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An arithmetic equivalence of the Riemann Hypothesis

Marc Deléglise, Jean-Louis Nicolas*

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Abstract

Let $h(n)$ denote the largest product of distinct primes whose sum is $\leq n$. The main result of this article is that the property “ for all $n \geq 1$, we have $\log h(n) < \sqrt{\text{li}^{-1}(n)}$ ” (where li^{-1} denotes the inverse function of the logarithmic integral) is equivalent to the Riemann Hypothesis.

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1 Introduction

If $n \geq 1$ is an integer, let us define $h(n)$ as the greatest product of a family of primes $q_1 < q_2 < \dots < q_j$ the sum of which does not exceed n . Let ℓ be the additive function such that $\ell(p^\alpha) = p^\alpha$ for p prime and $\alpha \geq 1$. In other words, if the standard factorization of M into primes is $M = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_j^{\alpha_j}$ we have $\ell(M) = q_1^{\alpha_1} + q_2^{\alpha_2} + \dots + q_j^{\alpha_j}$ and $\ell(1) = 0$. If μ denotes the Möbius function, $h(n)$ can also be defined by

$$h(n) = \max_{\substack{\ell(M) \leq n \\ \mu(M) \neq 0}} M. \quad (1.1)$$

The above equality implies $h(1) = 1$. Note that

$$\ell(h(n)) \leq n. \quad (1.2)$$

Landau [16, p. 222-229] introduced the function $g(n)$ as the maximal order of an element in the symmetric group \mathfrak{S}_n ; he proved that

$$g(n) = \max_{\ell(M) \leq n} M. \quad (1.3)$$

From (1.1) and (1.3), it follows that

$$h(n) \leq g(n), \quad (n \geq 1). \quad (1.4)$$

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Sequences $(h(n))_{n \geq 1}$ and $(g(n))_{n \geq 1}$ are sequences A159685 and A000793 in the OEIS (*On-line Encyclopedia of Integer Sequences*). One can find results about $h(n)$ in [7, 8] and about $g(n)$ in [17, 18, 6, 9]. In the introductions of [6, 9], other references are given. A fast algorithm to compute $h(n)$ and $g(n)$ is described in [7, §8] and [9] while in [8, (4.13)] it is proved that

$$\log h(n) \leq \log g(n) \leq \log h(n) + 5.68 (n \log n)^{1/4}, \quad n \geq 1. \quad (1.5)$$

Let li denote the logarithmic integral and li^{-1} its inverse function (cf. below §2.2). In [17, Theorem 1 (iv)], it is stated that, under the Riemann Hypothesis, the inequality

$$\log g(n) < \sqrt{\text{li}^{-1}(n)} \quad (1.6)$$

holds for n large enough. It is also proved (cf. [17, Theorem 1(i) and (ii)]) that under the Riemann Hypothesis,

$$\log g(n) = \sqrt{\text{li}^{-1}(n)} + \mathcal{O}((n \log n)^{1/4}) \quad (1.7)$$

while, if the Riemann hypothesis is not true, there exists $\xi > 0$ such that

$$\log g(n) = \sqrt{\text{li}^{-1}(n)} + (n \log n)^{1/4} \Omega_{\pm}((n \log n)^{\xi}). \quad (1.8)$$

With (1.5), (1.7) implies

$$\log h(n) = \sqrt{\text{li}^{-1}(n)} + \mathcal{O}((n \log n)^{1/4}), \quad (1.9)$$

while (1.8) yields

$$\log h(n) = \sqrt{\text{li}^{-1}(n)} + (n \log n)^{1/4} \Omega_{\pm}((n \log n)^{\xi}). \quad (1.10)$$

From the expansion of $\text{li}(x)$ given below in (2.7), the asymptotic expansion of $\sqrt{\text{li}^{-1}(n)}$ can be obtained by classical methods in asymptotic theory. A nicer method is given in [23]. From (1.7) and (1.9), it turns out that the asymptotic expansions of $\log g(n)$ and $\log h(n)$ do coincide with the one of $\sqrt{\text{li}^{-1}(n)}$ (cf. [17, Corollaire, p. 225]):

$$\left. \begin{array}{l} \log h(n) \\ \log g(n) \\ \sqrt{\text{li}^{-1}(n)} \end{array} \right\} = \sqrt{n \log n} \left(1 + \frac{\log \log n - 1}{2 \log n} - \frac{(\log \log n)^2 - 6 \log \log n + 9 + o(1)}{8 \log^2 n} \right). \quad (1.11)$$

Let us introduce the sequence (b_n) defined, for $n \geq 2$, by

$$\log h(n) = \sqrt{\text{li}^{-1}(n)} - b_n (n \log n)^{1/4} \quad \text{i.e. } b_n = \frac{\sqrt{\text{li}^{-1}(n)} - \log h(n)}{(n \log n)^{1/4}}, \quad (1.12)$$

and the constant

$$c = \sum_{\rho} \frac{1}{|\rho(\rho+1)|} = 0.046\,117\,644\,421\,509\dots \quad (1.13)$$

where ρ runs over the non trivial roots of the Riemann ζ function. The computation of the above numerical value is explained below in §2.4.2.

The aim of this article is to make more precise the estimate (1.9) and to prove the following result.

Theorem 1.1. *Under the Riemann Hypothesis, we have*

- (i) $\log h(n) < \sqrt{\text{li}^{-1}(n)}$ for $n \geq 1$.
- (ii) $b_{17} = 0.49795\dots \leq b_n \leq b_{1137} = 1.04414\dots$ for $n \geq 2$.
- (iii) $b_n \geq \frac{2}{3} - c - \frac{0.22 \log \log n}{\log n}$ for $n \geq 18$.
- (iv) $b_n \leq \frac{2}{3} + c + \frac{0.77 \log \log n}{\log n}$ for $n \geq 157\,933\,210$.
- (v) $\frac{2}{3} - c = 0.620\dots \leq \liminf b_n \leq \limsup b_n \leq \frac{2}{3} + c = 0.712\dots$
- (vi) *For n tending to infinity,*

$$\left(\frac{2}{3} - c\right) \left(1 + \frac{\log \log n + \mathcal{O}(1)}{4 \log n}\right) \leq b_n \leq \left(\frac{2}{3} + c\right) \left(1 + \frac{\log \log n + \mathcal{O}(1)}{4 \log n}\right).$$

Under the Riemann Hypothesis, the point (vi) of Theorem 1.1 shows that, for n large enough, $b_n > 2/3 - c$. We prove (cf. (5.46) below) that $b_n > 2/3 - c$ holds for $78 \leq n \leq \pi_1(10^{10}) = \sum_{p \leq 10^{10}} p$, and it is reasonable to think that it holds for all $n \geq 78$. In the point (iii), we have tried to replace the constant -0.22 by a positive one, but without success.

Corollary 1.2. *Each of the six points of Theorem 1.1 is equivalent to the Riemann Hypothesis.*

Proof. If the Riemann Hypothesis fails, (1.10) and (1.12) contradict (i), (ii), ..., (vi) of Theorem 1.1. □

Corollary 1.3. *The inequalities*

$$\sqrt{\text{li}^{-1}(n)} - 1.045(n \log n)^{1/4} \leq \log g(n) \leq \sqrt{\text{li}^{-1}(n)} + 5.19 (n \log n)^{1/4} \quad (1.14)$$

are true for each $n \geq 2$, if and only if the Riemann Hypothesis is true.

Proof. From (1.12) and from the point (ii) of Theorem 1.1, for $n \geq 2$,

$$\sqrt{\text{li}^{-1}(n)} - 1.045(n \log n)^{1/4} \leq \log h(n) \leq \sqrt{\text{li}^{-1}(n)} - 0.49 (n \log n)^{1/4}$$

which, with (1.5), proves (1.14). If the Riemann Hypothesis is false, (1.8) contradicts (1.14). □

1.1 Notation

- $\pi_r(x) = \sum_{p \leq x} p^r$. For $r = 0$, $\pi_0(x) = \pi(x) = \sum_{p \leq x} 1$ is the prime counting function.
- $\Pi_r(x) = \sum_{p^k \leq x} \frac{p^{rk}}{k} = \sum_{k=1}^{\kappa} \frac{\pi_{rk}(x^{1/k})}{k}$ with $\kappa = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$.
- $\theta(x) = \sum_{p \leq x} \log p$ and $\psi(x) = \sum_{p^k \leq x} \log p = \sum_{k=1}^{\kappa} \theta(x^{1/k})$ are the Chebyshev functions.
- $\Lambda(x) = \begin{cases} \log p & \text{if } x = p^k \\ 0 & \text{if not} \end{cases}$ is the von Mangoldt function.
- $(p_n)_{n \geq 1}$ is the sequence of prime numbers, where $p_1 = 2$.
- $\text{li}(x)$ denotes the logarithmic integral of x (cf. below §2.2), and li^{-1} the inverse function.
- $\gamma_0 = 0.57721566\dots$ is the Euler constant. The coefficients γ_m and δ_m are defined in §2.4.
- $\sum_{\rho} f(\rho) = \lim_{T \rightarrow \infty} \sum_{|\Im(\rho)| \leq T} f(\rho)$ where $f : \mathbb{C} \rightarrow \mathbb{C}$ is a complex function and ρ runs over the non-trivial roots of the Riemann ζ function.
- If $\lim_{n \rightarrow \infty} u_n = +\infty$, $v_n = \Omega_{\pm}(u_n)$ is equivalent to

$$\limsup_{n \rightarrow \infty} \frac{v_n}{u_n} > 0 \text{ and } \liminf_{n \rightarrow \infty} \frac{v_n}{u_n} < 0.$$

- We use the following constants:

$$x_0 = 10^{10} + 19 \text{ is the smallest prime exceeding } 10^{10}$$

$$n_0 = \pi_1(x_0) = 2\,220\,822\,442\,581\,729\,257 = 2.22\dots 10^{18}$$

$$L_0 = \log n_0 = 42.244\,409\,270\,801\,490\dots$$

$$\lambda_0 = \log L_0 = 3.743\,472\,020\,096\,020\dots \quad \nu_0 = \lambda_0/L_0 = 0.088614\dots$$

- Let us write $\sigma_0 = 0$, $N_0 = 1$, and, for $j \geq 1$,

$$N_j = p_1 p_2 \cdots p_j \quad \text{and} \quad \sigma_j = p_1 + p_2 + \cdots + p_j = \ell(N_j). \quad (1.15)$$

- For $n \geq 0$, let $k = k(n)$ denote the integer $k \geq 0$ such that

$$\sigma_k = p_1 + p_2 + \cdots + p_k \leq n < p_1 + p_2 + \cdots + p_{k+1} = \sigma_{k+1}. \quad (1.16)$$

In [7, Proposition 3.1], for $j \geq 1$, it is proved that

$$h(\sigma_j) = N_j. \quad (1.17)$$

We often implicitly use the following result: For u and v positive and w real, the function

$$t \mapsto \frac{(\log t - w)^u}{t^v} \quad \text{is decreasing for } t > \exp\left(w + \frac{u}{v}\right). \quad (1.18)$$

1.2 Plan of the article.

In §2, we recall several results and state some lemmas that are used in the proof of Theorem 1.1. §2.1 is devoted to effective estimates in prime number theory, §2.2 deals with the logarithmic integral while §2.3 give effective estimates for $\pi_r(x) = \sum_{p \leq x} p^r$ and more specially for $\pi_1(x)$. In §2.4 are recalled two explicit formulas (cf. (2.41) and (2.42)) of the Prime Number Theorem, some results about the roots of the Riemann ζ function, and the computation of the constant c (cf. (1.13)) is explained.

The computation of $h(n)$ plays an important role in the proof of our results. The algorithm described in [7] is shortly recorded in §3.

In §4, in preparation to the proof of Theorem 1.1, four lemmas about b_n (defined in (1.12)) will be given.

The proof of Theorem 1.1 is given in §5. It follows the lines of the proof of Theorem 1 of [17] about the asymptotic estimate, under the Riemann Hypothesis, of $\log g(n)$, starting from the explicit formula of $\Pi_1(x)$. But, here, we deal with effective estimates. The positive integers are split in three classes: the small ones ($\leq n_0 = \pi_1(10^{10} + 19)$) that are mainly treated by computation, the large ones $> n_0$ and, to prove the point (vi), those tending to infinity. In each class, the n 's belonging to the interval $[\sigma_k, \sigma_{k+1}]$ (where σ_k is defined by (1.16)) are considered globally because, from (1.17), $h(\sigma_k)$ is easy to evaluate, and, for $n \in [\sigma_k, \sigma_{k+1}]$, $h(n)$ remains close to $h(\sigma_k)$.

Effective estimates are more technical to get than the asymptotic ones. It was why Landau introduced his famous notation " \mathcal{O} " and " o ". But fortunately nowadays computer algebra systems help us.

On the web site [27], a Maple sheet is given, explaining the algebraic and numerical computations. The extensive computations described in §3.2 have been made in C^{++} .

2 Useful results

2.1 Effective estimates.

Platt and Trudgian [21] have shown by computation that

$$\theta(x) < (1 + \epsilon)x \text{ for } x \geq 2, \quad \text{with } \epsilon = 7.5 \times 10^{-7}, \quad (2.1)$$

so improving on results of Schoenfeld [24].

Without any hypothesis, one knows that

$$|\theta(x) - x| < \frac{\alpha x}{\log^3 x} \text{ for } x \geq x_1 = x_1(\alpha) \quad (2.2)$$

with

$$\alpha = \begin{cases} 1 & \text{and } x_1 = 89\,967\,803 & (\text{cf. [12, Theorem 4.2]}) \\ 0.5 & \text{and } x_1 = 767\,135\,587 & (\text{cf. [12, Theorem 4.2]}) \\ 0.15 & \text{and } x_1 = 19\,035\,709\,163 & (\text{cf. [3, Theorem 1.1]}) . \end{cases}$$

Under the Riemann Hypothesis, for $x \geq 599$, we shall use the upper bounds (cf. [24, (6.3)])

$$|\psi(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \log^2 x \quad \text{and} \quad |\theta(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \log^2 x. \quad (2.3)$$

Lemma 2.1. *Under the Riemann Hypothesis, for $x \geq 1$,*

$$\psi(x) - \sqrt{x} - \frac{4}{3}x^{1/3} \leq \theta(x) \leq \psi(x) - \sqrt{x} + 2.14. \quad (2.4)$$

Proof. In [20, Lemma 2.4] or in [22, Lemma 3], the above lower bound is given and $\theta(x) \leq \psi(x) - \sqrt{x}$ is proved for $x \geq 121$. It remains to check that, for $1 \leq x \leq 121$, $\theta(x) - \psi(x) + \sqrt{x} < \sqrt{8} - \log 2 = 2.1352\dots$ holds. \square

2.2 The logarithmic integral.

For x real > 1 , we define $\text{li}(x)$ as (cf. [1, p. 228])

$$\text{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \rightarrow 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right) = \int_2^x \frac{dt}{\log t} + \text{li}(2).$$

We have the following values:

x	1	1.45136...	1.96904...	2	e^2
$\text{li}(x)$	$-\infty$	0	1	1.04516...	4.95423...

From the definition of $\text{li}(x)$, it follows that

$$\frac{d}{dx} \text{li}(x) = \frac{1}{\log x} \quad \text{and} \quad \frac{d^2}{dx^2} \text{li}(x) = -\frac{1}{x \log^2 x}. \quad (2.5)$$

The function $t \mapsto \text{li}(t)$ is an increasing bijection from $(1, +\infty)$ onto $(-\infty, +\infty)$. We denote by $\text{li}^{-1}(y)$ its inverse function that is defined and increasing for all $y \in \mathbb{R}$. Note that $\text{li}^{-1}(y) > 1$ holds for all $y \in \mathbb{R}$.

