1-t+2\delta}{2}})_{\mathfrak{m}} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$  given by the entire cohomology of  $\mathcal{P}(t, \chi_v)$ . Roughly there exists some  $k$  with  $\Gamma := \Gamma_k$  and a short exact sequence  $\Gamma_0 \hookrightarrow \Gamma \twoheadrightarrow T$  where  $\Gamma_0$  is the lattice given by the entire cohomology of the associated Harris-Taylor perverse sheaf, and  $T$  is non trivial and torsion.

- We then play with the action of  $GL_d(F_w)$  by allowing infinite level at  $w$ . The main observation at the end of the section, cf. proposition 3.3.8, is that as a  $\bar{\mathbb{F}}_l$ -representation of  $GL_d(T_w)$ , all the irreducible sub-quotients of the  $l$ -torsion of  $T$ , up to multiplicities, are also sub-quotients of the  $l$ -torsion of the global cohomology. In particular, as  $v$  and  $w$  are playing symmetric roles, these sub-quotients are not generic, cf. corollary 3.3.9.
- Finally in §3.4, using theorem 3.4.1 which is a direct generalization of a classical result of Carayol, and playing with the monodromy operator, we prove that the  $l$ -torsion of some of the cokernels  $T$  of the actual lattices of the free quotient of  $H^0(\mathrm{Sh}_{I^w(\infty), \bar{\eta}}, V_{\bar{\mathbb{Z}}_l, \xi})_{\mathfrak{m}^{v,w}}$  relatively to the natural lattices given by the  $H^0(\mathrm{Sh}_{I^{v,w}(\infty), \bar{\eta}}, \mathcal{P}_{\xi}(t, \chi_v))_{\mathfrak{m}^{v,w}}$ , should contain an irreducible generic sub-quotient as a  $\bar{\mathbb{F}}_l[GL_d(F_w)]$ -module.

The last point gives us the expected contradiction so that  $H^0(\mathrm{Sh}_{I, \bar{\eta}}, V_{\bar{\mathbb{Z}}_l, \xi})_{\mathfrak{m}}$  has to be torsion free.

**3.1. Torsion cohomology classes of Harris-Taylor perverse sheaves.**— We focus now on the torsion in the cohomology groups of the Harris-Taylor perverse sheaves  $\mathcal{P}_{\xi}(\chi_v, t)$  when the level at  $v$  is infinite.

**3.1.1. Notation.** — We will denote by  $I^v(\infty) \in \mathcal{I}$  a finite<sup>(4)</sup> level outside  $v$ , and let  $\mathfrak{m}^v$  be the maximal ideal of  $\mathbb{T}_{\xi}^{S \cup \{v\}}$  associated to  $\mathfrak{m}$ . We also denote by

$$H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \bar{\mathbb{Z}}_l)_{\mathfrak{m}^v} := \varinjlim_{I_v} H^i(\mathrm{Sh}_{I, \bar{s}_v}, \bar{\mathbb{Z}}_l)_{\mathfrak{m}^v},$$

<sup>(4)</sup>and morally infinite at  $v$

which can be viewed as a  $\overline{\mathbb{Z}}_l[GL_d(F_v)]$ -module.

Start first from the following resolution of  ${}^p j_{!*}^{=tg} HT(\chi_v, \Pi_t)$

$$\begin{aligned} 0 \rightarrow j_!^{=d} HT(\chi_v, \Pi_t \left\{ \frac{t-s}{2} \right\}) \times \text{Speh}_{d-t}(\chi_v \{t/2\}) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \dots \\ \longrightarrow j_!^{=t+1} HT(\chi_v, \Pi_t \{-1/2\} \times \chi_v \{t/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow \\ j_!^{=t} HT(\chi_v, \Pi_t) \longrightarrow {}^p j_{!*}^{=t} HT(\chi_v, \Pi_t) \rightarrow 0. \end{aligned} \quad (3.1.2)$$

*Remark.* This result is proved in full generality over  $\overline{\mathbb{Q}}_l$  for any irreducible cuspidal representation in replacement of  $\chi_v$ . Over  $\overline{\mathbb{Z}}_l$ , it's proved in [7] for any irreducible representation cuspidal representation. In the case of a character, the argument is trivial as we just have to notice that the strata  $\text{Sh}_{F_v, \bar{s}_v, 1}^{\geq h}$  are smooth so that the constant sheaf, up to shift, is perverse and so equals to the intermediate extension of the constant sheaf, shifted by  $d-h$ , on  $\text{Sh}_{F_v, \bar{s}_v, 1}^=h$ . The previous resolution is then just the induced version of this.

By the adjunction property, the map

$$\begin{aligned} j_!^{=t+\delta} HT(\chi_v, \Pi_t \left\{ \frac{-\delta}{2} \right\}) \times \text{Speh}_{\delta}(\chi_v \{t/2\}) \otimes \Xi^{\delta/2} \\ \longrightarrow j_!^{=t+\delta-1} HT(\chi_v, \Pi_t \left\{ \frac{1-\delta}{2} \right\}) \times \text{Speh}_{\delta-1}(\chi_v \{t/2\}) \otimes \Xi^{\frac{\delta-1}{2}} \end{aligned} \quad (3.1.3)$$

is given by

$$\begin{aligned} HT(\chi_v, \Pi_t \left\{ \frac{-\delta}{2} \right\}) \times \text{Speh}_{\delta}(\chi_v \{t/2\}) \otimes \Xi^{\delta/2} \longrightarrow \\ p_i^{t+\delta, !} j_!^{=t+\delta-1} HT(\chi_v, \Pi_t \left\{ \frac{1-\delta}{2} \right\}) \times \text{Speh}_{\delta-1}(\chi_v \{t/2\}) \otimes \Xi^{\frac{\delta-1}{2}} \end{aligned} \quad (3.1.4)$$

From [7], we have

$$\begin{aligned} p_i^{t+\delta, !} j_!^{=t+\delta-1} HT(\chi_v, \Pi_t \left\{ \frac{1-\delta}{2} \right\}) \times \text{Speh}_{\delta-1}(\chi_v \{t/2\}) \otimes \Xi^{\frac{\delta-1}{2}} \\ \simeq HT\left(\chi_v, \Pi_t \left\{ \frac{1-\delta}{2} \right\}\right) \times \left(\text{Speh}_{\delta-1}(\chi_v \{-1/2\}) \times \chi_v \left\{ \frac{\delta-1}{2} \right\} \{t/2\}\right) \otimes \Xi^{\delta/2} \end{aligned} \quad (3.1.5)$$

*Fact.* In particular, up to homothety, the map (3.1.5), and so those of (3.1.4), is unique. Finally as the map of (3.1.2) are strict, the given

maps (3.1.3) are uniquely determined, that is if we forget the infinitesimal parts, these maps are independent of the chosen  $t$  in (3.1.2).

We want now to copy the arguments of §2.1.

**3.1.6. Notation.** — For every  $1 \leq h \leq d$ , let denote by  $i_{I^v}(h)$  the smaller index  $i$  such that  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, {}^p j_{I^v, *}{=}^h HT_\xi(\chi_v, \Pi_h))_{\mathfrak{m}^v}$  has non trivial torsion: if it doesn't exists then set  $i_{I^v}(h) = +\infty$ .

Let first state two remarks.

- By duality, as  ${}^p j_{I^v, *}{=}^h = {}^{p+} j_{I^v, *}{=}^h$  for Harris-Taylor local systems associated to character, note that when  $i_I(h)$  is finite then  $i_{I^v}(h) \leq 0$ .
- Using the classical determinant map, it is in fact independent of the character  $\chi_v$ . Note also that we do not really need this fact in the following.

**3.1.7. Notation.** — Suppose there exists  $I \in \mathcal{I}$  such that there exists  $1 \leq h \leq d$  with  $i_{I^v}(h)$  finite and denote by  $h_0(I^v)$  the bigger such  $h$ .

