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ON FANO SCHEMES OF COMPLETE INTERSECTIONS

C. CILIBERTO, M. ZAIDENBERG

ABSTRACT. We provide enumerative formulas for the degrees of varieties parameterizing hypersurfaces and complete intersections which contain projective subspaces and conics. Besides, we find all cases where the Fano scheme of the general complete intersection is irregular of dimension at least 2, and for the Fano surfaces we deduce formulas for their holomorphic Euler characteristic.

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INTRODUCTION

The study of hypersurfaces in projective space, or more generally, of complete intersection, and specifically of varieties contained in them, is a classical subject in algebraic geometry. The present paper is devoted to this subject, and in particular to some enumerative aspects of it.

Recall that the *Fano scheme* $F_k(X)$ of a projective variety $X \subset \mathbb{P}^r$ is the Hilbert scheme of k -planes (that is, linear subspaces of dimension k) contained in X ; see [1] or [21, 14.7.13]. For a hypersurface $X \subset \mathbb{P}^r$ of degree d the integer

$$\delta(d, r, k) = (k + 1)(r - k) - \binom{d + k}{k}$$

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is called the *expected dimension* of $F_k(X)$. Let $\Sigma(d, r)$ be the projective space of dimension $\binom{d+r}{r} - 1$ which parameterizes the hypersurfaces of degree $d \geq 3$ in \mathbb{P}^r . If either $\delta(d, r, k) < 0$ or $2k \geq r$ then $F_k(X) = \emptyset$ for a general $X \in \Sigma(d, r)$. Otherwise $F_k(X)$ has dimension δ for a general $X \in \Sigma(d, r)$ ([8, 15, 31, 39]). Setting

$$\gamma(d, r, k) = -\delta(d, r, k) > 0$$

the general hypersurface of degree $d \geq 3$ in \mathbb{P}^r contains no k -plane. Let $\Sigma(d, r, k)$ be the subvariety of $\Sigma(d, r)$ of points corresponding to hypersurfaces which carry k -planes. Then $\Sigma(d, r, k)$ is a nonempty, irreducible, proper subvariety of codimension $\gamma(d, r, k)$ in $\Sigma(d, r)$ (see [37]), and its general point corresponds to a hypersurface of degree d which carries a unique k -plane (see [4]). In Section 1 we compute the degree of this subvariety of the projective space $\Sigma(d, r)$. This degree is the total number of k -planes in the members of the general linear system \mathcal{L} of degree d hypersurfaces, provided $\dim(\mathcal{L}) = \gamma(d, r, k)$. We interpret $\deg(\Sigma(d, r, k))$ as the top Chern number of a vector bundle, and explore three different techniques for computing it:

- the Schubert calculus;
- a trick due to Debarre-Manivel;
- the Bott residue formula and the localization in the equivariant Chow rings.

In Section 2 we extend these computations to the Fano schemes of complete intersections in \mathbb{P}^r .

In Sections 3-5 we turn to the opposite case $\gamma(d, r, k) < 0$, that is, the expected dimension of the Fano scheme is positive. In Section 3 we compute certain Chern classes related to the Fano scheme. In Section 4 we apply these computations in the case where the Fano scheme is a surface, and provide several concrete examples. The main result of Section 5 describes all the cases where the Fano scheme of the general complete intersection has dimension ≥ 2 and a positive irregularity. This happens only for the general cubic threefolds in \mathbb{P}^4 ($k = 1$), the general cubic fivefolds in \mathbb{P}^6 ($k = 2$), and the general intersections of two quadrics in \mathbb{P}^{2k+3} , $k \geq 1$; see Theorem 5.1.

In the final Section 6 we turn to the conics in degree d hypersurfaces in \mathbb{P}^r . Let

$$\epsilon(d, r) = 2d + 2 - 3r.$$

Let $\Sigma_c(d, r)$ be the subvariety of $\Sigma(d, r)$ consisting of the degree d hypersurfaces which contain conics. We show that $\Sigma_c(d, r)$ is irreducible of codimension $\epsilon(d, r)$ in $\Sigma(d, r)$, provided $\epsilon(d, r) \geq 0$. Then we prove that the general hypersurface in $\Sigma_c(d, r)$ contains a unique (smooth) conic if $\epsilon(d, r) > 0$. Our main results in this section are formulas (37)-(38) which express the degree of $\Sigma_c(d, r)$ via Bott's residue formula. Notice that there exists already a formula for $\deg(\Sigma_c(d, r))$ in the case $r = 3$, $d \geq 5$, that is, for the surfaces in \mathbb{P}^3 , see [33, Prop. 7.1]. It expresses this degree as a polynomial in d .

Let us finish with a few comments on the case $\epsilon(d, r) < 0$. It is known (see [24]) that for $2d \leq r + 1$, given a general hypersurface $X \subset \mathbb{P}^r$ of degree d and any point $x \in X$, there is a family of dimension $e(r + 1 - d) - 2 \geq ed$ of degree e rational curves containing x . In particular, X carries a $2(r - d)$ -dimensional family of smooth conics through an arbitrary point. Moreover (see

[6]), for $3d \leq 2r - 1$ the Hilbert scheme of smooth rational curves of degree e on a general X is irreducible of the *expected dimension* $e(r - d + 1) + r - 4$. In particular, the Hilbert scheme of smooth conics in X is irreducible of dimension $3r - 2d - 2 = -\epsilon(d, r)$. Analogs of the latter statements hold as well for general complete intersections (see [6]).

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1. HYPERSURFACES CONTAINING LINEAR SUBSPACES

Recall that the *Fano scheme* $F_k(X)$ of a projective variety $X \subset \mathbb{P}^r$ is the Hilbert scheme of linear subspaces of dimension k contained in X ; see [1] or [21, 14.7.13]. For a hypersurface $X \subset \mathbb{P}^r$ of degree d the integer

$$\delta(d, r, k) = (k + 1)(r - k) - \binom{d + k}{k}$$

is called the *expected dimension* of $F_k(X)$. Let $\Sigma(d, r)$ be the projective space of dimension $\binom{d+r}{r} - 1$ which parameterizes the hypersurfaces of degree $d \geq 3$ in \mathbb{P}^r . If either $\delta(d, r, k) < 0$ or $2k \geq r$ then $F_k(X) = \emptyset$ for a general $X \in \Sigma(d, r)$. Otherwise $F_k(X)$ has dimension δ for a general $X \in \Sigma(d, r)$ ([8, 15, 31, 39]). We assume in the sequel that

$$(1) \quad \gamma(d, r, k) := -\delta(d, r, k) > 0.$$

Then the general hypersurface of degree $d \geq 3$ in \mathbb{P}^r contains no linear subspace of dimension k . Let $\Sigma(d, r, k)$ be the subvariety of $\Sigma(d, r)$ of points corresponding to hypersurfaces which do contain a linear subspace of dimension k . The following statement is a particular case of Theorem 1.1 in [4]; see Proposition 2.1 below.

Proposition 1.1. *Assume $\gamma(d, r, k) > 0$. Then $\Sigma(d, r, k)$ is a nonempty, irreducible and rational subvariety of codimension $\gamma(d, r, k)$ in $\Sigma(d, r)$. The general point of $\Sigma(d, r, k)$ corresponds to a hypersurface which contains a unique linear subspace of dimension k and has singular locus of dimension $\max\{-1, 2k - r\}$ along its unique k -dimensional linear subspace (in particular, it is smooth provided $2k < r$).*

For instance, take $d = 3$, $r = 5$, and $k = 2$. Then $\Sigma(3, 5)$ parameterizes the cubic fourfolds in \mathbb{P}^5 , and $\Sigma(3, 5, 2)$ parameterizes those cubic fourfolds which contain a plane. Since $\gamma(3, 5, 2) = 1$, we conclude that $\Sigma(3, 5, 2)$ is a divisor in $\Sigma(3, 5)$, and the general point of $\Sigma(3, 5, 2)$ corresponds to a smooth cubic fourfold which contains a unique plane.

Our aim is to compute the degree of $\Sigma(d, r, k)$ in the projective space $\Sigma(d, r)$ in the case $\gamma(d, r, k) > 0$.

1.1. On the Grassmannian $\mathbb{G}(k, r)$ of k -subspaces of \mathbb{P}^r , consider the dual \mathcal{S}^* of the tautological vector bundle \mathcal{S} of rank $k + 1$. Let $\Pi \in \mathbb{G}(k, r)$ correspond to a k -subspace of \mathbb{P}^r . Then the fiber of \mathcal{S}^* over Π is $H^0(\Pi, \mathcal{O}_\Pi(1))$. It is known ([19, Sect. 5.6.2], [21]) that

$$c(\mathcal{S}^*) = 1 + \sum_{i=1}^{k+1} \sigma_{(1^i)},$$

where (1^i) stays for the vector $(1, \dots, 1)$ of length i , and $\sigma_{(1^i)}$ is the (Poincaré dual of the) class of the Schubert cycle $\Sigma_{(1^i)}$. This cycle has codimension i in $\mathbb{G}(k, r)$, therefore, $\Sigma_{(1^i)} \in A^i(\mathbb{G}(k, r))$ in the Chow ring $A^*(\mathbb{G}(k, r))$.

The *splitting principle* (see [19, Sect. 5.4]) says that any relation among Chern classes which holds for all split vector bundles holds as well for any vector bundle. So, we can write formally

$$\mathcal{S}^* = L_0 \oplus \dots \oplus L_k,$$

the L_i s being (virtual) line bundles. In terms of the *Chern roots* $x_i = c_1(L_i)$ one can express

$$c(\mathcal{S}^*) = 1 + c_1(\mathcal{S}^*) + \dots + c_{k+1}(\mathcal{S}^*) = (1 + x_0) \cdots (1 + x_k).$$

Hence $\sigma_{(1^i)}$ is the i -th elementary symmetric polynomial in x_0, \dots, x_k , i.e.,

$$\sigma_{(1)} = x_0 + \dots + x_k, \quad \sigma_{(1^2)} = \sum_{0 \leq i < j \leq k} x_i x_j, \quad \dots, \quad \sigma_{(1^{k+1})} = x_0 \dots x_k.$$

Consider further the vector bundle $\text{Sym}^d(\mathcal{S}^*)$ on $\mathbb{G}(k, r)$ of rank

$$\binom{d+k}{k} > (k+1)(r-k) = \dim(\mathbb{G}(k, r)).$$

To compute the Chern class of $\text{Sym}^d(\mathcal{S}^*)$ one writes

$$\text{Sym}^d(\mathcal{S}^*) = \bigoplus_{v_0 + \dots + v_k = d} L_0^{v_0} \cdots L_k^{v_k}.$$

Since $c_1(L_0^{v_0} \cdots L_k^{v_k}) = v_0 x_0 + \dots + v_k x_k$ one obtains

$$(2) \quad c(\text{Sym}^d(\mathcal{S}^*)) = \prod_{v_0 + \dots + v_k = d} (1 + v_0 x_0 + \dots + v_k x_k).$$

The following lemma is standard. For the reader's convenience we include the proof. As usual, the integral of the top degree cohomology class stands for its value on the fundamental cycle. The integral of the dual of a zero cycle α coincides with the degree of α .

Lemma 1.2. *Suppose (1) holds. Then one has*

$$\deg(\Sigma(d, r, k)) = \int_{\mathbb{G}(k, r)} c_{(k+1)(r-k)}(\text{Sym}^d(\mathcal{S}^*)).$$

Proof. Let $p : V(k, r) \rightarrow \mathbb{G}(k, r)$ be the tautological \mathbb{P}^k -bundle over the Grassmanian $\mathbb{G}(k, r)$. Consider the composition

$$\varphi : V(k, r) \xrightarrow{\phi} \mathbb{P}^r \times \mathbb{G}(k, r) \xrightarrow{\pi} \mathbb{P}^r,$$

where ϕ is the natural embedding and π stands for the projection to the first factor. Letting $\mathcal{T} = \mathcal{O}_{\mathbb{P}^r}(-1)$ and $\mathcal{S}_d = \text{Sym}^d(\mathcal{T}^*)$ one obtains

$$\mathcal{S}^* = R^0 p_* \varphi^*(\mathcal{T}^*) \quad \text{and} \quad \text{Sym}^d(\mathcal{S}^*) = R^0 p_* \varphi^*(\mathcal{S}_d).$$

Any $F \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ defines a section σ_F of $\text{Sym}^d(\mathcal{S}^*)$ such that $\sigma_F(\Pi) = F|_{\Pi} \in H^0(\Pi, \mathcal{O}_{\Pi}(d))$. Consider the hypersurface X_F of degree d on \mathbb{P}^r with equation $F = 0$. The support of X_F contains a linear subspace $\Pi \in \mathbb{G}(k, r)$ if and only if $\sigma_F(\Pi) = 0$, i.e., the subspaces $\Pi \in \mathbb{G}(k, r)$ lying in $\text{Supp}(X_F)$ correspond to the zeros of σ_F in $\mathbb{G}(k, r)$, which have a natural scheme structure.

Let $\rho = \dim(\mathbb{G}(k, r)) = (k+1)(r-k)$. By our assumption one has

$$\text{rk}(\text{Sym}^d(\mathcal{S}^*)) - \rho = \gamma(r, k, d) > 0.$$

Choose a general linear subsystem $\mathcal{L} = \langle X_0, \dots, X_{\gamma} \rangle$ in $\Sigma(d, r) = |\mathcal{O}_{\mathbb{P}^r}(d)|$ of dimension $\gamma = \gamma(r, k, d)$, where $X_i = \{F_i = 0\}$. By virtue of Proposition 1.1, \mathcal{L} meets $\Sigma(d, r, k) \subset \Sigma(d, r)$ transversally in $\deg(\Sigma(d, r, k))$ simple points, and to any such point $X \in \Sigma(d, r, k) \cap \mathcal{L}$ corresponds a unique k -dimensional subspace $\Pi \in \mathbb{G}(k, r)$ such that $\Pi \subset X$.

