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Calabi-Yau algebras viewed as deformations of Poisson algebras

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Abstract

From any algebra A defined by a single non-degenerate homogeneous quadratic relation f , we prove that the quadratic algebra B defined by the potential $w = fz$ is 3-Calabi-Yau. The algebra B can be viewed as a 3-Calabi-Yau completion of Keller. The algebras A and B are both Koszul. The classification of the algebras B in three generators, i.e., when A has two generators, leads to three types of algebras. The second type (the most interesting one) is viewed as a deformation of a Poisson algebra S whose Poisson bracket is non-diagonalizable quadratic. Although the potential of S has non-isolated singularities, the homology of S is computed. Next the Hochschild homology of B is obtained.

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Key words. Koszul algebras, Calabi-Yau algebras, Poisson algebras, deformations of algebras, Poisson homology, Hochschild homology.

1 Introduction

The deformation theory offers a way to study non-commutative algebras A by examining associated Poisson algebras S . The idea is to get (more or less explicitly) invariants attached to S by the use of differential calculus on the commutative algebra S , and then to deduce from them some invariants attached to A . Invariants of interest in various domains are of homological nature. In particular, a natural question is the following (there is the same in cohomology).

(Q) Is the Hochschild homology $\mathrm{HH}_\bullet(A)$ of A isomorphic to the Poisson homology $\mathrm{HP}_\bullet(S)$ of S ? A positive answer was given by Kontsevich [19] in cohomology when S is the algebra of the C^∞ functions on \mathbb{R}^n endowed with any Poisson bracket π extended to the space of formal series $S[[\hbar]]$, and A is the space $S[[\hbar]]$ whose usual commutative product is replaced by the Kontsevich star product \star_π (see e.g. [9] for further developments). The question (Q) was initiated by Brylinski [8] when the algebra A is filtered such that $\mathrm{gr}(A)$ is assumed to be commutative and smooth. Then $\mathrm{gr}(A)$ is naturally a Poisson algebra S , and

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there is the Brylinski spectral sequence

$$E^2 = HP_{\bullet}(S) \implies HH_{\bullet}(A). \quad (1.1)$$

In this context, we can replace the question (Q) by the following one.

(Q') Does the Brylinski spectral sequence degenerate at E^2 ?

As shown by Kassel [16], this is the case if S is any polynomial algebra whose generators have all degree 1, so that the Poisson bracket of S is of degree ≤ 1 and A is a Sridharan enveloping algebra. Actually, Kassel proved more precisely that the symmetrization defines an isomorphism from the Poisson complex of S to the Koszul complex of A . The latter result was generalized (in a weaker form) in [15] to the crossed products of enveloping algebras.

Van den Bergh showed that the question (Q') has a positive answer if A is a generic Sklyanin algebra in three generators [31]. In this case, the Poisson bracket of S (which is quadratic) is derived from a "Poisson potential" ϕ such that

(IS) the origin is an isolated singularity of the polynomial ϕ .

In the paper of Van den Bergh, the property (IS) appears as essential in the computation of $HP_{\bullet}(S)$, and it is the same for the other examples (quadratic, cubic, or more generally quasi-homogeneous) considered further [22, 25, 24]. In this paper, we present an example of quadratic algebra, called B in the text, for which the previous programme is performed without the property (IS). In other words, in our example of quadratic algebra B , the Poisson bracket of S (which is non-diagonalizable quadratic) is derived from a potential ϕ having non-isolated singularities. Nevertheless, $HP_{\bullet}(S)$ is explicitly computed (Section 5 below). Next we prove that any Poisson cycle can be lifted to a Koszul cycle (since B is Koszul, it is more convenient to use Koszul complex, instead of Hochschild complex, in order to compute $HH_{\bullet}(B)$). Finally we get the computation of $HH_{\bullet}(B)$ as stated in the following.

Theorem 1.1 *Let B be the \mathbb{C} -algebra defined by the generators x, y, z , and the following relations*

$$zy = yz + 2xz, \quad zx = xz, \quad yx = xy + x^2.$$

Let S be the polynomial \mathbb{C} -algebra in x, y, z , endowed with the Poisson bracket derived from the potential $\phi = -x^2z$. Then the Hochschild homology of B is isomorphic by a specific canonical morphism (an edge morphism in the Brylinski spectral sequence) to the Poisson homology of S and is given by

$$\begin{aligned} HH_0(B) &\simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z]; \\ HH_1(B) &\simeq \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} \oplus \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} \oplus \bigoplus_{\substack{n \in \mathbb{N}^* \\ 0 \leq k \leq n}} \mathbb{C} \begin{pmatrix} 0 \\ ky^{k-1}z^{n-k} \\ (n-k)y^kz^{n-1-k} \end{pmatrix} \\ &\quad \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \begin{pmatrix} y^n \\ nxy^{n-1} \\ 0 \end{pmatrix} \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ 1 \leq k \leq n+1}} \mathbb{C} \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1}z^{n+1-k}, \end{aligned}$$

$$\begin{aligned}
HH_2(B) \simeq \mathbb{C}[\phi] \begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus (x\mathbb{C}[\phi] \oplus z\mathbb{C}[z]) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
\oplus \bigoplus_{\substack{n \in \mathbb{N} \\ 0 \leq k \leq n}} \mathbb{C} \begin{pmatrix} (k+1)x \\ (2(n-k)+1)y \\ -2(k+1)z \end{pmatrix} y^k z^{n-k};
\end{aligned}$$

$$\begin{aligned}
HH_3(B) &\simeq \mathbb{C}[\phi]; \\
HH_p(B) &\simeq 0 \text{ if } p \geq 4.
\end{aligned}$$

Since B is 3-Calabi-Yau, the Hochschild cohomology is then immediate: $HH^\bullet(B) = HH_{3-\bullet}(B)$. Remark that the duality $HP^\bullet(S) = HP_{3-\bullet}(S)$ holds since the Poisson bracket derives from a potential.

Our example of Koszul algebra B belongs to a large class of Koszul 3-Calabi-Yau algebras still denoted by B . These algebras B can be viewed as 3-Calabi-Yau completions of algebras A of global dimension 2 (consequently they are quasi-isomorphic to Ginzburg algebras [14]). Calabi-Yau completions were defined and studied by Keller in [17] (with an appendix by Van den Bergh). In particular we refer to Section 6.8 of [17]. The Calabi-Yau property of our algebras B is then immediate from Theorem 4.8 in [17]. Let us explain how we define these algebras B (see Section 2 below for details). For any non-degenerate quadratic relation $f = \sum_{i,j} f_{ij}x_i x_j$ in $n \geq 2$ non-commutative variables x_i , the algebra A defined by the single relation f is Koszul and AS-Gorenstein of global dimension 2 [12], and A is Calabi-Yau if and only if f is symplectic [3]. Add an extra generator z to the x_i 's and consider the quadratic algebra B defined by the potential $w = fz$ (see e.g. [14, 6, 5] for algebras defined by potentials). The properties of B that we shall obtain in Section 2 work out over any field and are stated in the following.

Theorem 1.2 *Let $f = \sum_{i,j} f_{ij}x_i x_j$ be any non-degenerate quadratic relation over a field k in non-commutative variables x_1, \dots, x_n ($n \geq 2$) and let z be an extra generator. Let A be the k -algebra defined by the generators x_1, \dots, x_n and the single relation f . Let B be the k -algebra defined by the generators x_1, \dots, x_n, z and the potential $w = fz$. The following hold*

- 1) B is a skew polynomial ring over A in the generator z and defined by an automorphism of A .
- 2) z is normal in B , that is $Bz = zB$.
- 3) If f is alternating, then z is central in B . The converse holds if the characteristic of k is $\neq 2$.
- 4) B is Koszul and 3-Calabi-Yau. Moreover, B is a domain.
- 5) The Hilbert series $h_B(t)$ of the graded algebra B is

$$h_B(t) = (1 - (n+1)t + (n+1)t^2 - t^3)^{-1}.$$

- 5) The Gelfand-Kirillov dimension $GK.\dim(B)$ of B is finite if and only if $n = 2$, and in this case $GK.\dim(B) = 3$.
- 6) B is left (or right) noetherian if and only if $n = 2$.

When $n = 2$, the graded \mathbb{C} -algebras B are classified in three types (Section 4 below): the polynomial algebra in x, y and z (classical type), the algebra of Theorem 1.1 (Jordan type), a family of quantum spaces in x, y, z (quantum type). The second type is the one of interest for us, since it is well-known that the question (Q) has a positive answer in the first or third type (if the quantum parameter q is not a root of unity). Hochschild homology in the quantum type can be deduced from Wambst's result ([33], Théorème 6.1). When

$n = 2$, the \mathbb{C} -algebras B are AS-regular of global dimension 3, and it is important to notice that their invariants j (in terminology of [1]) are *infinite*, unlike Sklyanin algebras. The first type and the third type can be considered as limits of Sklyanin algebras by vanishing the parameter c used in [31], but such a process is not possible for the second type which stands really apart. Recently, Smith has given a detailed study of a remarkable algebra having seven generators and defined in terms of octonions [28]. To avoid confusion with our notation, let us call C the Smith algebra. Actually, C does not belong to the class of algebras B defined in Theorem 1.2, since in characteristic zero C has no normal element except the elements of k ([28], Proposition 11.2). However C has many properties in common with our algebras B : C is Koszul and 3-Calabi-Yau, C is defined by an explicit potential, C is a skew polynomial ring over an algebra A in the last generator (but C is defined by a derivation of A , not an automorphism), and the single relation of A is symplectic in the first six generators. Suárez-Alvarez has obtained similar properties for a class of algebras containing C and defined from any oriented Steiner triple system [30]. It would be satisfactory to enlarge naturally the class of algebras B in order to include the Suárez-Alvarez algebras, and one would expect that for any AS-regular algebra there is a skew polynomial ring in one variable over it which is Calabi-Yau.

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2 A family of 3-Calabi-Yau algebras

A down-to-earth approach of non-commutative projective algebraic geometry consists in studying graded algebras defined by generators (assumed to be of degree 1) satisfying some homogeneous non-commutative polynomial relations. Following this naïve approach, the first class to study is certainly the class of non-commutative quadrics, i.e., the class of non-commutative graded algebras defined by a single quadratic relation. It is a bit surprising that this class can be used as a toy model (see [3]) in order to introduce to several duality theories (Koszul duality, AS-Gorenstein duality and Calabi-Yau duality) playing a basic role in more sophisticated approaches. Throughout the paper, the algebras of this class will be denoted by A , sometimes $A(f)$ or $A(M)$ if we want to specify the naïve non-commutative quadric which is just a quadratic relation f of matrix M in the non-commutative generators. Our first goal will be to show that such an algebra A is the quotient of a quadratic graded algebra B defined by a potential depending on f . The study of the algebras $B = B(f) = B(M)$ is the main purpose of this paper. Let us fix the notation. For $n \geq 1$, $k\langle x_1, \dots, x_n \rangle$ denotes the graded free associative algebra over a field k , generated by x_1, \dots, x_n assumed to be of degree 1. Let us give a non-zero element $f = \sum_{1 \leq i, j \leq n} f_{ij} x_i x_j$ homogeneous of degree 2 in this algebra, or equivalently a non-zero $n \times n$ matrix $M = (f_{ij})_{1 \leq i, j \leq n}$ with entries in k . Then $A = A(f) = A(M)$ denotes the quadratic graded algebra defined as the quotient of $k\langle x_1, \dots, x_n \rangle$ by the two-sided ideal generated by f . Let us recall the properties of A (see [3, 36] for the proofs and for the definitions of Koszul, AS-Gorenstein or Calabi-Yau algebras).

Proposition 2.1 *Let $A = A(f) = A(M)$ be as above.*

- 1) *The quadratic algebra A is Koszul.*
- 2) *The global dimension of A is equal to 2, except if f is symmetric of rank 1 (in this case, the global dimension is infinite).*
- 3) *A is AS-Gorenstein if and only if f is non-degenerate (i.e., M is invertible).*
- 4) *A is 2-Calabi-Yau if and only if f is non-degenerate and skew-symmetric.*

5) If the global dimension of A is equal to 2, the Hilbert series of the graded algebra A is given by

$$h_A(t) = (1 - nt + t^2)^{-1}.$$

Otherwise, one has

$$h_A(t) = (1 - nt + t^2 - t^3 + t^4 - \dots)^{-1}.$$

6) The Gelfand-Kirillov dimension $GK.\dim(A)$ of A is equal to 0 if $n = 1$, to ∞ if $n > 2$, and if $n = 2$, it is equal to the global dimension.

7) If f is non-degenerate, A is left (or right) noetherian if and only if $n = 1$ or $n = 2$.

8) If f is non-degenerate, A is a domain if and only if $n \geq 2$.

Set $F = k\langle x_1, \dots, x_n, z \rangle$ where z is an extra generator of degree 1. We refer to [14, 6, 5] for more details on the definitions and basic properties concerning on algebras defined by a potential. The elements of $F_{cyc} = F/[F, F]$ are called *potentials*. The k -vector space F_{cyc} is sometimes identified to the space of the cyclic sums $c(a)$ when a runs over F . Let us define our potential as being the class \bar{w} in F_{cyc} of $w \in F$ where

$$w = fz = \sum_{1 \leq i, j \leq n} f_{ij} x_i x_j z,$$

or as being the cyclic sum

$$c(w) = \sum_{1 \leq i, j \leq n} f_{ij} (x_i x_j z + z x_i x_j + x_j z x_i).$$

Let us denote by $B = B(f) = B(M)$ the algebra defined by this potential. This is the quotient of the free algebra F by the cyclic partial derivatives $\partial_{x_1}(w), \dots, \partial_{x_n}(w), \partial_z(w)$, where

$$\partial_{x_i}(w) = \sum_{1 \leq j \leq n} (f_{ij} x_j z + f_{ji} z x_j), \quad 1 \leq i \leq n, \quad (2.1)$$

$$\partial_z(w) = f. \quad (2.2)$$

So B is a quadratic graded algebra. The obvious morphism of graded algebras from A to the quotient of B by z is clearly an isomorphism. In all the following, we abbreviate this isomorphism by $A \cong B/(z)$. The following lemma allows us to consider A as a quadratic subalgebra of B .

Lemma 2.2 *The morphism of graded algebras $A \rightarrow B$ induced by the composite*

$$k\langle x_1, \dots, x_n \rangle \xrightarrow{\text{can}} F \xrightarrow{\text{can}} B$$

is injective.

Proof. It suffices to observe that the ideal which defines the algebra B is homogeneous with respect to the grading by \deg_z . ■

Denote by V_A and R_A (respectively V_B and R_B) the space of generators and the space of relations of A (resp. B). One has $V_A = kx_1 \oplus \dots \oplus kx_n$, $R_A = kf$, $V_B = V_A \oplus kz$ and $R_B = (\sum_{1 \leq i \leq n} k\partial_{x_i}(w)) \oplus R_A$.

Lemma 2.3 1) *We have $\dim(R_B) \geq \text{rk}(M) + 1$.*

2) *In particular, if f is non-degenerate, then $\dim R_B = n + 1$ and $\partial_{x_1}(w), \dots, \partial_{x_n}(w)$ are k -linearly independent in F .*

3) *If f is non-degenerate, then $Az = zA$ and the element z is normal in the algebra B , that is, $Bz = zB$.*

Proof. Write down the relations $\partial_{x_i}(w) = 0$, $1 \leq i \leq n$, as the linear system (in the space B)

$$M \begin{pmatrix} x_1 z \\ \vdots \\ x_n z \end{pmatrix} = - {}^t M \begin{pmatrix} z x_1 \\ \vdots \\ z x_n \end{pmatrix} \quad (2.3)$$

with unknowns $x_1 z, \dots, x_n z$ viewed in B . Reducing this system to a triangular form provides $p = rk(M)$ equalities in B beginning by p distinct pivots $x_i z$, so that these equalities viewed as elements of R_B are linearly independent. From this the inequality in 1) follows. This shows as well that if M is invertible, the elements $x_1 z, \dots, x_n z$ belong to zA , proving that $Az \subseteq zA$ and $Bz \subseteq zB$. The opposite inclusions are obtained similarly by reading the linear system (2.3) from the right to the left. ■

It is also easy to deduce the inequality in 1) from the following relation

$$\dim(R_{B(M)}) = rk \begin{pmatrix} M & {}^t M \end{pmatrix} + 1, \quad (2.4)$$

which is immediate from the following

$$\begin{pmatrix} \partial_{x_1}(w) \\ \vdots \\ \partial_{x_n}(w) \end{pmatrix} = \begin{pmatrix} M & {}^t M \end{pmatrix} \begin{pmatrix} x_1 z \\ \vdots \\ z x_n \end{pmatrix} \quad (2.5)$$

where the latter column is formed by the $2n$ elements $x_1 z, \dots, x_n z, z x_1, \dots, z x_n$. Note that as a consequence of (2.4), we have equality in 1) if M is symmetric or antisymmetric (not necessarily invertible), and that for $n = 2$ and $f = x_1^2 + x_1 x_2$, the inequality is strict.

