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# Sensitivity and Uncertainty Quantification for jet stability analysis

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## Abstract:

*The paper is twofold. It aims first to validate and compare adjoint-based sensitivity and other sensitivity methods and their possible relation to Uncertainty Quantification analysis. These methods, illustrated on a simple toy-models can be a lower cost tool for mathematical analysis of complex problems of industrial interests. The second objective is to propose an Uncertainty Quantification of the compressible single stream jet stability subjected to frequency variation. The governing equations are the low cost model called Parabolized Stability Equations. This objective is strongly related to the noise sensitivity and control studies since it has been demonstrated that noise is originated from K-H instability in such a flow. Later, the adjoint PSE approach will be used, to make the link between sensitivity and UQ analysis as this quantification for Large Eddy Simulations or Reynolds Average Navier-Stokes simulations can be considerably expensive. Envelope curves of the standard deviation are determined and compared with good agreement for small variations of the input parameters in the first toy model. For the jet stability, it has been found that the growth rate is almost insensitive to small frequency variations and, on the other hand, the phases of the amplitude functions of the disturbance are extremely sensitive to frequency.*

## Résumé:

*Le papier comporte deux objectifs. Le premier consiste à valider et à comparer, sur un modèle simple, diverses approches de sensibilité et de relier les résultats à ceux obtenus par une quantification d'incertitude. Les approches par sensibilité, numériquement moins coûteuses peuvent être une bonne alternative aux analyses UQ pour des problèmes industriels plus complexe. Le second objectif est de proposer une étude de quantification d'incertitude associée aux propriétés de stabilité d'un écoulement compressible simple flux par rapport à des variations de fréquences. L'étude est basé sur le modelés simple des des*

*équations de stabilité paraboliques (PSE). Cet objectif est associé aux études de sensibilités et de contrôle des émissions acoustiques dans les jets, celles-ci ont pour origine les instabilités de type K-H. En parallèle, les équations PSE adjointes seront utilisées pour faire le lien entre la sensibilité et l'analyse UQ. Ces analyses UQ couramment employées sur des simulation de type LES et RANS s'accroissent plus coûteuse numériquement qu'une étude de sensibilité. Des courbes enveloppes à partir des variances sont déterminées et comparées avec de très bon accord et pour des petites variations des paramètres d'entrée, pour le cas du modèle de référence. Pour la partie stabilité de jet, l'analyse UQ conclut que les taux d'amplification des ondes d'instabilité sont peu sensibles à la fréquence, alors que les phases des perturbations y sont à l'opposé très sensibles.*

**Keywords: sensitivity, adjoint, uncertainty quantification, stability, PSE, aeroacoustics**

## 1 Introduction

The generation of noise has been demonstrated to originate from Kelvin-Helmholtz instabilities amplifying in the jet stream. Some noise reduction can not be achieved before a deep analysis of the sensitivity of this instability to any flow perturbations or variations. In [3, 4] an adjoint -based sensitivity of the jet flow instability subjected to any forcing has been performed. This work is based on the Linearized Parabolized Stability Equations (LPSE) and their adjoint. Results are linked to uncertainty analysis. The Linear LPSE is used to analyse the jet instability [5, 6, 8]. The base flow under consideration is a semi-analytical, axisymmetric, inviscid and low-Mach number single jet [6]. In this paper, an Uncertainty Quantification of the jet flow stability subjected to frequency variation is proposed since noise emission are related to frequency. The remaining question is how sensitivity studies can be related to an Uncertainty Quantification? To answer we first propose a toy model on which both sensitivity and UQ analysis are carried out. UQ analysis is about determining how likely the outputs of a problem under investigation are when the inputs are not precise. In the present, paper such output uncertainties are represented by the standard deviation. For cases where the determination of uncertainties is of a high interest it is recommended to also look into the probabilistic or cumulate distribution functions. On the other hand, sensitivity analysis is about apportioning the uncertainty in the output with the different input uncertainties (gradients). In this paper the gradients are calculated by four ways: i) exactly calculating the sensitivity coefficients of the governing equations with the adjoint-based approach, ii) by integrating the standard deviation ratio from

many direct computation, iii) by a sensitivity indices and iv) by a variance-based sensitivity analysis. The approach ii) and iii) can be qualified as One-factor-At-a-Time (OAT), just on component of the parameter vector is modified when the others are kept constant. In addition we present a way to perform local sensitivity using gradient of the direct equations.