To compute numerical values of $\text{li}(x)$, we used the following formula, due to Ramanujan (cf. [4, p. 126-131]),

$$\text{li}(x) = \gamma_0 + \log \log x + \sqrt{x} \sum_{n=1}^{\infty} a_n (\log x)^n \text{ with } a_n = \frac{(-1)^{n-1}}{n! 2^{n-1}} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2m+1}.$$

Let N be a positive integer and $s \geq 1$ a real number. We have

$$\int \frac{t^{s-1}}{\log^N t} dt = \frac{1}{(N-1)!} \left(s^{N-1} \text{li}(t^s) - \sum_{k=1}^{N-1} \frac{(k-1)! s^{N-1-k} t^s}{\log^k t} \right) \quad (2.6)$$

and, for $x \rightarrow \infty$,

$$\text{li}(x) = \sum_{k=1}^N \frac{(k-1)! x}{(\log x)^k} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1}}\right). \quad (2.7)$$

We shall need the following lemmas that gives bounds for the logarithmic integral.

Lemma 2.2. For $t > 4$,

$$\text{li}(t) > \frac{t}{\log t}. \quad (2.8)$$

For $t > 1$,

$$\text{li}(t) < t - 0.82 < t \quad (2.9)$$

$$\text{li}(t) < 1.49 \frac{t}{\log t}. \quad (2.10)$$

For $t \geq 10^{10}$,

$$\text{li}(t) < \frac{t}{\log t} + 1.101 \frac{t}{\log^2 t}. \quad (2.11)$$

Proof.

- For $t > 1$, the function $t \mapsto \text{li}(t) - t/\log t$ is increasing and vanishes for $t = 3.846\dots$
- The function $t \mapsto t - \text{li}(t)$ is minimal for $t = e$ and $e - \text{li}(e) = 0.823\dots$
- The maximum of $t \mapsto \text{li}(t) - 1.49 t/\log t$ is $-0.04\dots$, obtained for $t = \exp(1.49/0.49)$.
- The function $t \mapsto \text{li}(t) - t/\log t - 1.101 t/\log^2 t$ is decreasing for $t > 2.95 \times 10^9$ and its value for $t = 10^{10}$ is $-5015.15\dots < 0$.

□

Lemma 2.3. For $t > 77$,

$$\text{li}(t) > \frac{t}{\log t} + \frac{t}{\log^2 t} + \frac{2t}{\log^3 t} + \frac{6t}{\log^4 t}, \quad (2.12)$$

for $t > 4.96 \times 10^{12}$

$$\text{li}(t) < \frac{t}{\log t} + \frac{t}{\log^2 t} + \frac{2t}{\log^3 t} + \frac{7t}{\log^4 t} \quad (2.13)$$

and for $t > 1$

$$\text{li}(t) < \frac{t}{\log t} + \frac{t}{\log^2 t} + \frac{2t}{\log^3 t} + \frac{40}{3} \frac{t}{\log^4 t}. \quad (2.14)$$

Proof. For $u \in \{6, 7, 40/3\}$, we set

$$f = \text{li}(t) - \frac{t}{\log t} - \frac{t}{\log^2 t} - \frac{2t}{\log^3 t} - u \frac{t}{\log^4 t}.$$

From (2.5), one gets

$$\frac{df}{dt} = \frac{(6-u)\log t + 4u}{\log^5 t}.$$

- For $u = 6$, f is increasing and vanishes for $t = 76.54\dots$ which proves (2.12).
- For $u = 7$, f is increasing for $t < t_0 = \exp(28) = 1.446\dots \times 10^{12}$ and decreasing for $t > t_0$. One computes $f(4.96 \times 10^{12}) = -259.07\dots < 0$ and (2.13) follows.
- For $u = 40/3$, f is increasing for $t < t_1 = \exp(80/11) = 1440.47\dots$ and decreasing for $t > t_1$. Therefore, (2.14) results from the negativity of $f(t_1) = -0.0033\dots$

□

Lemma 2.4. *If $t \geq 3.28$,*

$$\text{li}^{-1}(t) < t(\log t + \log \log t), \quad (2.15)$$

for $t > 41$,

$$\text{li}^{-1}(t) > t \log t \quad (2.16)$$

and, for $t > 12\,218$,

$$\text{li}^{-1}(t) > t(\log t + \log \log t - 1). \quad (2.17)$$

Proof.

- For $t \geq e$, let us consider the function $f(t) = \text{li}(t(\log t + \log \log t)) - t$. By noting $\log t$ by L , we have

$$\frac{df}{dt} = \frac{\log t + 1 + \log \log t + 1/\log t}{\log(t(\log t + \log \log t))} - 1 = \frac{L + 1 - L \log(1 + \log L/L)}{L^2 + L \log(L + \log L)}.$$

The denominator is ≥ 1 and the numerator is $\geq L + 1 - \log L \geq L + 1 - (L - 1) = 2 > 0$. So f is increasing and its value for $t = 3.28$ is $0.0073\dots > 0$, which completes the proof of (2.15).

- Now, let us consider $f(t) = \text{li}(t \log t) - t$. One has

$$f'(t) = \frac{\log t + 1}{\log(t \log t)} - 1 = \frac{1 - \log \log t}{\log t + \log \log t} < 0$$

for $t > e^e = 15.15\dots$, which shows that f is decreasing for $t > e^e$ and, from $f(41) = -0.048\dots < 0$, we get (2.16).

- Finally, for $t > 1$, we set $f(t) = t(\log t + \log \log t - 1)$. One has $f'(t) = \log t + \log \log t + 1/\log t$ which is positive for $t > e$ so that f is increasing for $t > e$. As $f(t_0) = 1$ for $t_0 = 3.1973\dots$, we assume $t > t_0$ so that $f(t) > 1$, $L = \log t > 1$ and $\log L > 0$ hold. We set

$$y = t - \text{li}(f(t)) = t - \text{li}(t(\log t + \log \log t - 1))$$

and, by using the inequality $\log(1 + u) \geq u/(1 + u)$ (for $u > -1$), one gets

$$\begin{aligned} y' \log f(t) &= \log \left(1 + \frac{\log L - 1}{L} \right) - \frac{1}{L} \\ &\geq \frac{\log L - 1}{L(1 + (\log L - 1)/L)} - \frac{1}{L} = \frac{(L - 1)(\log L - 2) - 1}{L(L + \log L - 1)}. \end{aligned}$$

For $t > e^{e^2} = 1618.17\dots$, the denominator is positive. The numerator is increasing, and positive for $t = 4678$. Therefore, y is increasing for $t > 4678$. It remains to calculate $y(12218) = 0.00106\dots > 0$ to prove (2.17).

□

Lemma 2.5. *The function $t \mapsto \sqrt{\text{li}^{-1}(t)}$ is defined and increasing for $t \in \mathbb{R}$.*

– *It is concave for $t > \text{li}(e^2) = 4.954\dots$*

– *Let $a \leq 1$ be a real number. For $t \geq 31$, the function $t \mapsto \sqrt{\text{li}^{-1}(t)} - a(t(\log t))^{1/4}$ is concave.*

Proof.

– Let us set $f_1 = \sqrt{\text{li}^{-1}(t)}$, $f_2 = (t(\log t))^{1/4}$, $F = f_1 - af_2$ and $u = \text{li}^{-1}(t)$ i.e. $t = \text{li}(u)$. We have

$$\frac{df_1}{dt} = \frac{\log u}{2\sqrt{u}}, \quad \frac{d^2f_1}{dt^2} = -\frac{\log u(\log u - 2)}{4u^{3/2}}, \quad \frac{df_2}{dt} = -\frac{3\log^2 t + 2\log t + 3}{16(t\log t)^{7/4}}.$$

Let us assume $t > \text{li}(e^2)$. We have $u > e^2$, $\log u > 2$ and $\frac{d^2f_1}{dt^2} < 0$ so that f_1 is concave.

– Further, $\frac{d^2f_2}{dt^2} < 0$ so that, if $a \leq 0$ then $F = f_1 - af_2$ is concave. Moreover, from (2.9) and (2.8), we have $u/\log u < t = \text{li} u < u$ and

$$0 < -\frac{d^2f_2}{dt^2} \leq \frac{3\log^2 u + 2\log u + 3}{16(u(1 - (\log \log u)/\log u))^{7/4}}.$$

If $0 < a \leq 1$ holds, it suffices to show that $\left| \frac{d^2f_2}{dt^2} / \frac{d^2f_1}{dt^2} \right| < 1$. By writing L for $\log u$, one gets

$$\begin{aligned} \left| \frac{d^2f_2}{dt^2} / \frac{d^2f_1}{dt^2} \right| &\leq \frac{1}{4u^{1/4}} \left(1 - \frac{\log L}{L} \right)^{-7/4} \left(\frac{3L^2 + 2L + 3}{L(L - 2)} \right) \\ &= \frac{1}{4u^{1/4}} \left(1 - \frac{\log L}{L} \right)^{-7/4} \left(3 + \frac{8}{L} + \frac{19}{L(L - 2)} \right). \quad (2.18) \end{aligned}$$

The three factors of the right handside of (2.18) are positive and decreasing on u so that their product is decreasing, and for $u = 103$, $t = 30.77\dots$, it is < 1 . □

Remark. By using more accurate inequalities, it would be possible to replace the bound $t \geq 31$ by $t \geq 8.42\dots$

2.3 Study of $\pi_r(x) = \sum_{p \leq x} p^r$.

Without any hypothesis, improving on results of Massias and Robin about the bounds of $\pi_r(x) = \sum_{p \leq x} p^r$ (cf. [19, Théorème D]), by using recent improvements on effective estimates of $\theta(x)$, we prove

Proposition 2.6. *Let $\alpha, x_1 = x_1(\alpha)$ be two real numbers such that $0 < \alpha \leq 1, x_1 \geq 89\,967\,803$ and $|\theta(x) - x| < \alpha x / \log^3 x$ for $x \geq x_1$. Then, for $r \geq 0.6$ and $x \geq x_1$,*

$$\pi_r(x) \leq C_0 + \frac{x^{r+1}}{(r+1)\log x} + \frac{x^{r+1}}{(r+1)^2 \log^2 x} + \frac{2x^{r+1}}{(r+1)^3 \log^3 x} + \frac{(51\alpha r^4 + 176\alpha r^3 + 222\alpha r^2 + 120\alpha r + 23\alpha + 168)x^{r+1}}{24(r+1)^4 \log^4 x} \quad (2.19)$$

with

$$C_0 = \pi_r(x_1) - \frac{x_1^r \theta(x_1)}{\log x_1} - \frac{3\alpha r^4 + 8\alpha r^3 + 6\alpha r^2 + 24 - \alpha}{24} \text{li}(x_1^{r+1}) + \frac{(3\alpha r^3 + 5\alpha r^2 + \alpha r + 24 - \alpha)x_1^{r+1}}{24 \log x_1} + \frac{\alpha(3r^2 + 2r - 1)x_1^{r+1}}{24 \log^2 x_1} + \frac{\alpha(3r - 1)x_1^{r+1}}{12 \log^3 x_1} - \frac{\alpha x_1^{r+1}}{4 \log^4 x_1}. \quad (2.20)$$

Let $r_0(\alpha)$ be the unique positive root of the equation $3r^4 + 8r^3 + 6r^2 - 24\alpha - 1 = 0$. One has $r_0(\alpha) \geq r_0(1) = 1.1445\dots$ and,

for $0.06 \leq r \leq r_0(\alpha)$ and $x \geq x_1(\alpha)$, we have

$$\pi_r(x) \geq \widehat{C}_0 + \frac{x^{r+1}}{(r+1)\log x} + \frac{x^{r+1}}{(r+1)^2 \log^2 x} + \frac{2x^{r+1}}{(r+1)^3 \log^3 x} - \frac{(2\alpha r^4 + 7\alpha r^3 + 9\alpha r^2 + 5\alpha r + \alpha - 6)x^{r+1}}{(r+1)^4 \log^4 x} \quad (2.21)$$

while, if $r > r_0(\alpha)$ and $x \geq x_1(\alpha)$,

$$\pi_r(x) \geq \widehat{C}_0 + \frac{x^{r+1}}{(r+1)\log x} + \frac{x^{r+1}}{(r+1)^2 \log^2 x} + \frac{2x^{r+1}}{(r+1)^3 \log^3 x} - \frac{(51\alpha r^4 + 176\alpha r^3 + 222\alpha r^2 + 120\alpha r + 23\alpha - 168)x^{r+1}}{24(r+1)^4 \log^4 x}, \quad (2.22)$$

with

$$\widehat{C}_0 = \pi_r(x_1) - \frac{x_1^r \theta(x_1)}{\log x_1} + \frac{3\alpha r^4 + 8\alpha r^3 + 6\alpha r^2 - \alpha - 24}{24} \text{li}(x_1^{r+1}) - \frac{(3\alpha r^3 + 5\alpha r^2 + \alpha r - \alpha - 24)x_1^{r+1}}{24 \log x_1} - \frac{\alpha(3r^2 + 2r - 1)x_1^{r+1}}{24 \log^2 x_1} - \frac{\alpha(3r - 1)x_1^{r+1}}{12 \log^3 x_1} + \frac{\alpha x_1^{r+1}}{4 \log^4 x_1}. \quad (2.23)$$

Proof. It is convenient to set

$$s = r + 1.$$

By Stieltjes integral, we have

$$\pi_r(x) = \sum_{p \leq x} p^r = \pi_{s-1}(x) = \pi_{s-1}(x_1) + \int_{x_1}^x \frac{t^{s-1}}{\log t} d[\theta(t)]$$

and, by partial integration,

$$\begin{aligned} \pi_{s-1}(x) = \pi_{s-1}(x_1) + \frac{x^{s-1}\theta(x)}{\log x} - \frac{x_1^{s-1}\theta(x_1)}{\log x_1} \\ - \int_{x_1}^x \left(\frac{(s-1)t^{s-2}}{\log t} - \frac{t^{s-2}}{\log^2 t} \right) \theta(t) dt. \end{aligned} \quad (2.24)$$

Since $x \geq x_1(\alpha)$ holds, in (2.24), from our assumption, we have $\theta(x) \leq x + \alpha x / \log^3 x$. Under the integral sign, as $s \geq 1 + 1/\log x_1(\alpha) \geq 1 + 1/\log(89967803) = 1.054\dots$, the parenthesis is positive and $\theta(t) \geq t - \alpha t / \log^3 t$, which implies

$$\pi_{s-1}(x) \leq \pi_{s-1}(x_1) - \frac{x_1^{s-1}\theta(x_1)}{\log x_1} + \frac{x^s}{L} + \frac{\alpha x^s}{L^4} - (s-1)I_1 + I_2 + (s-1)\alpha I_4 - \alpha I_5, \quad (2.25)$$

with $L = \log x$ and, for $i \geq 1$, $I_i = \int_{x_1}^x \frac{t^{s-1}}{\log^i t} dt = f_i(x) - f_i(x_1)$ with $f_i(t) = \int \frac{t^{s-1}}{\log^i t}$. By (2.6), one gets

$$\begin{aligned} f_1 &= \text{li}(t^s), & f_2 &= s \text{li}(t^s) - \frac{t^s}{\log t}, & f_3 &= \frac{s^2}{2} \text{li}(t^s) - \frac{s t^s}{2 \log t} - \frac{t^s}{2 \log^2 t}, \\ f_4 &= \frac{s^3 \text{li}(t^s)}{6} - \frac{s^2 t^s}{6 \log t} - \frac{s t^s}{6 \log^2 t} - \frac{t^s}{3 \log^3 t}, \\ f_5 &= \frac{s^4 \text{li}(t^s)}{24} - \frac{s^3 t^s}{24 \log t} - \frac{s^2 t^s}{24 \log^2 t} - \frac{s t^s}{12 \log^3 t} - \frac{t^s}{4 \log^4 t}. \end{aligned}$$

Let us set

$$\begin{aligned} f(t) = -(s-1)f_1 + f_2 + (s-1)\alpha f_4 - \alpha f_5 = \frac{3\alpha s^4 - 4\alpha s^3 + 24}{24} \text{li}(t^s) \\ - \frac{(3\alpha s^3 - 4\alpha s^2 + 24)t^s}{24 \log t} - \frac{\alpha s(3s-4)t^s}{24 \log^2 t} - \frac{\alpha(3s-4)t^s}{12 \log^3 t} + \frac{\alpha t^s}{4 \log^4 t}. \end{aligned} \quad (2.26)$$

From (2.25), one has

$$\pi_{s-1}(x) \leq C_0 + \frac{x^s}{L} + \frac{\alpha x^s}{L^4} + f(x) \quad \text{with} \quad C_0 = \pi_{s-1}(x_1) - \frac{x_1^{s-1}\theta(x_1)}{\log x_1} - f(x_1).$$

Now, $s = r + 1 \geq 1.6$, $x^s \geq x_1(\alpha)^{1.6} > 89967803^{1.6} > 4.96 \cdot 10^{12}$ and one may use the upper bound (2.13) of $\text{li}(x^s)$ in (2.26) to get

$$\pi_{s-1}(x) \leq C_0 + \frac{x^s}{sL} + \frac{x^s}{s^2 L^2} + \frac{2x^s}{s^3 L^3} + \frac{(51\alpha s^4 - 28\alpha s^3 + 168)x^s}{24s^4 L^4} \quad (2.27)$$

which, by substituting $r + 1$ to s , proves (2.19) and (2.20).