Example 2.4 Assume that f is symplectic, i.e., non-degenerate and alternating. It follows that $n = 2p$ is even and we can choose generators $x_1, \dots, x_p, y_1, \dots, y_p$ such that $f = \sum_{1 \leq i \leq p} (x_i y_i - y_i x_i)$. Then $\partial_{x_i}(w) = y_i z - z y_i$ and $\partial_{y_i}(w) = z x_i - x_i z$ for $1 \leq i \leq p$. In this case, z is *central* in B and one has

$$c(w) = \sum_{1 \leq i \leq p} Ant(x_i, y_i, z) \quad (2.6)$$

where $Ant(a, b, c)$ denotes the antisymmetrizer of a, b, c .

We continue the study of the algebra $B = B(f)$. Throughout the rest of this section, we assume that f is *non-degenerate*. Lemma 2.3 shows that $Az = zA$ is a sub- A -bimodule of B . Let $T_A(Az)$ denote the tensor algebra of the A -bimodule Az (on tensor algebras of bimodules, see e.g. [18], p.485). The k -algebra $T_A(Az)$ is generated by $A \oplus Az$, hence by x_1, \dots, x_n, z . Let

$$u : T_A(Az) \rightarrow B$$

be the natural morphism of k -algebras extending the inclusion $A \oplus Az \hookrightarrow B$. The inclusions $x_1, \dots, x_n \hookrightarrow A$ and $z \hookrightarrow Az$ define a morphism of k -algebras

$$F \rightarrow T_A(Az)$$

factoring out the relations of B and providing the morphism of k -algebras

$$v : B \rightarrow T_A(Az)$$

which leaves “fixed” x_1, \dots, x_n, z . Then $v \circ u$ and $u \circ v$ coincide with the respective identity maps since the k -algebras $T_A(Az)$ and B are both generated by x_1, \dots, x_n, z . Thus u and v are isomorphisms, inverse to each other. The following lemma is crucial.

Lemma 2.5 *The left (resp. right) sub- A -module Az (resp. zA) of B is free generated by z .*

Proof. Let $a \in k\langle x_1, \dots, x_n \rangle$ be homogeneous of degree $\ell \geq 0$ such that az belongs to the two-sided ideal of F generated by $\partial_{x_1}(w), \dots, \partial_{x_n}(w), f$. Our aim is to prove that a belongs to the two-sided ideal of $k\langle x_1, \dots, x_n \rangle$ generated by f . Write

$$az = \sum_{1 \leq i \leq n, \alpha, \beta \in \mathcal{M}} \lambda_{i, \alpha, \beta} \alpha \partial_{x_i}(w) \beta + \sum_{\alpha, \beta \in \mathcal{M}'} \mu_{\alpha, \beta} \alpha f \beta, \quad (2.7)$$

where \mathcal{M} denotes the set of (non-commutative) monomials in x_1, \dots, x_n and \mathcal{M}' is the union of \mathcal{M} with the singleton $\{z\}$, where $\lambda_{i, \alpha, \beta}$ and $\mu_{\alpha, \beta}$ are in k , and where the sums are finitely supported. In the second sum, one has $\deg_z \alpha + \deg_z \beta = 1$ where \deg_z denotes the degree in z . Viewing the linear system (2.3) in the free algebra F and reducing it to the triangular form, we see that z in α or β can be put on the right modulo the elements $\partial_{x_i}(w)$. Moreover the non-commutative Euler relation

$$\sum_{1 \leq i \leq n} \partial_{x_i}(w) x_i + fz = \sum_{1 \leq i \leq n} x_i \partial_{x_i}(w) + zf = c(w) \quad (2.8)$$

associated to the potential w shows that the z appearing in each $\alpha f \beta$ can be put completely on the right modulo the elements $\partial_{x_i}(w)$. So we can write

$$az = \sum_{1 \leq i \leq n, \alpha, \beta \in \mathcal{M}} \lambda_{i, \alpha, \beta} \alpha \partial_{x_i}(w) \beta + \left(\sum_{\alpha, \beta \in \mathcal{M}} \mu_{\alpha, \beta} \alpha f \beta \right) z, \quad (2.9)$$

by keeping the same notation for the coefficients. Denote by S the first sum. In order to conclude, it is sufficient to prove that $S = 0$. We have

$$S = \sum_{1 \leq i, j \leq n, \alpha, \beta \in \mathcal{M}} (\lambda_{i, \alpha, \beta} f_{ij} \alpha x_j z \beta + \lambda_{i, \alpha, \beta} f_{ji} \alpha z x_j \beta), \quad (2.10)$$

where $\deg \alpha + \deg \beta = \ell - 1$ and \deg denotes the total degree in x_1, \dots, x_n .

From $S = (a - \sum_{\alpha, \beta \in \mathcal{M}} \mu_{\alpha, \beta} \alpha f \beta)z$, we are going to deduce that the coefficients $\lambda_{i, \alpha, \beta}$ vanish inductively. Choose β with the maximal degree $\ell - 1$, so that $\alpha = 1$. Since $\deg(x_j \beta) > \deg(\beta')$ for any α', β' appearing in a term $\alpha' x_k z \beta'$ such that $\deg \alpha' + \deg \beta' = \ell - 1$, the coefficient of $z x_j \beta$ vanishes, hence $\sum_{1 \leq i \leq n} \lambda_{i, 1, \beta} f_{ji} = 0$ for $j = 1, \dots, n$. As M is invertible, this implies that $\lambda_{i, 1, \beta} = 0$ for $i = 1, \dots, n$. Thus we can remove all the elements $\lambda_{i, 1, \beta}$ when $\deg \beta = \ell - 1$. Next, choose β of degree $\ell - 2$ and α of degree 1. By the same argument, the coefficient of $\alpha z x_j \beta$ vanishes, hence $\sum_{1 \leq i \leq n} \lambda_{i, \alpha, \beta} f_{ji} = 0$ for $j = 1, \dots, n$, and we conclude again that all the elements $\lambda_{i, \alpha, \beta}$ vanish when $\deg \beta = \ell - 2$. Continuing the process, we arrive to $S = 0$. One proves similarly that the right A -module zA is free generated by z . ■

Lemma 2.5 has the following consequence: for any $a \in A$, there exists a unique element $\sigma(a) \in A$ (resp. $\sigma'(a) \in A$) such that $za = \sigma(a)z$ (resp. $az = z\sigma'(a)$). Clearly, σ and σ' are automorphisms of the k -algebra A , inverse to each other. Now we are going to describe B as a skew polynomial algebra with coefficients in A . It suffices to do it for $T_A(Az) = \bigoplus_{p \geq 0} (Az)^{\otimes AP}$. Proceeding by induction on p , we prove easily the following.

Lemma 2.6 *The left (or right) A -module $(Az)^{\otimes AP}$ is free generated by $z^{\otimes AP} = z \otimes_A \cdots \otimes_A z$ (p times) for any $p \geq 1$.*

Consequently, any element of $T_A(Az)$ is uniquely written as a finitely supported sum $\sum_{p \geq 0} a_p z^p$ where $a_i \in A$ and where we set $z^p = z^{\otimes Ap}$. The product in $T_A(Az)$ is determined by the product in A and by the relations

$$z^p a = \sigma^p(a) z^p,$$

for any $a \in A$ and $p \geq 1$. We have obtained the following proposition (on skew polynomial rings, see e.g. [7] p. 8-9).

Proposition 2.7 *For any non-degenerate f , the k -algebra $B = B(f)$ is isomorphic to the skew polynomial k -algebra $A[z; \sigma]$ defined over the k -algebra A by z and the k -automorphism σ of A .*

The isomorphism $B \cong A[z; \sigma]$ is an isomorphism of graded algebras, knowing that A is graded and z has degree 1. For any non-zero element $a = \sum_{p \geq 0} a_p z^p$ in B , its degree $\deg_z(a)$ in z is the highest p such that $a_p \neq 0$. One has $\deg_z(az) = \deg_z(za) = \deg_z(a) + 1$, so that z is not a zero-divisor in B . Moreover it is easy to deduce the Hilbert series of B from the Hilbert series of A :

$$h_B(t) = \frac{h_A(t)}{1-t}. \quad (2.11)$$

Using Proposition 2.1, we get

$$h_B(t) = (1 - (n+1)t + (n+1)t^2 - t^3)^{-1} \quad \text{if } n \geq 2, \quad (2.12)$$

$$h_B(t) = (1 - 2t + 2t^2 - 2t^3 + \dots)^{-1} \quad \text{if } n = 1. \quad (2.13)$$

Proposition 2.8 *Assume that f is non-degenerate. The quadratic algebra $B = B(f)$ is Koszul. The global dimension of B is equal to 3 if $n \geq 2$, to ∞ if $n = 1$. Moreover B is a domain if and only if $n \geq 2$.*

Proof. The global dimension of the graded algebra B is immediately derived from the expression (2.12) or (2.13) of its Hilbert series. According to a standard result on Koszul algebras [2], the Koszulity of B comes from the Koszulity of $A = B/(z)$ because the element z of degree 1 in B is normal and is not a zero-divisor (see also Example 1, p. 33 in [26]). The last statement is clear from $B \cong A[z; \sigma]$ and from 8) in Proposition 2.1. ■

Assume that f is non-degenerate as before, with $n \geq 2$. The Hilbert series (2.12) of the Koszul algebra B shows that

$$\dim((R_B \otimes V_B) \cap (V_B \otimes R_B)) = 1.$$

Since $c(w)$ is an element of $(R_B \otimes V_B) \cap (V_B \otimes R_B)$ by the non-commutative Euler relation (2.8), one has

$$(R_B \otimes V_B) \cap (V_B \otimes R_B) = k c(w).$$

Therefore the bimodule Koszul complex of B is the following complex K_w

$$0 \longrightarrow B \otimes k c(w) \otimes B \xrightarrow{d_3} B \otimes R_B \otimes B \xrightarrow{d_2} B \otimes V_B \otimes B \xrightarrow{d_1} B \otimes B \quad (2.14)$$

and $K_w \xrightarrow{\mu} B$ is the Koszul resolution of B where $\mu : B \otimes B \rightarrow B$ is the multiplication. To simplify notation, set $x_{n+1} = z$ and $r_i = \partial_{x_i}(w)$ for $i = 1, \dots, n+1$. Beside the cyclic partial

derivative ∂_{x_i} defined on F_{cyc} , there is the “ordinary” partial derivative $\frac{\partial}{\partial x_i} : F \rightarrow F \otimes F$ (see [32]) defined on any monomial a by

$$\frac{\partial a}{\partial x_i} = \sum_{a=ux_iv} u \otimes v,$$

which will be written as

$$\frac{\partial a}{\partial x_i} = \sum_{1,2} \left(\frac{\partial a}{\partial x_i} \right)_1 \otimes \left(\frac{\partial a}{\partial x_i} \right)_2.$$

Then the differential of the Koszul complex K_w is given by

$$d_1(1 \otimes x_i \otimes 1) = x_i \otimes 1 - 1 \otimes x_i, \quad (2.15)$$

$$d_2(1 \otimes r_i \otimes 1) = \sum_{1 \leq j \leq n+1} \sum_{1,2} \left(\frac{\partial r_i}{\partial x_j} \right)_1 \otimes x_j \otimes \left(\frac{\partial r_i}{\partial x_j} \right)_2 \quad (2.16)$$

$$d_3(1 \otimes c(w) \otimes 1) = \sum_{1 \leq j \leq n+1} x_j \otimes r_j \otimes 1 - 1 \otimes r_j \otimes x_j. \quad (2.17)$$

The following theorem is an immediate consequence of Keller’s theorem (Theorem 4.8 in [17]), since the algebra $B(f)$ is a 3-Calabi-Yau completion in sense of Keller (see 6.8 in [17]). The reader could be interested in a proof not using Keller’s completions, so we present below such a proof based on elementary homological algebra.

Theorem 2.9 *For any non-degenerate f and for any $n \geq 2$, the algebra $B = B(f)$ is 3-Calabi-Yau. In other words, the potential $w = fz$ is 3-Calabi-Yau.*

Proof. It suffices to prove that the complex K_w is self-dual with respect to the functor $(\cdot)^\vee = \text{Hom}_{B-B}(\cdot, B \otimes B)$ ([5], Lemma 3.7). Denote by E^* the dual space of a k -linear space E . It is easy to compute the complex K_w^\vee :

$$B \otimes B \xrightarrow{d_1^*} B \otimes V_B^* \otimes B \xrightarrow{d_2^*} B \otimes R_B^* \otimes B \xrightarrow{d_3^*} B \otimes kc(w)^* \otimes B \longrightarrow 0 \quad (2.18)$$

where the differential is given on the dual basis by

$$d_1^*(1 \otimes 1) = \sum_{1 \leq j \leq n+1} x_j \otimes x_j^* \otimes 1 - 1 \otimes x_j^* \otimes x_j, \quad (2.19)$$

$$d_2^*(1 \otimes x_i^* \otimes 1) = \sum_{1 \leq j \leq n+1} \sum_{1,2} \left(\frac{\partial r_j}{\partial x_i} \right)_2 \otimes r_j^* \otimes \left(\frac{\partial r_j}{\partial x_i} \right)_1 \quad (2.20)$$

$$d_3^*(1 \otimes r_i^* \otimes 1) = x_i \otimes c(w)^* \otimes 1 - 1 \otimes c(w)^* \otimes x_i. \quad (2.21)$$

Consider the diagram (in which we have omitted the symbols \otimes)

$$\begin{array}{ccccccc} B(kc(w))B & \xrightarrow{d_3} & BR_B B & \xrightarrow{d_2} & BV_B B & \xrightarrow{d_1} & Bk B \\ f_3 \downarrow & & f_2 \downarrow & & f_1 \downarrow & & f_0 \downarrow \\ Bk B & \xrightarrow{d_1^*} & BV_B^* B & \xrightarrow{d_2^*} & BR_B^* B & \xrightarrow{d_3^*} & B(kc(w)^*)B \end{array} \quad (2.22)$$

where the B -bimodule isomorphisms f_i are given by

$$f_0(1) = c(w)^*, \quad f_1(x_i) = r_i^*, \quad f_2(r_i) = x_i^*, \quad f_3(c(w)) = 1. \quad (2.23)$$

The diagram (2.22) is *commutative*. In fact, it is immediate to check that the left square and the right square are commutative. Moreover the commutativity of the central square is a straightforward consequence of the following non-commutative Hessian formula [32]

$$\tau \left(\frac{\partial^2 w}{\partial x_i \partial x_j} \right) = \frac{\partial^2 w}{\partial x_j \partial x_i} \quad (2.24)$$

where we set $\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \circ \partial_{x_j}$ and where $\tau : F \otimes F \rightarrow F \otimes F$ is the flip. Thus $f : K_w \rightarrow K_w^\vee$ is a complex isomorphism, so that K_w is self-dual. ■

We complete the properties of $B = B(f)$ by the following.

Proposition 2.10 *Assume that f is non-degenerate and $n \geq 2$.*

- 1) B is AS-Gorenstein.
- 2) If f is alternating, then z is central in B . The converse holds if the characteristic of k is $\neq 2$.
- 3) The Gelfand-Kirillov dimension $GK.\dim(B)$ of B is finite if and only if $n = 2$, and in this case $GK.\dim(B) = 3$.
- 4) B is left (or right) noetherian if and only if $n = 2$.

Proof. 1) It is an immediate consequence of the Calabi-Yau property ([5], Proposition 4.3).
 2) The first claim comes from Example 2.4. For the converse, if z is central in B , then we have in B the equalities $\sum_{1 \leq j \leq n} (f_{ij} + f_{ji})x_j z = 0$ for $i = 1, \dots, n$. But the A -module Az is free, hence $x_1 z, \dots, x_n z$ are linearly independent in B , and we conclude that $f_{ij} + f_{ji} = 0$ for any i and j .

3) For $n = 2$, we have $h_B(t) = (1-t)^{-3}$, hence $GK.\dim(B) = 3$. If $n > 2$, the polynomial $1 - (n+1)t + (n+1)t^2 - t^3$ has a real root between 0 and 1, hence $GK.\dim(B) = \infty$.