In the following, a toy model is proposed as a test where the input uncertainties are set arbitrarily. This simple example wants to manifest the relation between adjoint-based sensitivity and other sensitivity methods that can also include an uncertainty analysis. The next section is an application on the stability of compressible single-jet flow.

## 2 A model toy problem

### a - ODE direct equations

The toy model is a set of two coupled ODE formulated as a classical optimization problem. Given the state vector  $\mathbf{y} = [y_1, y_2]^t \in \mathbb{C}^2$  and the control or design parameter  $\lambda = [\lambda_1, \lambda_2]^t \in \mathbb{C}^2$ , where the exponent  $t$  always stands for transpose. The goal is to get, quantitatively or qualitatively, the variation of the cost function  $E(\mathbf{y}, \lambda)$  with respect to the variation of the control parameter  $\lambda$ . Moreover the state vector  $\mathbf{y}$  and the control parameter  $\lambda$  are directly related by the constrained state equation  $F(\mathbf{y}, \lambda) = \mathbf{0}$ . The state equation of the toy model (1) includes its initial condition and is given by:

$$F(\mathbf{y}, \lambda) = \dot{\mathbf{y}}(t, \lambda) - A(\lambda)\mathbf{y}(t, \lambda) - \mathbf{b}(\lambda)u(t) = \mathbf{0}, \quad \mathbf{y}(0) = \mathbf{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (1)$$

with

$$A = \begin{bmatrix} -2I & \lambda_1 \\ \lambda_2\lambda_1^2 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 + \lambda_2 \\ \lambda_1\lambda_2^2 \end{bmatrix}$$

where  $I$  stands for the square root of  $-1$  and  $u(t)$  can be any forcing or control fixed in this simple model as a cosine function,  $u(t) = \cos(3t)$ .

The quadratic cost function (eq. 2) is defined as:

$$E(\lambda) = \frac{1}{2} \langle \mathbf{y}(t, \lambda), \mathbf{y}(t, \lambda) \rangle_t. \quad (2)$$

Given  $\mathbf{v}$  and  $\mathbf{u}$  as two arbitrary complex vectors the brackets  $\langle \mathbf{v}, \mathbf{u} \rangle_t$  indicates an integral inner product in the complex plane defined by over the time domain  $[0, T]$ :

$$\langle \mathbf{v}, \mathbf{u} \rangle_t = \int_0^T \mathbf{v}^h \mathbf{u} dt,$$

where the superscript  $h$  denotes transpose conjugate.

The solution can be integrated numerically or analytically and the stability will strongly depend on the values of  $\lambda$  and the eigenvalues of  $A$ .

In order to perform an UQ or variance-based sensitivity analysis a variation of the control parameter  $\lambda$  has to be done. We decompose each component of the vector  $\lambda$  into a reference value  $\lambda_{i0}$  and a small perturbation  $\Delta\lambda_i(\eta)$ :

$$\lambda_i = \lambda_{i0} + \Delta\lambda_i(\eta), \quad \text{with } i = 1, 2 \quad (3)$$

where the parameter  $\eta$  is defined in the range  $[-1 \ 1]$ . The small perturbation  $\Delta\lambda_i$  is defined by:

$$\Delta\lambda_i = \varepsilon_i f(\eta) \quad (4)$$

where the amplitude  $\varepsilon_i$  is a constant and  $f(\eta)$  is set as a sine or a Gaussian function.

In equation (3), and in the following  $\eta$  in  $\lambda_1$  and  $\eta$  in  $\lambda_2$  are considered as independent variables.