**3.1.8. Lemma.** — For  $1 \leq h \leq h_0(I^v)$  then  $i_{I^v}(h) = h - h_0(I^v)$ .

*Proof.* — Note first that for every  $h_0(I^v) \leq h \leq s$ , then the cohomology groups of  $j_{I^v, *}{=}^h HT_\xi(\chi_v, \Pi_h)$  are torsion free. Indeed there exists a filtration

$$(0) = \mathrm{Fil}^0(\chi_v, h) \hookrightarrow \mathrm{Fil}^{-d}(\chi_v, h) \hookrightarrow \dots \hookrightarrow \mathrm{Fil}^{-h}(\chi_v, h) = j_{I^v, *}{=}^h HT(\chi_v, \Pi_h)$$

with graded parts

$$\mathrm{gr}^{-k}(\chi_v, h) \simeq {}^p j_{I^v, *}{=}^k HT(\chi_v, \Pi_h\{\frac{h-k}{2}\}) \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\})\left(\frac{h-k}{2}\right).$$

The  $\xi$ -associated spectral sequence localized at  $\mathfrak{m}^v$  is then concentrated in middle degree and torsion free. Then the spectral sequence associated to (3.1.2) has all its  $E_1$  terms torsion free and degenerates at its  $E_1$  terms.

Consider then the spectral sequence associated to the resolution (3.1.2): its  $E_1$  terms are torsion free and it degenerates at  $E_2$ . As by hypothesis the aims of this spectral sequence is free and equals to only



one  $E_2$  terms, we deduce that all the maps

$$\begin{aligned} H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{=h+\delta} HT_\xi(\chi_v, \Pi_h\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{t/2\})) \otimes \Xi^{\delta/2} \Big|_{\mathfrak{m}^v} \longrightarrow \\ H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{=h+\delta-1} HT_\xi(\chi_v, \Pi_h\{\frac{1-\delta}{2}\}) \\ \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \Big|_{\mathfrak{m}^v} \end{aligned} \quad (3.1.9)$$

are strict. Then from the previous fact stressed after (3.1.5), this property remains true when we consider the associated spectral sequence for  $1 \leq h' \leq h_0$ .

Consider now  $h = h_0(I^v)$  where we know the torsion to be non trivial. From what was observed above we then deduce that the map

$$\begin{aligned} H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{=h_0(I^v)+1} HT_\xi(\chi_v, \Pi_{h_0(I^v)}\{\frac{-1}{2}\}) \times \chi_v\{h_0(I^v)/2\}) \otimes \Xi^{1/2} \Big|_{\mathfrak{m}^v} \\ \longrightarrow H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{=h_0(I^v)} HT_\xi(\chi_v, \Pi_{h_0(I^v)})) \Big|_{\mathfrak{m}^v} \end{aligned} \quad (3.1.10)$$

has a non trivial torsion cokernel so that  $i_I(h_0(I^v)) = 0$ .

Finally for any  $1 \leq h \leq h_0$ , the map like (3.1.10) for  $h + \delta - 1 < h_0$  are strict so that the  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_{!*}^{=h} HT_\xi(\chi_v, \Pi_h)) \Big|_{\mathfrak{m}^v}$  are zero for  $i < h - h_0$  while when  $h + \delta - 1 = h_0$  its cokernel has non trivial torsion which gives then a non trivial torsion class in  $H^{h-h_0}(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_{!*}^{=h} HT_\xi(\chi_v, \Pi_h)) \Big|_{\mathfrak{m}^v}$ .  $\square$

**3.1.11. Lemma.** — *With the notation of 2.1.4, we have  $h_0(I^v) \geq h_0(I)$ .*

*Proof.* — Consider the previous map (3.1.10) by replacing  $h_0(I^v)$  by  $h_0(I)$ . As by hypothesis the order of  $q_v$  modulo  $l$  is strictly greater than  $q$ , then the pro-order of the local component  $I_v$  of  $I$  at  $v$ , is invertible modulo  $l$ , so that the functor of invariants under  $I_v$  is exact. Note then that, as the  $I_v$ -invariants of the map (3.1.10) when replacing  $h_0(I^v)$  by  $h_0(I)$ , has a cokernel which is not free, then the cokernel of (3.1.10), for  $h_0(I)$ , is also not free.  $\square$

From the previous proof, we also deduce that all cohomology classes of any of the  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathcal{P}_\xi(t, \chi_v)) \Big|_{\mathfrak{m}^v}$  comes from non strictness of some

of the map (3.1.10) where  $\Pi_v := \text{St}_t(\chi_v)$ . In the following we will focus on  $H^i(\text{Sh}_{I^v(\infty), \bar{s}_v}, \mathcal{P}_\xi(t, \chi_v))_{\mathfrak{m}^v}[l]$  as a  $\overline{\mathbb{F}}_l$ -representation of  $GL_d(F_v)$ . More precisely we are interested in irreducible such sub-quotients which have maximal non-degeneracy level at  $v$ .

**3.1.12. Notation.** — Let first fix such non degeneracy level  $\underline{\lambda}$  for  $GL_d(F_v)$  in the sense of notation 1.1.9, which is maximal for torsion classes in  $H^0(\text{Sh}_{I^v(\infty), \bar{s}_v}, \mathcal{P}_\xi(t, \chi_v))_{\mathfrak{m}^v}[l]$  for various  $1 \leq t \leq d$  and  $\chi_v \in \text{Cusp}_\rho(-1)$ .

*Remark.* As mentioned after 3.1.6, for the definition of  $\lambda$ , you could also consider any fixed  $\chi_v \in \text{Cusp}_\rho(-1)$ .

**3.1.13. Lemma.** — Let  $\rho$  be a  $\overline{\mathbb{F}}_l$ -character of  $F_v^\times$  and  $\pi_v \in \text{Cusp}_\rho$ . Then all  $\overline{\mathbb{F}}_l[GL_d(F_v)]$ -irreducible sub-quotients of  $H^i(\text{Sh}_{I^v(\infty), \bar{s}_v}, \mathcal{P}_\xi(t, \pi_v))_{\mathfrak{m}^v}[l]$ , for  $i \neq 0, 1$ , have a level of non degeneracy strictly less than  $\underline{\lambda}$ .

*Remark.* Recall that we only know, a priori, that the  $\mathcal{P}(t, \pi_v)$  only verify

$${}^p j_{l,*}^{-tg} HT(\pi_v, \text{St}_t(\pi_v)) \hookrightarrow_+ \mathcal{P}(t, \pi_v) \hookrightarrow_+ {}^{p+} j_{l,*}^{-tg} HT(\pi_v, \text{St}_t(\pi_v)).$$

*Proof.* — If  $\pi_v \in \text{Cusp}_\rho(-1)$  is a character then the result follows from above. In particular, as the  $\mathfrak{m}$ -localized  $\overline{\mathbb{Q}}_l$ -cohomology is concentrated in middle degree, the result remains true for  ${}^p j_{l,*}^{-t} HT_{\xi, \overline{\mathbb{F}}_l}(\rho, \Pi_t)[d-t]$ , i.e. if  $H^i(\text{Sh}_{I^v(\infty), \bar{s}_v}, {}^p j_{l,*}^{-t} HT_{\xi, \overline{\mathbb{F}}_l}(\rho, \Pi_t)[d-t])_{\mathfrak{m}^v}$  has an irreducible subquotient with level of non-degeneracy greater or equal to  $\underline{\lambda}$  then  $i = 0, 1$ .<sup>(5)</sup> We then conclude using the equality of proposition 2.2.6 and its following remark.  $\square$

**3.2. Global torsion and genericity.** — Recall that  $v \in \text{Spl}$  is such that the order of  $q_v$  modulo  $l$  is strictly greater than  $d$ . Let denote by  $I^v$  the component of  $I$  outside  $v$ . We then simply denote by  $\Psi_v$  and  $\Psi_{v, \xi}$ , the inductive system of perverse sheaves indexed by the finite level  $I^v I_v \in \mathcal{I}$  for varying  $I_v$ .