4) If $n = 2$, the algebras A are noetherian [36]. Thus $B \cong A[z; \sigma]$ is noetherian by a standard result [7]. If $n > 2$, then $GK.\dim(B) = \infty$, so we conclude that B is not (left or right) noetherian by using a theorem due to Stephenson and Zhang [29]. Let us note that the equivalence in 4) can be deduced from the analogous equivalence for A (see 7) in Proposition 2.1) and from a general result ([20], Corollary 2.3). ■

3 Classification of the algebras B

The aim of this section is to classify the algebras $B = B(M)$ up to isomorphisms of graded algebras. Denote by $A(V, R)$ the quadratic graded algebra defined by a space of generators V and a space of relations R (subspace of $V \otimes V$). We write $A(V, R) \cong A(V', R')$ if there exists an isomorphism of graded algebras from $A(V, R)$ to $A(V', R')$. It is equivalent to say that there exists a linear isomorphism $\varphi : V \rightarrow V'$ such that $(\varphi \otimes \varphi)(R) = R'$ (this property is the definition of isomorphisms between two quadratic graded algebras given in [21, 4]). When $M \in M_n(k)$, we begin to classify up to isomorphism the graded algebras $A(M)$ defined at the beginning of Section 2. Using the same notation, let $A(M)$ be the quadratic algebra defined as the quotient of $k\langle x_1, \dots, x_n \rangle$ by the two-sided ideal generated by f , where $f = \sum_{1 \leq i, j \leq n} f_{ij} x_i x_j$ and $M = (f_{ij})_{1 \leq i, j \leq n}$. It is clear that for any matrices M_1 and M_2 , the equality $A(M_1) = A(M_2)$ holds if and only if there exists a non-zero scalar ν such that $M_1 = \nu M_2$. We have

$$f = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \quad (3.1)$$

The isomorphism class of $A(M)$ is the class of the algebras $A(N)$ which can be defined by M with respect to another basis (x'_1, \dots, x'_n) . It means that $A(N)$ is the quotient

$$k\langle x_1, \dots, x_n \rangle / (f')$$

where

$$f' = \begin{pmatrix} x'_1 & \dots & x'_n \end{pmatrix} M \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}, \quad (3.2)$$

in which f' has to be written in the variables x_1, \dots, x_n . Considering $P \in GL_n(k)$ such that

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

we obtain

$$f' = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} {}^t P M P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \quad (3.3)$$

Thus the matrices N are the matrices of the form $N = \nu {}^t P M P$ for any non-zero scalar ν and any $P \in GL_n(k)$. Two matrices M and N in $M_n(k)$ are said to be *congruent* if there exists $P \in GL_n(k)$ such that $N = {}^t P M P$. Note that we can choose $\nu = 1$ in $N = \nu {}^t P M P$ if any element of k is a square in k , but if ν is not a square in k , ν is not congruent to 1 (for $n = 1$). We have obtained the following result (when $k = \mathbb{C}$, see [3], end of Section 5).

Proposition 3.1 *Let k be a field and $n \geq 1$. Let M and N be in $M_n(k)$. Then $A(M) \cong A(N)$ if and only if M is congruent to a non-zero scalar multiple of N . If any element of k is a square in k , then $A(M) \cong A(N)$ if and only if M and N are congruent.*

Now we are interested in the classification of the graded algebras B up to isomorphism. We shall prove that this classification is the same as for the algebras A . Keeping the algebra $A(M)$ as above, we define the quadratic algebra $B(M)$ by the potential $w = fz$ as at the beginning of Section 2. The isomorphism class of $B(M)$ is the class of the algebras $B(N)$ which can be defined by M with respect to another basis (x'_1, \dots, x'_n, z') . To be more precise, let $\Lambda \in GL_{n+1}(k)$ be such that

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \\ z' \end{pmatrix} = \Lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ z \end{pmatrix}. \quad (3.4)$$

Note that z can occur in x'_1, \dots, x'_n and that x_1, \dots, x_n can occur in z' . Write f' given by (3.2) and $w' = f'z'$ in the variables x_1, \dots, x_n, z . Consider the quadratic algebra

$$\Lambda \cdot B(M) := k\langle x_1, \dots, x_n, z \rangle / (\partial_{x'_1}(w'), \dots, \partial_{x'_n}(w'), f').$$

Then we have to find the algebras $B(N)$ which are equal to $\Lambda \cdot B(M)$ for some $\Lambda \in GL_{n+1}(k)$. Clearly the isomorphism class of $B(0) = k\langle x_1, \dots, x_n, z \rangle$ is the singleton $\{B(0)\}$.

Proposition 3.2 *If $N = \nu {}^t P M P$ for some non-zero scalar ν and $P \in GL_n(k)$, then $B(N) = \Lambda \cdot B(M)$, where $\Lambda = \begin{pmatrix} P & 0 \\ 0 & \nu \end{pmatrix}$.*

Proof. Let (x'_1, \dots, x'_n, z') be as in (3.4) with $\Lambda = \begin{pmatrix} P & 0 \\ 0 & \nu \end{pmatrix}$. According to (2.5), we have

$$\begin{pmatrix} \partial_{x'_1}(w') \\ \vdots \\ \partial_{x'_n}(w') \end{pmatrix} = \begin{pmatrix} M & {}^t M \end{pmatrix} \begin{pmatrix} x'_1 z' \\ \vdots \\ z' x'_n \end{pmatrix}. \quad (3.5)$$

Consequently, from

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and $z' = \nu z$, we obtain

$${}^t P \begin{pmatrix} \partial_{x'_1}(w') \\ \vdots \\ \partial_{x'_n}(w') \end{pmatrix} = \begin{pmatrix} N & {}^t N \end{pmatrix} \begin{pmatrix} x_1 z \\ \vdots \\ z x_n \end{pmatrix}, \quad (3.6)$$

and

$$f' = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} {}^t P M P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \quad (3.7)$$

Thus the spaces of relations of $\Lambda \cdot B(M)$ and $B(N)$ are equal. ■

For any $\Lambda \in GL_{n+1}(k)$, write $\Lambda = \begin{pmatrix} \tilde{\Lambda} & c \\ \ell & \lambda \end{pmatrix}$ where $\tilde{\Lambda} \in M_n(k)$. In order to illustrate the proof of Theorem 3.4 below when $\tilde{\Lambda}$ is not invertible, the next proposition provides some explicit examples.

Proposition 3.3 *Let Λ be as in (3.4) with the basis change $x'_1 = z$, $x'_2 = x_2$, \dots , $x'_n = x_n$, $z' = x_1$. For any matrix*

$$M = \begin{pmatrix} 0 & f_{12} & \dots & f_{1n} \\ f_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & 0 & \dots & 0 \end{pmatrix}, \quad (3.8)$$

we have $\Lambda \cdot B(M) = B({}^t M)$.

Proof. Performing the basis change in f' given by (3.2) and in $w' = f' z'$, we obtain $f' = \sum_{2 \leq j \leq n} (f_{1j} z x_j + f_{j1} x_j z)$ and $w' = \sum_{2 \leq j \leq n} (f_{1j} z x_j x_1 + f_{j1} x_j z x_1)$. Therefore the relations of $\Lambda \cdot B(M)$ are

$$\begin{aligned} \partial_{x'_1}(w') &= \partial_z(w') = \sum_{2 \leq j \leq n} (f_{1j} x_j x_1 + f_{j1} x_1 x_j), \\ \partial_{x'_i}(w') &= \partial_{x_i}(w') = f_{1i} x_1 z + f_{i1} z x_1, \quad 2 \leq i \leq n, \end{aligned}$$

$$f' = \sum_{2 \leq j \leq n} (f_{1j}zx_j + f_{j1}x_jz).$$

Since $B({}^tM)$ is defined by the potential $gz = \sum_{2 \leq j \leq n} (f_{1j}x_jx_1z + f_{j1}x_1x_jz)$, the relations of $B({}^tM)$ are

$$\begin{aligned} \partial_{x_1}(gz) &= \sum_{2 \leq j \leq n} (f_{1j}zx_j + f_{j1}x_jz), \\ \partial_{x_i}(gz) &= f_{1i}x_1z + f_{i1}zx_1, \quad 2 \leq i \leq n, \\ g &= \sum_{2 \leq j \leq n} (f_{1j}x_jx_1 + f_{j1}x_1x_j). \end{aligned}$$

Thus the spaces of relations of $\Lambda \cdot B(M)$ and $B({}^tM)$ are equal. ■

Actually, for the matrix Λ of Proposition 3.3, one can show that any matrix M such that there exists N satisfying $\Lambda \cdot B(M) = B(N)$ is of the form (3.8), and then N is a non-zero scalar multiple of tM . For the proof of the classification theorem of the algebras $B(M)$ (Theorem 3.4 below), one could probably do things intrinsically by using matrices Λ , but following the suggestion of the referee, we prefer a straightforward approach which is structurally simpler.

Theorem 3.4 *Let k be a field and $n \geq 1$. Let $M = (f_{i,j})$ and $N = (f'_{i,j})$ be in $M_n(k)$. Then $B(M) \cong B(N)$ if and only if M is congruent to a non-zero scalar multiple of N . If any element of k is a square in k , then $B(M) \cong B(N)$ if and only if M and N are congruent.*

Proof. According to Proposition 3.2, it remains to prove the necessity of the condition when no assumption is made on k . Assume that there exists an isomorphism of graded algebras $\phi : B(M) \rightarrow B(N)$. As noted just before Proposition 3.2, $M = 0$ if and only if $N = 0$, so that we assume that M and N are non-zero. Summing implicitly over repeated indices, we have $\phi(x_i) = \alpha_{i,j}x_j + \beta_i z$ and $\phi(z) = \gamma_j x_j + \delta z$ for some well-determined scalars $\alpha_{i,j}$, β_i , γ_j and δ . Denote by $\bar{\phi} : B(N) \rightarrow B(M)$ the map inverse of ϕ , and let $\bar{\alpha}_{i,j}$, $\bar{\beta}_i$, $\bar{\gamma}_j$, $\bar{\delta}$ be the corresponding scalars for the map $\bar{\phi}$. As $\bar{\phi}\phi = 1_{B(M)}$, we have

$$\begin{aligned} \alpha_{i,j}\bar{\alpha}_{j,k} + \beta_i\bar{\gamma}_k &= \delta_{i,k}, & \alpha_{i,j}\bar{\beta}_j + \beta_i\bar{\delta} &= 0, \\ \gamma_i\bar{\alpha}_{i,j} + \delta\bar{\gamma}_j &= 0, & \gamma_i\bar{\beta}_i + \delta\bar{\delta} &= 1; \end{aligned} \tag{3.9}$$

of course, there are similar equations coming from $\phi\bar{\phi} = 1_{B(N)}$.

As $f_{i,j}x_i x_j = 0$ in $B(M)$, we have $f_{i,j}(\alpha_{i,k}x_k + \beta_i z)(\alpha_{j,l}x_l + \beta_j z) = 0$ in $B(N)$. Since the algebra $B(N)$ is graded by \deg_z , looking at the homogeneous components of degrees zero and two of this equality tells us that in $B(N)$

$$f_{i,j}\alpha_{i,k}\alpha_{j,l}x_k x_l = 0, \quad f_{i,j}\beta_i\beta_j z^2 = 0.$$

The first of these tells us that there is a scalar λ such that

$$f_{i,j}\alpha_{i,k}\alpha_{j,l} = \lambda f'_{k,l} \tag{3.10}$$

and the second one – since $z^2 \neq 0$ in $B(N)$ – that

$$f_{i,j}\beta_i\beta_j = 0. \tag{3.11}$$

If the matrix $(\alpha_{i,j})$ is invertible, then λ cannot be zero, for we would then have that $M = 0$ in view of equation (3.10), and that same equation tells us that M and λN are

congruent. We may therefore assume that it is not and, by symmetry, that neither is the matrix $(\bar{\alpha}_{i,j})$. This assumption implies that there exist scalars ζ_i , not all zero, such that $\zeta_i \alpha_{i,j} = 0$ for all j , and then

$$\zeta_j x_j = \bar{\phi} \phi(\zeta_i x_i) = \bar{\phi}(\zeta_i \alpha_{i,j} x_j + \zeta_i \beta_i z) = \zeta_i \beta_i (\bar{\gamma}_j x_j + \bar{\delta} z).$$

It follows that $\zeta_i \beta_i \bar{\gamma}_j = \zeta_j$ for all j and that $\zeta_i \beta_i \bar{\delta} = 0$; since not all of the ζ_j are zero, this implies that $\zeta_i \beta_i \neq 0$ and therefore $\bar{\delta} = 0$. By symmetry, we also have $\delta = 0$. As $f_{i,j} x_j z + f_{j,i} z x_j$ is zero in $B(M)$ and the homogeneous component of degree zero of its image under ϕ is $(f_{i,j} \alpha_{j,k} \gamma_l + f_{j,i} \alpha_{j,l} \gamma_k) x_k x_l$, this must be a multiple of $f'_{k,l} x_k x_l$, and we see that there exist scalars μ_i such that

$$f_{i,j} \alpha_{j,k} \gamma_l + f_{j,i} \alpha_{j,l} \gamma_k = \mu_i f'_{k,l}. \quad (3.12)$$

If we multiply (3.10) by $\bar{\beta}_k$ and sum over k , we see that $\lambda f'_{k,l} \bar{\beta}_k = f_{i,j} \alpha_{i,k} \alpha_{j,l} \bar{\beta}_k = 0$, because of the second equation in (3.9). If we had $\lambda \neq 0$, so that $f'_{k,l} \bar{\beta}_k = 0$, we could multiply equation (3.12) by $\bar{\beta}_k$, sum over k , and – using that $f'_{k,l} \bar{\beta}_k = 0$ and $\gamma_k \bar{\beta}_k = 1$ – we would have that $f_{j,i} \alpha_{j,l} = 0$; equation (3.10) would then tell us that in fact $f' = 0$, against our hypothesis. Therefore $\lambda = 0$. Let $\omega_{i,j} = \alpha_{i,j} + \beta_i \gamma_j$ and similarly for $\bar{\omega}_{i,j}$. Using equations (3.9) and the fact that $\delta = \bar{\delta} = 0$, we immediately see that the matrix $(\omega_{i,j})$ is invertible with inverse $(\bar{\omega}_{i,j})$. Also we have

$$f_{i,j} \omega_{i,k} \omega_{j,l} = f_{i,j} \alpha_{i,k} \alpha_{j,l} + f_{i,j} (\alpha_{i,k} \beta_j \gamma_l + \alpha_{j,l} \beta_i \gamma_k) + f_{i,j} \beta_i \beta_j \gamma_k \gamma_l. \quad (3.13)$$

The first term is zero because of (3.10) and the third one vanishes according to (3.11), so that we have

$$f_{i,j} \omega_{i,k} \omega_{j,l} = f_{i,j} \alpha_{i,k} \beta_j \gamma_l + f_{i,j} \alpha_{j,l} \beta_i \gamma_k = (f_{j,i} \alpha_{j,k} \gamma_l + f_{i,j} \alpha_{j,l} \gamma_k) \beta_i = \mu_i \beta_i f'_{l,k},$$

in view of equation (3.12). Since the matrix $(\omega_{i,j})$ is invertible and not all the $f_{i,j}$ are zero, this implies that $\nu := \mu_i \beta_i \neq 0$ and we conclude that M and the transpose of νN are congruent. Since any matrix is congruent to its transpose [11], it follows that M is congruent to νN itself. ■

4 Special properties of $B(M)$ when $n = 2$

Throughout the sequel of this paper, we assume that $k = \mathbb{C}$ (actually, it would be sufficient to assume that k is algebraically closed of characteristic zero). An explicit parameter space for the $GL(2, \mathbb{C})$ -orbits of congruent matrices in $GL(2, \mathbb{C})$ is given by Dubois-Violette in [12] (for the classification of matrices under congruence in general, the reader may consult [10, 27]). According to the previous section, this parameter space forms a “moduli space” for the algebras $B(M)$ when M runs over $GL(2, \mathbb{C})$. Recall the description in three types of this parameter space as stated in ([12], end of Section 2). In each type, we give the potential w and the relations of $B(M)$. The generators are denoted by x, y and z . The types depend on the rank \mathbf{rk} of the symmetric part $s(M) = \frac{1}{2}(M + {}^t M)$ of M . It turns out that the corresponding algebras $A(M)$ are exactly the AS-regular algebras of global dimension 2 [1]. Therefore the three types classify the AS-regular algebras of global dimension 2 as well, and we have kept the same standard terminology for the classification of our algebras $B(M)$.