## b - Sensitivity analysis

A possible way to quantify uncertainties is to refer to the standard deviation of a variable under interest. The ratio between the variance of the cost function when varying one component  $\lambda_i$  of the parameter  $\lambda$ , and the standard deviation of such parameter can be written as

$$\frac{\sigma_{E_{\lambda_i}}^2}{\sigma_{\lambda_i}^2} = \frac{\int_{-1}^1 [E(\lambda_i) - \bar{E}_{\lambda_i}]^2 d\eta}{\int_{-1}^1 [\lambda_i - \bar{\lambda}_i]^2 d\eta} \quad (5)$$

The idea of using such ratio is to see whether a relation with local sensitivity exists. The over-bar denotes the mean value of the function or of the parameter. In the previous equation,  $E(\lambda_i)$  is the cost function where the component  $\lambda_i$  may vary whilst the other components are kept constant.  $\bar{E}_{\lambda_i}$  is the mean value and  $\sigma_{E_{\lambda_i}}$  is the variance when considering  $\lambda_i$  as the varying parameter.

In this paper, as a first step, the integrals over  $\eta$  are numerically computed over a constant mesh grid for  $\eta = (\eta_k)_{k=1, N}$ . To evaluate the integral of the numerator of equation (5),  $N$  computations of the ODE system (eq. (1)) and  $N$  evaluations of  $E$  are required. The ODE is solved with a fourth-order Runge-Kutta scheme.

As a second step, the sensitivity is now performed in order to get the gradients of the cost function  $E(\lambda)$  with respect to  $\lambda_i$ ,  $\frac{\partial E}{\partial \lambda_i}$ . The sensitivity  $\frac{\partial E}{\partial \lambda_i}$  is calculated by solving the adjoint of the state model, see [1, 3]. The gradients are determined with a Lagrangian functional  $\mathcal{L}$  where

$$\mathcal{L} = E - \langle \mathbf{y}^*, F(\mathbf{y}, \lambda) \rangle_t$$

As a result, the gradients and therefore the sensitivity function are the adjoint states:

$$\frac{\partial E}{\partial \lambda_i} = \langle \mathbf{y}^*, \mathbf{z} \rangle_t, \quad \text{with} \quad \mathbf{z} = \frac{\partial A}{\partial \lambda_i} \mathbf{y}(t, \lambda) + \frac{\partial \mathbf{b}}{\partial \lambda_i} u(t) \quad (6)$$

where  $\mathbf{y}^*$  is the adjoint state solution of:

$$\dot{\mathbf{y}}^*(t, \lambda) = -A(\lambda)^h \mathbf{y}^*(t, \lambda) - \mathbf{y}(t, \lambda), \quad \mathbf{y}^*(T, \lambda) = \mathbf{0} \quad (7)$$

The adjoint equations are solved in reverse time, and a terminal condition at  $t = T$  has to be set.

Only two computations of equations (1) and (7) are required to solve the sensitivity functions with the adjoint approach.

Finally, the numerical results  $\sigma_E/\sigma_{\lambda_i}$  of equation (5) and the gradients  $\frac{\partial E}{\partial \lambda_i}$  from (6) are compared and a very good agreement has been found as shown in table 1. It shows that for small uncertainties in the inputs, an adjoint-based sensitivity analysis provides the same numerical results than equation (5), which is computationally much more expensive and less exact. These results can be explained by the linearity that the response may have in such a small range. To explore such property, two more sensitivity analysis have been developed: a One-factor-At-a-Time (OAT) with sensitivity indices  $SI$  and a Variance-Based Sensitivity Analysis. Sensitivity indices provide a relatively good measure of the sensitivity of an output with respect to the variation in the inputs in the cases that the model is linear additive and the parameters are independent each other. This method also represents an inefficient sampling of the random space, as only the initial and last values of the interval are computed. The sensitivity indices can be obtained by:

$$SI_i = \frac{E(\lambda_{i_{\max}}) - E(\lambda_{i_{\min}})}{\lambda_{i_{\max}} - \lambda_{i_{\min}}} \quad (8)$$

In the literature, this coefficient  $SI_i$  is often normalized by the mean quantities :  $\bar{\lambda}_i/\bar{E}$ . In Table 1 it can be seen that they are relatively close to the ones calculated by the previous two methods. This is a sign about the linear property our model is experiencing in the small range of uncertainties.