Consider now the sections $\sigma_i := \sigma_{F_i}$, $i = 0, \dots, \gamma$, of $\text{Sym}^d(\mathcal{S}^*)$. The intersection of \mathcal{L} with $\Sigma(d, r, k)$ is exactly the scheme of points $\Pi \in \mathbb{G}(k, r)$ where there is a linear combination of $\sigma_0, \dots, \sigma_{\gamma}$ vanishing on Π . This is the zero dimensional scheme of points of $\mathbb{G}(k, r)$ where the sections $\sigma_0, \dots, \sigma_{\gamma}$ are linearly dependent. This zero dimensional scheme represents the top Chern class $c_{\rho}(\text{Sym}^d(\mathcal{S}^*))$ (see [19, Thm. 5.3]). Its degree (which is equal to $\deg(\Sigma(d, r, k))$) is the required Chern number $\int_{\mathbb{G}(k, r)} c_{(k+1)(r-k)}(\text{Sym}^d(\mathcal{S}^*))$. \square

Let us explain now three methods for computing $\deg(\Sigma(d, r, k))$.

1.1. Schubert calculus. In order to compute $c_{(r-k)(k+1)}(\text{Sym}^d(\mathcal{S}^*))$, one computes the polynomial in x_0, \dots, x_k appearing in (2) and extracts the homogeneous component $\tau_{(d, r, k)}$ of degree $(k+1)(r-k)$. The latter homogeneous polynomial in x_0, \dots, x_k is symmetric, hence it can be expressed via a polynomial in the elementary symmetric functions $\sigma_{(1^i)}$, $i = 0, \dots, k+1$:

$$(3) \quad \tau_{(d, r, k)} = \sum_{j_1 + 2j_2 + \dots + (k+1)j_{k+1} = (k+1)(r-k)} \phi_{d, r}(j_1, j_2, \dots, j_{k+1}) \sigma_{(1)}^{j_1} \sigma_{(1^2)}^{j_2} \cdots \sigma_{(1^{k+1})}^{j_{k+1}}$$

with suitable coefficients $\phi_{d, r}(j_1, j_2, \dots, j_{k+1})$. In this way the top Chern number

$$\tau_{(d, r, k)} = c_{(k+1)(r-k)}(\text{Sym}^d(\mathcal{S}^*))$$

in Lemma 1.2 is expressed in terms of the Chern numbers

$$\sigma_{(1)}^{j_1} \sigma_{(1^2)}^{j_2} \cdots \sigma_{(1^{k+1})}^{j_{k+1}}$$

appearing in (3). By computing the intersection products among Schubert classes and plugging these in (3) one obtains the desired degree

$$\deg(\Sigma(d, r, k)) = \tau_{(d, r, k)}.$$

1.2. Debarre–Manivel’s trick. This trick (applied for a similar purpose by van der Waerden [47]) allows to avoid passing to the elementary symmetric polynomials, which requires to compute the coefficients in (3). Let us recall the basics on the Chow ring of the Grassmannian $\mathbb{G}(k, r)$ following [34].

A *partition* λ of length $k + 1$ is a (non-strictly) decreasing sequence of non-negative integers $(\lambda_0, \dots, \lambda_k)$. To such a partition λ there corresponds a homogeneous symmetric *Schur polynomial* $s_\lambda \in \mathbb{Z}[x_0, \dots, x_k]$ of degree $|\lambda| = \lambda_0 + \dots + \lambda_k$. These polynomials form a base of the \mathbb{Z} -module Λ_{k+1} of symmetric polynomials in x_0, \dots, x_k . One writes $\lambda \subset (k+1) \times (r-k)$ if $r-k \geq \lambda_0 \geq \dots \geq \lambda_k \geq 0$. This inclusion means that the corresponding *Ferrers diagram* of λ is inscribed in the rectangular matrix of size $(k+1) \times (r-k)$ occupying λ_{i-1} first places of the i th line for $i = 1, \dots, k+1$. To any $\lambda \subset (k+1) \times (r-k)$ there correspond:

- a Schubert variety $\Sigma_\lambda \subset \mathbb{G}(k, r)$ of codimension $|\lambda|$;
- the corresponding Schubert cycle $[\Sigma_\lambda]$ in the Chow group $A_*(\mathbb{G}(k, r))$;
- the corresponding dual Schubert class σ_λ of degree $|\lambda|$ in the Chow ring $A^*(\mathbb{G}(k, r))$.

The nonzero Schubert classes form a base of the free \mathbb{Z} -module $A^*(\mathbb{G}(k, r))$ ([34, Cor. 3.2.4]). There is a unique partition $\lambda_{\max} = (r-k, \dots, r-k) \subset (k+1) \times (r-k)$ of maximal weight $|\lambda_{\max}| = (k+1)(r-k)$. Its Ferrers diagram coincides with the whole rectangle $(k+1) \times (r-k)$. The corresponding Schur polynomial is $s_{\lambda_{\max}} = (x_0 \cdots x_k)^{r-k}$. The corresponding Schubert cycle is a reduced point, and the corresponding Schubert class $\sigma_{\lambda_{\max}}$ generates the \mathbb{Z} -module $A^{(k+1)(r-k)}(\mathbb{G}(k, r)) \simeq \mathbb{Z}$.

Let as before $x_0, \dots, x_k \in A^1(\mathbb{G}(k, r))$ be the Chern roots of the vector bundle \mathcal{S}^* over $\mathbb{G}(k, r)$. A homogeneous symmetric polynomial $\tau \in \mathbb{Z}[x_0, \dots, x_k]$ of degree $(k+1)(r-k)$ admits a unique decomposition as an integral linear combination of Schur polynomials s_λ of the same degree. The corresponding Schubert classes σ_λ vanish except for $\sigma_{\lambda_{\max}}$. If τ corresponds to an effective zero cycle on $\mathbb{G}(k, r)$, then the degree of this cycle equals the coefficient of $s_{\lambda_{\max}} = (x_0 \cdots x_k)^{r-k}$ in the decomposition of τ as a linear combination of Schur polynomials. Multiplying τ by the Vandermonde polynomial

$$V = V(x_0, \dots, x_k) = \prod_{0 \leq i < j \leq k} (x_i - x_j),$$

this coefficient becomes the coefficient of the monomial $x_0^r x_1^{r-1} \cdots x_k^{r-k}$ in the product $\tau \cdot V$, see the proofs of [15, Thm. 4.3] and [34, Thm. 3.5.18].

Let $P(x_0, \dots, x_k)$ be a polynomial, and let $x_0^{i_0} \cdots x_k^{i_k}$ be a monomial, which we identify with the lattice vector $\mathbf{i} = (i_0, \dots, i_k) \in \mathbb{Z}^{k+1}$. We write $\psi_{\mathbf{i}}(P)$ for the coefficient of $x_0^{i_0} \cdots x_k^{i_k}$ in $P = \sum_{\mathbf{i}} \psi_{\mathbf{i}}(P) x_0^{i_0} \cdots x_k^{i_k}$. Summarizing the preceding discussion and taking into account Lemma 1.2 one arrives at the following conclusion.

Proposition 1.3. *One has*

$$\deg(\Sigma(d, r, k)) = \psi_{(r, r-1, \dots, r-k)}(V \cdot \tau_{(d, r, k)}),$$

that is, the degree $\deg(\Sigma(d, r, k))$ equals the coefficient of $x_0^r x_1^{r-1} \cdots x_k^{r-k}$ in the product $\tau \cdot V$, where $\tau = \tau_{(d, r, k)}(x_0, \dots, x_k)$ is as in (3).

1.3. Bott's residue formula. Bott's residue formula [9, Thms. 1, 2] says, in particular, that one can compute the degree of a zero-dimensional cycle class on a smooth projective variety X in terms of local contributions given by the fixed point loci of a torus action on X . Here we follow the treatment in [25] based on [10], [18], and [36] and adapted to our setting.

We consider the diagonal action of $T = (\mathbb{C}^*)^{r+1}$ on \mathbb{P}^r given in coordinates by

$$(t_0, \dots, t_r) \cdot (x_0 : \dots : x_r) = (t_0 x_0 : \dots : t_r x_r).$$

This induces an action of T on $\mathbb{G}(k, r)$, with $\binom{r+1}{k+1}$ isolated fixed points corresponding to the coordinate k -subspaces in \mathbb{P}^r , which are indexed by the subsets I of order $k+1$ of the set $\{0, \dots, r\}$. We let \mathcal{I}_{k+1} denote the set of all these subsets, and $\Pi_I \in \mathbb{G}(k, r)$ denote the subspace which corresponds to $I \in \mathcal{I}_{k+1}$. Bott's residue formula, applied in our setting, has the form

$$\deg(\Sigma(d, r, k)) = \int_{\mathbb{G}(k, r)} c_{(r-k)(k+1)}(\text{Sym}^d(\mathcal{S}^*)) = \sum_{I \in \mathcal{I}_{k+1}} \frac{c_I}{e_I},$$

where c_I results from the local contribution of $c_{(r-k)(k+1)}(\text{Sym}^d(\mathcal{S}^*))$ at Π_I , and e_I is determined by the torus action on the tangent space to $\mathbb{G}(k, r)$ at Π_I .

As for the computation of e_I , this goes exactly as in [25, p. 116], namely

$$e_I = (-1)^{(k+1)(r-k)} \prod_{i \in I} \prod_{j \notin I} (t_i - t_j).$$

Also the computation of c_I is similar to the one made in [25, p. 116]. Recalling (2), for a given $I \in \mathcal{I}_{k+1}$, consider the polynomial

$$\prod_{v_0 + \dots + v_k = d} (1 + \sum_{i \in I} v_i t_i)$$

and extract from this its homogeneous component $\tau_{(d, r, k)}^I$ of degree $(k+1)(r-k)$. Then

$$c_I = \tau_{(d, r, k)}^I(-t_i)_{i \in I} = (-1)^{(r-k)(k+1)} \tau_{(d, r, k)}^I(t_i)_{i \in I}.$$

In conclusion we have

$$\deg(\Sigma(d, r, k)) = \sum_{I \in \mathcal{I}_{k+1}} \frac{\tau_{(d, r, k)}^I(t_i)_{i \in I}}{\prod_{i \in I} \prod_{j \notin I} (t_i - t_j)}.$$

As in [25, p. 111], we notice that the right hand side of this formula is, a priori, a rational function in the variables t_0, \dots, t_k . As a matter of fact, it is a constant and a positive integer.

2. FANO SCHEMES OF COMPLETE INTERSECTIONS

In this section we extend the considerations of Section 1 to complete intersections in projective space. We consider the case in which a general complete intersection X of type $\mathbf{d} := (d_1, \dots, d_m)$ in \mathbb{P}^r , where $\prod_{i=1}^m d_i > 2$, does not contain any linear subspace of dimension k . Like in the case of hypersurfaces, the latter happens if and only if either $2k > r - m = \dim(X)$, or

$$\gamma(\mathbf{d}, r, k) := \sum_{j=1}^m \binom{d_j + k}{k} - (k+1)(r-k) > 0,$$

see [8, 15, 31, 37, 38, 39].

Let $\Sigma(\mathbf{d}, r)$ be the parameter space for complete intersections of type \mathbf{d} in \mathbb{P}^r . This is a tower of projective bundles over a projective space, hence a smooth variety. Consider the subvariety $\Sigma(\mathbf{d}, r, k)$ of $\Sigma(\mathbf{d}, r)$ parameterizing complete intersections which contain a linear subspace of dimension k . One has ([4, Thm. 1.1]):

Proposition 2.1. *Assume $\gamma(\mathbf{d}, r, k) > 0$. Then $\Sigma(\mathbf{d}, r, k)$ is a nonempty, irreducible and rational subvariety of codimension $\gamma(\mathbf{d}, r, k)$ in $\Sigma(\mathbf{d}, r)$. The general point of $\Sigma(\mathbf{d}, r, k)$ corresponds to a complete intersection which contains a unique linear subspace of dimension k and has singular locus of dimension $\max\{-1, 2k + m - 1 - r\}$ along its unique k -dimensional linear subspace (in particular, it is smooth provided $r \geq 2k + m$).*

Next we would like to make sense of, and to compute, the degree of $\Sigma(\mathbf{d}, r, k)$ inside $\Sigma(\mathbf{d}, r)$ when $\gamma(\mathbf{d}, r, k) > 0$. To do this we consider the general complete intersection X of type (d_1, \dots, d_{m-1}) , and the complete linear system

$$\Sigma(d_m, X) = |\mathcal{O}_X(d_m)|.$$

We assume that the Fano scheme $F_k(X)$ of linear subspaces of dimension k contained in X is non-empty. This implies that

$$(4) \quad \dim(F_k(X)) = (k+1)(r-k) - \sum_{j=1}^{m-1} \binom{d_j + k}{k} = \binom{d_m + k}{k} - \gamma(\mathbf{d}, r, k) \geq 0$$

(see [8, 31, 39]). Moreover, assume

$$(5) \quad \dim(\Sigma(d_m, X)) > \gamma(\mathbf{d}, r, k) \geq 0.$$

Notice that (4) and (5) do hold if $\gamma(\mathbf{d}, r, k)$ is sufficiently small, e.g., if $\gamma(\mathbf{d}, r, k) = 1$.

Let now $\Sigma(d_m, X, k)$ be the set of points in $\Sigma(d_m, X)$ corresponding to complete intersections of type $\mathbf{d} = (d_1, \dots, d_m)$ contained in X and containing a subspace of dimension k . As an immediate consequence of Proposition 2.1, we have

Corollary 2.2. *Assume $\gamma(\mathbf{d}, r, k) > 0$ and (4) holds. Let X be a general complete intersection of type (d_1, \dots, d_{m-1}) verifying (5). Then $\Sigma(d_m, X, k)$ is irreducible of codimension $\gamma(\mathbf{d}, r, k)$ in $\Sigma(d_m, X)$. The general point of $\Sigma(d_m, X, k)$ corresponds to a complete intersection of type $\mathbf{d} = (d_1, \dots, d_m)$ which contains a unique subspace of dimension k .*

Next we would like to compute the degree of $\Sigma(d_m, X, k)$ inside the projective space $\Sigma(d_m, X)$.