For  $\pi_v \in \text{Cusp}_\rho$ , let denote by

$$\text{Fil}_{l, \pi_v}^1(\Psi_v) \hookrightarrow \text{Fil}_l^1(\Psi_\rho)$$

<sup>(5)</sup>Recall that the  $\overline{\mathbb{Q}}_l$ -cohomology groups localized at  $\mathfrak{m}$  are concentrated in degree  $i = 0$ .

such that  $\mathrm{Fil}_{!,\pi_v}^1(\Psi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathrm{Fil}_!^1(\Psi_{\pi_v})$  where  $\Psi_{\pi_v}$  is the direct factor of  $\Psi_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  associated to  $\pi_v$ , cf. [4].

*Remark.* In the following, we will mainly be concerned with the case where  $\pi_v$  is a character  $\chi_v$ . We will then write the main statement in this case.

From the main result of [7], which can also be deduced easily from [16] and the comparison theorem of Faltings-Fargues cf. [14], we have then the following resolution of  $\mathrm{Fil}_{!,\chi_v}^1(\Psi)$

$$\begin{aligned} 0 \rightarrow j_!^{\bar{d}} HT(\chi_v, \mathrm{Speh}_d(\chi_v)) \otimes \mathbb{L}(\chi_v(\frac{d-1}{2})) \longrightarrow \\ j_!^{\bar{d}-1} HT(\chi_v, \mathrm{Speh}_{d-1}(\chi_v)) \otimes \mathbb{L}(\chi_v(\frac{d-2}{2})) \longrightarrow \\ \cdots \longrightarrow j_!^{\bar{1}} HT(\chi_v, \chi_v) \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{Fil}_{!,\chi_v}^1(\Psi) \rightarrow 0 \end{aligned} \quad (3.2.1)$$

As in notation 3.1.1, let  $\mathfrak{m}^v$  be the maximal ideal of  $\mathbb{T}_\xi^{S \cup \{v\}}$  associated to  $\mathfrak{m}$ . We can then apply the arguments of the previous section so that  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{Fil}_{!,\chi_v}^1(\Psi_{v,\xi}))_{\mathfrak{m}^v}$  has non trivial torsion for  $i = 1 - t_0$  and with free quotient zero for  $i \neq 0$ . Clearly we can also repeat the same arguments for the other  $\mathrm{gr}_{!,\chi_v}^t(\Psi) \hookrightarrow \mathrm{gr}_!^t(\Psi_\rho)$  with

$$\begin{aligned} 0 \rightarrow j_!^{\bar{d}} HT(\chi_v, LT_{\chi_v}(t-1, d-t)) \otimes \mathbb{L}(\chi_v(\frac{d-2t+1}{2})) \longrightarrow \\ j_!^{\bar{d}-1} HT(\chi_v, LT_{\chi_v}(t-1, d-t-1)) \otimes \mathbb{L}(\chi_v(\frac{d-2t}{2})) \longrightarrow \\ \cdots \longrightarrow j_!^{\bar{t}} HT(\chi_v, \mathrm{St}_t(\chi_v)) \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{gr}_{!,\chi_v}^t(\Psi) \rightarrow 0. \end{aligned} \quad (3.2.2)$$

Finally all the torsion cohomology classes of the  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{gr}_{!,\chi_v}^t(\Psi))_{\mathfrak{m}^v}$  come from the non strictness of the maps

$$H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{\bar{h}+1} HT(\chi_v, \Pi_{h+1}))_{\mathfrak{m}^v} \longrightarrow H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{\bar{h}} HT(\chi_v, \Pi_h))_{\mathfrak{m}^v} \quad (3.2.3)$$

where  $(\Pi_h, \Pi_{h+1})$  is of the shape  $(LT_{\chi_v}(t-1, h-t), LT_{\chi_v}(t-1, h+1-t))$ .

We can then copy the proof of lemma 3.1.8 which gives us the following statement.

**3.2.4. Lemma.** — For every  $1 \leq h \leq h_0$ , the number  $i_I(h) = h - h_0$  of notation 3.1.6, is also the lowest integer  $i$  so that the torsion of  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{gr}_!^h(\Psi_\xi))_{\mathfrak{m}^v}$  is non zero.

**3.2.5. Lemma.** — As a  $\overline{\mathbb{F}}_l[GL_d(F_v)]$ -module, for every  $i$ , the  $l$ -torsion of  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$ , does not have an irreducible generic sub-quotient whose cuspidal support is made of characters.

*Remark.* Note that when the order of  $q_v$  modulo  $l$  is strictly greater than  $d$ , then there is no difference between cuspidal or supercuspidal support made of characters.

*Proof.* — Recall first that, as by hypothesis  $\overline{\rho}_{\mathfrak{m}}$  is irreducible, the  $\overline{\mathbb{Q}}_l$ -version of the spectral sequence (2.3.2) degenerates at  $E_1$  so that in particular all the torsion cohomology classes appear in the  $E_1$  terms. As we are only interested in representations with cuspidal support made of characters, we have only to deal with the perverse sheaves  $\mathcal{P}(t, \chi_v)$  so that the result follows from the previous maps (3.2.3) and the fact that for any  $r > 0$ , the modulo  $l$  reduction of  $LT_{\chi_v}(t - 1, r)$  does not admit any irreducible generic sub-quotient.  $\square$

*Remark.* We could also prove the same result without restriction on the cuspidal support but then we would have to deal with the problem mentioned in the remark after lemma 3.1.13 which is the main subject of [7].

**3.3. Torsion and modified lattices.** — Before studying the filtration of the free quotient of  $H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$  given by the spectral sequence of vanishing cycles, we focus in this section on the cohomology of  $\mathrm{gr}_!^{h_0(I^v)}(\Psi_{\varrho, \xi})$ . To do so, consider first a filtration of it with successive graded parts  $\mathrm{gr}^i(\mathrm{gr}_!^{h_0(I^v)}(\Psi_{\varrho}))$  which are  $\overline{\mathbb{Z}}_l$ -structures of the following  $\overline{\mathbb{Q}}_l$ -perverse sheaf

$$\bigoplus_{\substack{k: g_k(\varrho) | h_0 + i \\ h_0 + i = g_k(\varrho) t_k(i)}} \bigoplus_{\pi_v \in \mathrm{Cusp}_{\varrho}(k)} \mathcal{P}(t_k(i), \pi_v) \left( \frac{1 - 2t_k(0) + t_k(i)}{2} \right),$$

for  $i \geq 0$ . To be more precise, note that  $\mathrm{gr}_!^{h_0(I^v)}(\Psi_{\varrho, \xi}) \twoheadrightarrow \mathrm{gr}^0(\mathrm{gr}_!^{h_0(I^v)}(\Psi_{\varrho, \xi}))$ , and its kernel has  $\mathrm{gr}^1(\mathrm{gr}_!^{h_0(I^v)}(\Psi_{\varrho, \xi}))$  as a quotient and so on.

*Remark.* As it might be disturbing and irrelevant in the following, we decide not to write anymore the shifts  $(\frac{1-2t_k(0)+t_k(i)}{2})$  in our formulas.

For any finite level  $I \in \mathcal{I}$ , we then now introduce two  $\overline{\mathbb{Z}}_l$ -lattices of

$$\bigoplus_{\substack{k: g_k(\varrho)|h_0+i \\ h_0+i=g_k(\varrho)t_k(i)}} \bigoplus_{\pi_v \in \text{Cusp}_\varrho(k)} H^0(\text{Sh}_{I, \overline{s}_v}, \mathcal{P}_\xi(t_k(i), \pi_v))_{\mathfrak{m}^v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l.$$

- The first one denoted by  $\Gamma_{\xi, \varrho}(I, h_0 + i)$  is given by the free quotient of  $H^0(\text{Sh}_{I, \overline{s}_v}, \text{gr}^i(\text{gr}_!^{h_0(I^v)}(\Psi_\varrho)))_{\mathfrak{m}^v}$ .
- The spectral sequence associated to the previous filtration of  $\text{gr}_!^{h_0(I^v)}(\Psi_{\xi, \varrho})$ , provides the free quotient of  $H^0(\text{Sh}_{I, \overline{s}_v}, \text{gr}_!^{h_0(I^v)}(\Psi_{\xi, \varrho}))_{\mathfrak{m}^v}$  with a  $\overline{\mathbb{Z}}_l$ -filtration. One of its graded part is such that its tensor product with  $\otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  is isomorphic to  $H^0(\text{Sh}_{I, \overline{s}_v}, \text{gr}^i(\text{gr}_!^{h_0(I^v)}(\Psi_\varrho)))_{\mathfrak{m}^v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ . We then denote by  $\Gamma_{\xi, \varrho, !}(I, h_0, i)$  the associated graded part.