In addition to the previous methods, a Variance-Based Sensitivity Analysis has been carried out in order to have a global understanding of the contributions of the input uncertainties. To develop such analysis, a decomposition of the variance is shown in equation (9), and sensitivity coefficients, defined differently from the previous approaches, are computed in equation (10) from its proportion with respect to the total variance.  $S_i$  and  $S_{T_i}$ , in equation (11), are the first-order and total sensitivities index respectively. In the following equations the

| Method              | $\frac{\partial E}{\partial \lambda_1}$ | $\frac{\partial E}{\partial \lambda_2}$ | $\frac{\partial E}{\partial \lambda_2} / \frac{\partial E}{\partial \lambda_1}$ |
|---------------------|---|---|---|
| Adjoint approach    | 0.27262                                 | 1.7992                                  | 6.5997  |
| OAT $\sigma$ -ratio | 0.27084                                 | 1.8006                                  | 6.6482  |
| OAT SI              | 0.20670                                 | 1.3772                                  | 6.6628  |

Table 1: Comparison among sensitivity methods.

| $\lambda_1$        | $\lambda_2$        | $\sqrt{S_2/S_1}$ |
|--------------------|--------------------|------------------|
| $S_1 = 0.0224$     | $S_2 = 0.9776$     | 6.606            |
| $S_{T_1} = 0.0228$ | $S_{T_2} = 0.9772$ | 6.547            |

Table 2: Results from Variance-Based Sensitivity Analysis.

multiple subscripts refer to second, third or higher order interactions, depending on the number of subscripts, see [10].

The results of such analysis are shown in table 2. Note that  $m$  represents the dimension of the stochastic space.

$$\sigma_E^2 = \sum_{i=1}^m \sigma_{E\lambda_i}^2 + \sum_{i=1, j>i}^m \sigma_{E\lambda_{ij}}^2 + \sum_{i=1, k>j>i}^m \sigma_{E\lambda_{ijk}}^2 + \dots \quad (9)$$

$$1 = \sum_{i=1}^m S_i + \sum_{i=1, j>i}^m S_{ij} + \sum_{i=1, k>j>i}^m S_{ijk} + \dots + S_{ijk, \dots, m} \quad (10)$$

$$S_{T_i} = S_i + S_{ij} + S_{ijk} + \dots + S_{ijk, \dots, m} \quad (11)$$

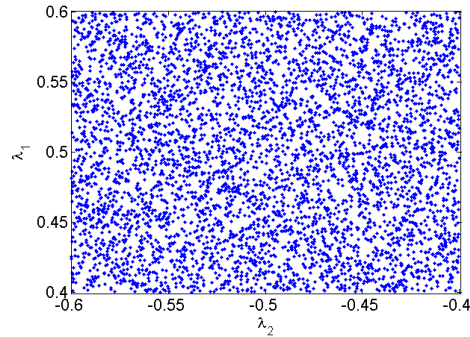


Figure 1: Latin Hypercube Sampling with 5000 samples.



The value of  $\sigma_E^2$  has been calculated by Latin Hypercube Sampling with a total of 5000 samples, as shown in Fig. 1, and it has been found that, by comparing with the data from equation (5) and with the third columns of table 1 and 2, we get

$$\sigma_E^2 \approx \sum_{i=1}^2 \sigma_{E_{\lambda_i}}^2 \quad \text{or} \quad \sqrt{\frac{S_2}{S_1}} \approx \frac{\partial E}{\partial \lambda_2} / \frac{\partial E}{\partial \lambda_1}$$

This reflects the weak interaction of the higher-order sensitivities, again due to the linearised behaviour of the model.

### c - Local sensitivity or gradient

Sensitivity based on global (integral of time or space) value is referred as a global sensitivity and in this section, local sensitivity means variation of a local (at a given time  $t$ ) quantity.

Regarding the representation of the variation of the output, the uncertainty quantification can be localized at each time  $t$ . It is usual to plot the stochastic response with its mean and its standard deviation to display envelope curves in cases where the probabilistic or cumulate distribution functions are not of a high interest.