First type (classical type): $\mathbf{rk} = 0$. There is only one orbit, which is the orbit of $M =$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One has $w = (yx - xy)z$. The relations of $B(M)$ are the following: $zy = yz$, $xz = zx$, $yx = xy$. Hence $B(M) = \mathbb{C}[x, y, z]$ the commutative polynomial algebra. Note that it is the unique orbit such that z is central in $B(M)$ (Proposition 2.10), or such that $A(M)$ is Calabi-Yau (Proposition 2.1).

Second type (Jordan type): $\mathbf{rk} = 1$. There is only one orbit, which is the orbit of $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. One has $w = (yx - xy - x^2)z$. The relations of $B(M)$ are the following: $zy = yz + 2xz$, $xz = zx$, $yx = xy + x^2$.

Third type (quantum type): $\mathbf{rk} = 2$. The orbits are parametrized by the set $\{q \in \mathbb{C} \setminus \{0, 1\}\}/(q \sim q^{-1})$. These are the orbits of $M = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$. One has $w = (yx - q^{-1}xy)z$. The relations of $B(M)$ are the following: $xy = qyx$, $yz = qzy$, $xz = qxz$. So $B(M)$ is a quantum affine space.

Proposition 4.1 *For any $M \in GL(2, \mathbb{C})$, the algebra $B(M)$ is a quadratic AS-regular algebra of global dimension 3, of type A non-generic (following the classification of Artin and Schelter [1]). For any M in the list of the above classification, $(x^i y^j z^k)_{i \geq 0, j \geq 0, k \geq 0}$ is a basis of the vector space $B(M)$.*

Proof. According to Proposition 2.10, $B(M)$ is AS-Gorenstein with a finite Gelfand-Kirillov dimension, hence is AS-regular. Since $B(M)$ is Calabi-Yau (Theorem 2.9), $B(M)$ is of type A ([5], Proposition 5.4). Moreover, $B(M)$ is a skew polynomial ring over $A(M)$ in z (Proposition 2.7) and it is clear from each relation f listed above that $A(M)$ is a skew polynomial ring in x, y . Therefore $B(M)$ is a skew polynomial ring in x, y, z (hence the basis of monomials $x^i y^j z^k$), and the invariant j of $B(M)$ is infinite ([1], Theorem 6.11). ■ Among the AS-regular algebras of type A, the generic ones correspond to a finite invariant j and are called Sklyanin algebras [31]. Recall that Sklyanin algebras are defined by three generators x, y, z , and three relations $\alpha yz + \beta zy + \gamma x^2 = 0$, $\alpha zx + \beta xz + \gamma y^2 = 0$, $\alpha xy + \beta yx + \gamma z^2 = 0$, where $\gamma \neq 0$. So the algebras $B(M)$ of the first or third type can be considered as limits of Sklyanin algebras by vanishing the parameter γ , but such a process is not possible for the second type.

In the classification of their quadratic regular algebras of global dimension 3, Artin and Schelter define the invariant j as the invariant of a certain cubic curve \mathcal{C} in \mathbb{P}^2 . As we shall see, the equation $\phi = 0$ of \mathcal{C} is easily deduced from the potential w of $B(M)$. More importantly, we shall interpret ϕ as a Poisson potential whose associated Poisson bracket (defined as usual by $\nabla\phi$) is exactly the semi-classical limit of $B(M)$ viewed as a deformation of $\mathbb{C}[x, y, z]$. Let us begin by the description of the curve \mathcal{C} as in [1]. To avoid confusion with our notation, let us replace the notation M, w, Q used in [1] by $\mathcal{M}, \mathcal{W}, \mathcal{Q}$. The 3×3 matrix \mathcal{M} is defined by

$$\begin{pmatrix} \partial_x(w) \\ \partial_y(w) \\ f \end{pmatrix} = \mathcal{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (4.1)$$

so that we have

$$\mathcal{M} = \begin{pmatrix} {}^t M z & M \begin{pmatrix} x \\ y \end{pmatrix} \\ (x \ y) M & 0 \end{pmatrix}.$$

The element \mathcal{W} is defined by $\mathcal{W} = X\mathcal{M}^tX$, where $X = \begin{pmatrix} x & y & z \end{pmatrix}$. From (4.1) and the non-commutative Euler relation (2.8), we deduce that

$$\mathcal{W} = c(w), \quad (4.2)$$

that is, \mathcal{W} is precisely the potential \bar{w} once identified to its cyclic sum. The matrix \mathcal{Q} is defined by $X\mathcal{M} = \begin{pmatrix} \partial_x(w) & \partial_y(w) & f \end{pmatrix}^t\mathcal{Q}$. Then (2.8) implies that \mathcal{Q} is equal to the identity matrix and we recover the fact that $B(M)$ is of type A (by definition). Next, define the subvariety \mathcal{C} of \mathbb{P}^2 by its equation $S(\mathcal{W}) = 0$, where $S(\mathcal{W})$ denotes the symmetrization of the element \mathcal{W} . The symmetrization consists in replacing products of variables in the free algebra by the same products in the polynomial algebra. In our situation, $S(\mathcal{W}) = 3S(w) = 3S(f)z$. Set $\phi = S(f)z$. The above classification in three types allows us to limit ourselves to the following matrices

$$M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, \quad b \neq 0, \quad (4.3)$$

so that

$$w = (ax^2 + bxy + yx)z, \quad \phi = (ax^2 + (b+1)xy)z. \quad (4.4)$$

In the classical type, one has $\mathcal{C} = \mathbb{P}^2$, and otherwise \mathcal{C} is the union of three straight lines, two of which coincide in the Jordan type.

Definition 4.2 *The polynomial $\phi = \phi(M)$ is called the Poisson potential associated to the algebra $B(M)$.*

This definition will be more natural when $B(M)$ will be viewed as a Gerstenhaber deformation whose Poisson bracket $\{\cdot, \cdot\}$ will be defined on $\mathbb{C}[x, y, z]$ by the formulas

$$\{x, y\} = \frac{\partial\phi}{\partial z}, \quad \{y, z\} = \frac{\partial\phi}{\partial x}, \quad \{z, x\} = \frac{\partial\phi}{\partial y},$$

$$\text{i.e., } \{\cdot, \cdot\} = \frac{\partial\phi}{\partial z} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial\phi}{\partial x} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial\phi}{\partial y} \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

Definition 4.3 *The so-defined Poisson bracket $\{\cdot, \cdot\}$ is called the Poisson bracket derived from the Poisson potential ϕ .*

It is clear that ϕ belongs to the Poisson center of $\{\cdot, \cdot\}$ (because $\{x, \phi\} = \{y, \phi\} = \{z, \phi\} = 0$), i.e., ϕ is a Casimir element of the Poisson bracket. In the classical type, $\{\cdot, \cdot\} = 0$. For the other types, the next proposition shows that the Casimir element ϕ lifts to a non-zero element Φ of the center of $B(M)$. Note that the potential w lifts ϕ in the free algebra, but w vanishes in B (it is a difference with Sklyanin algebras for which the potential w does not vanish in the algebra). Note that $c(w)$ always vanishes in the algebra according to the non-commutative Euler relation (2.8).

Most calculations in the algebra $B(M)$ where M is given by (4.3) are based on the following relations

$$zy = \left(-a + \frac{a}{b}\right)xz - byz, \quad zx = -\frac{1}{b}xz, \quad yx = -ax^2 - bxy. \quad (4.5)$$

It is easy to verify that these relations are confluent. Therefore, Bergman's diamond lemma provides the basis $(x^i y^j z^k)_{i \geq 0, j \geq 0, k \geq 0}$ of B , as already stated in Proposition 4.1.

Proposition 4.4 *The element Φ of $B(M)$ defined by*

$$\Phi = (ax^2 + (b+1)xy)z$$

belongs to the center of $B(M)$.

Proof. Relations (4.5) allow us to decompose $x\Phi$ and Φx in the basis $(x^i y^j z^k)_{i \geq 0, j \geq 0, k \geq 0}$. The computations are straightforward and they show that $x\Phi = ax^3z + (b+1)x^2yz = \Phi x$. In the same manner, we get $y\Phi = a^2(b-1)x^3z + a(b^2 - b - 1)x^2yz - b(b+1)xy^2z = \Phi y$ and $z\Phi = ax^2z^2 + (b+1)xyz^2 = \Phi z$. ■

We are now interested in the comparison between the Hochschild homology of $B(M)$ and the Poisson homology of $(\mathbb{C}[x, y, z], \{\cdot, \cdot\})$. In the classical type, the Hochschild homology of $\mathbb{C}[x, y, z]$ is well-known and coincides with the Poisson homology for the corresponding $\{\cdot, \cdot\}$ (which is zero in this case). As proved in the next proposition, it is the same for the quantum type if q is not a root of unity. We are grateful to Wambst for the explicit formulas of Hochschild homology.

Proposition 4.5 *Let q be a non-zero complex number which is not a root of unity. Let B be the \mathbb{C} -algebra defined by the potential $w = (yx - q^{-1}xy)z$, i.e. defined by the generators x, y, z , and the following relations*

$$xy = qyx, \quad yz = qzy, \quad zx = qxz.$$

Let $S = \mathbb{C}[x, y, z]$ be the polynomial \mathbb{C} -algebra in x, y, z , endowed with the Poisson bracket derived from the Poisson potential $\phi = (1 - q^{-1})xyz$. Denote by $Z = \mathbb{C}[xyz]$ the subalgebra of S generated by xyz . Then the Hochschild homology of B is isomorphic to the Poisson homology of S and is given by

$$\begin{aligned} HH_0(B) &\cong Z \oplus x\mathbb{C}[x] \oplus y\mathbb{C}[y] \oplus z\mathbb{C}[z], \\ HH_1(B) &\cong ((\mathbb{C}[x] \oplus yzZ) \otimes x) \oplus ((\mathbb{C}[y] \oplus xzZ) \otimes y) \oplus ((\mathbb{C}[z] \oplus xyZ) \otimes z), \\ HH_2(B) &\cong (xZ \otimes (y \wedge z)) \oplus (yZ \otimes (z \wedge x)) \oplus (zZ \otimes (x \wedge y)), \\ HH_3(B) &\cong Z \otimes (x \wedge y \wedge z), \\ HH_p(B) &= 0 \text{ for any } p \geq 4. \end{aligned}$$

Proof. On one hand, setting $x_1 = x, x_2 = y, x_3 = z, q_{12} = q_{23} = q_{31} = q$ and applying Théorème 6.1 in [33], we obtain the result for $HH_p(B)$. diagonalizable in sense of Monnier, and the Poisson cohomology is computed in [23]. Moreover, the duality $HP^\bullet(S) \cong HP_{3-\bullet}(S)$ holds *a priori*, because a Poisson bracket deriving from a potential is always unimodular, i.e., its modular class vanishes (see the definition of the modular vector field (and its class) in [35], which is called the curl vector field in [13]). The example 2.4 (i) of [23], then gives bases of the Poisson homology vector spaces of S and the fact that they are isomorphic to the Hochschild homology vector spaces of B . ■

We have seen that the duality $HP^\bullet(S) \cong HP_{3-\bullet}(S)$ holds since the Poisson bracket derives from a potential. Since B is 3-Calabi-Yau (Theorem 2.9), one also has the duality $HH^\bullet(B) \cong HH_{3-\bullet}(B)$.

5 Poisson homology and Hochschild homology for the second type

In this section, the field k is \mathbb{C} and $n = 2$. The aim of this section is to prove Theorem 1.1 of the Introduction, i.e., to determine the Hochschild homology of the algebra B of the second type (Jordan type):

$$B = \mathbb{C}\langle x, y, z \rangle / (zy = yz + 2xz, zx = xz, yx = x^2 + xy).$$

Let us recall the notation: $f = x^2 + xy - yx$, $w = fz$, $r_1 = \partial_x(w)$, $r_2 = \partial_y(w)$ and $r_3 = \partial_z(w)$. We have

$$r_1 = xz + zx + yz - zy, \quad r_2 = zx - xz, \quad r_3 = x^2 + xy - yx.$$

In order to obtain the Hochschild homology of B , we will first see B as the deformation of a Poisson algebra for which we will determine the Poisson homology. To do this, let us consider the filtration F of B given by the degree of y . In other words, for this filtration, the degree of y is 1 while the degrees of x and z are 0. It is clear from the monomial basis of B (Proposition 4.1) that $gr_F(B) \simeq \mathbb{C}[x, y, z]$, so the filtered algebra B is almost commutative [16]. Moreover, $gr_F(B)$ is equipped with the Poisson bracket defined by:

$$\{z, y\} = 2xz, \quad \{z, x\} = 0, \quad \{y, x\} = x^2, \quad (5.1)$$

which is the Poisson structure derived from the Poisson potential $\phi = -x^2z$. In the sequel, we will denote this Poisson algebra by $T = (\mathbb{C}[x, y, z], \{, \})$.

From the Koszul resolution K_w (2.14), we easily get the following Koszul complex $B \otimes_{B^e} K_w$ associated to B (where we have omitted the symbols \otimes):

$$0 \longrightarrow B(\mathbb{C}c(w)) \xrightarrow{\tilde{d}_3} BR_B \xrightarrow{\tilde{d}_2} BV_B \xrightarrow{\tilde{d}_1} B \longrightarrow 0$$

where $c(w) = x^2z + xzx + zx^2 + xyz + yzx + zxy - yxz - xzy - zyx$ and where the differentials are given, for $a \in B$, by:

$$\tilde{d}_1(a \otimes x) = [a, x] = ax - xa, \quad \tilde{d}_1(a \otimes y) = [a, y], \quad \tilde{d}_1(a \otimes z) = [a, z],$$

while

$$\begin{aligned} \tilde{d}_2(a \otimes r_1) &= (za + az) \otimes x + (za - az) \otimes y + (ax + xa + ay - ya) \otimes z, \\ \tilde{d}_2(a \otimes r_2) &= (az - za) \otimes x + (xa - ax) \otimes z, \\ \tilde{d}_2(a \otimes r_3) &= (xa + ax + ya - ay) \otimes x + (ax - xa) \otimes y, \end{aligned}$$

and

$$\tilde{d}_3(a \otimes c(w)) = [a, x] \otimes r_1 + [a, y] \otimes r_2 + [a, z] \otimes r_3.$$

This complex computes the Hochschild homology $HH_\bullet(B)$ of B . Notice that one can naturally identify the spaces of chains of this complex with B or B^3 . Indeed, one can write

$$B(\mathbb{C}c(w)) \simeq B, \quad \text{while} \quad BR_B \simeq B^3 \quad \text{and} \quad BV_B \simeq B^3, \quad (5.2)$$

by identifying, for any $a_1, a_2, a_3 \in B$, the element $a_1 \otimes r_1 + a_2 \otimes r_2 + a_3 \otimes r_3 \in BR_B$ or the element $a_1 \otimes x + a_2 \otimes y + a_3 \otimes z \in BV_B$ with $(a_1, a_2, a_3) \in B^3$.

As in B , we assume that in V_B , R_B and $c(w)$, y has degree 1 while x and z have degree 0, so that we can consider the total filtration degree in $B \otimes_{B^e} K_w$. In Proposition 5.2 below, we show that \tilde{d}_1 , \tilde{d}_2 and \tilde{d}_3 have degree -1 for the total filtration degree. So $B \otimes_{B^e} K_w$ will be a filtered complex (with differentials preserving the degree) for the following shifts on the total filtration degree:

$$0 \longrightarrow B(\mathbb{C}c(w))(3) \xrightarrow{\tilde{d}_3} BR_B(2) \xrightarrow{\tilde{d}_2} BV_B(1) \xrightarrow{\tilde{d}_1} B \longrightarrow 0 \quad (5.3)$$

whose filtration is still denoted by F . Recall now the definition of the Poisson homology complex of the Poisson algebra T . First, the space of Poisson 1-chains is the T -module of Kähler differentials of T , denoted by $\Omega^1(T)$ and generated, as a T -module by the three

elements dx, dy, dz . Then, for $p \in \mathbb{N}^*$, the T -module of Kähler p -differentials is the space of Poisson p -chains and is given by $\Omega^p(T) = \bigwedge^p \Omega^1(T)$. Of course, one has $\Omega^p(T) = \{0\}$, as soon as $p \geq 4$. Notice that an element $F_1 dx + F_2 dy + F_3 dz \in \Omega^1(T)$ can naturally be identified with the element $(F_1, F_2, F_3) \in T^3$. Similarly, an element $F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy \in \Omega^2(T)$ is identified with the element $(F_1, F_2, F_3) \in T^3$, and an element $F dx \wedge dy \wedge dz \in \Omega^3(T)$ with the element $F \in T$. (Notice that in the following we will write elements in T^3 both as row vectors or as column vectors, depending on the context.)