**3.3.1. Notation.** — Let denote by

$$\text{Iw}(h_0) := \left\{ g \in GL_d(\mathcal{O}_v) \text{ such that } (g \pmod{\varpi_v}) \in P_{1,2,\dots, h_0, d}(\kappa(v)); \right\}.$$

**3.3.2. Lemma.** — With the previous notations, there exists a finite level  $I \in \mathcal{I}$  with  $I_v \simeq \text{Iw}(h_0(I^v))$ , and a short exact sequence

$$0 \rightarrow \Gamma_{\xi, \varrho}(I, h_0(I^v) + 1) \left( \frac{2 - h_0(I^v)}{2} \right) \longrightarrow \Gamma_{\xi, \varrho, !}(I, h_0(I^v), 1) \longrightarrow T \rightarrow 0$$

where every irreducible sub-quotient of  $T(\frac{h_0(I^v)-2}{2}) \neq (0)$ , as a  $\mathbb{T}_{\xi, \mathfrak{m}^v}^S \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ -module, can be obtained as a sub-quotient of the torsion submodule of the cokernel of some

$$\begin{aligned} H^0(\text{Sh}_{I, \overline{s}_v}, j_!^{=h_0(I^v)+1} HT_\xi(\chi_v, \text{St}_{h_0(I^v)+1}(\chi_v)))_{\mathfrak{m}^v} \\ \longrightarrow H^0(\text{Sh}_{I, \overline{s}_v}, j_!^{=h_0(I^v)} HT_\xi(\chi_v, \text{St}_{h_0(I^v)}(\chi_v)))_{\mathfrak{m}^v}, \end{aligned} \quad (3.3.3)$$

for  $\chi_v \in \text{Cusp}_\varrho(-1)$ .

*Proof.* — For simplicity denote  $h_0(I^v)$  by  $h_0$ . Consider the  $I^v(\infty)$ -version of (3.3.3). Note then that it is non strict if and only if the same is true for its non induced version in the next formula, whatever is  $\Pi_{h_0}$  a

representation of  $GL_{h_0}(F_v)$

$$\begin{aligned} H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_{\overline{1_{h_0}}}^{=h_0+1} HT_{\overline{1_{h_0}}, \xi}(\chi_v, \Pi_{h_0} \otimes \chi_v))_{\mathfrak{m}^v} \\ \longrightarrow H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_{\overline{1_{h_0}}}^{=h_0} HT_{\overline{1_{h_0}}, \xi}(\chi_v, \Pi_{h_0}))_{\mathfrak{m}^v}, \end{aligned} \quad (3.3.4)$$

where we denote by  $\mathrm{Sh}_{I^v(\infty), \bar{s}_v, \overline{1_{h_0}}}^{=h_0+1}$  as the disjoint union of the pure strata, cf. notation 2.2.1,  $\mathrm{Sh}_{I^v(\infty), \bar{s}_v, g}^{=h_0+1}$  contained in  $\mathrm{Sh}_{I^v(\infty), \bar{s}_v, \overline{1_{h_0}}}^{\geq h_0}$ . As usual the notation  $j_{\overline{1_{h_0}}}^{=h_0+1}$  designates the closed embedding of  $\mathrm{Sh}_{I^v(\infty), \bar{s}_v, \overline{1_{h_0}}}^{=h_0+1}$  in  $\mathrm{Sh}_{I^v(\infty), \bar{s}_v}^{\geq 1}$ . In particular, as a  $\overline{\mathbb{Z}}_l[P_{h_0, D}(F_v)]$ -module, for  $\Pi_{h_0}$  a character, we know

- by definition of  $h_0$  in §2.1,
- and using the fact that  $q_v$  modulo  $l$  is of order  $> d$  so that the functor of  $P_{h_0, d}(\mathcal{O}_v)$ -invariants is exact,

that the cokernel of (3.3.4) has non trivial vectors invariant under  $P_{h_0, d}(\mathcal{O}_v)$ , so that the cokernel of the  $I^v(\infty)$ -version of (3.3.3) has non trivial vectors invariant under  $\mathrm{Iw}(h_0)$ .

We then compute the  $\mathfrak{m}^v$ -localized cohomology of  $\mathrm{gr}_!^{h_0}(\Psi_{\xi, \varrho})$  in level  $I^v(\infty)$  having non trivial invariant under  $I$  with  $I_v \simeq \mathrm{Iw}(h_0)$ . Recall that

$$\mathrm{gr}_!^{h_0}(\Psi_{\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}_{\varrho}} \mathrm{gr}_!^{h_0}(\Psi_{\pi_v})$$

so that we can find a filtration of  $\mathrm{gr}_!^{h_0}(\Psi_{\varrho})$  whose graded parts are free and isomorphic, after tensoring with  $\overline{\mathbb{Q}}_l$ , to  $\mathrm{gr}_!^{h_0}(\Psi_{\pi_v})$ . For any  $i \geq 0$  and  $\pi_v \in \mathrm{Cusp}_{\varrho}(i)$ , then  $\mathrm{gr}_!^{h_0}(\Psi_{\pi_v})$  has trivial cohomology in level  $I^v \mathrm{Iw}(h_0)$  so we can restrict ourselves with characters  $\chi_v \in \mathrm{Cusp}_{\varrho}(-1)$ .

By maximality of  $h_0$ , note that for  $h_0 < t \leq d$ , the cohomology groups of  $\mathcal{P}_{\xi}(t, \chi_v)$  and  $j_!^{=t} HT_{\xi}(\pi_v, \Pi_t)$ , are all free after localization by  $\mathfrak{m}^v$ . Using the same argument as before through the spectral sequence associated to (3.2.2), we then deduce that  $H^i(\mathrm{Sh}_{I^v, \bar{s}_v}, \mathrm{gr}_!^{h_0}(\Psi_{\xi, \chi_v}))_{\mathfrak{m}^v}$  are all free for  $i \neq 0, 1$  while for  $i = 0$  the torsion is non trivial given by the non strictness of

$$\begin{aligned} H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{=h_0+1} HT_{\xi}(\chi_v, LT_{\chi_v}(h_0 - 1, 1)))_{\mathfrak{m}^v} \\ \longrightarrow H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, j_!^{=h_0} HT_{\xi}(\chi_v, \mathrm{St}_{h_0}(\chi_v)))_{\mathfrak{m}^v}. \end{aligned} \quad (3.3.5)$$

Concerning  $H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathcal{P}_\xi(\chi_v, h_0))_{\mathfrak{m}^v}$ , its torsion submodule is parabolic induced, so that beside those coming from the non strictness of (3.3.5), there is also the contribution given by the non strictness of (3.3.3), which contains in particular a subquotient, denoted  $\tilde{T}$ , such that  $\tilde{T}[l]$  is of level of non degeneracy strictly greater than those appearing in (3.3.5).

Consider then the cohomology of  $\mathrm{gr}_!^{h_0}(\Psi_{\xi, \chi_v})$  computing through its filtration of stratification with graded parts, up to Galois torsion, the  $H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathcal{P}_\xi(h_0 + k, \chi_v))_{\mathfrak{m}^v}$ , for  $0 \leq k \leq d - h_0$ , and more particularly the induced filtration of the free quotient of  $H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{gr}_!^{h_0}(\Psi_{\xi, \chi_v}))_{\mathfrak{m}^v}$  as before. As the level of non degeneracy of  $\tilde{T}[l]$  is higher than those of the  $l$ -torsion of  $H^0(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{Fil}_!^{h_0}(\Psi_{\xi, \chi_v}))_{\mathfrak{m}^v}$ , computing through the spectral sequence associated to (3.2.2), we then have a filtration of this free quotient where both appears

- torsion modules such as  $\tilde{T}$ ,
- and the free sub-quotients are given by the lattices  $\Gamma_{\xi, \chi_v}(I^v, h_0 + i)$  of the free quotient of the localized cohomology of  $\mathcal{P}_\xi(\chi_v, h_0 + i)$  for  $0 \leq i \leq d - h_0$ .