To get a lower bound for $\pi_{s-1}(x)$, in (2.24), we use inequalities $\theta(x) \geq x - \alpha x / L^3$, and $\theta(t) \leq t + \alpha t / \log^3 t$. One gets

$$\begin{aligned} \widehat{f}(t) = -(s-1)f_1 + f_2 - (s-1)\alpha f_4 + \alpha f_5 = \frac{-3\alpha s^4 + 4\alpha s^3 + 24}{24} \text{li}(t^s) \\ + \frac{(3\alpha s^3 - 4\alpha s^2 - 24)t^s}{24 \log t} + \frac{\alpha s(3s-4)t^s}{24 \log^2 t} + \frac{\alpha(3s-4)t^s}{12 \log^3 t} - \frac{\alpha t^s}{4 \log^4 t} \end{aligned} \quad (2.28)$$

(note that $\widehat{f}(t)$ is obtained by substituting $-\alpha$ to α in (2.26)) and

$$\pi_{s-1}(x) \geq \widehat{C}_0 + \frac{x^s}{L} - \frac{\alpha x^s}{L^4} + \widehat{f}(x) \quad \text{with} \quad \widehat{C}_0 = \pi_{s-1}(x_1) - \frac{x_1^{s-1}\theta(x_1)}{\log x_1} - \widehat{f}(x_1). \quad (2.29)$$

Let us set $\varphi(r) = 3r^4 + 8r^3 + 6r^2 - 24/\alpha - 1$, we have $\varphi'(r) = 12r(r+1)^2$, φ is minimal and negative for $r = 0$ and has one negative and one positive root, $r_0(\alpha)$. Note that $r_0(\alpha)$ is decreasing on α . One computes $r_0(1) = 1.1445\dots$, $r_0(0.5) = 1.4377\dots$ and $r_0(0.15) = 2.1086\dots$

The coefficient of $\text{li}(x^s)$ in $\widehat{f}(x)$ is

$$\frac{-3\alpha s^4 + 4\alpha s^3 + 24}{24} = \frac{-3\alpha r^4 - 8\alpha r^3 - 6\alpha r^2 + \alpha + 24}{24} = -\frac{\alpha \varphi(r)}{24}$$

and changes of sign for $r = r_0(\alpha)$. For $0.06 \leq r \leq r_0(\alpha)$ we have $x^s \geq x_1^s \geq x_1^{1.06} > 77$ and we use the lower bound (2.12) of $\text{li}(x^s)$ in $\widehat{f}(x)$ to get (2.21), while, for $r > r_0(\alpha)$, $x^s \geq x_1(\alpha)^{2.14} > 89967803^{2.14} > 4.96 \cdot 10^{12}$ and we use (2.13) to get (2.22). \square

Corollary 2.7. For $x \geq 110\,117\,910$,

$$\pi_1(x) \leq \frac{x^2}{2 \log x} + \frac{x^2}{4 \log^2 x} + \frac{x^2}{4 \log^3 x} + \frac{107 x^2}{160 \log^4 x} \quad (2.30)$$

and, for $x \geq 905\,238\,547$,

$$\pi_1(x) \geq \frac{x^2}{2 \log x} + \frac{x^2}{4 \log^2 x} + \frac{x^2}{4 \log^3 x} + \frac{3x^2}{20 \log^4 x}. \quad (2.31)$$

Proof. We choose $r = 1$, $\alpha = 0.15$, $x_1 = 19\,035\,709\,163$ and, from (2.2), we apply (2.19). By computation we get $\pi_1(x_1) = 7\,823\,414\,443\,039\,054\,263$,

$$\theta(x_1) = 19\,035\,493\,858.482\,419\,137\dots, \quad f(x_1) = -7.485\,421\,258\dots \times 10^{18}$$

and C_0 , defined by (2.20) with $r = 1$ is equal to $-1.586\dots \times 10^{13} < 0$ so that (2.30) follows from (2.19) for $x \geq x_1$ and, by computation, for $110\,117\,909 \leq x < x_1$.

Similarly, \widehat{C}_0 defined by (2.23) is equal to $1.655\dots \times 10^{14} > 0$ which implies (2.31) from (2.21) for $x \geq x_1$ and by computation for $905\,238\,546 \leq x < x_1$. \square

Remark. In [2, Theorem 6.7 and Proposition 6.9], C. Axler gives similar estimates for $\pi_1(x)$.

Lemma 2.8. Let us assume that $x \geq x_0 = 10^{10} + 19$ and $n = \pi_1(x)$ hold. Then x satisfies

$$\sqrt{n \log n} \left(1 + 0.365 \frac{\log \log n}{\log n} \right) \leq x \leq \sqrt{n \log n} \left(1 + \frac{\log \log n}{2 \log n} \right). \quad (2.32)$$

Proof. When $x \rightarrow \infty$, from $n = \pi_1(x) = \text{li}(x^2) + \mathcal{O}(x^2 \exp(-a \log x))$ with $a > 0$ (cf. [17, Lemme B]), one can see that the asymptotic expansion of x is given by (1.11). In particular,

$$x = \sqrt{n \log n} \left(1 + \frac{\log \log n - 1 + o(1)}{2 \log n} \right), \quad n \rightarrow \infty. \quad (2.33)$$

Now, we have to prove the effective bounds (2.32) of x . For convenience, we write L for $\log n$ and λ for $\log \log n$. We suppose $x \geq x_0 = 10^{10} + 19$. We have $n \geq n_0 = \pi_1(x_0) = 2.22 \dots 10^{18}$, $L = \log n > 42.24$ and $\lambda = \log \log n > 3.74$.

The upper bound. Let us note $f(n) = \sqrt{nL} \left(1 + \frac{\lambda}{2L}\right)$.

Since $\frac{t^2}{2 \log t} \left(1 + \frac{1}{2 \log t}\right)$ is increasing as a function of t for $t > e$, the inequality $x \leq f(n)$ is equivalent to

$$\frac{x^2}{2 \log x} \left(1 + \frac{1}{2 \log x}\right) \leq \frac{f(n)^2}{2 \log f(n)} \left(1 + \frac{1}{2 \log f(n)}\right). \quad (2.34)$$

From (2.31), for $x \geq x_0$, $\frac{x^2}{2 \log x} \left(1 + \frac{1}{2 \log x}\right) \leq \pi_1(x) = n$. Note that this result has been proved in [3, Corollary 6.10] for $x \geq 302971$. Thus to ensure (2.34) it suffices to prove

$$n < \frac{f(n)^2}{2 \log f(n)} \left(1 + \frac{1}{2 \log f(n)}\right).$$

As we have $2 \log f(n) = L + \lambda + 2 \log(1 + \lambda/(2L)) \leq L + \lambda + \lambda/L$, it suffices to show that

$$nL \frac{(1 + \lambda/(2L))^2}{L + \lambda + \lambda/L} \left(1 + \frac{1}{L + \lambda + \lambda/L}\right) > n$$

or, equivalently that

$$L(1 + \lambda/(2L))^2(L + \lambda + \lambda/L + 1) - (L + \lambda + \lambda/L)^2 > 0.$$

But the above left hand side is equal to

$$L + \frac{\lambda^2}{4} \left(1 - \frac{3}{L} - \frac{4}{L^2}\right) + \frac{\lambda^3}{4L} \left(1 + \frac{1}{L}\right),$$

which is positive for $L \geq 4$, i.e. for $n \geq e^4$.

The lower bound. First, from (2.30), for $x \geq x_0$,

$$\begin{aligned} n = \pi_1(x) &\leq \frac{x^2}{2 \log x} + \frac{x^2}{4 \log^2 x} \left(1 + \frac{1}{\log x_0} + \frac{107}{40 \log^2 x_0}\right) \\ &\leq \frac{x^2}{2 \log x} \left(1 + \frac{a}{2 \log x}\right) \end{aligned} \quad (2.35)$$

with $a = 1.049$. This time, we set $f(n) = \sqrt{nL}(1 + b\lambda/L)$, with $b = 0.365$. One has $2 \log f(n) = L + \lambda + 2 \log(1 + b\lambda/L)$. By using the inequality $\log(1 + u) \geq u/(1 + u_0)$ valid for $0 \leq u \leq u_0$, one has

$$2 \log f(n) \geq L + \lambda + c_0 \lambda/L \quad \text{with} \quad c_0 = 0.7 < 2b/(1 + b\lambda_0/L_0) = 0.707 \dots \quad (2.36)$$

We have to prove that $x \geq f(n)$ holds for $n \geq n_0$. From the increasingness of the mapping $t \mapsto \frac{t^2}{2 \log t} \left(1 + \frac{a}{2 \log t}\right)$, it suffices to show that

$$\frac{x^2}{2 \log x} \left(1 + \frac{a}{2 \log x}\right) \geq \frac{f(n)^2}{2 \log f(n)} \left(1 + \frac{a}{2 \log f(n)}\right). \quad (2.37)$$

From (2.35) and (2.36), to prove (2.37), it suffices to prove

$$n \geq \frac{nL(1 + b\lambda/L)^2}{L + \lambda + c_0\lambda/L} \left(1 + \frac{a}{L + \lambda + c_0\lambda/L} \right)$$

i.e.

$$L \left(1 + \frac{b\lambda}{L} \right)^2 \left(L + \lambda + \frac{c_0\lambda}{L} + a \right) - \left(L + \lambda + \frac{c_0\lambda}{L} \right)^2 \leq 0 \quad (2.38)$$

and equivalently, by expanding (2.38) and dividing by λL , that

$$2b - 1 + \frac{a}{\lambda} + \frac{(b^2 + 2b - 1)\lambda + 2ab}{L} + \frac{b^2\lambda^2 + ab^2\lambda}{L^2} + c_0 \left(-\frac{1}{L} + \frac{2\lambda(b-1)}{L^2} + \frac{b^2\lambda^2}{L^3} \right) - \frac{c_0^2\lambda}{L^3} \leq 0. \quad (2.39)$$

The coefficient of c_0 in (2.39) satisfies

$$c_0 \left(-\frac{1}{L} + \frac{2\lambda(b-1)}{L^2} + \frac{b^2\lambda^2}{L^3} \right) \leq -\frac{c_0}{L} + \frac{c_0\lambda}{L^2} \left(2b + \frac{b^2\lambda_0}{L_0} - 2 \right) \leq -\frac{c_0}{L} - \frac{d\lambda}{L^2}$$

with $d = 0.88 < c_0(2 - 2b - b^2\lambda_0/L_0) = 0.8807\dots$ so that it suffices to show that

$$B = 2b - 1 + \frac{a}{\lambda} + \frac{(b^2 + 2b - 1)\lambda + 2ab}{L} + \frac{b^2\lambda^2 + (ab^2 - d)\lambda}{L^2} - \frac{c_0}{L} \leq 0,$$

for $L = \exp(\lambda)$ and $\lambda \geq \lambda_0$. For that, one writes $c_0 = c_1 + c_2 + c_3$ with $c_1 = 0.44$ and $c_2 + c_3 = 0.26$.

Then,

$$B = \left[2b - 1 + \frac{a}{\lambda} - \frac{c_1}{L} \right] + \frac{(b^2 + 2b - 1)\lambda + 2ab - c_2}{L} + \frac{b^2\lambda^2 + (ab^2 - d)\lambda - c_3L}{L^2}. \quad (2.40)$$

It is easy to see that $a/\lambda - c_1/L = 1.049/\lambda - 0.44e^{-\lambda}$ is decreasing for $\lambda > 0$ and its value for $\lambda = \lambda_0$ is equal to $0.2698\dots$, so that the square bracket in (2.40) is negative.

For $\lambda_0 \leq \lambda \leq 4.3$ one chooses $c_2 = 0.26$, $c_3 = 0$ and one has

$$(b^2 + 2b - 1)\lambda + 2ab - c_2 \leq (b^2 + 2b - 1)\lambda_0 + 2ab - c_2 = -0.0062\dots < 0$$

and $b^2\lambda + (ab^2 - d) \leq 4.3b^2 + (ab^2 - d) = -0.167\dots$, so that B is negative.

For $\lambda > 4.3$, one chooses $c_2 = 0.18$, $c_3 = 0.08$ and one has

$$(b^2 + 2b - 1)\lambda + 2ab - c_2 < 4.3(b^2 + 2b - 1) + 2ab - c_2 = -0.0023\dots < 0.$$

The inequality $\lambda^2 \leq 4e^{\lambda-2} = 4L/e^2$ implies

$$b^2\lambda^2 - c_3L \leq (4b^2e^{-2} - c_3)L = -0.0078\dots L < 0$$

and, as we also have $ab^2 - d = -0.74\dots < 0$, we conclude that B is still negative, which completes the proof of Lemma 2.8. \square

2.4 The Riemann ζ function and explicit formulas for ψ and Π_1 .

2.4.1 Explicit formulas.

We shall use the two explicit formulas

$$\psi(x) = x + \frac{\Lambda(x)}{2} - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2} \log \left(1 - \frac{1}{x^2} \right), \quad x > 1 \quad (2.41)$$

(cf. [16, p. 334 and p. 353] with $r = 0$ and $\zeta'(0)/\zeta(0) = \log(2\pi)$) and

$$\Pi_1(x) = \text{li}(x^2) + \frac{x\Lambda(x)}{2 \log x} - \sum_{\rho} \int_{-1}^{\infty} \frac{x^{\rho-t}}{\rho-t} dt - \log 12 + \int_x^{\infty} \frac{dt}{(t^2-1) \log t}, \quad x > 1 \quad (2.42)$$

(cf. [16, p. 360 and 361], with $R = 1$ and $\zeta(-1) = -1/12$).

In connection with (2.41) we shall use the following lemma (cf. [15, p. 169 Théorème 5.8.(b)] or [14, p. 162 Theorem 5.8.(b)]):

Lemma 2.9. *If a, b are fixed real numbers satisfying $1 \leq a < b < \infty$, and g any function with a continuous derivative on the interval $[a, b]$, then*

$$\int_a^b g(t) \psi(t) dt = \int_a^b g(t) \left[t - \log(2\pi) - \frac{1}{2} \log \left(1 - \frac{1}{t^2} \right) \right] dt - \sum_{\rho} \int_a^b g(t) \frac{t^{\rho}}{\rho} dt. \quad (2.43)$$

We also have (cf. [13, p. 67] or [5, p. 272])

$$\sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma_0}{2} - \frac{1}{2} \log \pi - \log 2 = 0.023\,095\,708\,966\,121\,033\dots$$

and

$$\sum_{\rho} \frac{1}{\rho(1-\rho)} = \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{1-\rho} \right) = 2 \sum_{\rho} \frac{1}{\rho} = 0.046\,191\,417\,932\,2420\dots \quad (2.44)$$

The coefficients γ_m are defined by the Laurent expansion of $\zeta(s)$ around 1 (cf. [5, §10.3.5]) :

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=0}^{\infty} \frac{\gamma_m}{m!} (s-1)^m.$$

The first values of γ_m are

$m =$	0	1	2	3	4
$\gamma_m =$	0.57721...	-0.07281...	-0.00969...	0.00205...	0.00232...