Next, the Poisson boundary operator, $\delta_p : \Omega^p(T) \rightarrow \Omega^{p-1}(T)$, called the Brylinski or Koszul differential, is given, for $F_0, F_1, \dots, F_p \in T$, by (see [8]):

$$\begin{aligned} \delta_p(F_0 dF_1 \wedge \dots \wedge dF_p) &= \sum_{i=1}^p (-1)^{i+1} \{F_0, F_i\} dF_1 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge dF_p \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{i+j} F_0 d\{F_i, F_j\} \wedge dF_1 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge \widehat{dF_j} \wedge \dots \wedge dF_p, \end{aligned}$$

where the symbol $\widehat{dF_i}$ means that we omit the term dF_i .

Using the previous identifications, the Poisson homology complex of the Poisson algebra T can be written as:

$$0 \longrightarrow T \xrightarrow{\delta_3} T^3 \xrightarrow{\delta_2} T^3 \xrightarrow{\delta_1} T \longrightarrow 0$$

with the differentials given, for $F \in T$, and $\vec{F} := (F_1, F_2, F_3) \in T^3$, by:

$$\begin{aligned} \delta_1(\vec{F}) &= \nabla \phi \cdot (\nabla \times \vec{F}) = \text{Div}(\vec{F} \times \nabla \phi), \\ \delta_2(\vec{F}) &= -\nabla(\vec{F} \cdot \nabla \phi) + \text{Div}(\vec{F}) \nabla \phi, \\ \delta_3(F) &= -\nabla F \times \nabla \phi, \end{aligned} \tag{5.4}$$

where $\text{Div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \in T$, while, for all element $G \in T$, we write $\nabla G := (\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}) \in T^3$. We also have respectively denoted by \cdot, \times and $\nabla \times$, the usual inner and cross products in T^3 and the curl operator. Note that $\nabla \phi = (-2xz, 0, -x^2)$.

In proposition 5.2, we show that $gr_F(B \otimes_{B^e} K_w)$ is identified to the Poisson homology complex of the Poisson algebra T . The next lemma will be useful.

Lemma 5.1 *For all $k \in \mathbb{N}$, the following identities hold in B :*

$$yx^k = x^k y + kx^{k+1}, \quad z^k y = yz^k + 2kxz^k, \tag{5.5}$$

while

$$y^k x = \sum_{j=0}^k \frac{k!}{j!} x^{k-j+1} y^j, \tag{5.6}$$

and

$$zy^k = \sum_{\ell=0}^k (k-\ell+1) \frac{k!}{\ell!} x^{k-\ell} y^\ell z. \tag{5.7}$$

Proof. Straightforward by induction on $k \in \mathbb{N}$. ■

Proposition 5.2 *Let us consider the filtration F on the algebra B given by the degree in y . Then the differential of the Koszul complex $B \otimes_{B^e} K_w$ has degree -1 for the total filtration degree. Moreover, the graded complex associated to the filtered complex (5.3) is isomorphic to the Poisson homology complex of the Poisson algebra T .*

Proof. As we have already seen, $gr_F(B) \simeq T$. As the monomials $x^i y^j z^k$, with $i, j, k \in \mathbb{N}$, form a \mathbb{C} -basis of B (Proposition 4.1), the identifications of the spaces of Hochschild chains with B or B^3 and their analogs for the spaces of Poisson chains with T and T^3 show that the spaces of Hochschild and Poisson chains are isomorphic. Now, we have to compare the images of \tilde{d}_ℓ and of δ_ℓ , $1 \leq \ell \leq 3$. To do this, we consider the basis $(x^i y^j z^k)_{i,j,k}$ of $\mathbb{C}[x, y, z]$ and of B . In fact, using (5.4), it is easy to compute the elements $\delta_\ell(x^i y^j z^k e)$, where e is one of the symbols dx , dy and dz when $\ell = 1$, $dy \wedge dz$, $dz \wedge dx$ and $dx \wedge dy$ when $\ell = 2$, or $dx \wedge dy \wedge dz$, when $\ell = 3$. Also, using lemma 5.1, it is straightforward to obtain $\tilde{d}_1(x^i y^j z^k \otimes x)$, $\tilde{d}_1(x^i y^j z^k \otimes y)$, $\tilde{d}_1(x^i y^j z^k \otimes z)$, $\tilde{d}_2(x^i y^j z^k \otimes r_1)$, $\tilde{d}_2(x^i y^j z^k \otimes r_2)$, $\tilde{d}_2(x^i y^j z^k \otimes r_3)$ and $\tilde{d}_3(x^i y^j z^k \otimes c(w))$ and to verify that

$$\begin{aligned} gr_F(\tilde{d}_1(x^i y^j z^k \otimes x)) &= \delta_1(x^i y^j z^k dx) = jx^{i+2}y^{j-1}z^k, \\ gr_F(\tilde{d}_1(x^i y^j z^k \otimes y)) &= \delta_1(x^i y^j z^k dy) = (2k - i)x^{i+1}y^j z^k, \\ gr_F(\tilde{d}_1(x^i y^j z^k \otimes z)) &= \delta_1(x^i y^j z^k dz) = -2jx^{i+1}y^{j-1}z^{k+1}, \end{aligned}$$

while:

$$\begin{aligned} gr_F(\tilde{d}_2(x^i y^j z^k \otimes r_1)) &\simeq \delta_2(x^i y^j z^k dy \wedge dz) = \begin{pmatrix} 2x^i y^j z^{k+1} \\ 2jx^{i+1}y^{j-1}z^{k+1} \\ (2k - i + 2)x^{i+1}y^j z^k \end{pmatrix}, \\ gr_F(\tilde{d}_2(x^i y^j z^k \otimes r_2)) &\simeq \delta_2(x^i y^j z^k dz \wedge dx) = \begin{pmatrix} -2jx^{i+1}y^{j-1}z^{k+1} \\ 0 \\ -jx^{i+2}y^{j-1}z^k \end{pmatrix}, \\ gr_F(\tilde{d}_2(x^i y^j z^k \otimes r_3)) &\simeq \delta_2(x^i y^j z^k dx \wedge dy) = \begin{pmatrix} (i + 2 - 2k)x^{i+1}y^j z^k \\ jx^{i+2}y^{j-1}z^k \\ 0 \end{pmatrix}, \end{aligned}$$

and:

$$gr_F(\tilde{d}_3(x^i y^j z^k \otimes c(w))) \simeq \delta_3(x^i y^j z^k dx \wedge dy \wedge dz) = \begin{pmatrix} jx^{i+2}y^{j-1}z^k \\ (2k - i)x^{i+1}y^j z^k \\ -2jx^{i+1}y^{j-1}z^{k+1} \end{pmatrix}.$$

This also shows that the differential \tilde{d}_ℓ has degree -1 for the total filtration degree. \blacksquare

In order to obtain the Poisson homology of the Poisson algebra T , we need to compute the homology of a certain complex, called *Koszul complex associated to the polynomial ϕ* . By definition, this complex is given by $0 \rightarrow \Omega^0(T) \rightarrow \Omega^1(T) \rightarrow \Omega^2(T) \rightarrow \Omega^3(T) \rightarrow 0$, where the differential is the map $\wedge d\phi : \Omega^k(T) \rightarrow \Omega^{k+1}(T)$. This complex will be denoted by $(\Omega^\bullet(T), \wedge d\phi)$ in the following, while for $0 \leq p \leq 3$, the p -th homology space of this complex will be denoted by $H_p^\phi(T)$.

Remark 5.2 Let us recall from [25] that if $\varphi \in T$ is a weight-homogeneous polynomial with an isolated singularity at the origin, then the homology of the Koszul complex associated to φ is given by: $H_p^\varphi(T) = \{0\}$, for $p = 0, 1, 2$, while $H_3^\varphi(T) = \frac{T}{\{\bar{F}, \nabla\varphi\bar{F} \in T^3\}}$ is the so-called Milnor algebra associated to φ and is, in this case, a finite dimensional vector space. In the following lemma, we see that the homology of the Koszul complex associated to $\phi = -x^2z$ (admitting a non-isolated singularity at the origin) does not satisfy the same properties.

Lemma 5.3 *Let $\phi = -x^2z$. The homology of the Koszul complex $(\Omega^\bullet(T), \wedge d\phi)$ associated*

to ϕ is given by:

$$\begin{aligned} H_0^\phi(T) &= \{0\}; \\ H_1^\phi(T) &= \mathbb{C}[y, z] (2z dx + x dz); \\ H_2^\phi(T) &= \mathbb{C}[y, z] (x dz + 2z dx) \wedge dy \oplus (x\mathbb{C}[y] \oplus \mathbb{C}[y, z]) dz \wedge dx; \\ H_3^\phi(T) &= (x\mathbb{C}[y] \oplus \mathbb{C}[y, z]) dx \wedge dy \wedge dz. \end{aligned}$$

Proof. The fact that $H_0^\phi(T) \simeq \{0\}$ is clear. In order to compute $H_1^\phi(T)$, $H_2^\phi(T)$ and $H_3^\phi(T)$, we first point out that there exists an isomorphism of complexes between $(\Omega^\bullet(T), \wedge d\phi)$ and the tensor product of two analogous but simpler complexes:

$$(\Omega^\bullet(T), \wedge d\phi) \simeq (\Omega^\bullet(\mathbb{C}[y]), 0) \otimes (\Omega^\bullet(\mathbb{C}[x, z]), \wedge d\phi). \quad (5.8)$$

Moreover, denoting $V = \mathbb{C}[x, z]$, the complex $(\Omega^\bullet(V), \wedge d\phi)$ can be written as follows:

$$\Omega^0(V) \xrightarrow{\delta_V^0} \Omega^1(V) \xrightarrow{\delta_V^1} \Omega^2(V) \longrightarrow 0$$

where $\Omega^0(V) = V$, $\Omega^1(V) = V dx \oplus V dz$ and $\Omega^2(V) = V dz \wedge dx$, while for every $(F, G) \in V^2$, $\delta_V^0(F) = F(2xz dx + x^2 dz)$ and $\delta_V^1(F dx + G dz) = (Fx^2 - 2xzG) dz \wedge dx$. Then, it is clear that the 0-th cohomology space associated to $(\Omega^\bullet(V), \wedge d\phi)$ is $\{0\}$ and the 2-nd cohomology space is given by $\mathbb{C}[x, z]/(x^2, 2xz) \simeq (\mathbb{C}[z] \oplus \mathbb{C} \cdot x) dz \wedge dx$.

In order to compute the first cohomology space associated to $(\Omega^\bullet(V), \wedge d\phi)$, let $F dx + G dz$ be a 1-cochain of $(\Omega^\bullet(V), \wedge d\phi)$. This element is a cocycle if and only if $Fx^2 - 2xzG = 0$, i.e., if and only if there exists $H \in V$ such that $F = 2zH$ and $G = xH$. Now, writing H as $H = xK + L$, with $K \in V$ and $L \in \mathbb{C}[z]$, one has $F dx + G dz = \delta_V^0(K) + L(2z dx + x dz)$. Finally, for every $R \in \mathbb{C}[z]$, if $R(2z dx + x dz) \in \text{Im}(\delta_V^0)$, then necessarily $R = 0$. This allows us to conclude that the first cohomology space associated to the Koszul complex of V can be written as $\mathbb{C}[z](2z dx + x dz)$.

Since we are working over a field, from the isomorphism (5.8) and the Künneth formula, we immediately get $H_1^\phi(T)$, $H_2^\phi(T)$ and $H_3^\phi(T)$. ■

Remark 5.3 In [25], for every weight-homogeneous polynomial $\varphi \in \mathbb{C}[x, y, z]$ admitting an isolated singularity, it is shown that $\{\vec{F} \in T^3 \mid \nabla\varphi \cdot (\nabla \times \vec{F}) = 0\} = \{\nabla G + H\nabla\varphi \mid G, H \in T\}$, using the fact that the Koszul complex associated to φ is exact in degree 2. In the following lemma, this is not true anymore if φ is replaced by $\phi = -x^2z$.

In the following, we will say that an element $\vec{F} = (F_1, F_2, F_3) \in T^3$ is homogeneous of degree $d \in \mathbb{Z}$, if F_1 , F_2 , and F_3 are three homogeneous polynomials of the same degree d . (Notice that we use the convention that a polynomial of degree $d < 0$ is zero).

Lemma 5.4 *Let $\phi = -x^2z \in T = \mathbb{C}[x, y, z]$. If $\vec{F} \in T^3$ (homogeneous or not) is such that $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$, then there exist polynomials $G, H \in T$, $C, P \in \mathbb{C}[y, z]$ and $f \in \mathbb{C}[\phi]$, satisfying*

$$\frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z} \quad (5.9)$$

and

$$\vec{F} = \nabla G + H\nabla\phi + \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} C + \begin{pmatrix} 2yz \\ -3xz \\ xy \end{pmatrix} P + f \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix}. \quad (5.10)$$

If \vec{F} is homogeneous of degree $n \in \mathbb{N}$, then the polynomials G, H, C, P and f can be taken to be homogeneous of degrees $n+1$, $n-2$, $n-1$, $n-2$ and $n-2$, respectively. In particular,

if $n \notin 2 + 3\mathbb{N}$, then f can be taken to be zero, and if $n = 2 + 3k$, with $k \in \mathbb{N}$, then f can be taken to be $f = \alpha(x^2z)^k$, with $\alpha \in \mathbb{C}$. Conversely, elements $\vec{F} \in T^3$ of the form (5.10) satisfy the equation $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$.

Proof. First of all, the last statement is straightforward.

Suppose that \vec{F} is homogeneous of degree $n \in \mathbb{N}$ and satisfying $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$. We will prove this result by recursion on $n \in \mathbb{N}$. As we have to distinguish whether $n \in 2 + 3\mathbb{N}$ or not, we first have to show the desired result, for $n = 0, 1, 2$.

◦ If $n = 0$, then there exist $a, b, c \in \mathbb{C}$, such that $\vec{F} = (a, b, c)$, and it is clear that $\vec{F} = \nabla G$, with $G = ax + by + cz \in T$.

◦ If $n = 1$, there exist $a_i, b_i, c_i \in \mathbb{C}$, for $i = 1, 2, 3$, such that $\vec{F} = \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix}$ and the condition $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$ is equivalent to: $b_3 = c_2$ and $a_2 = b_1$. Using this, it is easy to verify that we can write

$$\vec{F} = \nabla G + \begin{pmatrix} zC \\ 0 \\ -xC \end{pmatrix},$$

with $G = \frac{1}{2}(a_1x^2 + 2b_1xy + (c_1 + a_3)xz + b_2y^2 + 2c_2yz + c_3z^2) \in T$ and $C = \frac{1}{2}(c_1 - a_3)$. Note also that $\frac{\partial C}{\partial y} = 0$, so that the equation (5.9) is satisfied (here $P = 0$).

◦ If $n = 2$, there exist $a_{ij}, b_{ij}, c_{ij} \in \mathbb{C}$, for $1 \leq i, j \leq 3$, such that

$$\vec{F} = \begin{pmatrix} a_{11}x^2 + a_{12}xy + a_{13}xz + a_{23}yz + a_{22}y^2 + a_{33}z^2 \\ b_{11}x^2 + b_{12}xy + b_{13}xz + b_{23}yz + b_{22}y^2 + b_{33}z^2 \\ c_{11}x^2 + c_{12}xy + c_{13}xz + c_{23}yz + c_{22}y^2 + c_{33}z^2 \end{pmatrix}.$$

The condition $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$ is equivalent to the following identities: $2c_{12} - b_{13} - a_{23} = 0$, $2c_{22} - b_{23} = 0$, $c_{23} - 2b_{33} = 0$, $2b_{11} - a_{12} = 0$ and $b_{12} - 2a_{22} = 0$. Using this, it is straightforward to verify that

$$\vec{F} = \nabla G + \begin{pmatrix} zC + 2yzP \\ -3xzP \\ -xC + xyP \end{pmatrix} + \alpha \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix},$$

with

$$G = \frac{1}{3}a_{11}x^3 + \frac{1}{2}a_{12}x^2y + \frac{1}{3}(a_{13} + c_{11})x^2z + \frac{1}{2}(a_{23} + b_{13})xyz + a_{22}xy^2 \\ + \frac{1}{3}(a_{33} + c_{13})xz^2 + \frac{1}{2}b_{23}y^2z + b_{33}yz^2 + \frac{1}{3}b_{22}y^3 + \frac{1}{3}c_{33}z^3,$$

$\alpha = \frac{1}{3}(a_{13} - 2c_{11})$, $C = \frac{1}{6}(a_{23} - b_{13})y + \frac{1}{3}(2a_{33} - c_{13})z$ and $P = \frac{1}{6}(a_{23} - b_{13})$. Notice that the equation (5.9) is clearly satisfied in this case.