Consider the vector bundle $\text{Sym}^{d_m}(\mathcal{S}^*)$ on $\mathbb{G}(k, r)$ and set

$$\rho := \binom{d_m + k}{k} - \gamma(\mathbf{d}, r, k) = \dim(F_k(X)).$$

Similarly as in Lemma 1.2, one sees that

$$(6) \quad \deg(\Sigma(d_m, X, k)) = \int_{\mathbb{G}(k, r)} c_\rho(\text{Sym}^{d_m}(\mathcal{S}^*)) \cdot [F_k(X)],$$

where $[F_k(X)]$ stands for the dual class of $F_k(X)$ in the Chow ring $A^*(\mathbb{G}(k, r))$.

2.1. Schubert calculus. The Chern class $c_\rho(\mathrm{Sym}^{d_m}(\mathcal{S}^*))$ is the homogeneous component θ of degree ρ of the polynomial

$$\prod_{v_0+\dots+v_k=d_m} (1 + v_0x_0 + \dots + v_kx_k).$$

As usual, θ can be written as a polynomial in the elementary symmetric functions of the Chern roots x_0, \dots, x_k , which can be identified with the $\sigma_{(1^i)}$ s. Eventually, one has a formula of the form

$$c_\rho(\mathrm{Sym}^{d_m}(\mathcal{S}^*)) = \sum_{j_1+2j_2+\dots+(k+1)j_{k+1}=\rho} \phi_{d_m,r}(j_1, j_2, \dots, j_{k+1}) \sigma_{(1)}^{j_1} \sigma_{(1^2)}^{j_2} \cdots \sigma_{(1^{k+1})}^{j_{k+1}}.$$

In conclusion one has

$$\begin{aligned} \deg(\Sigma(d_m, X, k)) &= \\ (7) \quad &= \int_{\mathbb{G}(k,r)} [F_k(X)] \cdot \sum_{j_1+2j_2+\dots+(k+1)j_{k+1}=\rho} \phi_{d_m,r}(j_1, \dots, j_{k+1}) \sigma_{(1)}^{j_1} \cdots \sigma_{(1^{k+1})}^{j_{k+1}}. \end{aligned}$$

2.2. Debarre–Manivel’s trick. Formula (7) is rather unpractical, since both, the computation of the coefficients and of the intersection products appearing in it are rather complicated, in general. A better result can be gotten using again Debarre–Manivel’s idea as in §1.2. Taking into account (6) one sees that $\deg(\Sigma(d_m, X, k))$ equals the coefficient of the monomial $x_0^r x_1^{r-1} \cdots x_k^{r-k}$ in the product of the following polynomials in x_0, \dots, x_k :

(i) the product $Q_{k,d} = \prod_{i=1}^{m-1} Q_{k,d_i}$ of the polynomials

$$Q_{k,d_i} = \prod_{v_0+\dots+v_k=d_i} (v_0x_0 + \dots + v_kx_k);$$

(ii) the polynomial θ ;

(iii) the Vandermonde polynomial $V(x_0, \dots, x_k)$.

Notice ([21, 14.7], [34, 3.5.5]) that $Q_{k,d}$ in (i) corresponds to the class $[F_k(X)]$ of degree $(k+1)(r-k) - \rho$ in the Chow ring $A^*(\mathbb{G}(k, r))$, whereas θ in (ii) corresponds to the class of $c_\rho(\mathrm{Sym}^{d_m}(\mathcal{S}^*))$ of degree ρ . In conclusion,

$$\deg(\Sigma(d_m, X, k)) = \psi_{(r,r-1,\dots,r-k)}(Q \cdot \theta \cdot V).$$

The Bott residue formula does not seem to be applicable in this case.

3. NUMERICAL INVARIANTS OF FANO SCHEMES

In this section we consider the complete intersections whose Fano schemes have positive expected dimension

$$(8) \quad \delta(\mathbf{d}, r, k) := -\gamma(\mathbf{d}, r, k) = (k+1)(r-k) - \sum_{j=1}^m \binom{d_j + k}{k} > 0$$

where $\mathbf{d} = (d_1, \dots, d_m)$. We may and we will assume $d_i \geq 2$, $i = 1, \dots, m$. If also $r \geq 2k + m + 1$ then, for a general complete intersection X of type \mathbf{d} in \mathbb{P}^r , the Fano variety $F_k(X)$ of linear subspaces of dimension k contained in X is a smooth, irreducible variety of dimension $\delta(\mathbf{d}, r, k)$ (see [8, 14, 15, 31, 38, 39]). We will compute some numerical invariants of $F_k(X)$. If $\delta(\mathbf{d}, r, k) = 1$ then $F_k(X)$ is a smooth curve; its genus was computed in [26]. In the next section

we treat the case where $F_k(X)$ is a surface, that is, $\delta(\mathbf{d}, r, k) = 2$; our aim is to compute the Chern numbers of this surface. Actually, we deduce formulas for $c_1(F_k(X))$ and $c_2(F_k(X))$ for the general case $\delta(\mathbf{d}, r, k) > 0$. To simplify the notation, we set in the sequel $F = F_k(X)$, $\mathbb{G} = \mathbb{G}(k, r)$, $\delta = \delta(\mathbf{d}, r, k)$, and we let \mathfrak{h} be the hyperplane section class of \mathbb{G} in the Plücker embedding.

Recall the following fact (cf. Proposition 1.3).

Proposition 3.1. ([15, Thm. 4.3]) *In the notation and assumptions as before, one has*

$$\deg(F) = \psi_{(r, r-1, \dots, r-k)}(Q_{k, \mathbf{d}} \cdot e^\delta \cdot V) \quad \text{where} \quad e(\mathbf{x}) := x_0 + \dots + x_k,$$

that is, the degree of the Fano scheme F under the Plücker embedding equals the coefficient of the monomial $x_0^r x_1^{r-1} \dots x_k^{r-k}$ of the product of $Q_{k, \mathbf{d}} \cdot e^\delta \cdot V$ where V stands for the Vandermonde polynomial (see Subsection 2.2 for the notation).

Remark 3.2. An alternative expression for $\deg(F)$ based on the Bott residue formula can be found in [25, Thm. 1.1] and [26, Thm. 2]; cf. also [21, Ex. 14.7.13] and [34, Sect. 3.5].

The next lemma is known in the case of the Fano scheme of lines on a general hypersurface, that is, for $k = m = 1$, see [1], [30, Ex. V.4.7].

Lemma 3.3. *In the notation and assumptions as before, one has*

$$(9) \quad c_1(T_F) = \left(r + 1 - \sum_{i=1}^m \binom{d_i + k}{k + 1} \right) \mathfrak{h}|_F$$

and

$$(10) \quad K_F \sim \mathcal{O}_F \left(\sum_{i=1}^m \binom{d_i + k}{k + 1} - (r + 1) \right)$$

where $\mathcal{O}_F(1)$ corresponds to the Plücker embedding. In particular, F is a smooth Fano variety provided $\sum_{i=1}^m \binom{d_i + k}{k + 1} \leq r$.

Proof. From the exact sequence

$$0 \rightarrow T_F \rightarrow T_{\mathbb{G}|F} \rightarrow N_{F|\mathbb{G}} \rightarrow 0$$

one obtains

$$c(T_{\mathbb{G}|F}) = c(T_F) \cdot c(N_{F|\mathbb{G}}).$$

Expanding one gets

$$(11) \quad c_1(T_F) = c_1(T_{\mathbb{G}|F}) - c_1(N_{F|\mathbb{G}})$$

and, for the further usage,

$$(12) \quad c_2(T_F) = c_2(T_{\mathbb{G}|F}) - c_2(N_{F|\mathbb{G}}) - c_1(T_{\mathbb{G}|F}) \cdot c_1(N_{F|\mathbb{G}}) + c_1(N_{F|\mathbb{G}})^2.$$

Notice ([19, Thm. 3.5]) that $T_{\mathbb{G}} = \mathcal{S}^* \otimes \mathcal{Q}$, where, as usual, $\mathcal{S} \rightarrow \mathbb{G}$ is the tautological vector bundle of rank $k + 1$ and $\mathcal{Q} \rightarrow \mathbb{G}$ is the tautological quotient bundle. Furthermore ([26, Lemma 3]), F is the zero scheme of a section of the vector bundle $\oplus_{i=1}^m \text{Sym}^{d_i}(\mathcal{S}^*)$ on \mathbb{G} . It follows that

$$(13) \quad N_{F|\mathbb{G}} \simeq \oplus_{i=1}^m \text{Sym}^{d_i}(\mathcal{S}^*)|_F.$$

By [26, Lemma 2] one has

$$(14) \quad c_1(T_{\mathbb{G}}) = (r + 1)\mathfrak{h}.$$

Taking into account (13), [26, Lemma 1] (see also Lemma 3.6 below), and the fact that $c_1(\mathcal{S}^*) = \mathfrak{h}$ (see [19, Sect. 4.1]), one gets

$$(15) \quad c_1(N_{F|\mathbb{G}}) = \sum_{i=1}^m c_1(\mathrm{Sym}^{d_i}(\mathcal{S}^*)|_F) = \left(\sum_{i=1}^m \binom{d_i + k}{k+1} \right) \mathfrak{h}|_F.$$

Plugging (14) and (15) in (11) we find (9) and then (10). \square

Corollary 3.4. *One has*

$$(16) \quad K_F^\delta = \left(\sum_{i=1}^m \binom{d_i + k}{k+1} - (r+1) \right)^\delta \deg(F),$$

where $\deg(F)$ is computed in Proposition 3.1.

Next we proceed to compute $c_2(T_F)$. Recalling (12), we need to compute $c_2(N_{F|\mathbb{G}})$ and $c_2(T_{\mathbb{G}})$. This requires some preliminaries. First of all, we need the following auxiliary combinatorial formula.

Lemma 3.5. *For any integers n, m, k where $n \geq m \geq 1$ and $k \geq 0$ one has*

$$\sum_{i=1}^n \binom{i-1}{m-1} \binom{n-i+k}{k} = \binom{n+k}{m+k}.$$

Proof. ¹ The choice of $m+k$ integers i_1, \dots, i_{m+k} among $\{1, \dots, n+k\}$, where $1 \leq i_1 < \dots < i_m < \dots < i_{m+k} \leq n+k$, can be done in two steps. At the first step one fixes the choice of $i_m = i$, where, clearly, $i \in \{1, \dots, n\}$. It remains to choose i_1, \dots, i_{m-1} among $\{1, \dots, i-1\}$ and i_{m+1}, \dots, i_{m+k} among $\{i+1, \dots, n+k\}$. \square

Lemma 3.6. *Let E be a vector bundle of rank $k+1$. Then*

$$(17) \quad c_2(\mathrm{Sym}^n(E)) = \alpha c_1(E)^2 + \beta c_2(E) \quad \text{and} \quad c_1(\mathrm{Sym}^n(E)) = \gamma c_1(E)$$

where ²

$$(18) \quad \begin{aligned} \alpha &= \frac{1}{2} \binom{n+k}{k+1}^2 - \frac{1}{2} \binom{n+k}{k+1} - \binom{n+k}{k+2}, \\ \beta &= \binom{n+k+1}{k+2}, \quad \text{and} \quad \gamma = \binom{n+k}{k+1}. \end{aligned}$$

Proof. We use the splitting principle. Write E as a formal direct sum of line bundles $E = L_0 \oplus \dots \oplus L_k$, with $c_1(L_i) = x_i$, for $0 \leq i \leq k$. From the equality

$$c(E) = (1+x_0) \cdots (1+x_k)$$

one deduces

$$(19) \quad c_1(E) = x_0 + \dots + x_k \quad \text{and} \quad c_2(E) = \sum_{0 \leq i < j \leq k} x_i x_j.$$

Since

$$\mathrm{Sym}^n(E) = \sum_{v_0 + \dots + v_k = n} L_0^{v_0} \cdots L_k^{v_k}$$

¹The authors are grateful to Roland Basher for communicating this beautiful, elementary argument.