We know go back to level  $I = I^v \mathrm{Iw}(h_0)$ : as  $q_v$  modulo  $l$  is of order strictly greater than  $d$ , the functor of  $\mathrm{Iw}(h_0)$ -invariants is exact. As only contributes the cohomology of  $\mathcal{P}_\xi(\chi_v, h_0 + i)$  for  $i = 0, 1$  the result follows from the fact that  $\tilde{T}$  has non trivial invariant under  $\mathrm{Iw}(h_0)$ .  $\square$

Arguing as in the proof of lemma 3.1.8, using (3.2.3), we have the following result.

**3.3.6. Lemma.** — *For every  $1 \leq t$ , let  $j(t)$  be the minimal integer  $j$  such that the torsion of  $H^j(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{gr}_!^t(\Psi_{\xi, \varrho}))_{\mathfrak{m}^v}$  is non trivial. Then*

$$j(t) = \begin{cases} +\infty & \text{if } t \geq h_0(I^v) + 1, \\ t - h_0(I^v) & \text{for } 1 \leq t \leq h_0(I^v). \end{cases}$$

Moreover as a  $\mathbb{T}_{\xi, \mathfrak{m}^v} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$ -module, up to multiplicities, the irreducible sub-quotients of  $H^{j(t)}(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{gr}_!^t(\Psi_{\xi, \varrho}))_{\mathfrak{m}^v}$  are independent of  $t$ .

3.3.7 — *Important fact:* Note that, up to multiplicities, the irreducible  $\mathbb{T}_{\xi, \mathfrak{m}^v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ -sub-quotients<sup>(6)</sup> of the  $l$ -torsion of the cohomology of  $\mathrm{gr}_{1, \chi_v}^{h_0(I^v)}(\Psi)$  and  $\mathcal{P}(h_0(I^v), \chi_v)$  are the same, given by the non strictness of the maps (3.2.3).

The idea is now to increase the level at another place  $w \in \mathrm{Spl}(I)$  verifying the same hypothesis than  $v$ , i.e.  $q_w$  modulo  $l$  is of order strictly greater than  $d$ . When the level at  $w$  is infinite, i.e. for  $I^w(\infty)$  with  $I_v \simeq \mathrm{Iw}(h_0(I^v))$ , using again the exactness of invariant with  $GL_d(\mathcal{O}_w)$ , from the previous observation we deduce that considering irreducible  $\overline{\mathbb{F}}_l$ -representations of  $GL_d(F_w)$ , the sets, i.e. without multiplicities, of irreducible sub-quotients of the  $l$ -torsion of the cohomology of respectively  $\mathcal{P}(h_0(I^v), \chi_v)$  and  $\mathrm{gr}_{1, \chi_v}^{h_0(I^v)}(\Psi)$ , are the same.

**3.3.8. Proposition.** — *Up to multiplicities, the set of irreducible  $\overline{\mathbb{F}}_l[GL_d(F_w)]$ -sub-quotients of the  $l$ -torsion of  $H^0(\mathrm{Sh}_{I^w(\infty), \bar{s}_v}, \mathrm{gr}_1^{h_0(I^v)}(\Psi_{\xi, \varrho}))_{\mathfrak{m}^v}$ , are the same as those of  $H^{d-h_0(I^v)}(\mathrm{Sh}_{I^w(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$ .*

*Proof.* — We compute  $H^{d-h_0(I^v)}(\mathrm{Sh}_{I^w(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$  using the filtration  $\mathrm{Fil}_1^\bullet(\Psi_{\xi, \varrho})$  through the spectral sequence (2.3.2). Recall that for every  $p + q \neq 0$ , the free quotient of  $E_{1, \varrho, 1}^{p, q}$  are zero. By definition of the filtration these  $E_{1, \varrho, 1}^{p, q}$  are trivial for  $p \geq 0$  while, thanks to the previous lemma, for any  $p \leq -1$  there are zero for  $p + q < j(p) := p - h_0(I^v)$ . Note then that  $E_{1, \varrho, 1}^{-1, j(1)+1}$  which is torsion and non zero, according to the previous lemma, is equal to  $E_{1, \varrho, \infty}^{j(1)} \simeq H^{d-h_0(I^v)}(\mathrm{Sh}_{I^w(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$ .  $\square$

Consider as before  $I^w(\infty)$  such that its local component at  $v$  is  $\mathrm{Iw}_v(h_0(I^v))$ . Then combining the result of lemma 3.3.2 in level  $I^w(\infty)$ , with the previous proposition, we then deduce that the cokernel  $T$  of 3.3.2 verifies the following property. As a  $\overline{\mathbb{F}}_l$ -representation of  $GL_d(F_w)$ , every irreducible sub-quotient of  $T[l]$  is also a sub-quotient of  $H^{d-h_0(I^v)}(\mathrm{Sh}_{I^w(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^v}$ . Then applying lemma 3.2.5 at the place  $w$  playing a symmetric role as  $v$ , we then deduce the following result.

<sup>(6)</sup>i.e. if one forget the action of  $GL_d(F_v)$



**3.3.9. Corollary.** — *As a  $\overline{\mathbb{F}}_l$ -representation of  $GL_d(F_w)$ , the  $l$ -torsion of the cokernel  $T$  of lemma 3.3.2 in level  $I^w(\infty)$  as above, does not contain any irreducible generic sub-quotient with cuspidal support made of characters.*

The aim of the following section is then to prove that if all the cokernel of maps (3.3.3), are such that, in infinite level at  $w$ , their  $l$ -torsion does not contain any irreducible generic sub-quotient with cuspidal support made of characters, then they should all be equal to zero, so that the torsion of every  $H^i(\mathrm{Sh}_{I, \overline{\eta}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}}$  is necessary trivial.

**3.4. Global lattices and generic representations.** — Start first with the following formal generalization of a previous result of Carayol in [12].

**3.4.1. Theorem.** — *(cf. [6] theorem 3.1.1 and [18] theorem 5.6) If  $\overline{\rho}_{\mathfrak{m}}$  is absolutely irreducible, then as a  $\mathbb{T}_{\xi, \mathfrak{m}}^S[\mathrm{Gal}_{F, S}]$ -module,*

$$H_{free}^{d-1}(\mathrm{Sh}_{I, \overline{\eta}}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}} \simeq \pi_{\xi, \mathfrak{m}} \otimes_{\mathbb{T}_{\xi, \mathfrak{m}}^S} \rho_{\xi, \mathfrak{m}},$$

where

- $\pi_{\xi, \mathfrak{m}}$  is a  $\mathbb{T}_{\xi, \mathfrak{m}}^S$ -module on which  $\mathrm{Gal}_{F, S}$  acts trivially,
- and  $\rho_{\xi, \mathfrak{m}} : \mathrm{Gal}_{F, S} \rightarrow GL_d(\mathbb{T}_{\xi, \mathfrak{m}}^S)$  is a Galois representation unramified outside  $S$  such that, cf. [17] V.4.4, for all  $u \notin S$ , then  $\det(1 - X \mathrm{Frob}_u | \rho_{\xi, \mathfrak{m}})$  is equal to the Hecke polynomial, cf. the end of §1.3.

Recall that

$$\mathbb{T}_{\xi, \mathfrak{m}}^S \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \prod_{\tilde{\mathfrak{m}} \subset \mathfrak{m}} (\mathbb{T}_{\xi}^S \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)_{\tilde{\mathfrak{m}}}$$

so that  $\rho_{\xi, \mathfrak{m}}$  is a stable lattice of

$$\rho_{\xi, \mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\tilde{\mathfrak{m}} \subset \mathfrak{m}} \rho_{\xi, \tilde{\mathfrak{m}}}.$$

For each  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ , denote by  $\Gamma_{\tilde{\mathfrak{m}}}^- := \rho_{\xi, \mathfrak{m}} \cap \rho_{\xi, \tilde{\mathfrak{m}}}$  and  $\Gamma^- := \bigoplus_{\tilde{\mathfrak{m}} \subset \mathfrak{m}} \Gamma_{\tilde{\mathfrak{m}}}^-$ .