The coefficients δ_m are defined by $\delta_1 = \gamma_0$, $\delta_2 = 2\gamma_1 + \gamma_0^2$, and, for $m \geq 1$,

$$\delta_{m+1} = (m+1) \frac{\gamma_m}{m!} + \sum_{j=0}^{m-1} \frac{\gamma_j \delta_{m-j}}{j!}.$$

These coefficients allow to compute the sums $\sum_{\rho} \frac{1}{\rho^m}$, see [5, p. 207 and 272]:

$$\sum_{\rho} \frac{1}{\rho^m} = 1 + \delta_m - \zeta(m) \left(1 - \frac{1}{2^m}\right), \quad m \geq 2. \quad (2.45)$$

For $m = 2$, we get

$$\sum_{\rho} \frac{1}{\rho^2} = 1 - \frac{\pi^2}{8} + 2\gamma_1 + \gamma_0^2 = -0.0461543172958046\dots$$

2.4.2 Computation of $\sum_{\rho} 1/|\rho(1+\rho)|$ and $\sum_{\rho} 1/|\Im\rho|^2$.

It is known (cf. [28]), that every non trivial root ρ of ζ satisfies

$$|\Im(\rho)| > 14.13472514173469379. \quad (2.46)$$

Lemma 2.10. *Under the Riemann Hypothesis, for $k \geq 2$,*

$$\sum_{\rho} \frac{1}{|\rho|^k} \leq \frac{10}{14^k}. \quad (2.47)$$

Proof. Under the Riemann Hypothesis, $\bar{\rho} = 1 - \rho$ and from (2.44),

$$\sum_{\rho} \frac{1}{|\rho|^2} = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 0.04619141\dots \leq \frac{1}{20}. \quad (2.48)$$

Using (2.46), we may write

$$\sum_{\rho} \frac{1}{|\rho|^k} \leq \frac{1}{14^{k-2}} \sum_{\rho} \frac{1}{|\rho|^2} \leq \frac{196}{20 \times 14^k}$$

which proves (2.47). □

Lemma 2.11. *Let t be a complex number satisfying $|t| < 1/2$. We have*

$$f(t) = ((1-t^2)(1-2t))^{-1/2} = \sum_{n=0}^{\infty} c_n t^n \quad \text{with} \quad 0 \leq c_n \leq \frac{4}{3} 2^n \quad (2.49)$$

and, if $|t| \leq 1/6$,

$$\Re(f(t)) \geq \frac{1}{3} \quad \text{and} \quad |\Im(f(t))| \leq \frac{2}{3}. \quad (2.50)$$

Proof. We have $(1-t)^{-1/2} = \sum_{n \geq 0} a_n t^n$ with

$$0 \leq a_n = (-1)^n \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-1}{2})}{n!} = \frac{1}{2^{2n}} \binom{2n}{n} \leq 1.$$

Therefore,

$$0 \leq c_n = \sum_{m=0}^{n/2} a_m (2^{n-2m} a_{n-2m}) \leq 2^n \sum_{m=0}^{\infty} \frac{1}{4^m} = \frac{2^{n+2}}{3},$$

which proves (2.49). If $|t| \leq 1/6$, then

$$\left| \sum_{n=1}^{\infty} c_n t^n \right| \leq \sum_{n=1}^{\infty} \frac{c_n}{6^n} \leq \frac{4}{3} \sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^n = \frac{2}{3}$$

whence

$$\Re(f(t)) = 1 + \Re\left(\sum_{n=1}^{\infty} c_n t^n\right) \geq 1 - \left|\sum_{n=1}^{\infty} c_n t^n\right| \geq 1 - \frac{2}{3} = \frac{1}{3}$$

and

$$|\Im(f(t))| = \left|\Im\left(\sum_{n=1}^{\infty} c_n t^n\right)\right| \leq \left|\sum_{n=1}^{\infty} c_n t^n\right| \leq \frac{2}{3}$$

which completes the proof of Lemma 2.11. \square

Lemma 2.12. *Under the Riemann Hypothesis, with the notation of (2.49), we have*

$$\sum_{\rho} \frac{1}{|\rho(1+\rho)|} = -\sum_{n=0}^{\infty} c_n \sum_{\rho} \frac{1}{\rho^{n+2}}. \quad (2.51)$$

Proof. Let $\rho = 1/2 + \nu\gamma$ ZZZ be a non trivial root of $\zeta(s)$ under the Riemann Hypothesis. First we observe that f defined by (2.49) satisfies

$$\begin{aligned} \left(-\frac{1}{\rho^2} f\left(\frac{1}{\rho}\right)\right)^2 &= \frac{1}{\rho^4(1-1/\rho^2)(1-2/\rho)} = \frac{1}{\rho(1-\rho)(\rho+1)(2-\rho)} \\ &= \frac{1}{|\rho(1+\rho)|^2} \end{aligned} \quad (2.52)$$

so that $-f(1/\rho)/\rho^2$ is real. Let us write

$$f\left(\frac{1}{\rho}\right) = a + bi.$$

As, by (2.46), $|1/\rho| < 1/14$, Lemma 2.11 gives $a \geq 1/3$, $|b| \leq 2/3$ and

$$-\frac{1}{\rho^2} f\left(\frac{1}{\rho}\right) = -\frac{a+bi}{(1/2+\nu\gamma)^2} = \frac{(\gamma^2 - 1/4 + \nu\gamma)(a+bi)}{(1/4 + \gamma^2)^2}.$$

Thus the sign of $-f(1/\rho)/\rho^2$ is the sign of $a(\gamma^2 - 1/4) - b\gamma$. As

$$\begin{aligned} a(\gamma^2 - 1/4) - b\gamma &\geq \frac{1}{3} \left(\gamma^2 - \frac{1}{4}\right) - \frac{2}{3} |\gamma| \\ &= \frac{1}{3} \left(|\gamma| - \frac{2+\sqrt{5}}{2}\right) \left(|\gamma| - \frac{2-\sqrt{5}}{2}\right) > 0 \end{aligned}$$

we have $-f(1/\rho)/\rho^2 > 0$, which, with (2.52), shows that

$$\frac{1}{|\rho(1+\rho)|} = -\frac{1}{\rho^2} f\left(\frac{1}{\rho}\right).$$

Therefore, from Lemma 2.11, we get

$$\sum_{\rho} \frac{1}{|\rho(1+\rho)|} = -\sum_{\rho} \frac{1}{\rho^2} \left(\sum_{n=0}^{\infty} \frac{c_n}{\rho^n}\right) \quad (2.53)$$

and, since from Lemmas 2.10 and 2.11, the sum $\sum_{\rho,n} \frac{c_n}{|\rho|^{n+2}}$ is finite, we may permute the summations in (2.53), which yields (2.51). \square

By using Lemmas 2.10, 2.11 and 2.12 together with formula (2.45), it is possible to compute c defined in (1.13) with a great precision.

Lemma 2.13. *Under the Riemann Hypothesis, $\sum_{\rho} \frac{1}{\Im(\rho)^2} \leq 0.0462493$.*

Proof. Let us set $\rho = 1/2 + i\gamma$. From (2.46) we have $|\gamma| \geq 14.134$ and from (2.48)

$$\begin{aligned} \sum_{\rho} \frac{1}{\gamma^2} &= \sum_{\rho} \frac{1 + 1/(4\gamma^2)}{1/4 + \gamma^2} \leq \sum_{\rho} \frac{1 + \frac{1}{4 \times 14.134^2}}{1/4 + \gamma^2} = \left(1 + \frac{1}{4 \times 14.134^2}\right) \sum_{\rho} \frac{1}{|\rho|^2} \\ &\leq 0.0462493 \end{aligned}$$

A more precise estimate can be obtained by writing $\gamma^2 = -(\rho - 1/2)^2$,

$$\sum_{\rho} \frac{1}{\gamma^2} = \sum_{\rho} -\frac{(1 - 1/(2\rho))^{-2}}{\rho^2} = -\sum_{m=0}^{\infty} \frac{m+1}{2^m} \left(\sum_{\rho} \frac{1}{\rho^{m+2}} \right).$$

To calculate the above series, choose some $M > 0$. For $m \leq M$, use (2.45) and, for $m > M$, use Lemma 2.10 to get an upper bound of the remainder. \square

3 Computation of $h(n)$

For n small, a table of $h(n)$ for $n \leq 10^6$ has been precomputed by the naive algorithm described in [7, §1.4].

For the computation of $h(n)$ for n large, the algorithm described in [7] is used. Let us recall some points about it.

3.1 Computing an isolated value of $h(n)$ or $\log h(n)$ for n possibly large.

– *The factorization of $h(n)$.* Let $k = k(n)$ be defined above by Eq. (1.16). The value $h(n)$ may be written as the product (cf.[7, §8]:

$$h(n) = N_k \cdot G(p_k, n - \sigma_k), \tag{3.1}$$

where $G(p, m)$ is defined in [9] by

$$G(p, m) = \max \frac{Q_1 Q_2 \cdots Q_s}{q_1 q_2 \cdots q_s},$$

the maximum being taken over primes $Q_1, Q_2, \dots, Q_s, q_1, q_2, \dots, q_s, s \geq 0$, satisfying

$$2 \leq q_s < q_{s-1} < \cdots < q_1 \leq p_k < Q_1 < Q_2 < \cdots < Q_s$$

$$\text{and } \sum_{i=1}^s (Q_i - q_i) \leq m.$$

Of course, $h(n)$ is an integer, and Equation (3.1) says that the prime factors of $h(n)$ are

$$(\{p_1, p_2, \dots, p_k\} \setminus \{q_1, q_2, \dots, q_s\}) \cup \{Q_1, Q_2, \dots, Q_s\}. \quad (3.2)$$

Thus the computation of p_k and $G(p_k, n - \sigma_k)$ gives the factorization of $h(n)$. Let us remark that, for large values of n , say $n \geq 10^{30}$, this factorization is not really effective because we are not able to enumerate the primes p_1, p_2, \dots, p_k .

– *Computing $G(p_k, n - \sigma_k)$.* The execution of the algorithm described in [9, §9] is relatively fast and shows that s is small and that, with the exception of the smallest one, q_s , all primes of $\{q_1, q_2, \dots, q_s\} \cup \{Q_1, Q_2, \dots, Q_s\}$ are very closed to p_k . But we are unable to prove this fact, nor evaluate the complexity of this algorithm, nor even its termination. The time for computing 1000 values $G(p_k, n - \sigma_k)$ for n close to 10^8 is about 4 seconds.

– *Computing p_k and σ_k .* For small values of n , say $n \leq 10^{18}$ the trivial method may be used : we add the first j primes until the sum σ_j exceeds n . If n is very large, say $n > 10^{24}$ this is impracticable. But the Lagarias-Miller-Odlysko algorithm for computing $\pi(x)$ improved by Deléglise-Rivat to cost $\mathcal{O}(x^{2/3}/\log^2 x)$ operations (cf. [10]), may be adapted to compute at the same cost sums of the form $S_f(x) = \sum_{p \leq x} f(p)$ where f is a completely multiplicative function. Choosing $f(x) = x$, we are able to compute $\pi_1(x) = \sum_{p \leq x} p$ with the same complexity, and also to compute p_k and s_k in time $\mathcal{O}(n^{1/3}/(\log n)^{5/3})$ (cf. [7, §8] for more details).

– *Computing $\log(h(n))$.* Once p_k , s_k and $G(p_k, n - s_k)$ are computed, from the prime factors (3.2) of $h(n)$ we get

$$\log h(n) = \theta(p_k) + \sum_{1 \leq j \leq s} \log(Q_j) - \sum_{1 \leq j \leq s} \log(q_j). \quad (3.3)$$

The last two terms of this sum are obtained by computing a small number of \log 's values, the $(\log q_i)_{1 \leq i \leq s}$ and $(\log Q_i)_{1 \leq i \leq s}$. It remains to compute $\theta(p_k)$. If p_k is small, say $p_k \leq 10^{10}$, we may use the naive algorithm, enumerate the primes up to p_k and add their logarithms. If p_k is large, the naive algorithm is too slow.

To compute $\theta(x)$ more efficiently, we first compute $\psi(x)$ in $\mathcal{O}(x^{2/3+\epsilon})$, using the algorithm given in [11], and then we add the difference $\psi(x) - \theta(x)$ which is easily computed in time $\mathcal{O}(x^{1/2+\epsilon})$ by the naive algorithm (cf. [25]). Some values of $\theta(x)$ for x up to 10^{18} are given in [26]. Figure 2 shows, for $2 \leq n \leq 18$, the largest prime $p_k < 10^n$, $\theta(p_k) = \log h(\sigma_k)$ and b_{σ_k} .

3.2 The computations we did for this work.

Computation of all the b_{σ_k} for $p_k \leq 10\,000\,000\,019$.

For the proof of (5.45) and (5.46) in Proposition 5.11 we need to compute b_{σ_k} for all the primes $p_k \leq 10^{10} + 19$. The sophisticated method presented in [25] to compute $\theta(p_k)$ is useless because each value $\theta(p_k)$ we need is obtained at once from the previous one $\theta(p_{k-1})$ by adding $\log p_k$.

We enumerate the 455 052 512 primes up to $p_{455052512} = 10\,000\,000\,019$, computing for each of them σ_k , $\log h(\sigma_k) = \theta(p_k)$ and b_{σ_k} . This was the most expansive computation we did. It took about 7 hours.

Computation of isolated values of $h(n)$.

For the proof of (5.47) in Proposition 5.11 we compute isolated values of b_n for $n \leq n_1 = 305\,926\,023$. Here also, for these small values of n we don't need the method presented in [25] to speedup the computations of the $\theta(p_k)$ values. We content ourselves by using a precomputed table of (σ_k, θ_k) values. The essential coast of each computation of $h(n)$ is then reduced to the coast of computation of $G(p_k, n - \sigma_k)$.

4 Estimates of b_n

In the proof of Theorem 1.1 we shall use Lemmas 4.1–4.4. The first of these establishes a concavity's property (cf. Figure 1 which displays the graph of (n, b_n) for $2 \leq n \leq 100$).