²See also [26, Lemma 1] for γ .

one has

$$c(\text{Sym}^n(E)) = \prod_{v_0+\dots+v_k=n} (1 + v_0x_0 + \dots + v_kx_k) = \prod_{|\mathbf{v}|=n} (1 + \langle \mathbf{v}, \mathbf{x} \rangle),$$

where $\mathbf{x} = (x_0, \dots, x_k)$, $\mathbf{v} = (v_0, \dots, v_k)$, and $|\mathbf{v}| = v_0 + \dots + v_k$. Therefore,

$$(20) \quad c_1(\text{Sym}^n(E)) = \sum_{|\mathbf{v}|=n} \langle \mathbf{v}, \mathbf{x} \rangle$$

and

$$(21) \quad c_2(\text{Sym}^n(E)) = \frac{1}{2} \sum_{|\mathbf{v}|=|\mathbf{w}|=n, \mathbf{v} \neq \mathbf{w}} \langle \mathbf{v}, \mathbf{x} \rangle \langle \mathbf{w}, \mathbf{x} \rangle.$$

The right hand sides of (20) and (21) are symmetric homogeneous polynomials in x_0, \dots, x_k of degree 1 and 2, respectively. Using (19) one deduces

$$c_1(\text{Sym}^n(E)) = \sum_{|\mathbf{v}|=n} \langle \mathbf{v}, \mathbf{x} \rangle = \gamma(x_0 + \dots + x_k) = \gamma c_1(E)$$

and

$$\begin{aligned} c_2(\text{Sym}^n(E)) &= \frac{1}{2} \sum_{|\mathbf{v}|=|\mathbf{w}|=n, \mathbf{v} \neq \mathbf{w}} \langle \mathbf{v}, \mathbf{x} \rangle \langle \mathbf{w}, \mathbf{x} \rangle \\ &= \alpha(x_0 + \dots + x_k)^2 + \beta \sum_{0 \leq i < j \leq k} x_i x_j = \alpha c_1(E)^2 + \beta c_2(E), \end{aligned}$$

cf. (17). In order to compute α, β and γ , we let in these relations $x_0 = 1, x_1 = \dots = x_k = 0$, so that the coefficient of β vanishes and the coefficients of α and γ become 1. Similarly, for $x_0 = x_1 = 1, x_2 = \dots = x_k = 0$ the coefficient of β in the decomposition of $c_2(\text{Sym}^n(E))$ is 1 and the coefficient of α is 4. So, one gets

$$(22) \quad \begin{aligned} \alpha &= \frac{1}{2} \sum_{|\mathbf{v}|=|\mathbf{w}|=n, \mathbf{v} \neq \mathbf{w}} v_0 w_0, \quad \beta + 4\alpha = \frac{1}{2} \sum_{|\mathbf{v}|=|\mathbf{w}|=n, \mathbf{v} \neq \mathbf{w}} (v_0 + v_1)(w_0 + w_1), \\ \gamma &= \sum_{|\mathbf{v}|=n} v_0. \end{aligned}$$

For $k = 1$, (22) yields

$$\begin{aligned} \alpha &= \frac{1}{2} \left(\sum_{i,j=1}^n ij - \sum_{i=1}^n i^2 \right) = \frac{1}{2} \left(\left(\sum_{i=1}^n i \right)^2 - \sum_{i=1}^n i^2 \right) \\ &= \frac{1}{2} \left(\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right) = \frac{(3n+2)}{4} \binom{n+1}{3}, \end{aligned}$$

and

$$\begin{aligned} \beta + 4\alpha &= \frac{1}{2} \sum_{v_0+v_1=w_0+w_1=n} n^2 - \frac{1}{2} \sum_{v_0+v_1=n} n^2 \\ &= \frac{1}{2} n^2 (n+1)^2 - \frac{1}{2} n^2 (n+1) = \frac{1}{2} n^3 (n+1). \end{aligned}$$

Plugging in the value of α gives

$$\beta = \frac{1}{2}n^3(n+1) - 4\alpha = \frac{1}{2}n^3(n+1) - (3n+2)\binom{n+1}{3} = \binom{n+2}{3}.$$

Similarly, if $k = 2$ one has

$$\alpha = \frac{1}{2} \sum_{i,j=1}^n ij(n-i+1)(n-j+1) - \sum_{i=1}^n i^2(n-i+1) = \frac{5}{3}(n+1)\binom{n+3}{5}$$

and

$$\beta = \frac{1}{2} \left(\sum_{i,j=1}^n i(i+1)j(j+1) - \sum_{i=1}^n i^2(i+1) \right) - 4\alpha = \binom{n+3}{4}.$$

In the general case, applying Lemma 3.5 with a suitable choice of parameters we find

$$\gamma = \sum_{i=1}^n i \binom{n-i+k-1}{k-1} = \binom{n+k}{k+1}$$

and

$$\begin{aligned} \alpha &= \frac{1}{2} \sum_{i,j=1}^n ij \binom{n-i+k-1}{k-1} \binom{n-j+k-1}{k-1} - \frac{1}{2} \sum_{i=1}^n i^2 \binom{n-i+k-1}{k-1} \\ &= \frac{1}{2} \left(\sum_{i=1}^n i \binom{n-i+k-1}{k-1} \right)^2 - \sum_{i=1}^n \binom{i+1}{2} \binom{n-i+k-1}{k-1} + \\ &\quad + \frac{1}{2} \sum_{i=1}^n i \binom{n-i+k-1}{k-1} \\ &= \frac{1}{2} \binom{n+k}{k+1}^2 - \binom{n+k+1}{k+2} + \frac{1}{2} \binom{n+k}{k+1} = \frac{1}{2} \binom{n+k}{k+1}^2 - \frac{1}{2} \binom{n+k}{k+1} - \\ &\quad - \binom{n+k}{k+2}, \end{aligned}$$

where at the last step one uses the standard identity

$$(23) \quad \binom{N+1}{k+1} = \binom{N}{k+1} + \binom{N}{k}.$$

Applying Lemma 3.5 and the identity

$$i^2(i+1) = 2 \binom{i+1}{2} + 6 \binom{i+1}{3},$$

for $k \geq 3$ we find:

$$\begin{aligned}
\beta + 4\alpha &= \frac{1}{2} \sum_{i,j=1}^n i(i+1)j(j+1) \binom{n-i+k-2}{k-2} \binom{n-j+k-2}{k-2} \\
&\quad - \frac{1}{2} \sum_{i=1}^n i^2(i+1) \binom{n-i+k-2}{k-2} = 2 \left(\sum_{i=1}^n \binom{i+1}{2} \binom{n-i+k-2}{k-2} \right)^2 \\
&\quad - \sum_{i=1}^n \binom{i+1}{2} \binom{n-i+k-2}{k-2} - 3 \sum_{i=1}^n \binom{i+1}{3} \binom{n-i+k-2}{k-2} \\
&= 2 \binom{n+k}{k+1}^2 - \binom{n+k}{k+1} - 3 \binom{n+k}{k+2}.
\end{aligned}$$

Using the formula for α and (23) we deduce

$$\beta = \binom{n+k}{k+1} + \binom{n+k}{k+2} = \binom{n+k+1}{k+2}.$$

□

Remark 3.7. The proof shows that for $k = 1, 2$, (18) can be simplified as follows:

$$(24) \quad (\alpha, \beta) = \begin{cases} \left(\frac{3n+2}{4} \binom{n+1}{3}, \binom{n+2}{3} \right), & k = 1, \\ \left(\frac{5(n+1)}{3} \binom{n+3}{5}, \binom{n+3}{4} \right), & k = 2. \end{cases}$$

One can readily check that the expressions for α in these formulas agree with the one in (18).

Lemma 3.8. *One has*

$$\begin{aligned}
c_1(\mathcal{Q}) &= c_1(\mathcal{S}^*) = \mathfrak{h} \quad \text{and} \\
c_2(\mathcal{Q}) &= c_1(\mathcal{S}^*)^2 - c_2(\mathcal{S}^*) = \mathfrak{h}^2 - c_2(\mathcal{S}^*).
\end{aligned}$$

Proof. One has $c(\mathcal{Q}) \cdot c(\mathcal{S}) = 1$. By expanding and taking into account that $c_i(\mathcal{S}) = (-1)^i c_i(\mathcal{S}^*)$ for all positive integers i and $c_1(\mathcal{S}^*) = \mathfrak{h}$ ([19, Sect. 4.1]), the assertion follows. □

Lemma 3.9. *One has*

$$(25) \quad c_2(T_{\mathbb{G}}) = \left(\binom{r+1}{2} + k \right) \mathfrak{h}^2 + (r - 2k - 1) c_2(\mathcal{S}^*).$$

Proof. We use again the splitting principle. Write

$$\mathcal{S}^* = L_0 \oplus \cdots \oplus L_k, \quad \mathcal{Q} = M_1 \oplus \cdots \oplus M_{r-k}$$

with $c_1(L_i) = x_i, c_1(M_j) = y_j$, for $0 \leq i \leq k$ and $1 \leq j \leq r - k$. Since $T_{\mathbb{G}} = \mathcal{Q} \otimes \mathcal{S}^*$, see [19, Thm. 3.5], one obtains

$$c(T_{\mathbb{G}}) = c(\mathcal{Q} \otimes \mathcal{S}^*) = \prod_{i=0}^k \prod_{j=1}^{r-k} (1 + x_i + y_j),$$

whence

$$c_2(T_{\mathbb{G}}) = \frac{1}{2} \sum_{\substack{\lambda, \mu=0, \dots, k \\ \sigma, \rho=1, \dots, r-k \\ (\lambda, \sigma) \neq (\mu, \rho)}} (x_{\lambda} + y_{\sigma})(x_{\mu} + y_{\rho}).$$

By expanding, we see that in $c_2(T_{\mathbb{G}})$ appear the following summands:

- $\xi = \sum_{i=0}^k x_i^2$ and $\eta = \sum_{i=1}^{r-k} y_i^2$, the former appearing $\binom{r-k}{2}$ times, the latter $\binom{k+1}{2}$ times;
- $c_2(\mathcal{S}^*) = \sum_{0 \leq i < j \leq k} x_i x_j$, $c_2(\mathcal{Q}) = \sum_{1 \leq i < j \leq r-k} y_i y_j$, the former appearing $(r-k)^2$ times, the latter $(k+1)^2$ times;
- $c_1(\mathcal{Q})c_1(\mathcal{S}^*) = \sum_{i=0}^k \sum_{j=1}^{r-k} x_i y_j$ appearing $(k+1)(r-k) - 1$ times.

Using Lemma 3.8 one obtains

$$\xi = \sum_{i=0}^k x_i^2 = (x_0 + \dots + x_k)^2 - 2 \sum_{0 \leq i < j \leq k} x_i x_j = c_1(\mathcal{S}^*)^2 - 2c_2(\mathcal{S}^*) = \mathfrak{h}^2 - 2c_2(\mathcal{S}^*),$$

and similarly

$$\eta = c_1(\mathcal{Q})^2 - 2c_2(\mathcal{Q}) = \mathfrak{h}^2 - 2(\mathfrak{h}^2 - c_2(\mathcal{S}^*)) = 2c_2(\mathcal{S}^*) - \mathfrak{h}^2.$$

Collecting these formulas and taking into account Lemma (3.8) one arrives at:

$$\begin{aligned} c_2(T_{\mathbb{G}}) &= \binom{r-k}{2} \xi + \binom{k+1}{2} \eta + (r-k)^2 c_2(\mathcal{S}^*) + (k+1)^2 c_2(\mathcal{Q}) \\ &\quad + ((k+1)(r-k) - 1) c_1(\mathcal{Q})c_1(\mathcal{S}^*) \\ &= \left[\binom{r-k}{2} - \binom{k+1}{2} \right] (\mathfrak{h}^2 - 2c_2(\mathcal{S}^*)) + (r-k)^2 c_2(\mathcal{S}^*) \\ &\quad + (k+1)^2 (\mathfrak{h}^2 - c_2(\mathcal{S}^*)) + ((k+1)(r-k) - 1) \mathfrak{h}^2 \\ &= \left[\binom{r-k}{2} - \binom{k+1}{2} + (k+1)^2 + (k+1)(r-k) - 1 \right] \mathfrak{h}^2 \\ &\quad + \left[2 \binom{k+1}{2} - 2 \binom{r-k}{2} + (r-k)^2 - (k+1)^2 \right] c_2(\mathcal{S}^*) \\ &= \left(\binom{r+1}{2} + k \right) \mathfrak{h}^2 + (r-2k-1) c_2(\mathcal{S}^*). \end{aligned}$$

□

Now we can deduce the following formulas.

Lemma 3.10. *Let α_i , β_i , and γ_i be obtained from α , β , and γ in (18) by replacing n by d_i , $i = 1, \dots, m$. Then one has*

$$(26) \quad c_2(F) = c_2(T_F) = \left(A\mathfrak{h}^2 + Bc_2(\mathcal{S}^*) \right) \cdot [F]$$

where $[F]$ is the class of F in the Chow ring $A^*(\mathbb{G})$, and ³

$$(27) \quad \begin{aligned} A &= \binom{r+1}{2} + k - \sum_{i=0}^m \alpha_i - \sum_{1 \leq i < j \leq m} \gamma_i \gamma_j \\ &\quad - (r+1) \cdot \sum_{i=1}^m \binom{d_i + k}{k+1} + \left(\sum_{i=1}^m \binom{d_i + k}{k+1} \right)^2, \end{aligned}$$

³The sum $\sum_{1 \leq i < j \leq m} \gamma_i \gamma_j$ disappears if $m = 1$.

and

$$(28) \quad B = r - 2k - 1 - \sum_{i=1}^m \beta_i.$$

Proof. Using (14) and (15) we deduce

$$(29) \quad c_1(T_{\mathbb{G}})|_F \cdot c_1(N_{F|\mathbb{G}}) = \left((r+1) \sum_{i=1}^m \binom{d_i+k}{k+1} \right) \mathfrak{h}^2 \cdot [F]$$

and

$$(30) \quad c_1(N_{F|\mathbb{G}})^2 = \left(\sum_{i=1}^m \binom{d_i+k}{k+1} \right)^2 \mathfrak{h}^2 \cdot [F].$$

Furthermore, the Whitney formula and Lemma 3.6 yield

$$\begin{aligned} c_2(N_{F|\mathbb{G}}) &= \sum_{i=1}^m c_2(\mathrm{Sym}^{d_i}(\mathcal{S}^*)|_F) + \sum_{1 \leq i < j \leq m} c_1(\mathrm{Sym}^{d_i}(\mathcal{S}^*)|_F) \cdot c_1(\mathrm{Sym}^{d_j}(\mathcal{S}^*)|_F) \\ &= \left(\sum_{i=1}^m (\alpha_i c_1(\mathcal{S}^*)^2 + \beta_i c_2(\mathcal{S}^*)) + \sum_{1 \leq i < j \leq m} \gamma_i \gamma_j c_1(\mathcal{S}^*)^2 \right) \cdot [F] \\ &= \left(\sum_{i=1}^m \alpha_i + \sum_{1 \leq i < j \leq m} \gamma_i \gamma_j \right) \mathfrak{h}^2 \cdot [F] + \left(\sum_{i=1}^m \beta_i \right) c_2(\mathcal{S}^*) \cdot [F]. \end{aligned}$$

Plugging this in (12) together with the values of the Chern classes from (25), (29), and (30) gives (26), (27), and (28). \square

Remark 3.11. The cycle F on \mathbb{G} is the reduced zero scheme of a section of the vector bundle $\mathcal{E}_F := \bigoplus_{i=1}^m \mathrm{Sym}^{d_i}(\mathcal{S}^*)$ on \mathbb{G} of rank

$$\mathrm{rk}(\mathcal{E}_F) = \binom{\mathbf{d}+k}{k} := \sum_{i=1}^m \binom{d_i+k}{k}.$$

The Poincaré dual $[F] \in A^{\binom{\mathbf{d}+k}{k}}(\mathbb{G})$ of the class of F in $A_\delta(\mathbb{G})$ is the top Chern class $c_{\binom{\mathbf{d}+k}{k}}(\mathcal{E}_F)$. The latter can be expressed in terms of the Chern roots as

$$[F] = Q_{k,\mathbf{d}}(x_0, \dots, x_k) = \prod_{i=1}^m Q_{k,d_i}(x_0, \dots, x_k) \in A^{\binom{\mathbf{d}+k}{k}}(\mathbb{G}),$$

see Section 2.2.