*Remark.* We could also choose any numbering of the set of  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  and define  $\Gamma_{\tilde{\mathfrak{m}}_1}^- := \Gamma_{\tilde{\mathfrak{m}}_1}^-$ , then  $\Gamma_{\tilde{\mathfrak{m}}_2}$  as the intersection of  $\rho_{\xi, \tilde{\mathfrak{m}}_2}$  with  $\rho_{\xi, \mathfrak{m}}/\Gamma_1$  and so on. Note then, as  $\overline{\rho}_{\xi, \mathfrak{m}}$  is supposed irreducible, then for all  $i$ , we have  $\Gamma_{\tilde{\mathfrak{m}}_i}^- \simeq \Gamma_{\tilde{\mathfrak{m}}_i}^-$ .

**3.4.2. Definition.** — Choose some basis of  $\Gamma_{\tilde{m}_1}^-$ ; as all the  $\Gamma_{\tilde{m}}^-$  are homothetic, this gives a basis of  $\Gamma^-$ . We then denote by  $M_{m,v}$  the matrix of the nilpotent monodromy operator  $N_{v,m}$  in this basis. Note that  $M_{m,v}$  is diagonal by  $d \times d$ -blocks, where the modulo  $l$  reduction of the nilpotent monodromy operator  $N_{v,m}$  on each block, are the same.

Recall that we can also compute  $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  through the spectral sequence of vanishing cycles. More precisely we want to construct a filtration of its free quotient, using the filtration

$$\mathrm{Fil}_1^1(\Psi_{\varrho}) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}_1^{d-1}(\Psi_{\varrho}) \hookrightarrow \mathrm{Fil}_1^d(\Psi_{\varrho}) \hookrightarrow \Psi_{\varrho}.$$

With the notations of the previous sections, we will consider the level  $I = I^v \mathrm{Iw}_v(h_0(I^v))$  such that, arguing by absurdity, the torsion of  $H^{d-1}(\mathrm{Sh}_{I^v GL_d(\mathcal{O}_v), \bar{\eta}}, V_{\xi,\bar{\mathbb{Z}}_l})_{\mathfrak{m}}$  is non trivial. For the definition of  $h_0 := h_0(I^v)$ , cf. notation 3.1.7. In the following, we will pay special attention to lattices of

$$V_{\xi,m}(h_0 + 1)(\delta) := \bigoplus_{\chi_v \in \mathrm{Cusp}_{\varrho}(-1)} H^i(\mathrm{Sh}_{I,\bar{s}}, \mathcal{P}_{\xi}(h_0 + 1, \chi_v)(\delta)_{\mathfrak{m}^v} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l),$$

for various  $\delta$ .

- Note first that the  $H^i(\mathrm{Sh}_{I,\bar{s}}, \Psi_{\xi,\varrho} / \mathrm{Fil}_1^{h_0+1}(\Psi_{\xi,\varrho}))_{\mathfrak{m}^v}$  are all zero. Indeed the torsion free graded parts  $\mathrm{gr}^k(\Psi_{\varrho})$  of any exhaustive filtration of  $\Psi_{\varrho} / \mathrm{Fil}_1^{h_0+1}(\Psi_{\varrho})$ , up to Galois torsion, are such that  $\mathrm{gr}^k(\Psi_{\varrho}) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l \simeq \mathcal{P}(t, \pi_v)$  which  $\pi_v \in \mathrm{Cusp}_{\varrho}$  an irreducible cuspidal representation of some  $GL_g(F_v)$  with  $tg > h_0 + 1$ . Then every irreducible constituent of  $H^i(\mathrm{Sh}_{I^v(\infty), \bar{s}_v}, \mathrm{gr}^k(\Psi_{\xi,\varrho}))_{\mathfrak{m}^v} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$ , as a  $\bar{\mathbb{F}}_l$ -representation of  $GL_d(F_v)$  is a sub-quotient of an induced representation  $r_l(\mathrm{St}_t(\pi_v))\{\delta/2\} \times \tau$  for some irreducible  $\bar{\mathbb{F}}_l$ -representation  $\tau$  of  $GL_{d-tg}(F_v)$ . In particular such a representation does not have non trivial invariants under  $\mathrm{Iw}_v(h_0)$ , so that, as the functor of  $\mathrm{Iw}_v(h_0)$ -invariants is exact, there is no cohomology in level  $I$  as stated.
- Recall the following short exact sequence

$$K_{\chi_v}(h_0 + 1) \hookrightarrow \mathrm{gr}_1^{h_0+1}(\Psi_{\chi_v}) \twoheadrightarrow \mathcal{P}(h_0 + 1, \chi_v)\left(\frac{-h_0}{2}\right)$$

where the irreducible sub-quotients of  $K_{\chi_v}(h_0 + 1)$  are, up to Galois torsion, the  $\mathcal{P}(t, \chi_v)$  with  $t > h_0 + 1$ . Arguing as before on the  $\text{Iw}(h_0)$ -invariants, we then conclude that  $H^i(\text{Sh}_{I, \bar{s}}, \text{gr}_!^{h_0+1}(\Psi_{\xi, \varrho}))_{\mathfrak{m}^v}$  is some lattice of  $V_{\xi, \mathfrak{m}}(h_0 + 1)(\frac{-h_0}{2})$ , denoted by

$$\Gamma_{\xi, \varrho}(I, h_0 + 1)(\frac{-h_0}{2})$$

in the previous section. Note also that

- as recalled in lemma 2.2.5, there is only one notion of intermediate extension for Harris-Taylor local systems associated to characters,
  - and, by definition of  $h_0$ , all these cohomology groups are torsion free and concentrated in degree zero.
- Similarly the kernel  $K_{\chi_v}(h_0)$  of  $\text{gr}_!^{h_0}(\Psi_{\chi_v}) \rightarrow \mathcal{P}(h_0, \chi_v)(\frac{1-h_0}{2})$ , verifies the same property as before, i.e.

$$L_{\chi_v}(h_0) \hookrightarrow K_{\chi_v}(h_0) \rightarrow \mathcal{P}(h_0 + 1, \chi_v)(\frac{2-h_0}{2})$$

where  $L_{\chi_v}(h_0)$  does not have non trivial cohomology in level  $I$  when  $I_v \simeq \text{Iw}_v(h_0)$ . With the notation of the previous section, we then have a short exact sequence

$$\begin{aligned} 0 \rightarrow \Gamma_{\xi, \varrho, !}(I, h_0, 1) &\longrightarrow H^0(\text{Sh}_{I, \bar{s}_v}, \text{gr}_!^{h_0}(\Psi_{\varrho}))_{\mathfrak{m}^v} \\ &\longrightarrow \Gamma_{\xi, \varrho}(I, h_0)(\frac{1-h_0}{2}) \rightarrow 0, \end{aligned} \quad (3.4.3)$$

where as before

$$\Gamma_{\xi, \varrho, !}(I, h_0, 1)$$

is a lattice of  $V_{\xi, \mathfrak{m}}(h_0 + 1)(\frac{2-h_0}{2})$ .

- For every  $1 \leq h < h_0$ , we can write

$$0 \rightarrow K_{h, !}(\Psi_{\chi_v}) \longrightarrow \text{gr}_!^h(\Psi_{\chi_v}) \longrightarrow Q_{h, !}(\Psi_{\chi_v}) \rightarrow 0,$$

with

$$0 \rightarrow KK_{h, !}(\Psi_{\chi_v}) \longrightarrow K_{h, !}(\Psi_{\chi_v}) \longrightarrow \mathcal{P}(h_0 + 1, \chi_v)(\frac{h_0 - 2h + 2}{2}) \rightarrow 0,$$

where the irreducible sub-quotients of  $KK_{h, !}(\Psi_{\chi_v}) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$  (resp.  $Q_{h, !}(\Psi_{\chi_v}) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$ ) are up to Galois torsion, the  $\mathcal{P}(t, \chi_v)$  with  $t > h_0 + 1$

(resp.  $t \leq h_0$ ). As before  $KK_{h,!}(\Psi_{\chi_v})$  has trivial cohomology in level  $I$  with  $I_v \simeq \text{Iw}_v(h_0)$ . We then denote by

$$\Gamma_{\xi,\varrho,!}(I, h_0, h_0 + 1 - h)$$

the lattice of  $V_{\xi,m}(h_0 + 1)(\frac{h_0 - 2h + 2}{2})$  given by the associated filtration of the free quotient of  $H^0(\text{Sh}_{I,\bar{s}_v}, \text{gr}_{!,\xi}^h(\Psi_{\chi_v}))_{\mathfrak{m}}$ .