◦ Let now $m \in \mathbb{N}$, such that $m \geq 3$, and suppose the lemma is proved for all $n \in \mathbb{N}$ such that $n < m$. Let now $\vec{F} \in T^3$ be a homogeneous element of degree m , satisfying $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$.

According to Lemma 5.3, this hypothesis implies that there exist homogeneous elements: $\vec{K} \in T^3$, $P, C \in \mathbb{C}[y, z]$ and $E \in \mathbb{C}[y]$ such that

$$\nabla \times \vec{F} = \vec{K} \times \nabla\phi + \begin{pmatrix} x \\ 0 \\ -2z \end{pmatrix} P + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (xE + C),$$

with the degrees of \vec{K} , P , E and C respectively equal to $m - 3$, $m - 2$, $m - 2$ and $m - 1$. Computing the divergence of this, we obtain

$$0 = \text{Div}(\nabla \times \vec{F}) = (\nabla \times \vec{K}) \cdot \nabla\phi - P - 2z \frac{\partial P}{\partial z} + xE' + \frac{\partial C}{\partial y},$$

where $E' = \frac{dE}{dy}$ and where we have used that $\text{Div}(\vec{K} \times \nabla\phi) = (\nabla \times \vec{K}) \cdot \nabla\phi$. This implies that x divides the polynomial $-P - 2z\frac{\partial P}{\partial z} + \frac{\partial C}{\partial y}$ which lies in $\mathbb{C}[y, z]$, so that it is zero:

$$\frac{\partial C}{\partial y} = P + 2z\frac{\partial P}{\partial z}.$$

It remains that $(\nabla \times \vec{K}) \cdot \nabla\phi + xE' = 0$, which gives $E' \in (xT + zT) \cap \mathbb{C}[y] = \{0\}$, and $E' = 0$. This fact, together with the hypothesis that E is supposed to be homogeneous of degree $m - 2$, and $m \geq 3$, imply $E = 0$. We have obtained:

$$\nabla \times \vec{F} = \vec{K} \times \nabla\phi + \begin{pmatrix} xP \\ C \\ -2zP \end{pmatrix}.$$

Now, \vec{K} is of degree $m - 3 < m$ and satisfies $(\nabla \times \vec{K}) \cdot \nabla\phi = 0$, so that we can apply the recursion hypothesis to obtain the existence of homogeneous elements $G, H \in T$, $D, Q \in \mathbb{C}[y, z]$, $\alpha \in \mathbb{C}$, and $k \in \mathbb{N}$, such that

$$\frac{\partial D}{\partial y} = Q + 2z\frac{\partial Q}{\partial z}, \quad (5.11)$$

and

$$\vec{K} = \nabla G + H\nabla\phi + \begin{pmatrix} zD+2yzQ \\ -3xzQ \\ -xD+xyQ \end{pmatrix} + \alpha(x^2z)^k \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix}.$$

The polynomials G, H, D and Q are respectively of degree $m - 2, m - 5, m - 4$ and $m - 5$. Notice that, by hypothesis, α is supposed to be zero, except if $m - 3 = 2 + 3k$. We now compute

$$\vec{K} \times \nabla\phi = \nabla G \times \nabla\phi + \begin{pmatrix} 3x^3zQ \\ 3x^2zD \\ -6x^2z^2Q \end{pmatrix} + \alpha(x^2z)^k \begin{pmatrix} 0 \\ 3x^3z \\ 0 \end{pmatrix}.$$

Denoting by \vec{e} the so-called Euler vector $\vec{e} := (x, y, z) \in T^3$, we use the following general result (Proposition 3.5 in [25]), which is due to the exactness of the De Rham complex of $\mathbb{C}[x, y, z]$: if $\vec{A} = (A_1, A_2, A_3) \in T^3$ is a homogeneous element of degree $d \in \mathbb{N}$, such that $\text{Div}(\vec{A}) = 0$, then the Euler formula $\nabla A_i \cdot \vec{e} = dA_i$ ($1 \leq i \leq 3$), implies that $(d + 2)\vec{A} = \nabla \times (\vec{A} \times \vec{e})$.

As

$$\begin{aligned} \text{Div} \left(\begin{pmatrix} 3x^3zQ \\ 3x^2zD \\ -6x^2z^2Q \end{pmatrix} + \alpha(x^2z)^k \begin{pmatrix} 0 \\ 3x^3z \\ 0 \end{pmatrix} \right) &= \text{Div} \left(\vec{K} \times \nabla\phi - \nabla G \times \nabla\phi \right) \\ &= \left(\nabla \times (\vec{K} - \nabla G) \right) \cdot \nabla\phi \\ &= 0, \end{aligned}$$

we can write $\vec{K} \times \nabla\phi = \nabla \times \vec{L}$, where \vec{L} is given by:

$$\begin{aligned} \vec{L} &= G\nabla\phi + \frac{1}{(m+1)} \begin{pmatrix} 3x^3zQ \\ 3x^2zD \\ -6x^2z^2Q \end{pmatrix} \times \vec{e} + \frac{1}{m+1} \alpha(x^2z)^k \begin{pmatrix} 0 \\ -3x^3z \\ 0 \end{pmatrix} \times \vec{e} \\ &= G\nabla\phi + \frac{1}{(m+1)} \begin{pmatrix} 3x^2z^2D+6x^2yz^2Q \\ -9x^3z^2Q \\ 3x^3yzQ-3x^3zD \end{pmatrix} + \frac{3}{m+1} \alpha(x^2z)^{k+1} \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix}. \end{aligned}$$

Moreover, we have also

$$\begin{pmatrix} xP \\ C \\ -2zP \end{pmatrix} = \frac{1}{m+1} \nabla \times \left(\begin{pmatrix} xP \\ C \\ -2zP \end{pmatrix} \times \vec{e} \right) = \frac{1}{m+1} \nabla \times \left(\begin{pmatrix} zC+2yzP \\ -3xzP \\ -xC+xyP \end{pmatrix} \right).$$

We finally obtain

$$\nabla \times \vec{F} = \nabla \times \left(\vec{L} + \frac{1}{m+1} \begin{pmatrix} zC+2yzP \\ -3xzP \\ -xC+xyP \end{pmatrix} \right).$$

This allows us to apply another general result (Proposition 3.5 in [25]): if $\vec{A} = (A_1, A_2, A_3) \in T^3$ is a homogeneous element of degree $d \in \mathbb{N}$, such that $\nabla \times \vec{A} = 0$, then the Euler formula $\nabla A_i \cdot \vec{e} = d A_i$ ($1 \leq i \leq 3$), implies that $(d+1)\vec{A} = \nabla(\vec{A} \cdot \vec{e})$.

This implies that there exists a homogeneous element $S \in T$ (of degree $m+1$), such that

$$\begin{aligned} \vec{F} &= \nabla S + \vec{L} + \frac{1}{m+1} \begin{pmatrix} zC+2yzP \\ -3xzP \\ -xC+xyP \end{pmatrix} \\ &= \nabla S + G\nabla\phi + \frac{1}{(m+1)} \begin{pmatrix} 3x^2z^2D+6x^2yz^2Q \\ -9x^3z^2Q \\ 3x^3yzQ-3x^3zD \end{pmatrix} \\ &\quad + \frac{3}{m+1} \alpha(x^2z)^{k+1} \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \frac{1}{m+1} \begin{pmatrix} zC+2yzP \\ -3xzP \\ -xC+xyP \end{pmatrix}. \end{aligned}$$

Let now $V := \frac{3}{(2m-7)}xz(D+2yQ)$ (as $2m-7 \neq 0$). The polynomial V is homogeneous of degree $m-2$. Now, using (5.11) and the Euler formulas $y\frac{\partial D}{\partial y} + z\frac{\partial D}{\partial z} = (m-4)D$ and $y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} = (m-5)Q$, it is straightforward to verify that we have :

$$\nabla \left(-\frac{3}{m+1}x^2zV \right) - V\nabla\phi = \frac{1}{(m+1)} \begin{pmatrix} 3x^2z^2D+6x^2yz^2Q \\ -9x^3z^2Q \\ 3x^3yzQ-3x^3zD \end{pmatrix}.$$

Denoting by $\tilde{\alpha} := \frac{3}{m+1}\alpha$, $\tilde{G} := G - V$ and by $\tilde{S} := S - \frac{3}{(m+1)}x^2zV$, $\tilde{C} = \frac{1}{m+1}C$ and $\tilde{P} = \frac{1}{m+1}P$, we can write:

$$\vec{F} = \nabla\tilde{S} + \tilde{G}\nabla\phi + \tilde{\alpha}(x^2z)^{k+1} \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \begin{pmatrix} z\tilde{C}+2yz\tilde{P} \\ -3xz\tilde{P} \\ -x\tilde{C}+xy\tilde{P} \end{pmatrix}.$$

Moreover, \tilde{S} and \tilde{G} are homogeneous polynomials and we have already seen that C and P satisfy the identity (5.9) so that this identity is also satisfied by \tilde{C} and \tilde{P} . ■

In the sequel, we will several times need the following technical result.

Lemma 5.5 *If $r \in \mathbb{N}$, $c \in \mathbb{C}$ and if $K \in \mathbb{C}[x, y, z]$ is a polynomial such that $c\phi^r \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \nabla K \times \nabla\phi$, then $K \in \mathbb{C}[\phi]$ and c is necessarily 0.*

Proof. The hypothesis on K is equivalent to $\frac{\partial K}{\partial y} = 0$ and $c\phi^r = -2z\frac{\partial K}{\partial z} + x\frac{\partial K}{\partial x}$, hence $K \in \mathbb{C}[x, z]$. Denote by \mathcal{D} the operator $\mathcal{D} = -2z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x}$. For every $i, j \in \mathbb{N}$, $\mathcal{D}(x^i z^j) = (i-2j)x^i z^j$, so that \mathcal{D} is diagonalizable on $\mathbb{C}[x, z]$ and its kernel is $\mathbb{C}[x^2 z]$. As $\mathcal{D}(K) = c\phi^r$, we have

$$0 = \mathcal{D}(c\phi^r) = \mathcal{D}^2(K),$$

hence $K \in \ker(\mathcal{D}) = \mathbb{C}[\phi]$. Finally, the fact that K is in $\mathbb{C}[\phi]$ implies $\nabla K \times \nabla\phi = 0$, and $c = 0$. ■

Corollary 5.6 Let $\phi = -x^2z \in T = \mathbb{C}[x, y, z]$. We have

$$\begin{aligned} \frac{\{\vec{F} \in T^3 \mid \nabla\phi \cdot (\nabla \times \vec{F}) = 0\}}{\{\nabla G + H\nabla\phi \mid G, H \in T\}} &\simeq \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} \oplus \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} \\ &\oplus \bigoplus_{\substack{n \in \mathbb{N} \\ 1 \leq k \leq n+1}} \mathbb{C} \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1} z^{n+1-k}. \end{aligned}$$

Proof. Lemma 5.4 gives:

$$\begin{aligned} &\frac{\{\vec{F} \in T^3 \mid \nabla\phi \cdot (\nabla \times \vec{F}) = 0\}}{\{\nabla G + H\nabla\phi \mid G, H \in T\}} \\ &= \left\{ \begin{pmatrix} zC+2yzP \\ -3xzP \\ -xC+xyP \end{pmatrix} + \alpha(x^2z)^k \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} \mid \alpha \in \mathbb{C}, k \in \mathbb{N}, \right. \\ &\quad \left. C, P \in \mathbb{C}[y, z] \text{ satisfying } \frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z} \right\}. \end{aligned}$$

Fix now $n \in \mathbb{N}$. Let $C, P \in \mathbb{C}[y, z]$ be homogeneous polynomials satisfying $\frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z}$. We suppose that P is of degree n , so that C is zero or of degree equal to $n+1$. We write C and P as

$$P = \sum_{k=0}^n a_k y^k z^{n-k}, \text{ and } C = \sum_{k=0}^{n+1} b_k y^k z^{n+1-k},$$

where $a_k, b_k \in \mathbb{C}$. Now, compute

$$0 = \frac{\partial C}{\partial y} - P - 2z \frac{\partial P}{\partial z} = \sum_{k=0}^n ((k+1)b_{k+1} - (2(n-k)+1)a_k) y^k z^{n-k},$$

so that, necessarily, for all $k = 1, \dots, n+1$, we have $b_k = \frac{(2(n-k)+3)}{k} a_{k-1}$. We then can write

$$\begin{aligned} \begin{pmatrix} zC+2yzP \\ -3xzP \\ -xC+xyP \end{pmatrix} &= b_0 \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} z^n + \sum_{k=1}^{n+1} \frac{1}{k} a_{k-1} \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1} z^{n-k+1} \\ &\in \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{k=1}^{n+1} \mathbb{C} \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1} z^{n+1-k} \end{aligned}$$

We finally have shown that

$$\begin{aligned} \frac{\{\vec{F} \in T^3 \mid \nabla\phi \cdot (\nabla \times \vec{F}) = 0\}}{\{\nabla G + H\nabla\phi \mid G, H \in T\}} &\simeq \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} \\ &+ \sum_{\substack{n \in \mathbb{N} \\ 1 \leq k \leq n+1}} \mathbb{C} \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1} z^{n+1-k}, \end{aligned}$$

and it remains to show that this sum is a direct one. To do this, it suffices to show that each homogeneous component of this sum is a direct sum. Notice that an element of the space of the right hand side of the previous equation is at least of degree 2, so that, we fix $n \in \mathbb{N}$, and we consider the degree $n+2$ component of the previous sum. Let $\alpha, c \in \mathbb{C}$ and

$r \in \mathbb{N}$, and for all $1 \leq k \leq n+1$, we consider $a_k \in \mathbb{C}$. Suppose that there exist homogeneous elements $G, H \in T$, of respective degrees equal to $n+3$ and n , satisfying:

$$\alpha \phi^r \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + cz^n \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{k=1}^{n+1} a_k \begin{pmatrix} (2n+3)y^k z^{n+2-k} \\ -3kxy^{k-1} z^{n+2-k} \\ (-2n+3(k-1))xy^k z^{n+1-k} \end{pmatrix} = \nabla G + H \nabla \phi. \quad (5.12)$$

Notice that $\alpha = 0$ if $n \neq 3r$. Applying the curl operator to this identity allows us to obtain

$$-\sum_{k=1}^{n+1} k(2n+6) a_k y^{k-1} z^{n+2-k} = 2xz \frac{\partial H}{\partial y},$$

which implies that $a_k = 0$, for all $1 \leq k \leq n+1$. This, together with (5.12) imply that $\frac{\partial G}{\partial y} = 0$, so that $G \in \mathbb{C}[x, z]$, while, this together with the result obtained by applying the curl operator to the previous identity (5.12), give:

$$3(r+1)\alpha x \phi^r + (n+3)cz^{n+1} = -2xz \frac{\partial H}{\partial z} + x^2 \frac{\partial H}{\partial x}.$$

This shows that x divides $(n+3)cz^{n+1}$, which means that $c = 0$ and the equation (5.12) becomes:

$$\alpha \phi^r \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} = \nabla G + H \nabla \phi. \quad (5.13)$$

Applying to this equation the cross product with $\nabla \phi$ leads to: $-3\alpha x \phi^{r+1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \nabla G \times \nabla \phi$. This, together with lemma 5.5, give $\alpha = 0$, which is what remains to prove. ■

We now determine the Poisson homology of the Poisson algebra $(T, \{\cdot, \cdot\})$.