To perform this new study, from a sensitivity point of view, we are looking for the variation of  $\mathbf{y}$  with respect to  $\lambda_i$ , that is  $\frac{\partial \mathbf{y}}{\partial \lambda_i} = \mathbf{w}_{\lambda_i}$ . In this case, we have two vectors of dimension two. The local sensitivities can be determined with the same ideas of the adjoint equation, by deriving equation (1) with respect to any  $\lambda_i$ :

$$\dot{\mathbf{w}}_{\lambda_i}(t) = A(\lambda)\mathbf{w}_{\lambda_i}(t) + \mathbf{z}, \quad \mathbf{w}_{\lambda_i}(0) = \mathbf{0} \quad (12)$$

where  $\mathbf{z}$  is the same as found in equation (6). The initial condition is null since the initial condition of the state  $\mathbf{y}$  does not depend on  $\lambda$ . This equation can be called "gradient" equation and it is not the same as the adjoint or state equations. It is then easy to write the standard deviation ratio:

$$\frac{\sigma_{y_{\lambda_i}}^{(n)}(t)}{\sigma_{\lambda_i}} = \mathbf{w}_{\lambda_i}^{(n)}(t), \quad (13)$$

where the exponent  $n$  indicates the component of the vector ( $n=1$  or  $2$  here).

The stochastic mean and standard deviation envelopes are plotted in figure 2. It is an example, where  $\lambda_1$  is fixed and  $\lambda_2$  is the varying parameter calculated by ranging  $\eta$  from  $-1$  to  $+1$  in equal step size (see equation (3)). The

graph are representing the real part, the imaginary part and the modulus of the first component ( $n = 1$ ) of the state vector  $\mathbf{y}$  solution of the ODEs (eq. 1) for a given value of  $\lambda_2$ . The black full-lines and the dash-lines are respectively the real part, the imaginary part and the modulus of  $\bar{y}_{\lambda_2}^{(1)}$  and  $\bar{y}_{\lambda_2}^{(1)} \pm \sigma_{y_{\lambda_2}}^{(1)}$ . Similar results are found with the second components of  $\mathbf{y}$  and for  $\mathbf{y}$  when  $\lambda_1$  varies.

In conclusion two types of sensitivity or variance sensitivity can be determined:

1. Sensitivity of global function ( $E$ ) or global variance sensitivity  $E_\lambda$ . One state and one adjoint equation computation are required from one hand. For UQ (and also for the presented Variance-Based Sensitivity) at minimum  $N$  state equation computations are necessary, when  $N$  is normally large and depends on the integration method (Monte Carlo, quadrature, finite difference, etc).
2. Local sensitivity or local variance sensitivity  $\mathbf{w}_{\lambda_i}^{(n)}(t)$ . Denote  $m$  as the dimension of the parameter  $\lambda$ , one state and  $m$  gradient equation computations are required on one hand. For UQ (and Variance-Based Sensitivity), a minimum of  $N$  computations of the state equation are necessary.

"Gradient" and "adjoint" approaches can be some low computational techniques to analyse sensitivity or uncertainties of a given problem subjected to varying parameters with a given distribution. These cheap approaches remain interesting as soon as the variations of the inputs remain small. For this reason, the aim of future work is to explore the possibility of using a costless model based on adjoint sensitivity theory, to apply UQ sampling techniques. This would represent a very efficient tool.

## 4 PSE and sensitivity in jet stability analysis

The approach described in the previous section is to be extended to analyse the sensitivity of the stability properties of a compressible single jet-stream flow. Usually the linear stability theory (LST) leads to some ODEs and an eigenvalue problem. The flow stability depends on the sign of the imaginary part of the eigenvalue. The PSE (for Parabolized Stability Equations [5]) approach used in that work consists on solving a set of Partial Differential Equations (PDE) mostly parabolic in the streamwise direction. Several advantages can be found in the use of the PSE. Indeed, contrary to LST where local parallel flow is assumed they take into account the small streamwise variations of the base flow and of the disturbances directly in the formulation. Since PSE are PDEs, it is simple to solve them by adding various boundary conditions and source terms. This leads to use them for receptivity and sensitivity analysis [1], in optimal flow control approaches [2] and for weakly nonlinear stability studies [5].