4. THE CASE OF FANO SURFACES

Let us turn to the case where the Fano scheme $F = F_k(X)$ of a general complete intersection $X \subset \mathbb{P}^r$ of type \mathbf{d} is an irreducible surface, that is, $\delta = 2$ and $r \geq 2k + m$. Let us make the following observations.

In the surface case, $\int_F c_2(F) = e(F)$ is the Euler–Poincaré characteristic of F . By Lemma 3.10 one can compute $e(F)$ once one knows $\int_{\mathbb{G}} \mathfrak{h}^2 \cdot [F]$ and $\int_{\mathbb{G}} c_2(\mathcal{S}^*) \cdot [F]$. As for the former, one can use the Debarre–Manivel formula for the degree $\deg(F) = \int_{\mathbb{G}} \mathfrak{h}^2 \cdot [F]$, see Proposition 3.1; cf. also Remark 3.2.

As for the latter, recall that $c_2(\mathcal{S}^*) = \sigma_{(1^2)}$ is the class of the Schubert cycle of the \mathbb{P}^k s in \mathbb{P}^r intersecting a fixed \mathbb{P}^{r-k-1} in a line. Computing $\int_{\mathbb{G}} c_2(\mathcal{S}^*) \cdot [F]$

geometrically is difficult. However, one can compute it using Debarre–Manivel’s trick. Indeed, arguing as in the proof of [15, Thm. 4.3], cf. Subsection 1.2, one can see that $\int_{\mathbb{G}} c_2(\mathcal{S}^*) \cdot [F]$ equals the coefficient of $x_0^r x_1^{r-1} \cdots x_k^{r-k}$ in the product of the three factors:

- $Q_{k,\mathbf{d}} = \prod_{i=1}^m Q_{k,d_i}$, see Subsection 2.2;
- $c_2(\mathcal{S}^*) = \sum_{0 \leq i < j \leq k} x_i x_j$;
- the Vandermonde polynomial $V(x_0, \dots, x_k) = \prod_{i < j} (x_i - x_j)$.

Notice that for $\delta = 2$ one has

$$\deg \left(Q_{k,\mathbf{d}} \cdot \sum_{0 \leq i < j \leq k} x_i x_j \right) = \binom{\mathbf{d} + k}{k} + 2 = (k+1)(r-k) = \dim(\mathbb{G}).$$

Putting together (26), (27) and (28) one finds a formula for the Euler characteristic $e(F) = \int_F c_2(F)$. Then, using (16) and the Noether formula

$$\chi(\mathcal{O}_F) = \frac{1}{12} \left(K_F^2 + e(F) \right) = \frac{1}{12} \left(c_1(F)^2 + c_2(F) \right)$$

one can compute the holomorphic Euler characteristic $\chi(\mathcal{O}_F)$, the arithmetic genus $p_a(F) = \chi(\mathcal{O}_F) - 1$, and the signature $\tau(F) = 4\chi(\mathcal{O}_F) - e(F)$.

Example 4.1. Let us apply these recipes to the well known case of the Fano surface $F = F_1(X)$ of lines on the general cubic threefold in \mathbb{P}^4 . Letting $r = 4, k = m = 1, d = 3$ one gets $\delta = 2$ and

$$Q_{1,(3)}(x_0, x_1) = 9x_0x_1(2x_0 + x_1)(x_0 + 2x_1), \quad V(x_0, x_1) = x_0 - x_1.$$

Therefore,

$$\deg(F) = \int_{\mathbb{G}(1,4)} \mathfrak{h}^2 \cdot [F] = \int_{\mathbb{G}(1,4)} c_1(\mathcal{S}^*)^2 \cdot [F] = \psi_{4,3}(Q_{1,(3)} \cdot (x_0 + x_1)^2 \cdot V) = 45$$

and

$$\int_{\mathbb{G}(1,4)} c_2(\mathcal{S}^*) \cdot [F] = \psi_{4,3}(Q_{1,(3)} \cdot x_0x_1 \cdot V) = 27.$$

Applying (24) and (26) one obtains

$$\alpha = 11, \quad \beta = 10, \quad A = 6, \quad \text{and} \quad B = -9.$$

Using the Noether formula and (16) one arrives at the classical values (see [1], [32])

$$e(F) = c_2(F) = 6 \deg(F) - 9 \int_{\mathbb{G}(1,4)} c_2(\mathcal{S}^*) \cdot [F] = 6 \cdot 45 - 9 \cdot 27 = 27$$

and

$$c_1(F)^2 = K_F^2 = \left(\binom{4}{2} - 5 \right)^2 \deg(F) = 45, \quad \chi(\mathcal{O}_F) = \frac{1}{12} (45 + 27) = 6.$$

Example 4.2. More generally, one can consider the Fano surface $F = F_1(X)$ of lines on a general hypersurface X of degree $d = 2r - 5$ in \mathbb{P}^r , $r \geq 4$. Plugging

in (26)-(28) the values of α and β from (24) one obtains

$$\begin{aligned} A &= \binom{2r-4}{2}^2 + \binom{r+1}{2} + 1 - \frac{6r-13}{4} \binom{2r-4}{3} - (r+1) \binom{2r-4}{2}, \\ &= \frac{1}{3}(6r^4 - 56r^3 + 177r^2 - 211r + 78), \\ B &= r - 3 - \binom{2r-3}{3}. \end{aligned}$$

Furthermore,

$$e(F) = c_2(F) = A \deg(F) + B \int_{\mathbb{G}(1,r)} c_2(\mathcal{S}^*) \cdot [F], \quad c_1^2(F) = \left(\binom{2r-4}{2} - (r+1) \right)^2,$$

and

$$\chi(\mathcal{O}_F) = \frac{1}{12} (c_1^2(F) + c_2(F)),$$

where

$$\deg(F) = \psi_{r,r-1} (Q_{1,(d)} \cdot (x_0 + x_1)^2 (x_0 - x_1))$$

and

$$\int_{\mathbb{G}(1,r)} c_2(\mathcal{S}^*) \cdot [F] = \psi_{r,r-1} (Q_{1,(d)} \cdot x_0 x_1 (x_0 - x_1))$$

with

$$Q_{1,(d)} = \prod_{v_0+v_1=d} (v_0 x_0 + v_1 x_1).$$

Consider, for instance, the Fano scheme F of lines on a general quintic four-fold in \mathbb{P}^5 . One has

$$r = 5, \quad d = 5, \quad k = m = 1, \quad \delta = 2.$$

One gets

$$\alpha = 85, \quad \beta = 35, \quad A = 66, \quad B = -33,$$

and further (cf. [15, Table 1] and [47])

$$\deg(F) = \psi_{5,4} (Q_{1,(5)} \cdot (x_0 + x_1)^2 (x_0 - x_1)) = 25 \cdot 245 = 6125$$

and

$$c_2(\mathcal{S}^*) \cdot [F] = \psi_{5,4} (Q_{1,(5)} \cdot x_0 x_1 (x_0 - x_1)) = 25 \cdot 115 = 2875.$$

Hence

$$e(F) = c_2(F) = 25 \cdot 33 \cdot 375 = 309375, \quad c_1^2(F) = 25 \cdot 81 \cdot 245 = 496125.$$

Finally,

$$\chi(\mathcal{O}_F) = \frac{1}{12} (c_1^2(F) + c_2(F)) = 25 \cdot 15 \cdot 179 = 67125.$$

Example 4.3. Consider further the Fano surface $F = F_1(X)$ of lines on the intersection X of two general quadrics in \mathbb{P}^5 . We have

$$r = 5, \quad m = 2, \quad \mathbf{d} = (2, 2), \quad k = 1, \quad \delta = 2,$$

$$Q_{1,(2,2)}(x_0, x_1) = 16x_0^2 x_1^2 (x_0 + x_1)^2, \quad \text{and} \quad V(x_0, x_1) = x_0 - x_1.$$

Hence

$$\deg(F) = \psi_{5,4}(Q_{1,(2,2)} \cdot (x_0 + x_1)^2 \cdot V) = 32$$

and

$$\int_{\mathbb{G}(1,5)} c_2(\mathcal{S}^*) \cdot [F] = \psi_{5,4}(Q_{1,(2,2)} \cdot x_0 x_1 \cdot V) = 16.$$

Furthermore,

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = \beta_2 = 4, \quad \gamma_1 = \gamma_2 = 3, \quad A = 3, \quad \text{and} \quad B = -6.$$

Therefore,

$$e(F) = 3 \deg(F) - 6 \int_{\mathbb{G}(1,5)} c_2(\mathcal{S}^*) \cdot [F] = 3 \cdot 32 - 6 \cdot 16 = 0,$$

$$c_1(F)^2 = \left(2 \binom{3}{2} - 6\right)^2 \deg(F) = 0, \quad \text{and so,} \quad \chi(\mathcal{O}_F) = 0.$$

In fact, F is an abelian surface ([40]).

Example 4.4. Let now $F = F_2(X)$ be the Fano scheme of planes on a general cubic fivefold X in \mathbb{P}^6 . Thus, one has

$$r = 6, \quad m = 1, \quad d = 3, \quad k = 2, \quad \text{and} \quad \delta = 2.$$

Letting

$$Q_{2,(3)} = 27x_0x_1x_2(2x_0+x_1)(2x_0+x_2)(x_0+2x_1)(x_0+2x_2)(2x_1+x_2)(x_1+2x_2)(x_0+x_1+x_2)$$

and

$$V(x_0, x_1, x_2) = (x_0 - x_1)(x_0 - x_2)(x_1 - x_2).$$

The Wolfram Alpha gives (cf. [15, Table 2])

$$\deg(F) = \psi_{6,5,4}(Q_{2,(3)} \cdot (x_0 + x_1 + x_2)^2 \cdot V) = 27 \cdot 105 = 2835$$

and

$$\int_{\mathbb{G}(2,6)} c_2(\mathcal{S}^*) \cdot [F] = \psi_{6,5,4}(Q_{2,(3)} \cdot (x_0x_1 + x_0x_2 + x_1x_2) \cdot V) = 27 \cdot 63 = 1701.$$

Standard calculations yield

$$\alpha = 40, \quad \beta = 15, \quad \gamma = 10, \quad A = 13, \quad \text{and} \quad B = -14.$$

So, one obtains

$$e(F) = 13 \deg(F) - 14c_2(\mathcal{S}^*) \cdot [F] = 13041,$$

$$K_F^2 = \left(\binom{5}{3} - 7\right)^2 \deg(F) = 9 \deg(F) = 9 \cdot 27 \cdot 105 = 25515,$$

and

$$\chi(\mathcal{O}_F) = \frac{13041 + 25515}{12} = 3213.$$

5. IRREGULAR FANO SCHEMES

In this section we study the cases in which the Fano scheme F of a general complete intersection is irregular, that is, $q(F) = h^1(\mathcal{O}_F) > 0$. As follows from the next proposition, for the Fano surfaces F this occurs only if F is one of the surfaces in Examples 4.1 (or, which is the same, in 4.2 for $r = 4$), 4.3, and 4.4. In all these cases one has $r = 2k + m + 1$.

Theorem 5.1. *Let X be a general complete intersection of type $\mathbf{d} = (d_1, \dots, d_m)$ in \mathbb{P}^r . Suppose that the Fano scheme $F = F_k(X)$ of k -planes in X , $k \geq 1$, is irreducible of dimension $\delta \geq 2$. Then F is irregular if and only if one of the following holds:*

- (i) F is the variety of lines on a general cubic threefold in \mathbb{P}^4 ($\dim(F) = 2$);
- (ii) F is the variety of planes on a general cubic fivefold in \mathbb{P}^6 ($\dim(F) = 2$);
- (iii) F is the variety of k -planes on the intersection of two general quadrics in \mathbb{P}^{2k+3} , $k \in \mathbb{N}$ ($\dim(F) = k + 1$).

Proof. By our assumption, $\delta \geq 2$. By [15, Thm. 3.4] one has $q(F) = 0$ if $r \geq 2k + m + 2$. By [15, Thm. 2.1], F being nonempty implies $r \geq 2k + m$. Therefore, $q(F) > 0$ leaves just two possibilities:

$$r = 2k + m \quad \text{and} \quad r = 2k + m + 1.$$

We claim that the first possibility is not realized. Indeed, let $r = 2k + m$. We may assume that $d_i \geq 2$ for all $i = 1, \dots, m$. From (8) one deduces:

$$(31) \quad (k+1)(k+m) = (k+1)(r-k) = \delta + \sum_{i=1}^m \binom{d_i+k}{k} \geq 2 + m \binom{k+2}{2}.$$

This implies the inequality

$$(32) \quad 4 \leq k(k+1)(2-m),$$

and so, $m = 1$, that is, X is a hypersurface in \mathbb{P}^{2k+1} . Letting $d = d_1$, (31) reads

$$\delta = (k+1)^2 - \binom{d+k}{k} \geq 2.$$

This inequality holds only when $d = 2$, that is, X is a smooth quadric of dimension $2k$. However, in the latter case $F = F_k(X)$ consists of two components ([17, Lemma 1.1]), contrary to our assumption. This proves our claim.