Recall that for any  $k \leq h_0 + 1$ , the nilpotent monodromy operator  $N_v$  induces a map

$$\begin{array}{ccc} \text{Fil}_!^k(\Psi_\varrho) & \longrightarrow & \text{Fil}_!^{k-1}(\Psi_\varrho) \\ \downarrow & & \downarrow \\ \text{gr}_!^k(\Psi_\varrho) & \dashrightarrow & \text{gr}_!^{k-1}(\Psi_\varrho). \end{array}$$

We then obtain a map on the cohomology groups

$$\begin{array}{ccc} \Gamma_{\xi,\varrho,!}(I, h_0, h_0 + 1 - k) & \dashrightarrow & \Gamma_{\xi,\varrho,!}(I, h_0, h_0 + 1 - k + 1) \\ \downarrow & & \downarrow \\ H^0(\text{Sh}_{I,\bar{s}_v}, \text{gr}_!^k(\Psi_{\xi,\varrho}))_{\mathfrak{m}^v} & \longrightarrow & H^0(\text{Sh}_{I,\bar{s}_v}, \text{gr}_!^{k-1}(\Psi_{\xi,\varrho}))_{\mathfrak{m}^v} \\ & & \downarrow \\ & & Q \end{array}$$

where the right arrow is exact. As the image of  $\Gamma_{\xi,\varrho,!}(I, h_0, h_0 + 1 - k) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  in  $Q \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  is zero, and  $Q$  is free by construction, then  $N_v$  induces a map

$$\Gamma_{\xi,\varrho,!}(I, h_0, h_0 + 1 - k) \rightarrow \Gamma_{\xi,\varrho,!}(I, h_0, h_0 + 1 - k + 1),$$

which is an embedding as it is so over  $\overline{\mathbb{Q}}_l$ . As by definition  $\Gamma_{\xi,\varrho,!}(I, h_0, 0) := \Gamma_{\xi,\varrho}(I, h_0 + 1)(-\frac{h_0}{2})$ , then  $N_v$  induces the following embeddings

$$\begin{aligned} \Gamma_{\xi,\varrho}(I, h_0 + 1)(-\frac{h_0}{2}) &\hookrightarrow \Gamma_{\xi,\varrho,!}(I, h_0, 1) \hookrightarrow \Gamma_{\xi,\varrho,!}(I, h_0, 2) \hookrightarrow \dots \\ &\hookrightarrow \Gamma_{\xi,\varrho,!}(I, h_0, h_0 - 1). \end{aligned} \quad (3.4.3)$$

**3.4.4. Lemma.** — *The previous embeddings of (3.4.3) induce*

$$l.\Gamma_{\xi,\varrho,!}(I, h_0, h_0 - 1)(-\frac{h_0}{2}) \hookrightarrow \Gamma_{\xi,\varrho}(I, h_0 + 1).$$

*Proof.* — Recall (2.4.1) that  $N_v$  induces isomorphisms

$$N_v : \mathcal{P}_{I, \bar{\mathbb{Z}}_l}(t, \chi_v)\left(\frac{-r}{2}\right) \longrightarrow \mathcal{P}_{I, \bar{\mathbb{Z}}_l}(t, \chi_v)\left(\frac{2-r}{2}\right),$$

for all characters  $\chi_v \in \text{Cusp}_\rho(-1)$ , all  $1 \leq t \leq d$  and all  $3 - t \leq r \leq t - 1$  with  $r \equiv t - 1 \pmod{2}$ . In particular  $N_v^r$  induces

$$\Gamma_{\xi, \rho}(I, h_0 + 1)\left(-\frac{h_0}{2}\right) \simeq \Gamma_{\xi, \rho}(I, h_0 + 1)\left(\frac{2r - h_0}{2}\right) \hookrightarrow \Gamma_{\xi, \rho, !}(I, h_0, r).$$

Recall moreover that the cokernel  $T$  of

$$\Gamma_{\xi, \rho}(I^w(\infty), h_0 + 1)\left(\frac{2r - h_0}{2}\right) \hookrightarrow \Gamma_{\xi, \rho, !}(I^w(\infty), h_0, r) \twoheadrightarrow T,$$

is given by the torsion of the  $E_\infty$  terms of the spectral sequence associated to the filtration  $\text{Fil}_!^\bullet(\Psi_{\xi, \rho})$ . Explicitly, there exists a filtration of  $T$  such that each of its graded part is a subquotient of some of the  $E_1^{p, -p} = H^0(\text{Sh}_{I^w(\infty), \bar{s}_v}, \text{gr}_!^{-p}(\Psi_{\xi, \rho}))_{\mathfrak{m}^v, w}$ .

From the property stated after corollary 3.3.9, we then deduce that  $T[l]$  as a  $\bar{\mathbb{F}}_l$ -representation of  $GL_d(F_w)$ , does not contain any irreducible generic sub-quotient. So as  $\Gamma_{\xi, \rho}(I^w(\infty), h_0 + 1) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$  has an irreducible generic sub-quotient, we then deduce that  $l \cdot \Gamma_{\xi, \rho, !}(I^w(\infty), h_0, r) \hookrightarrow \Gamma_{\xi, \rho}(I^w(\infty), h_0 + 1)\left(\frac{2r - h_0}{2}\right)$ . As before the same is then true in level  $I$ .  $\square$

*Remark.* As stressed in the previous proof, the cokernel

$$T_{\mathfrak{m}, I^w(\infty)}(N_v, h_0) := \Gamma_{\xi, \rho, !}(I^w(\infty), h_0, h_0 - 1) / \Gamma_{\xi, \rho}(I^w(\infty), h_0 + 1) \quad (3.4.4)$$

of the image of  $N_v^r$  in  $\text{Ker } N_v \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$ , has a filtration such that its graded pieces are sub-quotients of the cokernels of the maps (3.2.3) for various  $h$  and  $\Pi_h$ . So as before  $T_{\mathfrak{m}, I^w(\infty)}(N_v, h_0)[l]$ , as a  $\bar{\mathbb{F}}_l$ -representation of  $GL_d(F_w)$ , verifies the property of the corollary 3.3.9, i.e. it does not have any irreducible generic sub-quotient made of characters.

Consider then  $\tilde{\mathfrak{m}} \subset \mathfrak{m}$  such that  $\pi_{\tilde{\mathfrak{m}}}$  has non trivial invariants under  $I = I^v \text{Iw}_v(h_0)$ . The multi-set of Jordan block of the nilpotent monodromy operator  $N_{\tilde{\mathfrak{m}}}$  are then all of size  $\leq h_0 + 1$  and we denote by  $r$  the number of these Jordan blocks of size  $h_0 + 1$ .

*Remark.* As noticed in the remark following proposition 2.1.4, we can choose such  $\tilde{\mathfrak{m}}$  such that  $r > 0$ .