Lemma 4.1. *Let b_n be defined by (1.12) and $k = k(n)$ by (1.16). For each $n \geq 2$, if $\min(b_{\sigma_k}, b_{\sigma_{k+1}}) \leq 1$, we have*

$$b_n \geq \min(b_{\sigma_k}, b_{\sigma_{k+1}}).$$

Proof. Computation shows that $b_n \geq \min(b_{\sigma_k}, b_{\sigma_{k+1}})$ is satisfied if $n < 41 = \sigma_6$. Thus we may suppose $n \geq 41$. Let us set $\varepsilon = (\log p_{k+1})/p_{k+1}$. The function $\varphi(t) = \log t - \varepsilon t$ is concave for $t > 1$. For $k \geq 2$, one has $\varphi(2) = \log 2 - 2 \log p_{k+1}/p_{k+1} \geq \log 2 - 2 \log 5/5 > 0$ and $\varphi(p_{k+1}) = 0$. Let q denote an arbitrary prime number. Thus $\varphi(q)$ is ≥ 0 for $2 \leq q \leq p_k$ and ≤ 0 for $q \geq p_{k+1}$. Then, for each squarefree integer N ,

$$\begin{aligned} \log N - \varepsilon \ell(N) &= \sum_{q|N} \varphi(q) \leq \sum_{q|N, q \leq p_k} \varphi(q) \leq \sum_{q \leq p_k} \varphi(q) \\ &= \log N_k - \varepsilon \sigma_k = \log N_{k+1} - \varepsilon \sigma_{k+1}. \end{aligned} \quad (4.1)$$

We write

$$n = \alpha \sigma_k + \beta \sigma_{k+1} \quad \text{with} \quad 0 \leq \alpha \leq 1 \quad \text{and} \quad \beta = 1 - \alpha. \quad (4.2)$$

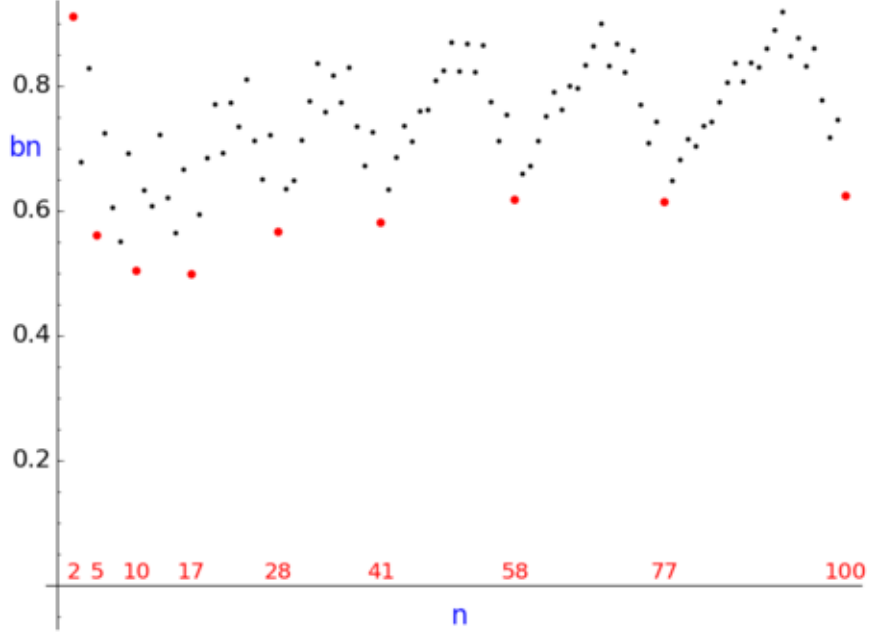


Figure 1: Graph of $(n, b_n)_{2 \leq n \leq 100}$. The red points are the (σ_k, b_{σ_k}) points.

From (1.1), $\ell(h(n)) \leq n$ holds and applying (4.1) to $N = h(n)$ yields

$$\begin{aligned}
 \log h(n) &\leq \varepsilon \ell(h(n)) + \log N_k - \varepsilon \sigma_k \leq \varepsilon n + \log N_k - \varepsilon \sigma_k \\
 &= \varepsilon(\alpha \sigma_k + \beta \sigma_{k+1}) + \alpha(\log N_k - \varepsilon \sigma_k) + \beta(\log N_{k+1} - \varepsilon \sigma_{k+1}) \\
 &= \alpha \log N_k + \beta \log N_{k+1}.
 \end{aligned} \tag{4.3}$$

Let us define $\Phi(t)$ on each interval $[\sigma_k, \sigma_{k+1}]$ by

$$\Phi(t) = \sqrt{\text{li}^{-1}(t)} - \min(b_{\sigma_k}, b_{\sigma_{k+1}})(t \log t)^{1/4}. \tag{4.4}$$

Since $\min(b_{\sigma_k}, b_{\sigma_{k+1}}) \leq 1$ and $\sigma_k \geq 31$ are assumed, from Lemma 2.5, Φ is concave on $[\sigma_k, \sigma_{k+1}]$. Moreover, from the definition of b_{σ_k} and $b_{\sigma_{k+1}}$, one has $\log N_k = \log h(\sigma_k) = \sqrt{\text{li}^{-1}(\sigma_k)} - b_{\sigma_k}(\sigma_k \log \sigma_k)^{1/4} \leq \Phi(\sigma_k)$ and $\log N_{k+1} = \log h(\sigma_{k+1}) \leq \Phi(\sigma_{k+1})$, which, from (4.3) and (4.2), implies

$$\begin{aligned}
 \log h(n) &\leq \alpha \log N_k + \beta \log N_{k+1} \leq \alpha \Phi(\sigma_k) + \beta \Phi(\sigma_{k+1}) \\
 &\leq \Phi(\alpha \sigma_k + \beta \sigma_{k+1}) = \Phi(n).
 \end{aligned}$$

With (1.12) defining b_n and (4.4), this gives $b_n \geq \min(b_{\sigma_k}, b_{\sigma_{k+1}})$. \square

Lemma 4.2. *Let n_1, n_2 be integers such that $2 \leq n_1 < n_2$. If $\sqrt{\text{li}^{-1}(n_2)} \geq \log h(n_1)$, for*

$n_1 \leq n \leq n_2$ we have

$$b_n \leq \frac{\sqrt{\text{li}^{-1}(n_2) - \log h(n_1)}}{(n_1 \log n_1)^{1/4}}. \quad (4.5)$$

Proof. It results from (1.12), defining b_n , and from the non-decreasingness of $\sqrt{\text{li}^{-1}}$, $\log h$ and $n \log n$. \square

Lemma 4.3. *Let $\mu > 0$, n_1, n_2 be integers such that $16 \leq n_1 < n_2$ and*

$$\frac{\sqrt{\text{li}^{-1}(n_2) - \log h(n_1)}}{(n_1 \log n_1)^{1/4}} \leq \frac{2}{3} + c + \mu \frac{\log \log n_2}{\log n_2}, \quad (4.6)$$

then the inequality

$$b_n < \frac{2}{3} + c + \mu \frac{\log \log n}{\log n} \quad (4.7)$$

is true for each $n \in [n_1, n_2]$.

Proof. We have $b_n \leq \frac{\sqrt{\text{li}^{-1}(n_2) - \log h(n_1)}}{(n \log n)^{1/4}}$. If $\sqrt{\text{li}^{-1}(n_2) - \log h(n_1)} \leq 0$, then $b_n \leq 0$ and (4.7) holds. If $\sqrt{\text{li}^{-1}(n_2) - \log h(n_1)} > 0$, (4.7) results from (4.6) and the decreasingness of $c + 2/3 + \mu \log \log n / \log n$ for $n \geq 16$. \square

Lemma 4.4. *Let p_k satisfy $p_k \geq x_0 = 10^{10} + 19$, $\sigma_k = \sum_{p \leq p_k} p \geq n_0 = \pi_1(x_0)$, and n be an integer such that $\sigma_k \leq n \leq \sigma_{k+1}$. Then*

$$\frac{1}{\log \sigma_k} \geq \frac{1}{\log n} > \frac{1}{(1 + 3 \times 10^{-10}) \log \sigma_k} \quad (4.8)$$

and

$$\sqrt{\text{li}^{-1}(n)} - \sqrt{\text{li}^{-1}(\sigma_k)} \leq 1.14 \log \sigma_k. \quad (4.9)$$

Proof. First, from Bertrand's postulate, we have $p_{k+1} < 2p_k$ and

$$n - \sigma_k \leq \sigma_{k+1} - \sigma_k = p_{k+1} < 2p_k.$$

From Lemma 2.8, as $\sigma_k = \pi_1(p_k)$ holds, we have

$$\begin{aligned} p_k &\leq \sqrt{\sigma_k \log \sigma_k} \left(1 + \frac{\log \log \sigma_k}{2 \log \sigma_k}\right) \leq \left(1 + \frac{\log \log n_0}{2 \log n_0}\right) \sqrt{\sigma_k \log \sigma_k} \\ &< 1.045 \sqrt{\sigma_k \log \sigma_k} \end{aligned}$$

so that

$$\begin{aligned} n &\leq \sigma_{k+1} < \sigma_k + 2p_k < \sigma_k + 2.09 \sqrt{\sigma_k \log \sigma_k} \\ &= \sigma_k \left(1 + 2.09 \sqrt{\frac{\log \sigma_k}{\sigma_k}}\right) \quad (4.10) \end{aligned}$$

holds. Further, one gets

$$\begin{aligned}\log n &\leq \log \sigma_k + 2.09 \sqrt{\frac{\log \sigma_k}{\sigma_k}} = \log \sigma_k \left(1 + \frac{2.09}{\sqrt{\sigma_k \log \sigma_k}}\right) \\ &\leq \log \sigma_k \left(1 + \frac{2.09}{\sqrt{n_0 \log n_0}}\right) < (1 + 3 \times 10^{-10}) \log \sigma_k\end{aligned}$$

which implies (4.8).

Let us set $f(t) = \sqrt{\text{li}^{-1}(t)}$. From Lemma 2.5, we know that $f'(t) = \frac{\log \text{li}^{-1}(t)}{2\sqrt{\text{li}^{-1}(t)}}$ is positive and decreasing for $\text{li}^{-1}(t) > e^2$. By the mean value theorem, one has $f(n) - f(\sigma_k) \leq (n - \sigma_k)f'(\sigma_k)$ and, from (4.10) and (2.16),

$$\begin{aligned}\sqrt{\text{li}^{-1}(n)} - \sqrt{\text{li}^{-1}(\sigma_k)} &\leq (n - \sigma_k) \frac{\log \text{li}^{-1}(\sigma_k)}{2\sqrt{\text{li}^{-1}(\sigma_k)}} \leq 2.09 \sqrt{\sigma_k \log \sigma_k} \frac{\log(\sigma_k \log \sigma_k)}{2\sqrt{\sigma_k \log \sigma_k}} \\ &= 1.045 \log \sigma_k \left(1 + \frac{\log \log \sigma_k}{\log \sigma_k}\right) \\ &\leq 1.045 \left(1 + \frac{\log \log n_0}{\log n_0}\right) \log \sigma_k = 1.1376 \dots \log \sigma_k.\end{aligned}$$

which proves (4.9). □

5 Proof of Theorem 1.1

Let x satisfy $p_k \leq x < p_{k+1}$. Then, from (1.15) and (1.17)

$$\sigma_k = \pi_1(x), \quad \log h(\sigma_k) = \log N_k = \theta(x)$$

and, from (1.12),

$$b_{\sigma_k} = \frac{\sqrt{\text{li}^{-1}(\pi_1(x))} - \theta(x)}{(\pi_1(x) \log \pi_1(x))^{1/4}}.$$

The aim of §5.1–5.4 is to obtain, under the Riemann Hypothesis, an effective estimate of the numerator of b_{σ_k} .

5.1 Estimate of $\text{li}(\theta^2(x))$.

Lemma 5.1. *Under the Riemann Hypothesis, for $x \geq x_0 = 10^{10} + 19$,*

$$\text{li}(\theta^2(x)) = \text{li}(x^2) + \frac{x}{\log x}(\theta(x) - x) + K_1(x) \tag{5.1}$$

with $0 \leq K_1(x) \leq 0.0008 x \log^3 x$.

Proof. Let us assume that $x \geq x_0$ holds. Applying Taylor's formula to the function $t \mapsto \text{li}(t^2)$ yields

$$\text{li}(\theta^2(x)) = \text{li}(x^2) + \frac{x}{\log x}(\theta(x) - x) + K_1(x)$$

with

$$K_1(x) = \left(\frac{1}{\log v} - \frac{1}{\log^2 v} \right) \frac{(\theta(x) - x)^2}{2} \quad (5.2)$$

where v satisfies $v \geq \min(x, \theta(x))$. From (2.3), we get

$$\frac{\theta(x)}{x} \geq 1 - \frac{\log^2 x}{8\pi\sqrt{x}} \geq 1 - \frac{\log^2 x_0}{8\pi\sqrt{x_0}} \geq 0.9997$$

and $v \geq 0.9997x$ holds. By setting $\varepsilon = -\log 0.9997$, one gets $\log v \geq \log x - \varepsilon$ and

$$\begin{aligned} 0 < \frac{1}{\log v} - \frac{1}{\log^2 v} &< \frac{1}{\log v} \leq \frac{1}{\log x - \varepsilon} = \frac{1}{\log x} \left(1 + \frac{\varepsilon}{\log x - \varepsilon} \right) \\ &\leq \frac{1}{\log x} \left(1 + \frac{\varepsilon}{\log x_0 - \varepsilon} \right) \leq \frac{1.000014}{\log x}. \end{aligned}$$

Finally, (2.3) and (5.2) imply

$$0 \leq K_1(x) \leq \frac{1.000014}{2 \log x} \left(\frac{1}{8\pi} \sqrt{x} \log^2 x \right)^2 \leq 0.000792 x \log^3 x$$

which completes the proof of (5.1). \square

5.2 Estimate of $\Pi_1(x) - \pi_1(x)$.

Lemma 5.2. *Under the Riemann Hypothesis, for $x \geq x_0 = 10^{10} + 19$,*

$$\Pi_1(x) = \sum_{p^m \leq x} \frac{p^m}{m} = \text{li}(x^2) - \sum_{\rho} \frac{x^{\rho+1}}{(\rho+1)\log x} + \frac{x\Lambda(x)}{2\log x} + K_2(x) \quad (5.3)$$

with $|K_2(x)| \leq 0.04625 \frac{x^{3/2}}{\log^2 x}$.

Proof. In view of (2.42), we first consider the integral $\int_{-1}^{\infty} \frac{x^{\rho-t}}{\rho-t} dt$ where ρ is a non trivial zero of ζ . By partial integration, one gets

$$\int_{-1}^{\infty} \frac{x^{\rho-t}}{\rho-t} dt = \frac{x^{\rho+1}}{(\rho+1)\log x} + J_{\rho}(x) \quad \text{with} \quad J_{\rho}(x) = \frac{x^{\rho}}{\log x} \int_{-1}^{\infty} \frac{e^{-t\log x}}{(\rho-t)^2} dt$$

and, since $\Re(\rho) = 1/2$,

$$|J_{\rho}(x)| \leq \frac{\sqrt{x}}{\log x} \int_{-1}^{\infty} \frac{e^{-t\log x}}{\Im(\rho)^2} dt = \frac{x^{3/2}}{(\log^2 x)\Im(\rho)^2}.$$

Let us set $J(x) = \sum_{\rho} J_{\rho}(x)$. Applying Lemma 2.13 yields

$$|J(x)| = \left| \sum_{\rho} J_{\rho}(x) \right| \leq \frac{x^{3/2}}{\log^2 x} \sum_{\rho} \frac{1}{\Im(\rho)^2} \leq 0.0462493 \frac{x^{3/2}}{\log^2 x}$$

and (2.42) imply

$$\Pi_1(x) = \text{li}(x^2) + \frac{x\Lambda(x)}{2\log x} - \sum_{\rho} \frac{x^{\rho+1}}{(\rho+1)\log x} + K_2(x) \quad (5.4)$$

with

$$K_2(x) = -\log 12 - J(x) + \int_x^\infty \frac{dt}{(t^2 - 1) \log t}.$$

For $t \geq x \geq 2$, $\frac{1}{(t^2 - 1) \log t} \leq \frac{4}{3t^2 \log x}$ and

$$\int_x^\infty \frac{dt}{(t^2 - 1) \log t} \leq \frac{4}{3 \log x} \int_x^\infty \frac{dt}{t^2} = \frac{4}{3x \log x}$$

so that

$$|K_2(x)| \leq \frac{x^{3/2}}{\log^2 x} \left(0.0462493 + \frac{4 \log x}{3x^{5/2}} + \frac{(\log 12) \log^2 x}{x^{3/2}} \right). \quad (5.5)$$

In (5.5), the parenthesis is decreasing for $x \geq x_0$ and its value for $x = x_0$ is < 0.04625 , which, together with (5.4), completes the proof of (5.3). \square

Lemma 5.3. For $x \geq 2$,

$$\Pi_1(x) - \pi_1(x) = \frac{x}{\log x} (\psi(x) - \theta(x)) - \sum_{k=2}^{\kappa} B_k \quad \text{with } \kappa = \left\lfloor \frac{\log x}{\log 2} \right\rfloor \quad (5.6)$$

and

$$B_k = \frac{1}{k} \int_2^{x^{1/k}} \frac{t^{k-1}}{\log^2 t} (k \log t - 1) \theta(t) dt. \quad (5.7)$$

Proof. From the definition of Π_1 ,

$$\Pi_1(x) - \pi_1(x) = \sum_{k=2}^{\kappa} \sum_{p \leq x^{1/k}} \frac{p^k}{k} = \sum_{k=2}^{\kappa} \frac{\pi_k(x^{1/k})}{k},$$

and, by Stieltjes integral,

$$\pi_k(y) = \int_{2^-}^y \frac{t^k}{\log t} d[\theta(t)] = \frac{\theta(y)y^k}{\log y} - \int_2^y \frac{t^{k-1}}{\log^2 t} (k \log t - 1) \theta(t) dt$$

so that

$$\begin{aligned} \Pi_1(x) - \pi_1(x) &= \sum_{k=2}^{\kappa} \frac{\theta(x^{1/k})x}{k(\log x)/k} - \sum_{k=2}^{\kappa} \frac{1}{k} \int_2^{x^{1/k}} \frac{t^{k-1}}{\log^2 t} (k \log t - 1) \theta(t) dt \\ &= \frac{x}{\log x} (\psi(x) - \theta(x)) - \sum_{k=2}^{\kappa} B_k. \end{aligned}$$

\square

5.3 Bounding $\sum_{k=2}^{+\infty} B_k$.

Proposition 5.4. Under the Riemann Hypothesis, for $x \geq x_0 = 10^{10} + 19$ and $\kappa = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$, B_k defined by (5.7) satisfies

$$\frac{2x^{3/2}}{3 \log x} - 0.327 \frac{x^{3/2}}{\log^2 x} \leq \sum_{k=2}^{\kappa} B_k \leq \frac{2x^{3/2}}{3 \log x} + 0.31 \frac{x^{3/2}}{\log^2 x}. \quad (5.8)$$

The proof of this proposition is rather technical. We begin by establishing some lemmata. For $k \leq \kappa$, one has $x^{1/k} \geq x^{\log 2 / \log x} = 2$ and, for $t \geq 2$ and $k \geq 2$, one has $k \log t > 1$, so that $B_k > 0$ holds.