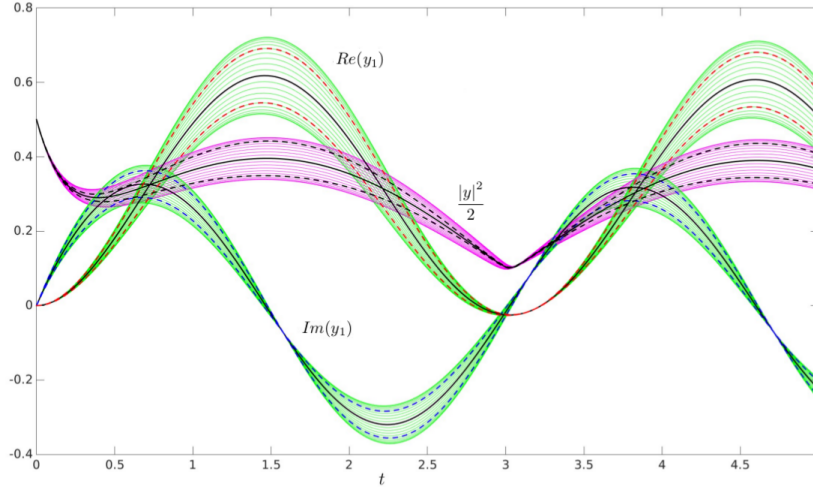


Figure 2: Mean and standard deviation envelopes of  $y_1$  and  $|y|^2/2$  with respect to a number of different values  $\lambda_2$  function of time. Dash line: from gradients (sensitivity analysis), green full line: direct computations of equation (1).

For this analysis, the PSE are derived from the Linearized Euler Equations (LEE) in cylindrical coordinates  $(x, r, \theta, t)$ . The total flow field with components respectively velocity, density and pressure, is decomposed into a base flow  $\bar{\mathbf{q}} = (\bar{u}_x, \bar{u}_r, \bar{u}_\theta, \bar{\rho}, \bar{p})^t$  and a small perturbation quantity  $\mathbf{q}' = (u'_x, u'_r, u'_\theta, \rho', p')^t$ . Any perturbation is assumed to have a like-wave exponential term  $\chi(x) \exp(i(m\theta - \omega t))$  and a  $x$ -slowly amplitude function  $\mathbf{q}(x, r)$  :

$$\mathbf{q}' = \mathbf{q}(x, r) \chi(x) \exp(i(m\theta - \omega t)), \quad \text{with} \quad \chi(x) = \exp\left[i \int_{x_0}^x \alpha(\xi) d\xi\right] \quad (14)$$

where  $\alpha(x)$  is the complex axial wavenumber. The imaginary part,  $\alpha_i(x)$ , can be defined as a growth rate and  $2\pi/\alpha_r(x)$  corresponds to a spatial wavelength.  $m$  is the fixed integer azimuthal wavenumber and the real number  $\omega$  is the fixed angular frequency of the disturbance.  $x_0$  is the inlet of the computational domain.

Substituting equation (14) into LEE we get the so-called Parabolized Stability equations (PSE):

$$L_{PSE} \mathbf{q} = \mathbf{0} \quad (15)$$

From equation (14) it can be noticed that the streamwise change of the distur-

bance can be described by the product of the amplitude function and of the exponential term. This ambiguity must be resolved by the introduction of an additional equation, called normalization or closure relationship, which imposes that the growth of the disturbance is absorbed by the wave function part of the decomposition  $\chi(x)$ , making sure that the shape function  $\mathbf{q}(x, r)$  stays slowly varying in  $x$  [5]. We set :

$$N(\mathbf{q}, x) = \int_0^\infty \bar{\mathbf{q}}^t \frac{\partial \mathbf{q}}{\partial x} r dr = 0 \quad (16)$$

Where  $t$  denotes transpose and over-bar, complex conjugate. Equations (15) and (16) are solved using a streamwise marching solution starting from the initial condition at  $x = x_0$  which is the solution of the local approach (LST). A compressible PSE solver designed by the ONERA stability team [6] has been used here.