**3.4.5. Lemma.** — *The rank of  $N_{\tilde{\mathfrak{m}}}^{h_0} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$  is equal to  $r > 0$ .*

*Proof.* — Fix any ordering of the set  $\{\tilde{\mathfrak{m}} \subset \mathfrak{m}\}$  and let define the corresponding filtration of  $\pi_{\xi, \mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\tilde{\mathfrak{m}} \subset \mathfrak{m}} \pi_{\xi, \tilde{\mathfrak{m}}}$  such that its graded parts are the  $\pi_{\xi, \tilde{\mathfrak{m}}_i}$  for  $i = 1, \dots, u$ . We then obtain a filtration of  $\pi_{\xi, \mathfrak{m}}$  which graded parts are stable lattices  $\Lambda_{\tilde{\mathfrak{m}}_i}$  of  $\pi_{\xi, \tilde{\mathfrak{m}}_i}$  which depend of the choice of the previous ordering. By theorem 3.4.1, we then obtain a filtration of  $H_{free}^0(\mathrm{Sh}_{I^w(\infty), \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}^w}$  with graded parts  $\Lambda_{\tilde{\mathfrak{m}}_i} \otimes \Gamma_{\tilde{\mathfrak{m}}_i}$ . Using the Hecke equivariance and denoting by  $K_{\mathfrak{m}}^{(u)}$  the kernel of  $\pi_{\xi, \mathfrak{m}} \otimes \rho_{\xi, \mathfrak{m}} \twoheadrightarrow \Lambda_{\tilde{\mathfrak{m}}_u} \otimes \Gamma_{\tilde{\mathfrak{m}}_u}$ , we then deduce that the image of  $K_{\mathfrak{m}}^{(u)}$  under  $N_{v, \mathfrak{m}}^{h_0}$ , remains in  $K_{\mathfrak{m}}^{(u)}$ :

$$\begin{array}{ccccc}
 K_{\mathfrak{m}}^{(u)} & \xrightarrow{N_{v, \mathfrak{m}}^{h_0}} & K_{\mathfrak{m}}^{(u)} & \twoheadrightarrow & Q^{(u)} \\
 \downarrow & & \downarrow & & \downarrow \\
 \pi_{\xi, \mathfrak{m}} \otimes \rho_{\xi, \mathfrak{m}} & \xrightarrow{N_{v, \mathfrak{m}}^{h_0}} & \pi_{\xi, \mathfrak{m}} \otimes \rho_{\xi, \mathfrak{m}} & \twoheadrightarrow & Q \\
 \downarrow & & \downarrow & & \downarrow \\
 \Lambda_{\tilde{\mathfrak{m}}_u} \otimes \Gamma_{\tilde{\mathfrak{m}}_u} & \xrightarrow{N_{v, \mathfrak{m}}^{h_0}} & \Lambda_{\tilde{\mathfrak{m}}_u} \otimes \Gamma_{\tilde{\mathfrak{m}}_u} & \twoheadrightarrow & Q_u
 \end{array}$$

If the rank of  $N_{\tilde{\mathfrak{m}}}^{h_0}$  were not maximal, then the torsion  $T_u$  of  $Q_u$  would be non trivial. More precisely  $T_u[l]$ , as a  $\mathbb{T}_{\mathfrak{m}, I}[GL_d(F_w)] \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ -module, would have, as a sub-quotient, any irreducible sub-quotients of the modulo  $l$  reduction of  $\Lambda_{\tilde{\mathfrak{m}}_u}$ .

Consider then the free quotient  $Q_f$  of  $Q$  and  $Q_f^{(u)}$  its sub-module coming from  $K_{\mathfrak{m}}^{(u)} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ . The image of  $N_{v, \mathfrak{m}}^{h_0}(Q_f^{(u)}) \subset \pi_{\xi, \mathfrak{m}} \otimes \rho_{\xi, \mathfrak{m}}$  in  $\Lambda_{\tilde{\mathfrak{m}}_u} \otimes \Gamma_{\tilde{\mathfrak{m}}_u}$ , by Hecke equivariance of  $N_{v, \mathfrak{m}}$ , is zero so that  $Q_f^{(u)}$  maps to zero in  $Q_u$ . We then deduce that  $T_u$  is a quotient of the torsion submodule  $T$  of  $Q$ . In particular with the previous notations, the  $l$ -torsion of  $T_{\mathfrak{m}, I^w(\infty)}(N_v, h_0)$  would have, as a  $\mathbb{T}_{\mathfrak{m}, I}[GL_d(F_w)] \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ -module, a sub-quotient for each of the irreducible sub-quotient of the reduction modulo  $l$  of  $\Lambda_{\tilde{\mathfrak{m}}_u}$ . This would contradicts the previous remark as the modulo  $l$  reduction of  $\Lambda_{\tilde{\mathfrak{m}}_u}$ , as a  $\overline{\mathbb{F}}_l$ -representation of  $GL_d(F_w)$ , has a generic irreducible sub-quotient.  $\square$

We then conclude that all the embeddings in (3.4.3) are isomorphisms. Meanwhile we proved in lemma 3.3.2 that, if there were non trivial torsion cohomology classes in any of  $H^i(\mathrm{Sh}_{I, \bar{s}_v}, V_{\xi, \overline{\mathbb{Z}}_l})_{\mathfrak{m}}$  then  $\Gamma_{\xi, \varrho}(I, h_0 + 1)(\frac{2-h_0}{2})$  and  $\Gamma_{\xi, \varrho}!(I, h_0, 1)$  could not be isomorphic.

## References

- [1] I.N. Bernstein and A.V. Zelevinsky. Induced representations of reductive  $p$ -adic groups. I. *Ann. Scient. de l'ENS 4e série, tome 10 n4*, pages 441–472, 1977.
- [2] P. Boyer. Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples. *Invent. Math.*, 177(2):239–280, 2009.
- [3] P. Boyer. Cohomologie des systèmes locaux de Harris-Taylor et applications. *Compositio*, 146(2):367–403, 2010.
- [4] P. Boyer. Filtrations de stratification de quelques variétés de Shimura simples. *Bulletin de la SMF*, 142, fascicule 4:777–814, 2014.
- [5] P. Boyer. Sur la torsion dans la cohomologie des variétés de Shimura de Kottwitz-Harris-Taylor. *Journal of the Institute of Mathematics of Jussieu*, pages 1–19, 2017.
- [6] P. Boyer. Ihara's lemma and level rising in higher dimension. *soumis*, pages 1–31, 2018.
- [7] P. Boyer. La cohomologie des espaces de Lubin-Tate est libre. *soumis*, 2018.
- [8] P. Boyer. Groupe mirabolique, stratification de Newton raffinée et cohomologie des espaces de Lubin-Tate. *Bull. SMF*, pages 1–18, 2019.
- [9] P. Boyer.  $p$ -stabilization in higher dimension. *Journal of the Ramanujan math. soc.*, pages 1–15, 2019.
- [10] P. Boyer. Principe de Mazur en dimension supérieure. *Journal de l'Ecole Polytechnique*, Tome 6:203–230, 2019.
- [11] A. Caraiani and P. Scholze. On the generic part of the cohomology of compact unitary Shimura varieties. *Ann. of Math. (2)*, 186(3):649–766, 2017.
- [12] H. Carayol. Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet.  *$p$ -adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, 165:213–237, 1994.
- [13] J.-F. Dat. Un cas simple de correspondance de Jacquet-Langlands modulo  $l$ . *Proc. London Math. Soc.* 104, pages 690–727, 2012.
- [14] L. Fargues, A. Genestier, and V. Lafforgue. *The isomorphism between Lubin-Tate and Drinfeld towers. (L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld.)*. Progress in Mathematics 262. Basel: Birkhäuser. xii, 406 p., 2008.
- [15] M. Harris, R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

- [16] P. Schneider and U. Stuhler. The cohomology of  $p$ -adic symmetric spaces. *Invent. Math.*, 105(1):47–122, 1991.
- [17] P. Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.
- [18] P. Scholze. On the  $p$ -adic cohomology of the Lubin-Tate tower. *Ann. Sci. c. Norm. Supér.*, (4) 51:no. 4, 811–863., 2018.
- [19] R. Taylor and T. Yoshida. Compatibility of local and global Langlands correspondences. *J.A.M.S.*, 20:467–493, 2007.
- [20] M.-F. Vignéras. *Représentations  $l$ -modulaires d'un groupe réductif  $p$ -adique avec  $l \neq p$* , volume 137 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1996.
- [21] A. V. Zelevinsky. Induced representations of reductive  $p$ -adic groups. II. On irreducible representations of  $GL(n)$ . *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980.

---

BOYER PASCAL • *E-mail* : [boyer@math.univ-paris13.fr](mailto:boyer@math.univ-paris13.fr), Université Sorbonne Paris Nord, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse (France), CoLoSS: ANR-19-PRC