In the case $r = 2k + m + 1$, (31) and (32) must be replaced, respectively, by

$$(k+1)(k+m+1) = (k+1)(r-k) = \delta + \sum_{i=1}^m \binom{d_i+k}{k} \geq \delta + m \binom{k+2}{2}$$

and

$$(33) \quad 4 \leq 2\delta \leq (k+1)[2(k+m+1) - m(k+2)] = (k+1)[(2-m)k + 2].$$

It follows from (33) that either $m = 1$ and $r = 2k + 2$, or $m = 2$ and $r = 2k + 3$.

In the hypersurface case (i.e., $m = 1$) one has

$$2 \leq \delta = (k+1)(k+2) - \binom{d+k}{k}.$$

This holds only if either $d = 2$, or $d \geq 3$ and $k \in \{1, 2\}$.

If $d = 2$, that is, X is a smooth quadric in \mathbb{P}^{2k+2} , then $\delta = \binom{k+2}{2}$, cf. [17, Lemma 1.1]. However, by [17, Lemma 1.2], in this case F is unirational, hence $q(F) = 0$, contrary to our assumption.

The possibility $d \geq 3$ realizes just in the following two cases:

- (i) $(d, r, k) = (3, 4, 1)$, that is, F is the Fano surface of lines on a smooth cubic threefold in \mathbb{P}^4 ;
- (ii) $(d, r, k) = (3, 6, 2)$, that is, F is the Fano surface of planes on a smooth cubic fivefold in \mathbb{P}^6 .

If further $m = 2$ then $r = 2k + 3$ and

$$2 \leq \delta = (k+1)(k+3) - \binom{d_1+k}{k} - \binom{d_2+k}{k}.$$

This inequality holds only for $\mathbf{d} = (2, 2)$, that is, only in the case where

- (iii) $F = F_k(X)$ is the Fano scheme of k -planes in a smooth intersection of two quadrics in \mathbb{P}^{2k+3} .

Notice that F as in (iii) is smooth, irreducible, of dimension $\delta = k + 1$, cf. [40, Ch. 4] and Remarks 5.2 below.

It remains to check that $q(F) > 0$ in (i)-(iii) indeed.

The Fano surface $F = F_1(X)$ of lines on a smooth cubic threefold $X \subset \mathbb{P}^4$ in (i) was studied by Fano ([20]) who found, in particular, that $q(F) = 5$. From Example 4.1, we deduce that $p_g(F) = 10$ (cf. also [5, Thm. 4], [7], [12], [23], [32], [40, Sect. 4.3], [41], [43], [44]). There is an isomorphism $\text{Alb}(F) \simeq J(X)$ where $J(X)$ is the intermediate Jacobian (see [12]). The latter holds as well for $F = F_2(X)$ where $X \subset \mathbb{P}^6$ is a smooth cubic fivefold as in (ii), see [13]. Thus, $q(F) > 0$ in (i) and (ii).

By a theorem of M. Reid [40, Thm. 4.8] (see also [16, Thm. 2], [46]), the Fano scheme $F = F_k(X)$ of k -planes on a smooth intersection X of two quadrics in \mathbb{P}^{2k+3} as in (iii) is isomorphic to the Jacobian $J(C)$ of a hyperelliptic curve C of genus $g(C) = k + 1$ (of an elliptic curve if $k = 0$). Hence, one has $q(F) = \dim(F) = k + 1 > 0$ for $k \geq 0$. Notice that there are isomorphisms $F \simeq J(C) \simeq J(X)$ where $J(X)$ is the intermediate Jacobian, see [17]. \square

Remarks 5.2. 1. The complete intersections in (i)-(iii) are Fano varieties. The ones in (i) are the Fano threefolds of index 2 with a very ample generator of the Picard group. The complete intersections Fano threefolds of index 1 with a very ample anticanonical divisor are the varieties $V_3^{2g-2} \subset \mathbb{P}^{g+1}$ of genera $g = 3, 4, 5$, that is, the smooth quartics V_3^4 in \mathbb{P}^3 ($g = 3$), the smooth intersections V_3^6 of a quadric and a cubic in \mathbb{P}^5 ($g = 4$), and the smooth intersections V_3^8 of three quadrics in \mathbb{P}^6 ($g = 5$), see [27, Ch. IV, Prop. 1.4]. The Fano scheme of lines $F = F_1$ on a general such Fano threefold V_3^{2g-2} is a smooth curve of a positive genus $g(F) > 0$. In fact, $g(F) = 801$ for $g = 3$, $g(F) = 271$ for $g = 4$, and $g(F) = 129$ for $g = 5$, see [35] and [26, Examples 1-3]. For these $X = V_3^{2g-2}$, the Abel-Jacobi map $J(F) \rightarrow J(X)$ to the intermediate Jacobian is an epimorphism, and $J(X)$ coincides with the Prym variety of X , see [27] and [45, Lect. 4, Sect. 1, Ex. 1 and Sect. 3].

2. Notice that the complete intersections whose Fano schemes of lines are curves of positive genera are not exhausted by the above Fano threefolds V_3^{2g-2} . The same holds, for instance, for a general hypersurface of degree $2r - 4$ in \mathbb{P}^r ,

$r \geq 4$, and for general complete intersections of types $\mathbf{d} = (r-3, r-2)$ and $\mathbf{d} = (r-4, r-4)$ in \mathbb{P}^r for $r \geq 5$ and $r \geq 6$, respectively, see [26, Examples 1-3], etc. One can find in [26] a formula for the genus of the curve F .

3. Let X be a smooth intersection of two quadrics in \mathbb{P}^{2k+2} . Then the Fano scheme $F_k(X)$ is reduced and finite of cardinality 2^{2k+2} ([40, Ch. 2]), whereas $F_{k-1}(X)$ is a rational Fano variety of dimension $2k$ and index 1, whose Picard number is $\rho = 2k + 4$, see [2], [11], and the references therein.

As for the Picard numbers of the Fano schemes of complete intersections, one has the following result (cf. also [15]).

Theorem 5.1. ([28, Thm. 03]) *Let X be a very general complete intersection in \mathbb{P}^r . Assume $\delta(\mathbf{d}, r, k) \geq 2$. Then $\rho(F_k(X)) = 1$ except in the following cases:*

- *X is a quadric in \mathbb{P}^{2k+1} , $k \geq 1$. Then $F_k(X)$ consists of two isomorphic smooth disjoint components, and the Picard number of each component is 1;*
- *X is a quadric in \mathbb{P}^{2k+3} , $k \geq 1$. Then $\rho(F_k(X)) = 2$;*
- *X is a complete intersection of two quadrics in \mathbb{P}^{2k+4} , $k \geq 1$. Then $\rho(F_k(X)) = 2k + 6$.*

The assumption “very general” of this theorem cannot be replaced by “general”; one can find corresponding examples in [28].

6. HYPERSURFACES CONTAINING CONICS

Recall (see [22, Thm. 1.1]) that for the general hypersurface X of degree d in \mathbb{P}^r , the variety $R_2(X)$ of smooth conics in X is smooth⁴ of the expected dimension $\mu(d, r) = 3r - 2d - 2$ provided $\mu(d, r) \geq 0$, and is empty otherwise. In this section we concentrate on the latter case.

6.1. The codimension count and uniqueness. Set

$$\epsilon(d, r) = 2d + 2 - 3r.$$

Consider the subvariety $\Sigma_c(d, r)$ of $\Sigma(d, r)$ whose points correspond to hypersurfaces containing plane conics. By abuse of language, in the sequel we say “conic” meaning “plane conic”; thus, a pair of skew lines does not fit in our terminology. A conic is smooth if it is reduced and irreducible.

Theorem 6.1. *Assume $d \geq 2$, $r \geq 3$, and $\epsilon(d, r) \geq 0$. Then the following hold.*

- (a) *$\Sigma_c(d, r)$ is irreducible of codimension $\epsilon(d, r)$ in $\Sigma(d, r)$.*
- (b) *If $\epsilon(d, r) > 0$ and $(d, r) \neq (4, 3)$ then the hypersurface corresponding to the general point of $\Sigma_c(d, r)$ contains a unique conic, and this conic is smooth. In the case $(d, r) = (4, 3)$ it contains exactly two distinct conics, and these conics are smooth and coplanar.*

Proof. (a) Let $\mathcal{H}_{c,r}$ be the component of the Hilbert scheme whose points parameterize conics in \mathbb{P}^r . There is an obvious morphism

$$\pi : \mathcal{H}_{c,r} \rightarrow \mathbb{G}(2, r)$$

⁴and connected provided $\mu_2 \geq 1$ and X is not a smooth cubic surfaces in \mathbb{P}^3 .

sending a conic Γ to the plane $\Pi = \langle \Gamma \rangle$. The fibers of π are projective spaces of dimension 5, hence $\mathcal{H}_{c,r}$ is a \mathbb{P}^5 -bundle over $\mathbb{G}(2, r)$. Therefore $\mathcal{H}_{c,r}$ is a smooth, irreducible projective variety of dimension $3r - 1$.

Consider the incidence relation

$$I = \{(\Gamma, X) \in \mathcal{H}_{c,r} \times \Sigma(d, r) \mid \Gamma \subset X\}$$

and the natural projections

$$p: I \rightarrow \mathcal{H}_{c,r} \quad \text{and} \quad q: I \rightarrow \Sigma(d, r).$$

It is easily seen that $q(I) = \Sigma_c(d, r)$ and that, for any $\Gamma \in \mathcal{H}_{c,r}$, $p^{-1}(\Gamma)$ is a linear subspace of $\{\Gamma\} \times \Sigma(d, r)$ of codimension $2d + 1$. Indeed, Γ being a complete intersection, it is projectively normal, hence the restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma(d)) \simeq \mathbb{C}^{2d+1}$$

is surjective. It follows that I and $\Sigma_c(d, r)$ are irreducible proper schemes. Moreover, one has

$$\begin{aligned} \dim(I) &= \dim(p^{-1}(\Gamma)) + \dim(\mathcal{H}_{c,r}) = \\ &= \binom{d+r}{d} - 1 - (2d + 2 - 3r) = \dim(\Sigma(d, r)) - \epsilon(d, r). \end{aligned}$$

Letting $\kappa(d, r)$ be the dimension of the general fiber of $q: I \rightarrow \Sigma_c(d, r)$, one obtains

$$\dim(\Sigma_c(d, r)) = \dim(I) - \kappa(d, r) = \dim(\Sigma(d, r)) - \epsilon(d, r) - \kappa(d, r),$$

and therefore

$$\text{codim}(\Sigma_c(d, r), \Sigma(d, r)) = \epsilon(d, r) + \kappa(d, r).$$

Next we prove that $\kappa(d, r) = 0$, which will accomplish the proof of part (a). To do this, we imitate the argument in [8, p. 29].

First of all, consider again the surjective morphism $q: I \rightarrow \Sigma_c(d, r)$. Since I is irreducible, the general element $(\Gamma, X) \in I$ maps to the general element $X \in \Sigma_c(d, r)$. Since $(\Gamma, X) \in I$ is general and $p: I \rightarrow \mathcal{H}_{c,r}$ is surjective, then Γ is smooth. Hence the general $X \in \Sigma_c(d, r)$ contains some smooth conic Γ . Moreover, the general fibre of q could be reducible, but, by Stein factorization, all components of it are of the same dimension and exchanged by monodromy. This implies that the general element (Γ, X) of any component of $q^{-1}(X)$ with $X \in \Sigma_c(d, r)$ general, is such that Γ is smooth (cf. Claim 6.2 below for an alternative argument).

By choosing appropriate coordinates, we may assume that if (Γ, X) is the general element of a component of $q^{-1}(X)$ with $X \in \Sigma_c(d, r)$ general, then Γ has equations

$$x_0x_1 - x_2^2 = x_3 = \cdots = x_r = 0$$

and X has equation $F = 0$ with

$$F = A(x_0x_1 - x_2^2) + B_3x_3 + \cdots + B_rx_r + R$$

where

$$A = \sum_{v_0+v_1+v_2=d-2} \alpha_{\mathbf{v}} x_0^{v_0} x_1^{v_1} x_2^{v_2}, B_i = \sum_{w_0+w_1+w_2=d-1} \beta_{\mathbf{w}} x_0^{w_0} x_1^{w_1} x_2^{w_2}, \quad \text{for } i = 3, \dots, r$$

and $R \in I_\Gamma^2$. By Bertini's theorem we may assume that X is smooth. We have the normal bundles sequence

$$0 \rightarrow N_{\Gamma|X} \rightarrow N_{\Gamma|\mathbb{P}^r} \simeq \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(1)^{\oplus r-3} \rightarrow N_{X|\mathbb{P}^r|_\Gamma} \simeq \mathcal{O}_\Gamma(d) \rightarrow 0$$

We want to show that $h^0(N_{\Gamma|X}) = 0$, which implies that $\kappa(d, r) = 0$, as desired. In order to prove this, we will prove that the map

$$\varphi : H^0(N_{\Gamma|\mathbb{P}^r}) \rightarrow H^0(N_{X|\mathbb{P}^r|_\Gamma})$$

is injective. Notice that $h^0(N_{\Gamma|\mathbb{P}^r}) = 3r - 1$ and $h^0(N_{X|\mathbb{P}^r|_\Gamma}) = 2d + 1$, and so, the assumption $\epsilon(d, r) \geq 0$ reads $h^0(N_{\Gamma|\mathbb{P}^r}) \leq h^0(N_{X|\mathbb{P}^r|_\Gamma})$.