Lemma 5.5. *For $x \geq x_0$, we have the bounds*

$$0 \leq \sum_{k=3}^{\kappa} B_k \leq 1.066 \frac{x^{4/3}}{\log x}. \quad (5.9)$$

Proof. First, by using (2.1) and (2.6),

$$B_k \leq \frac{1+\epsilon}{k} \int_2^{x^{1/k}} \frac{kt^k}{\log t} dt = (1+\epsilon)(\text{li}(x^{1+1/k}) - \text{li}(2^{k+1})) \leq (1+\epsilon) \text{li}(x^{1+1/k})$$

with $\epsilon = 7.5 \cdot 10^{-7}$. Now, by (2.11),

$$B_k \leq (1+\epsilon) \frac{x^{1+1/k}}{\log x^{1+1/k}} \left(1 + \frac{1.101}{\log x_0}\right) \leq \frac{1.05 x^{1+1/k}}{(1+1/k) \log x}. \quad (5.10)$$

Hypothesis $x \geq x_0$ implies $\kappa \geq 33$. Further, we have

$$\begin{aligned} \sum_{k=3}^{\kappa} B_k &\leq \frac{1.05 x^{4/3}}{\log x} \left(\sum_{k=3}^{26} \frac{x^{1/k-1/3}}{1+1/k} + \frac{\log x}{\log 2} x^{1/27-1/3} \right) \\ &\leq \frac{1.05 x^{4/3}}{\log x} \left(\sum_{k=3}^{26} \frac{x_0^{1/k-1/3}}{1+1/k} + \frac{\log x_0}{\log 2} x_0^{1/27-1/3} \right) < 1.066 \frac{x^{4/3}}{\log x}. \end{aligned}$$

□

The upper bound (5.10) is good for $k \geq 3$, but for $k = 2$ we need a better one. For $a \in \mathbb{C}$ let us define

$$I_a = \frac{1}{2} \int_2^{\sqrt{x}} F(t) t^a dt \quad \text{with} \quad F(t) = \frac{2t}{\log t} - \frac{t}{\log^2 t}. \quad (5.11)$$

Lemma 5.6. *For a belonging to $\left\{0, \frac{1}{3}, \frac{1}{2}, 1\right\}$ and $x \geq x_0 = 10^{10} + 19$ one has*

$$I_a = \frac{2}{a+2} \frac{x^{(a+2)/2}}{\log x} - \frac{2a\eta x^{(a+2)/2}}{(a+2)^2 \log^2 x} + \delta_a \quad (5.12)$$

with $1 < \eta < 1.101$ and $-3.15 < \delta_a < -2.88$.

Proof. From (2.6), we have $\int F(t) t^a dt = -a \text{li}(t^{2+a}) + t^{2+a} / \log t$ and

$$I_a = -\frac{a}{2} \text{li}(x^{(a+2)/2}) + \frac{x^{(a+2)/2}}{\log x} + \delta_a \quad \text{with} \quad \delta_a = \frac{a}{2} \text{li}(2^{a+2}) - \frac{2^{a+2}}{2 \log 2}$$

and δ_a satisfies $-3.15 < \delta_a < -2.88$. Further, by using inequalities (2.8) and (2.11), for $x \geq x_0$, one gets

$$\text{li}(x^{(a+2)/2}) = \frac{2 x^{(a+2)/2}}{(a+2) \log x} + \eta \frac{4 x^{(a+2)/2}}{(a+2)^2 \log^2 x}$$

with $1 < \eta < 1.101$ and, from there we get (5.12). □

In view of applying the explicit formula (2.41), we shall need an estimate of $S = \sum_{\rho} I_{\rho}/\rho$ where ρ is a non trivial zero of ζ .

Lemma 5.7. *Let us note $S = \sum_{\rho} \frac{I_{\rho}}{\rho}$. Under the Riemann Hypothesis, for $x \geq x_0$, $|S| \leq 0.148 \frac{x^{5/4}}{\log x}$.*

Proof. By partial integration, one gets

$$\begin{aligned} I_{\rho} &= \frac{1}{2} \int_2^{\sqrt{x}} F(t) t^{\rho} dt = \frac{1}{2} \int_2^{\sqrt{x}} \left(\frac{2t}{\log t} - \frac{t}{\log^2 t} \right) t^{\rho} dt \\ &= \frac{x^{(\rho+2)/2}}{\rho+1} \left(\frac{2}{\log x} - \frac{2}{\log^2 x} \right) - \frac{2^{\rho+1}}{\rho+1} \left(\frac{2}{\log 2} - \frac{1}{\log^2 2} \right) \\ &\quad - \int_2^{\sqrt{x}} \frac{t^{\rho+1}}{2(\rho+1)} F'(t) dt \end{aligned}$$

and, since $F'(t)$ satisfies for $t \geq 2$

$$0 \leq F'(t) = \frac{2 \log^2 t - 3 \log t + 2}{\log^3 t} \leq \frac{2 \log^2 t}{\log^3 t} = \frac{2}{\log t},$$

one has, from (2.6) and $\Re(\rho) = 1/2$,

$$\begin{aligned} |(\rho+1)I_{\rho}| &\leq \frac{2x^{5/4}}{\log x} + \frac{2^{3/2}(2-1/\log 2)}{\log 2} + \int_2^{\sqrt{x}} \frac{t^{3/2}}{\log t} dt \\ &= \frac{2x^{5/4}}{\log x} + \text{li}(x^{5/4}) - \text{li}(2^{5/2}) + \frac{2^{3/2}(2-1/\log 2)}{\log 2} \\ &\leq \frac{2x^{5/4}}{\log x} + \text{li}(x^{5/4}). \end{aligned} \tag{5.13}$$

Further, (5.13), (2.10) and (1.13) yield

$$\begin{aligned} |S| &= \left| \sum_{\rho} \frac{I_{\rho}}{\rho} \right| \leq \left(\sum_{\rho} \frac{1}{|\rho(\rho+1)|} \right) \left(\frac{2x^{5/4}}{\log x} + \text{li}(x^{5/4}) \right) \\ &\leq c \left(2 + \frac{1.49}{5/4} \right) \frac{x^{5/4}}{\log x} \leq 0.148 \frac{x^{5/4}}{\log x}. \end{aligned}$$

□

Now we come back to the proof of Proposition 5.4 From Lemma 2.1, it follows that

$$J - I_{1/2} - \frac{4}{3} I_{1/3} \leq B_2 \leq J - I_{1/2} + 2.14 I_0 \text{ with } J = \frac{1}{2} \int_2^{\sqrt{x}} F(t) \psi(t) dt. \tag{5.14}$$

Now, under the integral sign, we may replace $\psi(t)$ by its value in the explicit formula (2.41), and using equality (2.43) of Lemma 2.9, we get

$$\begin{aligned} J &= I_1 - S - J_1 \text{ with } S = \sum_{\rho} \frac{1}{\rho} I_{\rho} \\ \text{and } J_1 &= \frac{1}{2} \int_2^{\sqrt{x}} F(t) \left(\log(2\pi) + \frac{1}{2} \log \left(1 - \frac{1}{t^2} \right) \right) dt. \end{aligned}$$

For $t \geq 2$, one has $F(t) > 0$ and $0 < \log 2\pi + \frac{1}{2} \log \frac{3}{4} \leq \log 2\pi + \frac{1}{2} \log(1 - \frac{1}{t^2}) < \log 2\pi < 1.84$ whence

$$0 \leq J_1 \leq \log(2\pi)I_0 \leq 1.84 I_0$$

and, with the upper bound of B_2 given by (5.14), it gives

$$B_2 \leq I_1 + |S| - I_{1/2} + 2.14 I_0.$$

From Lemma 5.6 and Lemma 5.7, one gets

$$\begin{aligned} B_2 &\leq \frac{2x^{3/2}}{3\log x} - \frac{2x^{3/2}}{9\log^2 x} - 2.88 + 0.148 \frac{x^{5/4}}{\log x} - \frac{4x^{5/4}}{5\log x} \\ &\quad + \frac{4.404x^{5/4}}{25\log^2 x} + 3.15 + 2.14 \left(\frac{x}{\log x} - 2.88 \right) \\ &\leq \frac{2x^{3/2}}{3\log x} - \frac{2x^{3/2}}{9\log^2 x} + \frac{x^{5/4}}{\log x} \left(-\frac{4}{5} + 0.148 + \frac{17.616}{100\log x} + \frac{2.14}{x^{1/4}} \right) \end{aligned} \quad (5.15)$$

and, as the above parenthesis is decreasing for $x \geq x_0$ and its value for $x = x_0$ is negative, we get

$$B_2 \leq \frac{2x^{3/2}}{3\log x} - \frac{2x^{3/2}}{9\log^2 x}.$$

Now we use (5.9) to get

$$\begin{aligned} \sum_{k=2}^{\kappa} B_k &\leq \frac{2x^{3/2}}{3\log x} + \frac{x^{3/2}}{\log^2 x} \left(-\frac{2}{9} + \frac{1.066 \log x}{x^{1/6}} \right) \\ &\leq \frac{2x^{3/2}}{3\log x} + \frac{x^{3/2}}{\log^2 x} \left(-\frac{2}{9} + \frac{1.066 \log x_0}{x_0^{1/6}} \right) \leq \frac{2x^{3/2}}{3\log x} + \frac{0.31x^{3/2}}{\log^2 x}, \end{aligned} \quad (5.16)$$

which proves the upper bound of (5.8). Note that, for $x > 8.48 \times 10^{12}$, the parenthesis in (5.16) is negative and that $\sum_{k=2}^{\kappa} B_k \leq 2x^{3/2}/(3\log x)$. Similarly, we have the lower bound

$$\begin{aligned} B_2 &\geq J - I_{1/2} - \frac{4}{3}I_{1/3} \geq I_1 - |S| - J_1 - I_{1/2} - \frac{4}{3}I_{1/3} \\ &\geq I_1 - |S| - 1.84 I_0 - I_{1/2} - \frac{4}{3}I_{1/3} \\ &\geq \left(\frac{2x^{3/2}}{3\log x} - \frac{2.202x^{3/2}}{9\log^2 x} - 3.15 \right) - \left(\frac{4x^{5/4}}{5\log x} - \frac{4x^{5/4}}{25\log^2 x} - 2.88 \right) \\ &\quad - 0.148 \frac{x^{5/4}}{\log x} - \frac{4}{3} \left(\frac{6x^{7/6}}{7\log x} - \frac{6x^{7/6}}{49\log^2 x} - 2.88 \right) - 1.84 \left(\frac{x}{\log x} - 2.88 \right) \\ &\geq \frac{2x^{3/2}}{3\log x} - \frac{2.202x^{3/2}}{9\log^2 x} - 0.948 \frac{x^{5/4}}{\log x} - \frac{8x^{7/6}}{7\log x} - 1.84 \frac{x}{\log x} \\ &= \frac{2x^{3/2}}{3\log x} - \frac{x^{3/2}}{\log^2 x} \left(\frac{2.202}{9} + \frac{0.948 \log x}{x^{1/4}} + \frac{8 \log x}{7x^{1/3}} + \frac{1.84 \log x}{x^{1/2}} \right) \end{aligned}$$

and, as the last parenthesis is decreasing on x for $x > x_0$ and its value for $x = x_0$ is < 0.327 , we get

$$\sum_{k=2}^{\kappa} B_k \geq B_2 \geq \frac{2x^{3/2}}{3\log x} - 0.327 \frac{x^{3/2}}{\log^2 x}$$

which completes the proof of Proposition 5.4. \square

5.4 Estimate of $\text{li}(\theta^2(x)) - \pi_1(x)$.

Proposition 5.8. *Under the Riemann Hypothesis, for $x \geq x_0 = 10^{10} + 19$,*

$$\begin{aligned} \left(\frac{2}{3} - c\right) \frac{x^{3/2}}{\log x} - 0.426 \frac{x^{3/2}}{\log^2 x} &\leq \pi_1(x) - \text{li}(\theta^2(x)) \\ &\leq \left(\frac{2}{3} + c\right) \frac{x^{3/2}}{\log x} + 0.36 \frac{x^{3/2}}{\log^2 x} \end{aligned} \quad (5.17)$$

with c defined in (1.13).