Then the adjoint of PSE has been written specifically to investigate flow sensitivities to any disturbances of non-viscous jet and to define some new noise control strategies. Similarly to PSE, a PDE and a closure relation are found and solved. As usual, some sources terms are coming from the PSE solution and from the definition of the output. This output is a quadratic integral observation functional  $J$  defined from the perturbation field along the computational domain or only at the outlet of the computational domain. Adjoint PSE has been validated in case of subsonic single stream jet excited with a number of different external (source) forcing [3]. It is easy to demonstrate that adjoint PSE can be extended to sensitivity analysis of any parameters. The sensitivity with respect to the frequency  $\omega$  is investigated in the following. The result is the sensitivity function:

$$\frac{\partial J}{\partial \omega} = S_\omega(\bar{\rho}, M, \alpha, \mathbf{q}, \mathbf{q}^*)$$

where  $M$  is the Mach number and  $q^*$  is the solution of the adjoint PSE.

Similarly to the approach used in equation (12), it is possible to compute  $d\alpha/d\omega$ , indeed, equations (15) and (16) can be derived with respect to  $\omega$  and local sensitivity  $\mathbf{q}_\omega$  can be found.

$$L_{PSE} \mathbf{q}_\omega = -\frac{\partial L_{PSE}}{\partial \omega} \mathbf{q} = A_0 \mathbf{q}, \quad \int_0^\infty \left[ \bar{\mathbf{q}}_\omega^t \frac{\partial \mathbf{q}}{\partial x} + \bar{\mathbf{q}}^t \frac{\partial \mathbf{q}_\omega}{\partial x} \right] r dr = 0 \quad (17)$$

The numerical computation of equation (17) is currently in progress. In the following the outputs of the UQ analysis ( $\alpha, \mathbf{q}$ ) are referred as  $\Phi$ .

## 5 PSE and UQ in jet stability analysis

In this section, an UQ study is performed differently from section 1 and 2, since only the PSE are solved and considered as a black-box with uncertain input and outputs.

The input uncertainty associated to the frequency  $\omega$  has been modelled by an Uniform Probabilistic Distribution and then the Stochastic Collocation Method (SCM) with a Clenshaw- Curtis (C-C) Sparse Grid [9] has been implemented. SCM was developed by Mathelin and Hussaini [7] to improve the high costs of the Galerkin Polynomial Chaos method with non-linear equations. For each collocation point, the CFD problem is solved deterministically, and the solution can be constructed as an expansion:

$$\Phi(\mathbf{x}, \beta) \approx \sum_{i=1}^N \Phi_i(\mathbf{x}) \ell_i(\beta) \quad (18)$$

where  $\Phi_i(\mathbf{x})$  with  $\mathbf{x} = (x, r)$  stands for the deterministic solutions and  $\ell_i$  are the Lagrange interpolant polynomials. Statistical moments can be obtained by applying quadrature rules and SCM represents a very efficient option for lower dimension problems in comparison with sampling techniques such as Monte-Carlo. For higher dimension problems, sampling techniques use to be more suitable.

In this paper, the collocation points of the sparse grid have been determined according to the C-C quadrature nested rule. Special attention must be paid in the Probabilistic Density Function of the random variable,  $\xi \in \Xi$ , as we have to perform a mathematical transformation from the physical random variable space to an artificial stochastic space, called  $\alpha$ -domain but here referred to as  $\beta$ -domain ( $\beta = \mathcal{S}_\xi(\xi)$ ), as  $\alpha$  is used for the wavenumber. This transformation is an important difference with respect to other UQ methods.

Regarding the source of uncertainty, it has been based on a 10% of variation of the frequency, whose deterministic base value is imposed to  $\bar{\omega} = 1.2\pi$ , the mean value. An uniform distribution  $\omega \sim U(-0.1\bar{\omega}, 0.1\bar{\omega})$  is set.

In Figure 3, the mean and standard deviation of the real and imaginary parts of  $\alpha = \alpha_r + i \alpha_i$ , the complex axial wavenumber, is plotted for 33 Clenshaw-Curtis quadrature nodes. It can be observed that the imaginary part is not very sensitive with respect to the uncertain frequency in long distances, whilst the real one is very slowly changing along the  $x/D$  axis of the jet. In conclusion a small uncertainty on the frequency wave will modify the wavelength of the sound wave but the instability (growth rate) of the flow will remain almost unchanged.