We can interpret a section in $H^0(N_{\Gamma|\mathbb{P}^r})$ as the datum of (f, f_3, \dots, f_r) , where $f \in H^0(\mathcal{O}_\Gamma(2))$ is a homogeneous polynomial

$$f = \sum_{0 \leq i \leq j \leq 2} b_{ij} x_i x_j$$

taken modulo $x_0 x_1 - x_2^2$, and $f_i \in H^0(\mathcal{O}_\Gamma(1))$ is a linear form

$$f_i = a_{i0} x_0 + a_{i1} x_1 + a_{i2} x_2, \quad \text{for } i = 3, \dots, r.$$

Notice that the parameters on which (f, f_3, \dots, f_r) depends are indeed $3r - 1$, namely the $3(r - 2)$ coefficients a_{ij} s plus the 5 coefficients b_{ij} s. The map φ sends (f, f_3, \dots, f_r) to the restriction of $Af + B_3 f_3 + \dots + B_r f_r$ to Γ . By identifying Γ with \mathbb{P}^1 via the map sending $t \in \mathbb{P}^1$ to

$$(34) \quad x_0 = 1, x_1 = t^2, x_2 = t, x_3 = \dots = x_r = 0$$

the restriction of $Af + B_3 f_3 + \dots + B_r f_r$ to Γ identifies (after the substitution (34)) with a polynomial $P(t)$ of degree $2d$ in t . Let us order the $3(r - 2)$ coefficients a_{ij} s and the 5 coefficients b_{ij} s in such a way that the b_{ij} s come before the a_{ij} s, and inside each group they are ordered lexicographically. Then we can consider the matrix Φ of the map φ , which is of type $(2d + 1) \times (3r - 1)$. Indeed, each one of the $2d + 1$ coefficients of the polynomial $P(t)$ of degree $2d$ is in turn a polynomial of the b_{ij} and a_{ij} . Given b_{ij} or a_{ij} , its coefficients in those polynomials form the corresponding column of Φ .

Notice that the latter coefficients (that is, the entries of Φ) are linear functions of the $\alpha_{\mathbf{v}}$ s and the $\beta_{\mathbf{w}}$ s. Moreover, in each row and in each column of Φ a given $\alpha_{\mathbf{v}}$ and a given $\beta_{\mathbf{w}}$ appear at most once.

The map φ is injective if and only if Φ has rank $3r - 1$ for sufficiently general values of the $\alpha_{\mathbf{v}}$ s and the $\beta_{\mathbf{w}}$ s. We will in fact consider the $\alpha_{\mathbf{v}}$ s and the $\beta_{\mathbf{w}}$ s as indeterminates and prove that there is a maximal minor of Φ , e.g., the one Φ' determined by the first $3r - 1$ rows, which is a non-zero polynomial in these variables. This will finish our proof.

Consider, for example, the order of the $\alpha_{\mathbf{v}}$ s and the $\beta_{\mathbf{w}}$ s in which the former come before the latter and in each group they are ordered lexicographically. Let us order the monomials appearing in the expression of Φ' according to the following rule: the monomial m_1 comes before the monomial m_2 if for the smallest variable appearing in m_1 and in m_2 with different exponents, the exponent in m_1 is larger than the exponent in m_2 . The greatest monomial in this ordering will have coefficient ± 1 in Φ' , since in each row, the choice of the $\alpha_{\mathbf{v}}$ s and the $\beta_{\mathbf{w}}$ s entering in it is prescribed. This proves that $\Phi' \neq 0$.

(b) We have to show that, if $\epsilon(d, r) > 0$ and, except for $(d, r) = (4, 3)$, the hypersurface X corresponding to the general point of $\Sigma_c(d, r)$ contains a unique conic. To do this we use counts of parameters, which show that the codimension in $\Sigma(d, r)$ of the locus of hypersurfaces X containing at least two distinct conics is strictly larger than $\epsilon(d, r)$. The proof is a bit tedious, since it requires to consider a number of different possibilities, namely that two conics on X do not intersect, or they intersect in one, two or in four points (counting with multiplicity). We will not treat in detail all the cases, but only the former and the latter, leaving some easy details in the remaining two cases to the reader, which could profit from similarity with the dimension count we made at the beginning of this proof.

We start with the following two claims.

Claim 6.1. *The subset $\Sigma_{2l}(d, r)$ of all the $X \in \Sigma_c(d, r)$ such that X contains a double line is a proper subvariety of $\Sigma_c(d, r)$.*

Proof of Claim 6.1. Consider the closed subset $\mathcal{H}_{2l,r} \subset \mathcal{H}_{c,r}$ whose general point corresponds to a double line in \mathbb{P}^r . There is a natural \mathbb{P}^2 -fibration $\mathcal{H}_{2l,r} \rightarrow \mathbb{G}(2, r)$. Hence one has $\dim(\mathcal{H}_{2l,r}) = 3r - 4$. Consider further the incidence relation

$$I_{2l} = \{(\Gamma, X) \in \mathcal{H}_{2l,r} \times \Sigma(d, r) \mid \Gamma \in X\}$$

with projections p_{2l}, q_{2l} to the first and the second factors, respectively. The general fiber F_{2l} of p_{2l} is a linear subspace of $\Sigma(d, r)$ of codimension $2d + 1$. It follows that I_{2l} is an irreducible projective variety of dimension

$$\dim(I_{2l}) = \dim(\Sigma(d, r)) - (2d + 3 - 3r).$$

Therefore, the image $\Sigma_{2l}(d, r) = q_{2l}(I_{2l})$ is an irreducible proper subvariety of $\Sigma(d, r)$ of codimension at least

$$2d + 3 - 3r = \epsilon(r, d) + 1 = \text{codim}(\Sigma_c(d, r), \Sigma(d, r)) + 1,$$

see (a). □

We know by (a) that if $X \in \Sigma_c(d, r)$ is general, then X contains only finitely many conics (recall that the general fiber of $q : I \rightarrow \Sigma_c(d, r)$ has dimension $\kappa(d, r) = 0$). Our next claim is the following.

Claim 6.2. *The conics contained in the general $X \in \Sigma_c(d, r)$ are all smooth.*

Proof of Claim 6.2. The incidence variety I being irreducible, the monodromy group of the generically finite morphism $q : I \rightarrow \Sigma_c(d, r)$ acts transitively on the general fiber $q^{-1}(X)$. Its action on I lifts to the universal family of conics over I . The latter action by homeomorphisms of the general fiber (which consists of a finite number of conics) preserves the Euler characteristic. We know already that the general X contains a smooth conic. Due to Claim 6.1, X does not carry any double line. Since the Euler characteristic (equal 3) of the union of two crossing lines is different from the one of a smooth conic, all the conics in X are smooth. □

Suppose now the general $X \in \Sigma_c(d, r)$ contains more than one conic, and assume first it contains two conics which do not intersect. We will see this leads to a contradiction.

Let $\mathcal{H}_{cc,r}$ be the component of the Hilbert scheme whose general point corresponds to a pair of conics in \mathbb{P}^r which do not meet. It is easy to see that $\mathcal{H}_{cc,r}$ is an irreducible projective variety of dimension $6r - 2$.

Consider the incidence relation

$$I = \{(\Gamma, X) \in \mathcal{H}_{cc,r} \times \Sigma(d, r) \mid \Gamma \subset X\}$$

and the natural projections

$$p: I \rightarrow \mathcal{H}_{cc,r} \quad \text{and} \quad q: I \rightarrow \Sigma(d, r).$$

Claim 6.3. *For any $\Gamma \in \mathcal{H}_{cc,r}$ which corresponds to a pair of disjoint smooth conics, $p^{-1}(\Gamma)$ is a linear subspace of $\{\Gamma\} \times \Sigma(d, r)$ of codimension $4d + 2$.*

Proof of Claim 6.3. By our assumption, $r \geq 3$. Then the hypothesis $\epsilon(d, r) > 0$ implies $d \geq 4$. So, we have to prove that the restriction map

$$\rho: H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \rightarrow H^0(\Gamma, \mathcal{O}_{\Gamma}(d)) \simeq \mathbb{C}^{4d+2},$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 disjoint smooth conics, is surjective as soon as $d \geq 4$. Actually we will prove it for $d \geq 3$. By projecting generically into \mathbb{P}^3 , it suffices to prove the assertion if $r = 3$.

The restriction map

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \rightarrow H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}(d)) \simeq \mathbb{C}^{2d+1}$$

is surjective for all $d \geq 1$ because Γ_1 is projectively normal. Hence the kernel of this map, i.e., $H^0(\mathbb{P}^3, \mathcal{I}_{\Gamma_1|\mathbb{P}^3}(d))$ has codimension $2d + 1$ in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$. Consider now the restriction map

$$\rho': H^0(\mathbb{P}^3, \mathcal{I}_{\Gamma_1|\mathbb{P}^3}(d)) \rightarrow H^0(\Gamma_2, \mathcal{O}_{\Gamma_2}(d)) \simeq \mathbb{C}^{2d+1}.$$

We will prove that this map is also surjective. This will imply that its kernel, i.e., $H^0(\mathbb{P}^3, \mathcal{I}_{\Gamma|\mathbb{P}^3}(d))$ has codimension $2d + 1$ in $H^0(\mathbb{P}^3, \mathcal{I}_{\Gamma_1|\mathbb{P}^3}(d))$, hence it has codimension $4d + 2$ in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$, which proves that ρ is surjective.

Let $\Pi_i = \langle \Gamma_i \rangle$ be the plane spanned by Γ_i , for $i = 1, 2$. Consider the intersection scheme \mathfrak{D} of Γ_1 with Π_2 , so that \mathfrak{D} is a zero dimensional scheme of length 2, and \mathfrak{D} is not contained in Γ_2 . To prove that ρ' is surjective, notice that it is composed of the following two restriction maps

$$\rho_1: H^0(\mathbb{P}^3, \mathcal{I}_{\Gamma_1|\mathbb{P}^3}(d)) \rightarrow H^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d)),$$

$$\rho_2: H^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d)) \rightarrow H^0(\Gamma_2, \mathcal{O}_{\Gamma_2}(d)).$$

The map ρ_1 is surjective, because its cokernel is $H^1(\mathbb{P}^3, \mathcal{I}_{\Gamma_1|\mathbb{P}^3}(d-1))$ which is zero because Γ_1 is projectively normal. So, we are left to prove that the map ρ_2 is surjective. The kernel of ρ_2 is $H^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d-2))$, whose dimension is

$$h^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d-2)) = \binom{d}{2} - 2$$

as soon as $d \geq 3$. Similarly

$$h^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d)) = \binom{d+2}{2} - 2$$

for any $d \geq 1$. Hence the dimension of the image of ρ_2 is

$$h^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d)) - h^0(\Pi_2, \mathcal{I}_{\mathfrak{D}|\Pi_2}(d-2)) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1$$

which proves that ρ_2 is surjective. \square

By Claim 6.3, I has a unique component I' which dominates $\mathcal{H}_{cc,r}$, and

$$\begin{aligned} \dim(I') &= \dim(\mathcal{H}_{cc,r}) + \dim(\Sigma(d, r)) - (4d + 2) = \\ &= \dim(\Sigma(d, r)) - (4d - 6r + 4). \end{aligned}$$

Since we are assuming $q(I') = \Sigma_c(d, r)$, we have

$$\begin{aligned} \dim(\Sigma(d, r)) - (4d - 6r + 4) &= \dim(I') \geq \\ &\geq \dim(q(I')) = \dim(\Sigma(d, r)) - \epsilon(d, r) = \dim(\Sigma(d, r)) - (2d - 3r + 2), \end{aligned}$$

whence $\epsilon(d, r) = 2d + 2 - 3r \leq 0$, contrary to the assumption $\epsilon(d, r) > 0$.

A similar argument works also in the cases where the general $X \in \Sigma_c(d, r)$ contains two smooth conics which meet in one or two points, counting with multiplicity. The corresponding closed subset $\mathcal{H}_{cc,r}^{(i)} \subset \mathcal{H}_{cc,r}$ whose general point corresponds to a pair of smooth conics which meet in i points, where $i = 1, 2$, is an irreducible proper scheme of dimension $5r$ for $i = 1$ and $4r + 2$ for $i = 2$. Letting $I^{(i)} \subset \mathcal{H}_{cc,r}^{(i)} \times \Sigma_c(d, r)$ be the corresponding incidence relation and arguing as in the proof of Claim 6.3 one can easily show that any fiber of the projection $I' \rightarrow \mathcal{H}_{cc,r}^{(i)}$ over a point $\Gamma \in \mathcal{H}_{cc,r}^{(i)}$ representing a pair of conics with exactly i places in common counting with multiplicity, is a linear subspace of $\Sigma(d, r)$ of codimension $4d + 1$ if $i = 1$ and $4d$ if $i = 2$, where I' is the unique component of $I^{(i)}$ which dominates $\Sigma_c(d, r)$. Proceeding as before, this leads in both cases to the inequality $r \leq 2$, which contradicts the assumption $r \geq 3$.

Consider finally the remaining (extremal) case in which the general $X \in \Sigma_c(d, r)$ contains two conics which are coplanar, i.e., they intersect (counting with multiplicity) at 4 points.

We denote by $\mathcal{F}(= \mathcal{H}_{cc,r}^{(4)})$ the subvariety of the Hilbert scheme whose general point corresponds to a pair of coplanar conics in \mathbb{P}^r . It is easy to see that \mathcal{F} is an irreducible projective variety of dimension $3r + 4$.