Proof. From (5.1) and (5.3) we deduce

$$\begin{aligned} \text{li}(\theta^2(x)) &= \Pi_1(x) - \frac{x\Lambda(x)}{2\log x} + \sum_{\rho} \frac{x^{\rho+1}}{(\rho+1)\log x} \\ &\quad - K_2(x) + \frac{x}{\log x}(\theta(x) - x) + K_1(x) \\ &= \pi_1(x) + \sum_{\rho} \frac{x^{\rho+1}}{(\rho+1)\log x} + K_1(x) - K_2(x) + A(x) \end{aligned} \quad (5.18)$$

with

$$A(x) = \Pi_1(x) - \pi_1(x) - \frac{x\Lambda(x)}{2\log x} + \frac{x}{\log x}(\theta(x) - x).$$

Further, from Equation (5.6) of Lemma 5.3 and from the explicit formula (2.41) of $\psi(t)$,

$$\begin{aligned} A(x) &= \frac{x}{\log x}(\psi(x) - x) - \frac{x\Lambda(x)}{2\log x} - \sum_{k=2}^{\kappa} B_k \\ &= \frac{x}{\log x} \left(-\sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right) \right) - \sum_{k=2}^{\kappa} B_k \end{aligned}$$

and (5.18) implies $\text{li}(\theta^2(x)) = \pi_1(x) - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)\log x} + K_3(x)$ with

$$K_3(x) = K_1(x) - K_2(x) - \frac{x}{\log x} \left(\log(2\pi) + \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right) \right) - \sum_{k=2}^{\kappa} B_k.$$

For $x \geq x_0$, we have $0 < \log(2\pi) + \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right) < \log(2\pi) \leq 1.84$ and, from (5.1), (5.3) and (5.8), one gets the upper bound

$$\begin{aligned} K_3(x) &\leq 0.0008 x \log^3 x + \frac{0.04625 x^{3/2}}{\log^2 x} - \frac{2 x^{3/2}}{3 \log x} + \frac{0.327 x^{3/2}}{\log^2 x} \\ &= -\frac{2 x^{3/2}}{3 \log x} + \frac{x^{3/2}}{\log^2 x} \left(0.04625 + 0.327 + 0.0008 \frac{\log^5 x}{x^{1/2}} \right) \\ &\leq -\frac{2 x^{3/2}}{3 \log x} + \frac{0.426 x^{3/2}}{\log^2 x} \end{aligned}$$

for $x \geq x_0$. In the same way, one gets the lower bound for $x \geq x_0$:

$$\begin{aligned}
K_3(x) &\geq -\frac{0.04625 x^{3/2}}{\log^2 x} - \frac{1.84 x}{\log x} - \frac{2 x^{3/2}}{3 \log x} - \frac{0.31 x^{3/2}}{\log^2 x} \\
&= -\frac{2 x^{3/2}}{3 \log x} - \frac{x^{3/2}}{\log^2 x} \left(0.31 + 0.04625 + \frac{1.84 \log x}{x^{1/2}} \right) \\
&\geq -\frac{2 x^{3/2}}{3 \log x} - \left(0.31 + 0.04625 + \frac{1.84 \log x_0}{x_0^{1/2}} \right) \frac{x^{3/2}}{\log^2 x} \\
&\geq -\frac{2 x^{3/2}}{3 \log x} - 0.3567 \frac{x^{3/2}}{\log^2 x}
\end{aligned}$$

which completes the proof of Proposition 5.8. \square

5.5 Bounds of b_n for n large.

For convenience, in this and the next section we will use the following notation:

$$\begin{aligned}
x = p_k \geq x_0 = 10^{10} + 19, \quad \sigma = \sigma_k = \pi_1(x), \\
L = \log \sigma \geq L_0, \quad \lambda = \log L \geq \lambda_0, \quad \nu = \lambda/L \leq \nu_0. \quad (5.19)
\end{aligned}$$

Proposition 5.9. *Assume the Riemann hypothesis. Let $n \geq n_0$, b_n be defined by (1.12) and c by (1.13). Then we have*

$$\frac{2}{3} - c - 0.22 \frac{\log \log n}{\log n} < b_n < \frac{2}{3} + c + 0.77 \frac{\log \log n}{\log n}. \quad (5.20)$$

Proof. First, in §5.5.1 and §5.5.2, we consider the case $n = \sigma_k = \pi_1(x)$.

5.5.1 Lower bound of b_{σ_k} .

By (5.17), (5.19) and the fact that

$0.69(2/3 - c) > 0.426$ holds, we can write

$$\text{li}(\theta^2(x)) \leq \pi_1(x) - \delta = \sigma - \delta \quad \text{with} \quad \delta = \left(\frac{2}{3} - c \right) \frac{x^{3/2}}{\log x} \left(1 - \frac{0.69}{\log x} \right). \quad (5.21)$$

From (1.17), we have $\theta(x) = \log N_k = \log h(\sigma)$. As $\sigma = \sum_{p \leq x} p < x^2$, we have $\log \sigma < 2 \log x$ and $1 - 0.69/\log x > 1 - 1.38/\log \sigma \geq 1 - 1.38\lambda/(\lambda_0 L) = 1 - 1.38\nu/\lambda_0 > 1 - 0.37\nu$ so that

$$\delta \geq \left(\frac{2}{3} - c \right) \frac{x^{3/2}}{\log x} (1 - 0.37\nu). \quad (5.22)$$

Further, since the function $t \mapsto t^{3/2}/\log t$ is increasing, from (2.32), one gets

$$\frac{x^{3/2}}{\log x} \geq \frac{(\sigma \log \sigma)^{3/4} (1 + 0.365\nu)^{3/2}}{\frac{1}{2}L + \frac{1}{2}\lambda + \log(1 + 0.365\nu)} \geq \frac{(\sigma \log \sigma)^{3/4} (1 + 0.365\nu)^{3/2}}{\frac{1}{2}L + \frac{1}{2}\lambda + 0.365\nu}$$

which, as the denominator satisfies

$$\begin{aligned} \frac{L}{2} + \frac{\lambda}{2} + 0.365\nu &= \frac{L}{2} \left(1 + \nu \left(1 + \frac{0.73}{L} \right) \right) \leq \frac{L}{2} \left(1 + \nu \left(1 + \frac{0.73}{L_0} \right) \right) \\ &\leq \frac{L}{2} (1 + 1.018\nu), \end{aligned}$$

yields

$$\frac{x^{3/2}}{\log x} \geq 2 \left(\frac{\sigma^3}{L} \right)^{1/4} \frac{(1 + 0.365\nu)^{3/2}}{1 + 1.018\nu}. \quad (5.23)$$

For $t \geq \text{li}(e^2) = 4.54\dots$, the function $f(t) = \sqrt{\text{li}^{-1}(t)}$ is increasing and concave (cf. Lemma 2.5) and we have

$$f'(t) = \frac{\log(\text{li}^{-1}(t))}{2\sqrt{\text{li}^{-1}(t)}} \quad \text{and} \quad f''(t) = \frac{\log(\text{li}^{-1}(t))(2 - \log(\text{li}^{-1}(t)))}{4(\text{li}^{-1}(t))^{3/2}}.$$

Inequality (5.21) with the increasingness of f gives $f(\text{li}(\theta^2(x))) \leq f(\sigma - \delta)$. Applying Taylor's formula, with the concavity of f we get

$$\log h(\sigma) = \theta(x) = f(\text{li}(\theta^2(x))) \leq f(\sigma - \delta) \leq \sqrt{\text{li}^{-1}(\sigma)} - \delta f'(\sigma) \quad (5.24)$$

and we need a lower bound for $f'(\sigma)$. From (2.15), one has $\text{li}^{-1}(\sigma) < \sigma(L + \lambda)$. As the function $t \mapsto \log(t)/(2\sqrt{t})$ is decreasing on t , one gets

$$\begin{aligned} f'(\sigma) &= \frac{\log(\text{li}^{-1}(\sigma))}{2\sqrt{\text{li}^{-1}(\sigma)}} \geq \frac{\log(\sigma(L + \lambda))}{2\sqrt{\sigma(L + \lambda)}} = \frac{L + \lambda + \log(1 + \nu)}{2\sqrt{\sigma(L + \lambda)}} \\ &\geq \frac{L + \lambda}{2\sqrt{\sigma(L + \lambda)}} = \frac{\sqrt{L(1 + \nu)}}{2\sqrt{\sigma}} \end{aligned} \quad (5.25)$$

and (5.22), (5.23) and (5.25) imply

$$\delta f'(\sigma) \geq \left(\frac{2}{3} - c \right) (\sigma \log \sigma)^{1/4} \frac{(1 + 0.365\nu)^{3/2} (1 + \nu)^{1/2} (1 - 0.37\nu)}{1 + 1.018\nu}. \quad (5.26)$$

We observe that

$$\begin{aligned} &(1 + 0.365\nu)^3 (1 + \nu) (1 - 0.37\nu)^2 - (1 + 1.018\nu)^2 (1 - 0.3405\nu)^2 \\ &= 0.31552675\nu^2 + 0.09873042\nu^3 - 0.198647103641\nu^4 \\ &\quad + 0.0253884884125\nu^5 + 0.0066570534125\nu^6. \end{aligned}$$

The above polynomial is positive for $0 < \nu \leq 1$, which implies that in (5.26) the fraction is $> 1 - 0.3405\nu$ and

$$\delta f'(\sigma) \geq \left(\frac{2}{3} - c \right) (\sigma \log \sigma)^{1/4} (1 - 0.3405\nu).$$

Therefore, from the definition (1.12) of b_n and (5.24), for $p_k \geq x_0$, we have

$$\begin{aligned} b_{\sigma_k} = b_\sigma &\geq \frac{\delta f'(\sigma)}{(\sigma \log \sigma)^{1/4}} \geq \left(\frac{2}{3} - c \right) \left(1 - 0.3405 \frac{\log \log \sigma_k}{\log \sigma_k} \right) \\ &> \frac{2}{3} - c - 0.2113 \frac{\log \log \sigma_k}{\log \sigma_k}. \end{aligned} \quad (5.27)$$

5.5.2 Upper bound of b_{σ_k} .

The proof is similar to the one of the lower bound. Using (5.17)

$$\text{li}(\theta^2(x)) \geq \sigma - \eta \quad \text{with} \quad \eta = \left(\frac{2}{3} + c\right) \frac{x^{3/2}}{\log x} \left(1 + \frac{0.51}{\log x}\right). \quad (5.28)$$

Further, from the left handside inequality of (2.32), with $x = p_k$ and with the notation (5.19), one gets $x \geq \sqrt{\sigma \log \sigma}$ which implies $\log x \geq (L + \lambda)/2 > L/2$,

$$1 + \frac{0.51}{\log x} \leq 1 + \frac{1.02}{L} \leq 1 + \frac{1.02 \lambda}{\lambda_0 L} \leq 1 + 0.28 \nu$$

and, from the right handside inequality of (2.32) with the increasingness of $\frac{t^{3/2}}{\log t}$,

$$\frac{x^{3/2}}{\log x} \leq \frac{2(\sigma L)^{3/4}(1 + \nu/2)^{3/2}}{L + \lambda}.$$

The third derivative of $t \mapsto (1 + t)^{3/2}$ is negative so that

$$\left(1 + \frac{\nu}{2}\right)^{3/2} \leq 1 + \frac{3\nu}{4} + \frac{3\nu^2}{32} = 1 + \frac{3}{4}\nu \left(1 + \frac{\nu}{8}\right) \leq 1 + \frac{3}{4}\nu \left(1 + \frac{\nu_0}{8}\right) \leq 1 + 0.76 \nu$$

and

$$(1 + 0.76 \nu)(1 + 0.28 \nu) \leq 1 + \nu(1.04 + 0.2128 \nu_0) \leq 1 + 1.06 \nu$$

which implies

$$\eta \leq \left(\frac{2}{3} + c\right) \frac{2(\sigma L)^{3/4}}{L + \lambda} (1 + 1.06 \nu). \quad (5.29)$$

From (5.28) and Taylor's formula we get

$$\log h(\sigma) = \theta(x) \geq f(\sigma - \eta) = \sqrt{\text{li}^{-1}(\sigma)} - \eta f'(\sigma) + \frac{\eta^2}{2} f''(\xi) \quad \text{with} \quad \sigma - \eta \leq \xi \leq \sigma. \quad (5.30)$$

To estimate $(\eta^2/2)f''(\xi)$, we need a crude upper bound for η . From (5.29), one has

$$\eta \leq \left(\frac{2}{3} + c\right) \frac{2(\sigma L)^{3/4}}{L} (1 + 1.06 \nu_0) \leq 1.56 \frac{\sigma^{3/4}}{L^{1/4}} < \frac{\sigma}{2}. \quad (5.31)$$

As $\xi > \sigma - \sigma/2 = \sigma/2$ and $|f''(t)|$ is decreasing on t , we have

$$|f''(\xi)| \leq |f''(\sigma/2)| \leq \frac{\log^2(\text{li}^{-1}(\sigma/2))}{4(\text{li}^{-1}(\sigma/2))^{3/2}}.$$

But, from (2.16),

$$\text{li}^{-1}\left(\frac{\sigma}{2}\right) > \frac{\sigma}{2} \log\left(\frac{\sigma}{2}\right) = \frac{\sigma L}{2} \left(1 - \frac{\log 2}{L}\right) \geq \frac{\sigma L}{2} \left(1 - \frac{\log 2}{L_0}\right) > 0.49 \sigma L$$

and

$$|f''(\xi)| \leq \frac{\log^2(0.49 \sigma L)}{4(0.49 \sigma L)^{3/2}} < \frac{(L + \lambda)^2}{4(0.49)^2(\sigma L)^{3/2}} < 1.05 \frac{(L + \lambda)^2}{(\sigma L)^{3/2}}.$$

Therefore, from (5.31),

$$\frac{\eta^2}{2}|f''(\xi)| \leq \frac{(1.56)^2 \times 1.05}{2}(1 + \nu)^2 \leq 1.28(1 + \nu)^2 \leq 1.28(1 + \nu_0)^2 < 1.52. \quad (5.32)$$

Inequality (2.16), with the decreasingness of $\log t/\sqrt{t}$, implies

$$f'(\sigma) = \frac{\log(\text{li}^{-1}(\sigma))}{2\sqrt{\text{li}^{-1}(\sigma)}} \leq \frac{\log(\sigma \log \sigma)}{2\sqrt{\sigma \log \sigma}} = \frac{L + \lambda}{2\sqrt{\sigma L}} \quad (5.33)$$

and from (5.29)

$$\eta f'(\sigma) \leq \left(\frac{2}{3} + c\right) (\sigma \log \sigma)^{1/4} (1 + 1.06 \nu). \quad (5.34)$$

From (5.30), (5.32) and (5.34), one gets

$$\log h(\sigma) \geq$$

$$\sqrt{\text{li}^{-1}(\sigma)} - \left(\frac{2}{3} + c\right) (\sigma \log \sigma)^{1/4} \left(1 + \nu \left(1.06 + \frac{1.52}{(2/3 + c)\nu(\sigma L)^{1/4}}\right)\right).$$

But the above fraction is maximal for $\sigma = n_0$ and, therefore, is < 0.0003 , so that,

$$\begin{aligned} \log h(\sigma_k) &= \log(h(\sigma)) \\ &\geq \sqrt{\text{li}^{-1}(\sigma_k)} - \left(\frac{2}{3} + c\right) (\sigma_k \log \sigma_k)^{1/4} \left(1 + 1.061 \frac{\log \log \sigma_k}{\log \sigma_k}\right) \end{aligned}$$

and, from (1.12) and (1.13),

$$b_{\sigma_k} \leq \left(\frac{2}{3} + c\right) \left(1 + 1.061 \frac{\log \log \sigma_k}{\log \sigma_k}\right) < \frac{2}{3} + c + 0.757 \frac{\log \log \sigma_k}{\log \sigma_k}. \quad (5.35)$$

5.5.3 Bounds of b_n for $n \geq n_0$.

Let us recall that σ_k is defined by $\sigma_k \leq n < \sigma_{k+1}$. From (5.35), it follows that $b_{\sigma_k} < 2/3 + c + 0.757 \nu_0 < 0.78 < 1$ and we may apply Lemma 4.1 so that, from (5.27),

$$\begin{aligned} b_n &\geq \min\left(\frac{2}{3} - c - 0.2113 \frac{\log \log \sigma_k}{\log \sigma_k}, \frac{2}{3} - c - 0.2113 \frac{\log \log \sigma_{k+1}}{\log \sigma_{k+1}}\right) = \\ &\frac{2}{3} - c - 0.2113 \frac{\log \log \sigma_k}{\log \sigma_k} \geq \frac{2}{3} - c - 0.2113 \frac{\log \log n}{\log \sigma_k}. \end{aligned}$$

Now, from Lemma 4.4, $1/\log \sigma_k < (1 + 3 \times 10^{-10})/\log n$ holds, which proves the lower bound of (5.20).