In Figure 4, the mean value and envelopes of the real part of the amplitude function of the streamwise and radial velocity are displayed for  $r/D_j = 0.51$ . The uncertainty on the frequency generates strong variation on the phases of the amplitude functions, so that the real or imaginary parts are very sensitive to frequency variations. This results have been confirmed by the sensitivity analysis based on adjoint PSE.

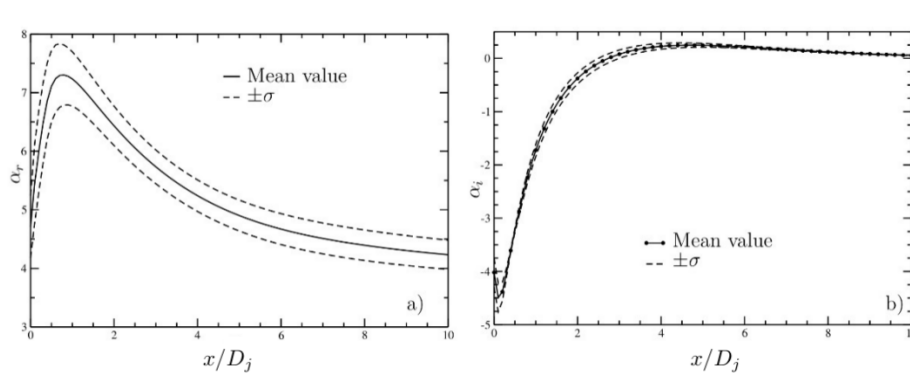


Figure 3: Mean and standard deviation envelopes of the wavenumber  $\alpha_r$  (a) and of the growth rate  $\alpha_i$  (b).

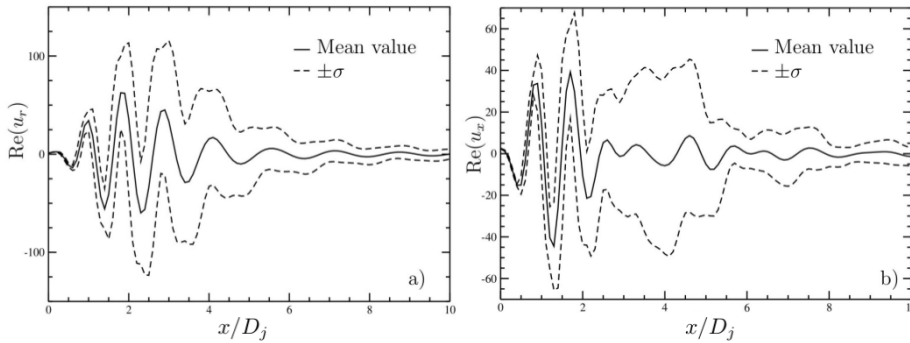


Figure 4: Mean and standard deviation envelopes of the real part of the amplitude functions  $u_r$  (a) and  $u_x$  (b),  $r/D_j = 0.51$ .

## 6 Conclusions

The relationship between a Uncertainty Quantification (standard deviation ratio) and an adjoint-based sensitivity analysis has been demonstrated, first on a toy ODE model and then on a PDE stability problem. The use of adjoint equations required few computations compared to some general usual UQ method, since if  $m$  is the number of parameters with uncertainties, only  $m + 1$  computations are necessary to get the standard deviation and the envelope curves. It has also been noticed that an uncertainty frequency wave modifies the wavelength of the sound wave but the growth rate will remain insensitive.

The next step will be to perform a propagation of instability by taking into account uncertainties of the mean flow into the stability of single- and double-stream jets and to propagate the uncertainties to the far-field noise emission.

Also, as here has been demonstrated that the sensitivity coefficients can be quickly obtained by the adjoint approach, the next step is to explore the possibility of quantify uncertainties by using a costless surrogate model based on the exactly calculated adjoint-based sensitivity coefficients.

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