Proposition 5.7 *Set $T = \mathbb{C}[x, y, z]$ and consider $\phi = -x^2 z \in T$. The algebra T becomes a Poisson algebra when equipped with the Poisson bracket $\{\cdot, \cdot\}$, defined by:*

$$\{y, z\} = \frac{\partial \phi}{\partial x} = -2xz, \quad \{z, x\} = \frac{\partial \phi}{\partial y} = 0, \quad \{x, y\} = \frac{\partial \phi}{\partial z} = -x^2.$$

Using the identifications $\Omega^1(T) \simeq T^3$, $\Omega^2(T) \simeq T^3$, $\Omega^3(T) \simeq T$ explained above, the Poisson homology spaces of the Poisson algebra T are given by:

$$HP_0(T) \simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z];$$

$$\begin{aligned} HP_1(T) \simeq & \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} \oplus \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} \oplus \bigoplus_{\substack{n \in \mathbb{N}^* \\ 0 \leq k \leq n}} \mathbb{C} \begin{pmatrix} 0 \\ ky^{k-1} z^{n-k} \\ (n-k)y^k z^{n-1-k} \end{pmatrix} \\ & \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \begin{pmatrix} y^n \\ nxy^{n-1} \\ 0 \end{pmatrix} \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ 1 \leq k \leq n+1}} \mathbb{C} \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1} z^{n+1-k}; \end{aligned}$$

$$\begin{aligned} HP_2(T) \simeq & \mathbb{C}[\phi] \begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus (x\mathbb{C}[\phi] \oplus z\mathbb{C}[z]) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ & \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ 0 \leq k \leq n}} \mathbb{C} \begin{pmatrix} (k+1)x \\ (2(n-k)+1)y \\ -2(k+1)z \end{pmatrix} y^k z^{n-k}; \end{aligned}$$

$$HP_3(T) \simeq \mathbb{C}[\phi].$$

Proof. Remark that it is not possible to apply the results of Monnier for the computation of Poisson homology as we have seen for the third type (proof of Proposition 4.5). Actually, it is easy to check that the Poisson bracket derived from the potential x^2z is not diagonalizable. To determine the Poisson homology spaces, notice that the polynomial ϕ is homogeneous of degree 3, so that, considering T graded by the total degree of the polynomials, the operator δ_p ($1 \leq p \leq 3$) is homogeneous of degree 1. This allows one to determine the Poisson homology spaces, degree by degree.

The 0-th Poisson homology space $HP_0(T)$.
According to (5.4),

$$HP_0(T) \simeq \frac{T}{\left\{ \delta_1(\vec{G}) \mid \vec{G} = (G_1, G_2, G_3) \in T^3 \right\}},$$

where $\delta_1(\vec{G}) = 2xz \left(\frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y} \right) + x^2 \left(\frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x} \right)$. Now let $G \in T$. We can write G as $G = xF + H$ with $F \in T$ and $H \in \mathbb{C}[y, z]$. Then, F can also be written as $F = xA + zB + C$, with $A, B \in T$ and $C \in \mathbb{C}[y]$, so that

$$G = x^2A + xzB + xC + H.$$

Now, there exist $G_1 \in T$ and $G_3 \in T$, such that $A = \frac{\partial G_1}{\partial y}$ and $B = -2\frac{\partial G_3}{\partial y}$, so that

$$G = x^2A + xzB + xC + H = -2xz\frac{\partial G_3}{\partial y} + x^2\frac{\partial G_1}{\partial y} + xC + H = \delta_1(\vec{G}) + xC + H,$$

with $\vec{G} = (G_1, 0, G_3) \in T^3$. This shows that $HP_0(T) \simeq x\mathbb{C}[y] + \mathbb{C}[y, z]$. It is also clear that this sum is a direct one, because an equality of the form $xC + H = \vec{F} \cdot \nabla\phi$, with $C \in \mathbb{C}[y]$, $H \in \mathbb{C}[y, z]$ and $\vec{F} \in T^3$ implies that x divides H , so $H = 0$ and then it remains that $C \in (xT + zT) \cap \mathbb{C}[y]$ which means that $C = 0$. We finally have obtained $HP_0(T) \simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z]$.

The third Poisson homology space $HP_3(T)$.

First, it is easy to see that $\mathbb{C}[\phi] \simeq \mathbb{C}[\phi]dx \wedge dy \wedge dz \subseteq HP_3(T)$. Conversely, we consider an element $F \in HP_3(T)$, i.e., $F \in T \simeq \Omega^3(T)$ satisfying $\nabla F \times \nabla(x^2z) = 0$. Lemma 5.5 (with $c = 0$) then implies that $F \in \mathbb{C}[\phi]$, so that $HP_3(T) \simeq \mathbb{C}[x^2z] = \mathbb{C}[\phi]$.

The first Poisson homology space $HP_1(T)$.

By (5.4),

$$HP_1(T) = \frac{\left\{ \vec{F} \in T^3 \mid \nabla\phi \cdot (\nabla \times \vec{F}) = 0 \right\}}{\left\{ -\nabla(\vec{G} \cdot \nabla\phi) + \text{Div}(\vec{G})\nabla\phi \mid \vec{G} \in T^3 \right\}}.$$

Let $\vec{F} \in T^3$ be an element satisfying $\nabla\phi \cdot (\nabla \times \vec{F}) = 0$. According to corollary 5.6, there exist $G, H \in T$ such that:

$$\vec{F} \in \nabla G + H\nabla\phi + \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \mathbb{C} \vec{u}_{n,k},$$

where $\vec{u}_{n,k} = \begin{pmatrix} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{pmatrix} y^{k-1} z^{n+1-k}$, for $n \in \mathbb{N}$ and $1 \leq k \leq n+1$.

Now, according to the determination of $HP_0(T)$, there exist $\vec{L}, \vec{K} \in T^3$, $A, \tilde{A} \in \mathbb{C}[y]$ and $B, \tilde{B} \in \mathbb{C}[y, z]$ such that

$$G = \nabla\phi \cdot (\nabla \times \vec{L}) + xA + B, \quad \text{and} \quad H = \nabla\phi \cdot (\nabla \times \vec{K}) + x\tilde{A} + \tilde{B}.$$

As $\delta_2(-\nabla \times \vec{L}) = \nabla(\nabla\phi \cdot (\nabla \times \vec{L}))$, and $\delta_2(\vec{K} \times \nabla\phi) = (\nabla\phi \cdot (\nabla \times \vec{K})) \nabla\phi$, we obtain

$$\begin{aligned} \vec{F} &\in \delta_2(-\nabla \times \vec{L} + \vec{K} \times \nabla\phi) + \nabla(xA + B) + (x\tilde{A} + \tilde{B})\nabla\phi \\ &+ \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \mathbb{C} \vec{u}_{n,k}. \end{aligned}$$

Now, let us consider $A_1 \in \mathbb{C}[y]$ satisfying $\frac{\partial A_1}{\partial y} = \tilde{A}$ and $B_1 \in \mathbb{C}[y, z]$ satisfying $\frac{\partial B_1}{\partial y} = \tilde{B}$. It is then straightforward to verify that

$$\delta_2 \left(\begin{pmatrix} -x\tilde{B} \\ xA_1 - 2z\frac{\partial B_1}{\partial z} \\ 2z\tilde{B} \end{pmatrix} \right) = (x\tilde{A} + \tilde{B})\nabla\phi,$$

so that

$$\vec{F} \in \text{Im}(\delta_2) + \nabla(xA + B) + \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \mathbb{C} \vec{u}_{n,k}.$$

This allows us to write

$$\begin{aligned} HP_1(T) &\simeq \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \mathbb{C} \vec{u}_{n,k} \\ &+ \{ \nabla(xA + B) \mid A \in \mathbb{C}[y], B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z] \}. \end{aligned} \quad (5.14)$$

Let us show that this sum is a direct one. Suppose that there exist $P \in \mathbb{C}[X]$ a polynomial in one variable, $Q \in \mathbb{C}[z]$, $A \in \mathbb{C}[y]$, and a polynomial $B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z]$ (i.e., $B \in \mathbb{C}[y, z]$ with no constant term), $\vec{K} \in T^3$, and for every $n \in \mathbb{N}$ and every $1 \leq k \leq n+1$, suppose that $\alpha_k^n \in \mathbb{C}$ are constants, such that:

$$\begin{aligned} \nabla(xA + B) + P(\phi) \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + Q(z) \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \alpha_k^n \vec{u}_{n,k} \\ = -\nabla(\vec{K} \cdot \nabla\phi) + \text{Div}(\vec{K})\nabla\phi. \end{aligned}$$

This implies that

$$P(\phi) \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} + Q(z) \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \alpha_k^n \vec{u}_{n,k} \in \{K\nabla\phi + \nabla L \mid K, L \in T\}.$$

According to Corollary 5.6, necessarily $P = 0$, $Q = 0$ and $\alpha_k^n = 0$, for all $n \in \mathbb{N}$ and all $1 \leq k \leq n+1$. It then remains $\nabla(xA + B + \vec{K} \cdot \nabla\phi) = \text{Div}(\vec{K})\nabla\phi$, which implies that $\nabla(xA + B + \vec{K} \cdot \nabla\phi) \times \nabla\phi = 0$ and according to the determination of the space $HP_3(T)$, this gives the existence of polynomial in one variable $R \in \mathbb{C}[X]$, such that $xA + B + \vec{K} \cdot \nabla\phi = R(\phi)$. As there is no constant term in B and for degree reason, necessarily $\phi = -x^2z$ divides $R(\phi)$ and x divides $B \in \mathbb{C}[y, z]$, which implies that $B = 0$. Moreover, last equation allows us to

obtain $A \in \mathbb{C}[y] \cap (xT + zT) = \{0\}$. This allows us to conclude that the sum in (5.14) is a direct one, i.e., we can write

$$\begin{aligned} HP_1(T) &\simeq \mathbb{C}[\phi] \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix} \oplus \mathbb{C}[z] \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix} \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{k=1}^{n+1} \mathbb{C} \vec{u}_{n,k} \\ &\oplus \{ \nabla(xA + B) \mid A \in \mathbb{C}[y], B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z] \}. \end{aligned}$$

Finally, it is clear that

$$\begin{aligned} \{ \nabla(xA + B) \mid A \in \mathbb{C}[y], B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z] \} = \\ \bigoplus_{n \in \mathbb{N}} \mathbb{C} \begin{pmatrix} y^n \\ nxy^{n-1} \\ 0 \end{pmatrix} \oplus \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{0 \leq k \leq n} \mathbb{C} \begin{pmatrix} ky^{k-1}z^{n-k} \\ (n-k)y^kz^{n-1-k} \end{pmatrix}, \end{aligned}$$

and this finishes the determination of $HP_1(T)$.

The second Poisson homology space $HP_2(T)$.

According to (5.4), we have

$$HP_2(T) = \frac{\{ \vec{F} \in T^3 \mid -\nabla(\vec{F} \cdot \nabla\phi) + \text{Div}(\vec{F})\nabla\phi = 0 \}}{\{ \nabla G \times \nabla\phi \mid G \in T \}}.$$

Let $\vec{F} \in T^3$ be a homogeneous element of degree $n \in \mathbb{N}$, satisfying

$$-\nabla(\vec{F} \cdot \nabla\phi) + \text{Div}(\vec{F})\nabla\phi = 0. \quad (5.15)$$

This implies that $\nabla(\vec{F} \cdot \nabla\phi) \times \nabla\phi = 0$. This, together with the determination of $HP_3(T)$ gives the existence of $\alpha \in \mathbb{C}$ and $r \in \mathbb{N}$ such that $\vec{F} \cdot \nabla\phi = \alpha\phi^r$. Notice that if $r = 0$, then for degree reasons, $\alpha = 0$. According to the Euler formula $\nabla\phi \cdot \vec{e} = 3\phi$ (where we recall that $\vec{e} = (x, y, z) \in T^3$), we get $\vec{F} \cdot \nabla\phi = \frac{\alpha}{3}\phi^{r-1}\vec{e} \cdot \nabla\phi$. With the help of the determination of $H_2^\phi(T)$ in Lemma 5.3, this gives the existence of a homogeneous element $\vec{G} \in T^3$ of degree $n - 2$, and homogeneous polynomials $B, D \in \mathbb{C}[y, z]$ and $A \in \mathbb{C}[y]$ such that

$$\vec{F} = \frac{\alpha}{3}\phi^{r-1}\vec{e} + \vec{G} \times \nabla\phi + \begin{pmatrix} xD \\ xA+B \\ -2zD \end{pmatrix}.$$

We now compute the divergence of \vec{F} :

$$\text{Div}(\vec{F}) = \alpha r\phi^{r-1} + (\nabla \times \vec{G}) \cdot \nabla\phi - D + xA' + \frac{\partial B}{\partial y} - 2z\frac{\partial D}{\partial z}.$$

The equation (5.15) then becomes

$$0 = (\nabla \times \vec{G}) \cdot \nabla\phi - D + xA' + \frac{\partial B}{\partial y} - 2z\frac{\partial D}{\partial z},$$

which shows that x divides the polynomial $\frac{\partial B}{\partial y} - 2z\frac{\partial D}{\partial z} - D \in \mathbb{C}[y, z]$. This implies that $\frac{\partial B}{\partial y} = D + 2z\frac{\partial D}{\partial z}$ and $0 = (\nabla \times \vec{G}) \cdot \nabla\phi + xA'$, so that we also have $A' \in (xT + zT) \cap \mathbb{C}[y] = \{0\}$. We have obtained that $A = \beta \in \mathbb{C}$ is a constant and

$$(\nabla \times \vec{G}) \cdot \nabla\phi = 0.$$

Lemma 5.4 leads to the existence of homogeneous polynomials $G, H \in T$, and $C, P \in \mathbb{C}[y, z]$, of respective degrees equal to $n - 3$ and $n - 4$, satisfying

$$\frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z} \quad (5.16)$$

and $\gamma \in \mathbb{C}$, $k \in \mathbb{N}$, such that

$$\vec{G} = \nabla G + H \nabla \phi + \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} C + \begin{pmatrix} 2yz \\ -3xz \\ xy \end{pmatrix} P + \gamma \phi^k \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix}.$$

Now, this enables us to write:

$$\vec{F} = \frac{\alpha}{3} \phi^{r-1} \vec{e} + \nabla G \times \nabla \phi + \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} + \begin{pmatrix} 3x^3 z P \\ 3x^2 z C \\ -6x^2 z^2 P \end{pmatrix} + (-3\gamma x \phi^{k+1} + \beta x) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Let us now fix a homogeneous polynomial $P_1 \in \mathbb{C}[y, z]$ verifying $\frac{\partial P_1}{\partial y} = P$. Then the equation (5.16) gives $\frac{\partial}{\partial y} (C - P_1 - 2z \frac{\partial P_1}{\partial z}) = 0$, i.e., $C - P_1 - 2z \frac{\partial P_1}{\partial z}$ is a homogeneous polynomial in $\mathbb{C}[z]$, of degree $n - 3$, which means that there exists $\eta \in \mathbb{C}$ such that $C = P_1 + 2z \frac{\partial P_1}{\partial z} + \eta z^{n-3}$. Now compute

$$\begin{aligned} \delta_3 \left(-3xzP_1 + \frac{-3\eta}{2n-5} xz^{n-2} \right) &= -3 \nabla \left(xzP_1 + \frac{\eta}{2n-5} xz^{n-2} \right) \times \nabla \phi \\ &= 3 \begin{pmatrix} zP_1 + \frac{\eta}{2n-5} z^{n-2} \\ xz \frac{\partial P_1}{\partial y} \\ xP_1 + xz \frac{\partial P_1}{\partial z} + \frac{\eta(n-2)}{2n-5} xz^{n-3} \end{pmatrix} \times \begin{pmatrix} 2xz \\ 0 \\ x^2 \end{pmatrix} \\ &= 3 \begin{pmatrix} x^3 z P \\ x^2 z P_1 + 2x^2 z^2 \frac{\partial P_1}{\partial z} + \eta x^2 z^{n-2} \\ -2x^2 z^2 P \end{pmatrix} \\ &= \begin{pmatrix} 3x^3 z P \\ 3x^2 z C \\ -6x^2 z^2 P \end{pmatrix}. \end{aligned}$$

This implies: $\vec{F} \in \text{Im}(\delta_3) + \mathbb{C}[\phi] \vec{e} + \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} + x\mathbb{C}[\phi] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Conversely, it is straightforward to see that an element of this space lies in the kernel of δ_2 . In other words,

$$HP_2(T) = \mathbb{C}[\phi] \vec{e} + x\mathbb{C}[\phi] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \left\{ \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} \mid D, B \in \mathbb{C}[y, z] \text{ satisfying } \frac{\partial B}{\partial y} = D + 2z \frac{\partial D}{\partial z} \right\}.$$

Let us show that the previous sum is a direct one. To do this, let us consider a homogeneous element of this sum. Let $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{C}$, and homogeneous polynomials $D, B \in \mathbb{C}[y, z]$ of respective degrees equal to $3n$ and $3n + 1$ satisfying $\frac{\partial B}{\partial y} = D + 2z \frac{\partial D}{\partial z}$ and a homogeneous $G \in T$ of degree $3n$, such that

$$\alpha \phi^n \vec{e} + \beta x \phi^n \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} = \nabla G \times \nabla \phi = \begin{pmatrix} -x^2 \frac{\partial G}{\partial y} \\ -2xz \frac{\partial G}{\partial z} + x^2 \frac{\partial G}{\partial x} \\ 2xz \frac{\partial G}{\partial y} \end{pmatrix}.$$

Computing the inner product of this identity with $\nabla \phi$ leads to $3\alpha \phi^{n+1} = 0$, so that $\alpha = 0$. Moreover, this gives $xD = -x^2 \frac{\partial G}{\partial y}$, so that $D \in xT \cap \mathbb{C}[y, z] = \{0\}$ and $\frac{\partial G}{\partial y} = 0$, i.e., $G \in \mathbb{C}[x, z]$. Moreover, the second row of the previous equation implies that $B \in xT \cap \mathbb{C}[y, z] = \{0\}$, and $B = 0$. It remains to show that $\beta = 0$, while we have $\beta x \phi^n \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \nabla G \times \nabla \phi$.