Note that $c + 0.22 \log \log n / \log n \leq c + 0.22 \nu_0 < 2/3$ which implies that the lower bound in (5.20) is positive so that, for $n \geq n_0$, $b_n > 0$ and $\sqrt{\text{li}^{-1}(n)} - \log h(n) > 0$ hold. Therefore, from the definition (1.12) of b_n , one has

$$\begin{aligned} b_n &= \frac{\sqrt{\text{li}^{-1}(n)} - \log h(n)}{(n \log n)^{1/4}} \leq \frac{\sqrt{\text{li}^{-1}(n)} - \log h(n)}{(\sigma_k \log \sigma_k)^{1/4}} \\ &\leq \frac{\sqrt{\text{li}^{-1}(\sigma_{k+1})} - \log h(\sigma_k)}{(\sigma_k \log \sigma_k)^{1/4}} = \tau_k + b_{\sigma_k} \quad (5.36) \end{aligned}$$

with, from (4.9),

$$\tau_k = \frac{\sqrt{\text{li}^{-1}(\sigma_{k+1})} - \sqrt{\text{li}^{-1}(\sigma_k)}}{(\sigma_k \log \sigma_k)^{1/4}} < 1.14 \frac{(\log \sigma_k)^{3/4}}{\sigma_k^{1/4}}. \quad (5.37)$$

Therefore, from (5.35) and (4.8), one gets

$$\begin{aligned} b_n &\leq \frac{2}{3} + c + \frac{\log \log \sigma_k}{\log \sigma_k} \left(0.757 + 1.14 \frac{(\log \sigma_k)^{7/4}}{\sigma_k^{1/4} \log \log \sigma_k} \right) \\ &< \frac{2}{3} + c + \frac{\log \log \sigma_k}{\log \sigma_k} \left(0.757 + 1.14 \frac{(\log n_0)^{7/4}}{n_0^{1/4} \log \log n_0} \right) \\ &< \frac{2}{3} + c + 0.763 \frac{\log \log \sigma_k}{\log \sigma_k} \\ &\leq \frac{2}{3} + c + 0.763 \frac{\log \log n}{\log \sigma_k} < \frac{2}{3} + c + 0.763(1 + 3 \times 10^{-10}) \frac{\log \log n}{\log n} \end{aligned}$$

which completes the proof of (5.20) and of Proposition 5.9. \square

5.6 Asymptotic bounds of b_n .

Proposition 5.10. *Under the Riemann Hypothesis, when k and σ_k tend to infinity,*

$$b_{\sigma_k} \geq \left(\frac{2}{3} - c \right) \left(1 + \frac{\log \log \sigma_k + \mathcal{O}(1)}{4 \log \sigma_k} \right) \quad (5.38)$$

and

$$b_{\sigma_k} \leq \left(\frac{2}{3} + c \right) \left(1 + \frac{\log \log \sigma_k + \mathcal{O}(1)}{4 \log \sigma_k} \right). \quad (5.39)$$

Proof. The proof follows the lines of the proof of Proposition 5.9 of which we keep the notation.

Lower bound.

First, from (2.33), with the notation of (5.19),

$$\begin{aligned} x^{3/2} &= (\sigma L)^{3/4} \left(1 + \frac{3(\log L + \mathcal{O}(1))}{4L} \right), \\ \log x &= \frac{1}{2}(L + \log L + \mathcal{O}(1)) = \frac{L}{2} \left(1 + \frac{\log L + \mathcal{O}(1)}{L} \right), \\ \frac{x^{3/2}}{\log x} &= 2 \frac{(\sigma L)^{3/4}}{L} \left(1 - \frac{\log L + \mathcal{O}(1)}{4L} \right), \\ \frac{x^{3/2}}{\log^2 x} &= \mathcal{O} \left(\frac{(\sigma L)^{3/4}}{L^2} \right), \end{aligned}$$

whence, from (5.21),

$$\delta = \left(\frac{2}{3} - c \right) \frac{2\sigma^{3/4}}{L^{1/4}} \left(1 - \frac{\lambda + \mathcal{O}(1)}{4L} \right). \quad (5.40)$$

Further, from (5.24) and (5.25), one gets

$$b_{\sigma_k} \geq \frac{\delta f'(\sigma)}{(\sigma L)^{1/4}} \geq \frac{\delta \sqrt{L(1+\nu)}}{2\sqrt{\sigma}(\sigma L)^{1/4}} = \frac{\delta L^{1/4}}{2\sigma^{3/4}} \left(1 + \frac{\lambda + \mathcal{O}(1)}{2L} \right)$$

which, with (5.40), yields (5.39).

Upper bound.

As for the lower bound, but using (5.28) instead of (5.21) one gets

$$\eta = \left(\frac{2}{3} + c\right) \frac{x^{3/2}}{\log x} = \left(\frac{2}{3} + c\right) \frac{2\sigma^{3/4}}{L^{1/4}} \left(1 - \frac{\lambda + \mathcal{O}(1)}{4L}\right). \quad (5.41)$$

Further, (5.30) and (5.32) yield

$$\log h(\sigma) = \sqrt{\text{li}^{-1}(\sigma)} - \eta f'(\sigma) + \mathcal{O}(1) \quad (5.42)$$

which implies

$$b_{\sigma_k} = \frac{\sqrt{\text{li}^{-1}(\sigma)} - \log h(\sigma)}{(\sigma L)^{1/4}} = \frac{\eta f'(\sigma) + \mathcal{O}(1)}{(\sigma L)^{1/4}}. \quad (5.43)$$

Here, for $f'(\sigma)$, we need a sharper upper bound than the one of (5.33). From (2.15) and (2.17), for σ tending to infinity, we have $\text{li}^{-1}(\sigma) = \sigma(L + \lambda + \mathcal{O}(1))$, $\log(\text{li}^{-1}(\sigma)) = L + \lambda + \log(1 + (\lambda + \mathcal{O}(1))/L) = L + \lambda + \mathcal{O}(1)$, $\sqrt{\text{li}^{-1}(\sigma)} = \sqrt{\sigma L}(1 + (\lambda + \mathcal{O}(1))/(2L))$ and

$$\begin{aligned} f'(\sigma) &= \frac{\log(\text{li}^{-1}(\sigma))}{2\sqrt{\text{li}^{-1}(\sigma)}} = \frac{L + \lambda + \mathcal{O}(1)}{2\sqrt{\sigma L}(1 + (\lambda + \mathcal{O}(1))/(2L))} \\ &= \frac{\sqrt{L}}{2\sqrt{\sigma}} \left(1 + \frac{\lambda + \mathcal{O}(1)}{2L}\right). \end{aligned} \quad (5.44)$$

Now, from (5.41) and (5.44), one gets

$$\eta f'(\sigma) = \left(\frac{2}{3} + c\right) (\sigma L)^{1/4} \left(1 + \frac{\lambda + \mathcal{O}(1)}{4L}\right).$$

As $(\sigma L)^{1/4}/L \rightarrow \infty$, with (5.43), this yields (5.39). \square

5.7 Bounds of b_n for n small.

Proposition 5.11. *Let us recall that $n_0 = \pi_1(10^{10} + 19)$ and that b_n is defined by (1.12). The following assertions hold*

1. For n , $2 \leq n < n_0$

$$b_{17} = 0.49795\dots \leq b_n \leq b_{1137} = 1.04414\dots \quad (5.45)$$

2. For $78 \leq n < n_0$,

$$b_n \geq b_{100} = b_{\sigma_9} = 0.62328\dots > 2/3 - c \quad (5.46)$$

3. For $157933210 \leq n \leq n_0$,

$$b_n < \frac{2}{3} + c + 0.77 \frac{\log \log n}{\log n}. \quad (5.47)$$

Proof. First, we calculate b_{σ_k} for $2 \leq \sigma_k < n_0$ (cf. §3.2). For $k \geq 9$,

$$b_{100} = b_{\sigma_9} = 0.62328\dots \leq b_{\sigma_k} \leq b_{31117} = b_{\sigma_{112}} = 0.88447\dots < 1.$$

Therefore, we may apply Lemma 4.1 which implies, for $100 \leq n < n_0$,

$$b_n \geq b_{100} = 0.62328\dots > 2/3 - c.$$

The computation of b_n for $2 \leq n < 100$ completes the proof of (5.46) and of the lower bound of (5.45).

To prove the upper bound of (5.45), for $\sigma_{253} = 186914 \leq \sigma_k < n_0$, we compute $b_{\sigma_k} + \tau_k$ (with τ_k defined by (5.37)) and observe that $b_{\sigma_k} + \tau_k < 1.044$ holds, which implies (cf. (5.36)) that b_n is smaller than 1.044 for $186914 \leq n < n_0$. It remains to calculate b_n for $2 \leq n \leq 186913$ to complete the proof of (5.45).

The proof of (5.47) is more complicated. If the following inequality

$$b_{\sigma_k} + \tau_k < \frac{2}{3} + c + \frac{0.77 \log \log \sigma_{k+1}}{\log \sigma_{k+1}} \quad (5.48)$$

holds, then, from (5.36), we have

$$b_n < \frac{2}{3} + c + \frac{0.77 \log \log \sigma_{k+1}}{\log \sigma_{k+1}} < \frac{2}{3} + c + \frac{0.77 \log \log n}{\log n} \quad (5.49)$$

for $\sigma_k \leq n < \sigma_{k+1}$. In the same time we compute all the b_{σ_k} for $2 \leq \sigma_k \leq n_0$ (cf. the beginning of this proof), we check that inequality (5.48) holds for $305926023 \leq \sigma_k < n_0$, so that one has (5.49) for $305926023 \leq n < n_0$.

It remains to compute the largest $n \leq n_1 = 305926023$ such that inequality (5.47) is wrong. This could be expensive because the computation of b_n is not very fast. Let us recall that for an n which is not of the form $n = \sigma_k$, for computing $h(n)$ we have to compute $G(p_k, n - \sigma_k)$, and this costs about 0.004 seconds. If we used the trivial method, computing $h(n)$ for $n = n_1 - 1, n_2 - 1, \dots$ until we find n not satisfying (5.47), we should have to compute about 1.5×10^8 values of $h(n)$, taking about one week of computation.

Lemma 4.3 gives us a test, proving in $\mathcal{O}(1)$ time that all the n 's in $[n_1, n_2]$ satisfy (5.47). Moreover there are a lot of intervals $[n_1, n_2]$ passing this test. The boolean function `good_interval(n1, n2)` returns true if and only if $[n_1, n_2]$ is such an interval, i.e. if (n_1, n_2) satisfy inequality (4.6) with $\mu = 0.77$.

Now, adopting Python's style, we define below, by a dichotomic recursion a boolean function `ok_rec(n1, n2)` which returns true if, and only if, every n in $[n_1, n_2]$ satisfies (5.47). Furthermore, when it return false, before returning, it prints the largest n in $[n_1, n_2]$ which doesn't satisfy this inequality.

```
def ok(n):
    if bn(n) >= 2/3 + c + 0.77 * log log n / log n :
        print n, ' does not satisfy inequality (iv) of Theorem 1.1 '
```

```

    return False
return True

def ok_rec(n1, n2):
    if n2 - n1 >= 2:
        if good_interval(n1,n2):
            return True
        nmed = (n1 + n2)//2
        if not ok_rec(nmed,n2):
            return False
        return ok_rec(n1,nmed)

    if n1==n2:
        return ok(n1)

    if n2 == n1 + 1:
        if ok(n2):
            return ok(n1)
        return False

```

The correctness of `ok_rec(n1, n2)` is proved by recursion about the size of $n_2 - n_1$. The largest n which doesn't satisfy (5.47), $n = 157\,933\,209$, is given by the call `ok_rec(2,305926023)`. It computed four values of `ok(n)` and 11 395 values of `good_interval(n1,n2)`, and took 35.27s. \square

5.8 Completing the proof of Theorem. 1.1

Proposition 5.9 implies that, for $n \geq n_0 = \pi_1(10^{10} + 19)$

$$0.6010\dots = \frac{2}{3} - c - 0.22 \frac{\log \log n_0}{\log n_0} < b_n < \frac{2}{3} + c + 0.77 \frac{\log \log n_0}{\log n_0} = 0.781\dots$$

which, together with inequality (5.45), proves the point (ii) of Theorem 1.1

- Point (i) is equivalent to $b_n > 0$ which follows from (ii).
- Inequalities (5.20) and (5.46) imply $b_n > \frac{2}{3} - c - 0.22 \frac{\log \log n}{\log n}$ for $n \geq 78$, and the computation of b_n for $2 \leq n < 78$ proves point (iii).
- Similarly, inequalities (5.20) and (5.47) imply $b_n < \frac{2}{3} + c + 0.77 \frac{\log \log n}{\log n}$ for $n \geq 157\,933\,210$.
- The point (v) follows from (iii) and (iv).

– To prove (vi), we assume $n \rightarrow \infty$ and $\sigma = \sigma_k \leq n \leq \sigma_{k+1}$ so that $n = \sigma + \mathcal{O}(p_k)$ holds. From Lemma 2.8, $\sigma = \sigma_k = \pi_1(p_k)$ yields

$$n = \sigma + \mathcal{O}\left(\sqrt{\sigma \log \sigma}\right) = \sigma\left(1 + \mathcal{O}\left(\sqrt{(\log \sigma)/\sigma}\right)\right) \sim \sigma.$$

This implies

$$\begin{aligned} \log n &= \log \sigma + \mathcal{O}\left(\sqrt{(\log \sigma)/\sigma}\right) = (\log \sigma)\left(1 + \mathcal{O}\left(1/\sqrt{\sigma \log \sigma}\right)\right) \\ \log \log n &= \log \log \sigma + \mathcal{O}(1) \end{aligned}$$

$$\begin{aligned} \frac{\log \log \sigma + \mathcal{O}(1)}{\log \sigma} &= \frac{\log \log n + \mathcal{O}(1)}{(\log n)\left(1 + \frac{\mathcal{O}(1)}{\log \log n}\right)} = \frac{\log \log n + \mathcal{O}(1)}{\log n} \\ &= \frac{\log \log \sigma_{k+1} + \mathcal{O}(1)}{\log \sigma_{k+1}}. \end{aligned}$$

From Lemma 4.1 and (5.38), we get

$$b_n \geq \min(b_{\sigma_k}, b_{\sigma_{k+1}}) \geq \left(\frac{2}{3} - c\right) \left(1 + \frac{\log \log n + \mathcal{O}(1)}{4 \log n}\right),$$

which proves the lower bound of (vi).

– From (5.36), (5.37) and (5.39), one gets

$$\begin{aligned} b_n \leq b_{\sigma_k} + \tau_k &= \left(\frac{2}{3} + c\right) \left(1 + \frac{\log \log \sigma_k + \mathcal{O}(1)}{4 \log \sigma_k}\right) + \mathcal{O}\left(\frac{\log^{3/4} \sigma}{\sigma^{1/4}}\right) \\ &= \left(\frac{2}{3} + c\right) \left(1 + \frac{\log \log n + \mathcal{O}(1)}{4 \log n}\right), \end{aligned}$$

which proves the upper bound of (vi). \square

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10^n	p_k	$\theta(10^n) = \theta(p_k) = \log(h(\sigma_k))$	b_{σ_k}
10^2	97	83.72839039906392294502	0.797141877
10^3	997	956.2452651200588678124	0.866433156
10^4	9973	9895.991379156987312668	0.825165752
10^5	99991	99685.38926861255083662	0.773752564
10^6	999983	998484.1750256342921339	0.736790483
10^7	9999991	9.995179317856311896844 e6	0.714394280
10^8	99999989	9.998773001802200438321 e7	0.714080633
10^9	999999937	9.999689785775661447991 e8	0.703113573
10^{10}	9999999967	9.999939830657757384159 e9	0.677576960
10^{11}	9999999977	9.999973765310744469485 e10	0.672240206
10^{12}	99999999989	9.99990303330962246369 e11	0.669158053
10^{13}	999999999971	9.99996988293034199653 e12	0.670195267
10^{14}	9999999999973	9.99999057324697853840 e13	0.675058840
10^{15}	99999999999989	9.99999657526609398407 e14	0.675161272
10^{16}	999999999999937	9.99999887717104034899 e15	0.663260174
10^{17}	999999999999997	9.99999970658237245237 e16	0.652185840
10^{18}	9999999999999989	9.99999991441156345121 e17	0.669367571

Figure 2: Values of p_k , $\theta(p_k)$, b_{σ_k} where p_k is the largest prime $< 10^n$.

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