Consider the incidence relation

$$I = I^{(4)} = \{(\Gamma, X) \in \mathcal{F} \times \Sigma(d, r) \mid \Gamma \subset X\}$$

and the projections

$$p: I \rightarrow \mathcal{F} \quad \text{and} \quad q: I \rightarrow \Sigma(d, r).$$

For any $\Gamma \in \mathcal{F}$, $p^{-1}(\Gamma)$ is a linear subspace of $\{\Gamma\} \times \Sigma(d, r)$ of codimension $4d - 2$. Indeed, since Γ is a complete intersection, it is projectively normal. Hence the restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma(d))$$

is surjective for all $d \geq 1$. On the other hand, Γ is a curve of arithmetic genus 3, and the dualizing sheaf of Γ is $\mathcal{O}_\Gamma(1)$. Hence, $h^0(\Gamma, \mathcal{O}_\Gamma(d)) = 4d - 3 + 1 = 4d - 2$, as soon as $d \geq 2$.

Thus, I is irreducible, and

$$\begin{aligned} \dim(I) &= \dim(\mathcal{F}) + \dim(\Sigma(d, r)) - (4d - 2) = \\ &= \dim(\Sigma(d, r)) - (4d - 3r - 6). \end{aligned}$$

Since we are assuming $q(I) = \Sigma_c(d, r)$, we have

$$\begin{aligned} \dim(\Sigma(d, r)) - (4d - 3r - 6) &= \dim(I) \geq \\ &\geq \dim(q(I)) = \dim(\Sigma(d, r)) - \epsilon(d, r) = \dim(\Sigma(d, r)) - (2d + 2 - 3r), \end{aligned}$$

whence $d \leq 4$. Since

$$0 < \epsilon(d, r) = 2d + 2 - 3r \leq 10 - 3r$$

we see that the only possibility is $d = 4, r = 3$. In this case a similar argument proves that the general $X \in \Sigma_c(4, 3)$ contains exactly two coplanar, smooth conics. \square

6.2. The degree count. Next we compute the degree of $\Sigma_c(d, r)$ in $\Sigma(d, r)$, provided $\epsilon(d, r) > 0$.

Let $f : \mathcal{C} \rightarrow \mathcal{H}_{c,r}$ be the universal family over $\mathcal{H}_{c,r}$, which is endowed with a map $g : \mathcal{C} \rightarrow \mathbb{P}^r$. We denote by \mathcal{E}_d the vector bundle $f_*(g^*(\mathcal{O}_{\mathbb{P}^r}(d)))$ over $\mathcal{H}_{c,r}$. If Γ is a conic, the fiber $\mathcal{E}_{d,\Gamma}$ of \mathcal{E}_d at (the point corresponding to) Γ is $H^0(\Gamma, \mathcal{O}_\Gamma(d))$. We set $\mathcal{E} = \mathcal{E}_1$. Notice that \mathcal{E}_d is a vector bundle of rank $2d + 1 > 3r - 1 = \dim(\mathcal{H}_{c,r})$; in particular, $\text{rk}(\mathcal{E}) = 3$.

Lemma 6.4. *If $\epsilon(d, r) > 0$ and $(d, r) \neq (4, 3)$, then*

$$\deg(\Sigma_c(d, r)) = \int_{\mathcal{H}_{c,r}} c_{3r-1}(\mathcal{E}_d).$$

Moreover

$$\deg(\Sigma_c(4, 3)) = \frac{1}{2} \int_{\mathcal{H}_{c,3}} c_8(\mathcal{E}_4).$$

Proof. Any homogeneous form F of degree d in $r + 1$ variables defines a section σ_F of \mathcal{E}_d such that $\sigma_F(\Gamma) = F|_\Gamma \in H^0(\Gamma, \mathcal{O}_\Gamma(d))$. Consider the effective divisor X_F of degree d on \mathbb{P}^r of zeros of F . The support of X_F contains Γ if and only if $\sigma_F(\Gamma) = 0$. Counting the conics $\Gamma \in \mathcal{H}_{c,r}$ lying in $\text{Supp}(X_F)$ is the same as counting the zeros of σ_F in $\mathcal{H}_{c,r}$ with their multiplicities.

Let further $\rho = \dim(\mathcal{H}_{c,r}) = 3r - 1$. By our assumption one has

$$\text{rk}(\mathcal{E}_d) - \rho = \epsilon(d, r) > 0.$$

Choose a general linear subsystem $\mathcal{L} = \langle X_0, \dots, X_\epsilon \rangle$ in $|\mathcal{O}_{\mathbb{P}^r}(d)|$ of dimension $\epsilon = \epsilon(d, r)$, where $X_i = \{F_i = 0\}$. By virtue of Theorem 6.1, \mathcal{L} meets $\Sigma(d, r, k) \subset \Sigma(d, r)$ transversally in $\deg(\Sigma_c(d, r))$ simple points.

Consider now the sections $\sigma_i := \sigma_{F_i}$, $i = 0, \dots, \epsilon$, of \mathcal{E}_d and assume $(d, r) \neq (4, 3)$. By Theorem 6.1, the intersection of \mathcal{L} with $\Sigma(d, r, k)$ is exactly the scheme of points $\Gamma \in \mathcal{H}_{c,r}$ where there is a linear combination of $\sigma_0, \dots, \sigma_\epsilon$ vanishing on Γ . This is the zero dimensional scheme of points of $\mathcal{H}_{c,r}$ where the sections $\sigma_0, \dots, \sigma_\epsilon$ are linearly dependent. This zero dimensional scheme represents the top Chern class $c_\rho(\mathcal{E}_d)$ (see [19, Thm. 5.3]). Its degree is the top Chern number $\int_{\mathcal{H}_{c,r}} c_{3r-1}(\mathcal{E}_d)$.

The case $(d, r) = (4, 3)$ is similar: one has to take into account again Theorem 6.1, which says that the general quartic surface in \mathbb{P}^3 contains exactly two smooth conics, and these conics are coplanar. \square

To compute $c_{3r-1}(\mathcal{E}_d)$ we proceed as follows. For a positive integer d consider $\text{Sym}^d(\mathcal{E})$. Note that the universal family \mathcal{C} over $\mathcal{H}_{c,r}$ is the zero set of a section ξ of $\text{Sym}^2(\mathcal{E})$. For any $d \geq 2$ one has the exact sequence

$$0 \rightarrow \text{Sym}^{d-2}(\mathcal{E}) \xrightarrow{\xi} \text{Sym}^d(\mathcal{E}) \rightarrow \mathcal{E}_d \rightarrow 0.$$

Hence

$$(35) \quad c(\mathcal{E}_d) = c(\text{Sym}^d(\mathcal{E})) \cdot c(\text{Sym}^{d-2}(\mathcal{E}))^{-1}.$$

To compute Chern classes, as usual, we use the splitting principle. We write formally

$$\mathcal{E} = L_1 \oplus L_2 \oplus L_3,$$

the L_i s being (virtual) line bundles. Consider the Chern roots $x_i = c_1(L_i)$ of \mathcal{E} . One has

$$c(\mathcal{E}) = (1 + x_1)(1 + x_2)(1 + x_3),$$

that is,

$$c_1(\mathcal{E}) = x_1 + x_2 + x_3, \quad c_2(\mathcal{E}) = x_1x_2 + x_1x_3 + x_2x_3, \quad c_3(\mathcal{E}) = x_1x_2x_3.$$

Furthermore,

$$\text{Sym}^d(\mathcal{E}) = \bigoplus_{v_1+v_2+v_3=d} L_1^{v_1} L_2^{v_2} L_3^{v_3}.$$

Letting

$$c_1(L_1^{v_1} L_2^{v_2} L_3^{v_3}) = v_1x_1 + v_2x_2 + v_3x_3 = \langle \mathbf{v}, \mathbf{x} \rangle$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ with $|\mathbf{v}| = v_1 + v_2 + v_3$ one obtains

$$c(\text{Sym}^d(\mathcal{E})) = \prod_{|\mathbf{v}|=d} (1 + \langle \mathbf{v}, \mathbf{x} \rangle)$$

and, by (35),

$$(36) \quad c(\mathcal{E}_d) = \left(\prod_{|\mathbf{v}|=d} (1 + \langle \mathbf{v}, \mathbf{x} \rangle) \right) \cdot \left(\prod_{|\mathbf{v}|=d-2} (1 + \langle \mathbf{v}, \mathbf{x} \rangle) \right)^{-1}.$$

Now, the top Chern class $c_{3r-1}(\mathcal{E}_d)$ is the homogeneous component $\eta(x_1, x_2, x_3)$ of degree $3r - 1$ in the right hand side of (36) written as a formal power series in x_1, x_2, x_3 . This is a symmetric form of degree $3r - 1$ in x_1, x_2, x_3 . It can be expressed via the elementary symmetric polynomials in x_1, x_2, x_3 , i.e., in terms of $c_1(\mathcal{E}), c_2(\mathcal{E}), c_3(\mathcal{E})$.

This time, in order to compute $c_{3r-1}(\mathcal{E}_d)$ effectively, one cannot take advantage from Schubert calculus, neither can one apply anything similar to Debarre–Manivel’s trick. One can however use the Bott residue formula.

The standard diagonal action of $T = (\mathbb{C}^*)^{r+1}$ on \mathbb{P}^r , see 1.3, induces an action of T on $\mathbb{G}(2, r)$ and on $\mathcal{H}_{c,r}$.

Lemma 6.5. *The action of T on $\mathcal{H}_{c,r}$ has exactly $r(r^2 - 1)$ isolated fixed points.*

Proof. Let Γ be a fixed point for the T -action on $\mathcal{H}_{c,r}$. Then $\Pi = \langle \Gamma \rangle$ is fixed under the action of T on $\mathbb{G}(2, r)$. Hence Π is one of the coordinate planes in \mathbb{P}^r , and these are $\binom{r+1}{3}$ in number. We let x, y, z be the three coordinate axes in Π . Then the only conics on Π fixed by the T -action are the singular conics

$x + y, x + z, y + z, 2x, 2y, 2z$. Thus, we get in total $6\binom{r+1}{3} = r(r^2 - 1)$ fixed points of T in $\mathcal{H}_{c,r}$. \square

We denote by \mathcal{F} the set of fixed points for the T -action on $\mathcal{H}_{c,r}$. Bott's residue formula, applied in our setting, has the form

$$\deg(\Sigma_c(d, r)) = \int_{\mathcal{H}_{c,r}} c_{3r-1}(\mathcal{E}_d) = \sum_{\Gamma \in \mathcal{F}} \frac{c_\Gamma}{e_\Gamma},$$

where $\frac{c_\Gamma}{e_\Gamma}$ is the local contribution of a fixed point $\Gamma \in \mathcal{F}$. Recall that c_Γ results from the local contribution of $c_{3r-1}(\mathcal{E}_d)$ at Γ , and e_Γ is determined by the torus action on the tangent space to $\mathcal{H}_{c,r}$ at the point corresponding to Γ , see 1.3.

To compute e_Γ we have to compute the characters of the T -action on the tangent space

$$T_\Gamma(\mathcal{H}_{c,r}) \simeq H^0(\Gamma, N_{\Gamma|\mathbb{P}^r}) \simeq H^0(\Gamma, \mathcal{O}_\Gamma(1))^{\oplus(r-2)} \oplus H^0(\Gamma, \mathcal{O}_\Gamma(2)) \simeq \mathcal{E}_\Gamma^{\oplus(r-2)} \oplus \mathcal{E}_{2,\Gamma}.$$

Let $\Pi = \langle \Gamma \rangle$. Then Π is a coordinate plane which corresponds to a subset $I = \{i, j, k\} \subset \{0, \dots, r\}$ consisting of 3 distinct elements. Let \mathcal{I}_3 be the set of all the $\binom{r+1}{3}$ such subsets I . The characters of the T -action on \mathcal{E}_Γ have weights $-t_\alpha$ with $\alpha \in I$. Let $I^{(2)}$ be the symmetric square of I ; it consists of 6 unordered pairs $\{\alpha, \beta\}$, $\alpha, \beta \in I$. The characters of the T -action on $\mathcal{E}_{2,\Gamma}$ have weights $t_\alpha + t_\beta$ with $\{\alpha, \beta\} \in I^{(2)} \setminus \{a, b\}$, where $x_a x_b = 0$ is the equation of Γ in Π . Then

$$e_\Gamma = (-1)^{3(r-2)} (t_i t_j t_k)^{r-2} \prod_{\{\alpha, \beta\} \in I^{(2)} \setminus \{a, b\}} (t_\alpha + t_\beta).$$

As for c_Γ , with the same notation as above we have

$$c_\Gamma = \eta(-t_i, -t_j, -t_k) = (-1)^{3r-1} \eta(t_i, t_j, t_k) \quad \text{where } I = \{i, j, k\}.$$

In conclusion we find the formula

$$(37) \quad \deg(\Sigma_c(d, r)) = - \sum_{I=\{i,j,k\} \in \mathcal{I}_3} \sum_{\{a,b\} \in I^{(2)}} \frac{\eta(t_i, t_j, t_k)}{(t_i t_j t_k)^{r-2} \prod_{\{\alpha, \beta\} \in I^{(2)} \setminus \{a, b\}} (t_\alpha + t_\beta)}.$$

Again, the right hand side of this formula is, a priori, a rational function in the variables t_0, \dots, t_r . In fact, this is a constant and a positive integer. Letting $t_i = 1$ for all $i = 0, \dots, r$ we arrive at the following conclusion.

Theorem 6.6. *Assuming that $\epsilon(d, r) = 2d + 2 - 3r > 0$ and $(d, r) \neq (4, 3)$ one has*

$$(38) \quad \deg(\Sigma_c(d, r)) = -\frac{5}{32} \binom{r+1}{3} \eta(1, 1, 1),$$

where η is the homogeneous form of degree $3r - 1$ in the formal power series decomposition of the right hand side of (36).

Remark 6.7. In the case of the surfaces in \mathbb{P}^3 , one can find in [33, Prop. 7.1] a formula for the degree of $\Sigma_c(d, 3)$ expressed as a polynomial in d for $d \geq 5$. This formula was deduced by applying Bott's residue formula. After dividing by 2, this formula gives also the correct value $\deg(\Sigma_c(4, 3)) = 2508$.

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