According to Lemma 5.5, G is in $\mathbb{C}[\phi]$ and $\beta = 0$.

We then have obtained

$$HP_2(T) = \mathbb{C}[\phi]\vec{e} \oplus x\mathbb{C}[\phi] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \oplus \left\{ \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} \mid D, B \in \mathbb{C}[y, z] : \frac{\partial B}{\partial y} = D + 2z \frac{\partial D}{\partial z} \right\},$$

so that it remains to show that

$$\begin{aligned} & \left\{ \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} \mid D, B \in \mathbb{C}[y, z] : \frac{\partial B}{\partial y} = D + 2z \frac{\partial D}{\partial z} \right\} = \\ & z\mathbb{C}[z] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ 0 \leq k \leq n}} \mathbb{C} \begin{pmatrix} (k+1)x \\ (2(n-k)+1)y \\ -2(k+1)z \end{pmatrix} y^k z^{n-k}. \end{aligned} \quad (5.17)$$

To do this, let us consider $n \in \mathbb{N}$ and homogeneous polynomials $D, B \in \mathbb{C}[y, z]$ of respective degrees n and $n+1$, satisfying $\frac{\partial B}{\partial y} = D + 2z \frac{\partial D}{\partial z}$. We have already seen in the proof of Corollary 5.6 that this implies the existence of complex numbers $a_k \in \mathbb{C}$, $0 \leq k \leq n$ and $b_0 \in \mathbb{C}$, such that

$$D = \sum_{k=0}^n a_k y^k z^{n-k}, \text{ and } B = b_0 z^{n+1} + \sum_{k=1}^{n+1} \frac{2(n-k)+3}{k} a_{k-1} y^k z^{n+1-k}.$$

This gives

$$\begin{aligned} \begin{pmatrix} xD \\ B \\ -2zD \end{pmatrix} &= b_0 z^{n+1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sum_{k=0}^n \frac{a_k}{k+1} \begin{pmatrix} (k+1)x y^k z^{n-k} \\ (2(n-k)+1)y^{k+1} z^{n-k} \\ -2(k+1)y^k z^{n+1-k} \end{pmatrix} \\ &\in z\mathbb{C}[z] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \oplus \bigoplus_{k=0}^n \mathbb{C} \begin{pmatrix} (k+1)x \\ (2(n-k)+1)y \\ -2(k+1)z \end{pmatrix} y^k z^{n-k}. \end{aligned}$$

Notice that the previous sum is a direct one, because of degree reasons. This allows us to conclude that (5.17) holds, which finishes the determination of $HP_2(T)$. \blacksquare

Remark 5.7 Recall that the Poisson structure which equips the algebra T is unimodular, i.e., its modular class (see [35]) vanishes (here, even the curl vector field (see [13]) is zero), which implies that there is a duality between the Poisson cohomology and the Poisson homology of the Poisson algebra T : $HP^\bullet(T) \simeq HP_{3-\bullet}(T)$.

Now, we show in an elementary way that each Poisson cycle can be lifted to a Koszul cycle. By definition, a *Koszul cycle* is a cycle of the filtered complex (5.3).

Proposition 5.8 *We keep the notations of the beginning of the section for the algebra B and the Poisson algebra $(T, \{\cdot, \cdot\})$. Let us consider the filtration F on the algebra B , given by the degree in y . For every Poisson cycle X , there exists a Koszul cycle (i.e. a cycle of the filtered complex (5.3)) \tilde{X} such that $gr_F(\tilde{X}) \simeq X$.*

Proof. Proposition 5.7 gives a basis for each Poisson homology vector space of T . For every Poisson boundary $\delta_k(X)$, we have seen in Proposition 5.2 that $gr_F(\tilde{d}_k(Y)) = \delta_k(X)$, where Y is the element X , viewed in the algebra B and written in the basis $(x^i y^j z^k)$, $i, j, k \in \mathbb{N}$. So that $\tilde{X} := \tilde{d}_k(Y)$ is a Koszul cycle satisfying $gr_F(\tilde{X}) \simeq \delta_k(X)$. This implies that it suffices to show that each element X of the bases of the Poisson homology spaces given in Proposition 5.7, can be lifted to a Koszul cycle, i.e., for each element X of the bases,

we will give a Koszul cycle \tilde{X} , satisfying $gr_F(\tilde{X}) = X$. Notice that we will use here the identifications explained before: $B(k(c(w))) \simeq B$, $BR_B \simeq B^3$, $BV_B \simeq B^3$, and $\Omega^3(T) \simeq T$, $\Omega^2(T) \simeq T^3$, $\Omega^1(T) \simeq T$ and $\Omega^0(T) \simeq T$.

Lifting of the Poisson 0-cycles.

As every element of B is a Koszul 0-cycle (and similarly for the Poisson 0-cycles), every Poisson 0-cycle can be lifted to a Koszul 0-cycle.

Lifting of the Poisson 3-cycles.

According to Proposition 5.7, $HP_3(T) \simeq \mathbb{C}[\phi] = \mathbb{C}[x^2z]$. For each $c \in \mathbb{C}$ and $k \in \mathbb{N}$, the definition of \tilde{d}_3 and lemma 5.1 clearly lead to $\tilde{d}_3(cx^{2k}z^k \otimes c(w)) = 0$, so that $cx^{2k}z^k \otimes c(w)$, which is identified to $cx^{2k}z^k$, is a Koszul 3-cycle satisfying $gr_F(cx^{2k}z^k) = cx^{2k}z^k$.

Lifting of the Poisson 1-cycles.

According to Proposition 5.7, the space $HP_1(T)$ is generated as a \mathbb{C} -vector space by the following elements: $A_k := (x^2z)^k \begin{pmatrix} xz \\ 0 \\ -x^2 \end{pmatrix}$, $B_r := z^r \begin{pmatrix} z^2 \\ 0 \\ -xz \end{pmatrix}$, $\vec{u}_{n,k} = \begin{pmatrix} (2n+3)y^k z^{n+2-k} \\ -3kxy^{k-1}z^{n+2-k} \\ (-2n+3(k-1))xy^k z^{n+1-k} \end{pmatrix}$, $\vec{v}_{m,s} := \begin{pmatrix} 0 \\ sy^{s-1}z^{m-s} \\ (m-s)y^s z^{m-1-s} \end{pmatrix}$, and $\vec{w}_p := \begin{pmatrix} y^p \\ pxy^{p-1} \\ 0 \end{pmatrix}$, where $n, p \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $1 \leq k \leq n+1$, $0 \leq s \leq m$.

Now, by definition of \tilde{d}_1 and because $xz = zx$ in B , it is clear that:

$$\tilde{d}_1(x^{2k+1}z^{k+1} \otimes x - x^{2k+2}z^k \otimes z) = 0, \quad \text{and} \quad \tilde{d}_1(z^{r+2} \otimes x - xz^{r+1} \otimes z) = 0,$$

so that $\tilde{A}_k := x^{2k+1}z^{k+1} \otimes x - x^{2k+2}z^k \otimes z$ and $\tilde{B}_r := z^{r+2} \otimes x - xz^{r+1} \otimes z$ are Koszul 1-cycles satisfying $gr_F(\tilde{A}_k) = A_k$ and $gr_F(\tilde{B}_r) = B_r$.

Next, we let

$$\tilde{U}_{n,k} :=$$

$$(2n+3)y^k z^{n+2-k} \otimes x - 3kxy^{k-1}z^{n+2-k} \otimes y + (-2n+3(k-1))xy^k z^{n+1-k} \otimes z$$

$$\left\{ \begin{array}{l} - \sum_{\ell=0}^{k-2} \frac{k!}{\ell!} \frac{a_{n,k,\ell}}{2(n+2-k)-(k-\ell)} \mathcal{X}_{k-\ell,\ell,n+2-k}, \quad \text{if } 3k-2(n+2) < 0; \\ + \sum_{\ell=0}^{\ell_{(n,k)}-1} \frac{k!}{\ell!} \frac{(a_{n,k,\ell_{(n,k)}} - a_{n,k,\ell})}{\ell - \ell_{(n,k)}} \mathcal{X}_{k-\ell,\ell,n+2-k} - \sum_{\ell=\ell_{(n,k)}+1}^{k-2} \frac{k!}{\ell!} \frac{a_{n,k,\ell}}{\ell - \ell_{(n,k)}} \mathcal{X}_{k-\ell,\ell,n+2-k} \\ - \frac{k!}{(\ell_{(n,k)}+1)!} a_{n,k,\ell_{(n,k)}} x^{2n+3-2k} y^{3k-3-2n} z^{n+2-k} \otimes x, \quad \text{if } 3k-2(n+2) \geq 0, \end{array} \right.$$

where for $1 \leq k \leq n+1$ and $0 \leq \ell \leq k-2$, $a_{n,k,\ell} := -2n-6+6k-3\ell+2n(k-\ell)-3k(k-\ell)$, for all $a, b, c \in \mathbb{N}$, $\mathcal{X}_{a,b,c} := x^a y^b z^c \otimes y - 2c x^a y^b z^c \otimes x$ and if $3k-2(n+2) \geq 0$, $\ell_{(n,k)} := 3k-4-2n$. Then, we have $gr_F(\tilde{U}_{n,k}) = \vec{u}_{n,k}$ and lemma 5.1 allows one to verify that $\tilde{U}_{n,k}$ is a Koszul 1-cycle.

Now, let $m \in \mathbb{N}^*$, $0 \leq s \leq m$ and consider

$$\begin{aligned} \tilde{V}_{m,s} := & sy^{s-1}z^{m-s} \otimes y + (m-s)y^s z^{m-1-s} \otimes z \\ & + \sum_{\ell=0}^{s-2} \frac{(m-s)}{2(m-s)-1} \frac{s!}{\ell!} x^{s-\ell-1} y^\ell z^{m-s} \otimes y. \end{aligned}$$

Then we have $gr_F(\tilde{V}_{m,s}) = \vec{v}_{m,s}$ and it is straightforward, using lemma 5.1, to verify that $\tilde{V}_{m,s}$ is a Koszul 1-cycle.

Finally, let $p \in \mathbb{N}$ and consider

$$\tilde{W}_p := \sum_{k=0}^p \frac{p!}{k!} x^{p-k} y^k \otimes x + \sum_{k=0}^{p-1} \frac{p!}{k!} x^{p-k} y^k \otimes y.$$

It is clear that $gr_F(\tilde{W}_p) = \vec{w}_p$ and moreover, using lemma 5.1, we obtain that $\tilde{d}_1(\tilde{W}_p) = 0$, i.e., \tilde{W}_p is a Koszul 1-cycle.

Lifting of the Poisson 2-cycles.

According to Proposition 5.7, the space $HP_2(T)$ is generated as a \mathbb{C} -vector space by the following elements: $C_r := (x^2 z)^r \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $D_s := x(x^2 z)^s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $E_t := z^{t+1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{o}_{n,k} := \begin{pmatrix} (k+1)xy^k z^{n-k} \\ (2(n-k)+1)y^{k+1} z^{n-k} \\ -2(k+1)y^k z^{n-k+1} \end{pmatrix}$, where $r, s, t, n \in \mathbb{N}$ and $0 \leq k \leq n$.

First, if $r \in \mathbb{N}$, then, using the definition of \tilde{d}_2 and lemma 5.1, we obtain easily:

$$\tilde{d}_2(x^{2r+1} z^r \otimes r_1 + x^{2r} y z^r \otimes r_2 + x^{2r} z^{r+1} \otimes r_3) = 0,$$

so that $\tilde{C}_r := x^{2r+1} z^r \otimes r_1 + x^{2r} y z^r \otimes r_2 + x^{2r} z^{r+1} \otimes r_3$ is a Koszul 2-cycle satisfying $gr_F(\tilde{C}_r) = C_r$.

Moreover, if $s, t \in \mathbb{N}$, then, because $xz = zx$ in B and by definition of \tilde{d}_2 , we have

$$\tilde{d}_2(x^{2s+1} z^s \otimes r_2) = 0, \quad \text{and} \quad \tilde{d}_2(z^{t+1} \otimes r_2) = 0,$$

so that $\tilde{D}_s := x^{2s+1} z^s \otimes r_2$ and $\tilde{E}_t := z^{t+1} \otimes r_2$ are Koszul 2-cycles and satisfy $gr_F(\tilde{D}_s) = D_s$ and $gr_F(\tilde{E}_t) = E_t$.

Now, let $n \in \mathbb{N}$ and $0 \leq k \leq n$ and

$$\begin{aligned} \tilde{O}_{n,k} := & (k+1)xy^k z^{n-k} \otimes r_1 + (2(n-k)+1)y^{k+1} z^{n-k} \otimes r_2 \\ & + \left(-2(k+1)y^k z^{n-k+1} - \sum_{j=0}^{k-1} \frac{(k+1)!}{j!} x^{k-j} y^j z^{n+1-k} \right) \otimes r_3. \end{aligned}$$

Then, using once more lemma 5.1, it is straightforward to verify that $\tilde{d}_2(\tilde{O}_{n,k}) = 0$. Moreover, we have of course, $gr_F(\tilde{O}_{n,k}) = \vec{o}_{n,k}$, which finishes the proof. ■

Proof of Theorem 1.1. Actually, it remains to prove that the Hochschild homology of the algebra B is isomorphic to the Poisson homology of T obtained in Proposition 5.7. Following the same method as in [31], we use the Brylinski spectral sequence of the almost commutative algebra B [8, 16]. Denote by C the filtered complex (5.3) and denote by $(F_p C)_{p \in \mathbb{Z}}$ its filtration. The complex $F_p C$ is the following

$$0 \longrightarrow F_{p+3}(B(\mathbb{C}c(w))) \xrightarrow{\tilde{d}_3} F_{p+2}(BR_B) \xrightarrow{\tilde{d}_2} F_{p+1}(BV_B) \xrightarrow{\tilde{d}_1} F_p(B) \longrightarrow 0 \quad (5.18)$$

where F_p denotes the filtration of the total degree. Let us consider the spectral sequence associated to the filtered complex C (Section 5.4 in [34]). The term E^0 of this spectral sequence is the graded complex naturally associated to the filtered complex C . By Proposition 5.2, E^0 is isomorphic to the Poisson complex of the Poisson algebra T (actually, the proof of

Proposition 5.2 provides an explicit isomorphism). Since the filtration F of the complex C is increasing, exhaustive and bounded below ($F_{-1}B = 0$), the spectral sequence converges to $H_{\bullet}(C)$ (Theorem 5.5.1.2 in [34]):

$$E_{pq}^1 = H_{p+q}(F_p C / F_{p-1} C) \implies H_{p+q}(C). \quad (5.19)$$

Thus, in order to conclude that the Hochschild homology of the algebra B is isomorphic to the Poisson homology of T , it is sufficient to prove the following.

Proposition 5.9 *The spectral sequence associated to the filtered complex C degenerates at E^1 .*

Proof. We apply a standard criterion for degeneration of spectral sequences (Lemma 5.2 in [31]) with $r = 1$ (we use the notation of [31]). This criterion consists in proving that the natural edge map

$$\phi_p^1 : H_{\bullet}(F_p C) \longrightarrow E_p^1$$

is surjective for any p . Since the term E^1 is isomorphic to the Poisson homology of T , surjectivity is given by Proposition 5.8. ■

Actually the proof of Proposition 5.8 provides an explicit section of the edge map ϕ_p^1 . Consequently we have an explicit isomorphism from $HP_{\bullet}(T)$ to $HH_{\bullet}(B)$ described as follows: Proposition 5.7 gives an explicit family (c) of Poisson cycles c such that the family $([c])$ of their classes form a basis of the space $HP_{\bullet}(T)$, and the proof of Proposition 5.8 lifts each Poisson cycle c to an explicit Koszul cycle \tilde{c} , so that the Hochschild classes $[\tilde{c}]$ form a basis of $HH_{\bullet}(B)$.

Since B is 3-Calabi-Yau (Theorem 2.9), we deduce Hochschild cohomology of B from Theorem 1.1: $HH^{\bullet}(B) \cong HH_{3-\bullet}(B)$.

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