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# A power plant valuation under an asymmetric risk criterion taking into account maintenance costs\*

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## Abstract

Power producers are interested in valuing their power plant production. By trading into forward contracts, we propose to reduce the contingency of the associated income considering the fixed costs and using an asymmetric risk criterion. In an asymptotic framework, we provide an optimal hedging strategy through a solution of a nonlinear partial differential equation. As a numerical experiment, we analyze the impact of the fixed costs structure on the hedging policy and the value of the assets.

**Keywords:** Hedging, asymmetric risk, fully nonlinear PDE, cost management, power plant.

**MSC2010:** Primary 60H30, 91G60; secondary 91G80.

## 1 Introduction

**Background** After the deregulation process of energy markets in early 1990, several spot and futures exchanges were established to trade an almost non-storable commodity, the electricity. Simultaneously, power producers faced the problem of valuing their power plants, whose future production depends on the electricity spot price.

In order to diminish the randomness of future earnings, practitioners generally use some discrete hedging strategies involving some forward contracts (see for example [Ai15] or [BGS14]). Here, we adopt a local asymmetric risk criterion distinct from the Black-Scholes (BS) approach. The aim of this approach is to control the distribution of the *residual risk*<sup>1</sup>, by penalizing the losses and considering the plant fixed costs.

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<sup>1</sup>In the following, residual risk is also called P&L (Profit and Loss).

**Residual risk** Here, we use an asymmetric risk criterion to define optimal pricing and hedging policies in an asymptotic framework (for more details, see [GPW18]). Power plant owners perform discrete hedging at trading dates  $t_0 = 0 < t_1 < \dots < t_N = T$  (for some  $N \geq 1$ ), using a hedging security  $X$  to reduce the risk associated to a contingent income  $G_T$  at the time  $T$ . This discrete hedging produces a *residual risk*  $\mathcal{E} = G_T - \sum_{n=0}^{N-1} \delta_{t_n} (X_{t_{n+1}} - X_{t_n})$ , where  $\delta_t$  denotes the number of units invested in the hedging security at time  $t$ . In the following, we consider the hedging security  $X$  is the forward contract with delivery time  $T$ .

Because the power producers perform a low-frequency hedging (once a week or twice a month, for example), they search to find the pricing and hedging strategies<sup>2</sup>  $(V, \delta)$  considering the *local balance*  $\mathcal{E}_n = V_{t_{n+1}} - V_{t_n} - \delta_{t_n} (X_{t_{n+1}} - X_{t_n})$ ,  $0 \leq n \leq N - 1$ , through *risk function*  $\ell$ . In this setting, we perform the asymptotic analysis of an integrated measure  $R^N[V, \delta] = \sum_{n=0}^{N-1} \mathbb{E}[\ell(\mathcal{E}_n)]/\Delta t$ .

Notice that power producers search to obtain the most concentrated residual risk distribution possible (perceived through the P&L histogram). Concurrently, they are averse to loss scenarios, which are locally described by negative local balance  $\mathcal{E}_n < 0$ . Looking to satisfy those preferences, we choose as risk function<sup>3</sup> the function  $\ell_\gamma$  such that  $\ell_\gamma(x) := (1 + \gamma \operatorname{sgn}(x))^2 x^2/2$ ,  $x \in \mathbb{R}$ , for any  $\gamma \in (-1, 0)$ , where the sign function is defined as  $\operatorname{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x<0}$ ,  $x \in \mathbb{R}$ . Here, the parameter  $\gamma$  corresponds to the producer aversion to extreme local losses (see Figure 1). When the risk aversion parameter  $\gamma$  gets closer to minus one, the probability of extreme local losses is diminished compared to  $\gamma$  equals to zero. In a residual risk perspective, this reduction in extreme local losses raises the likelihood of small losses globally.

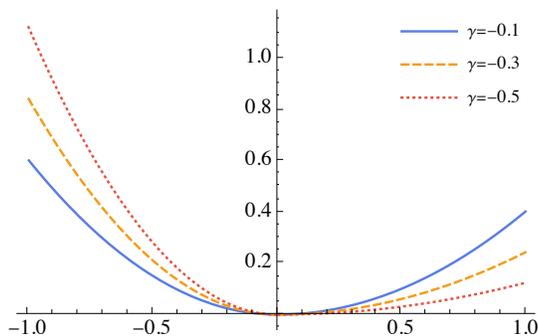


Figure 1: Risk level as a function  $\ell_\gamma$  of the local balance  $\mathcal{E}_n$  for different values of  $\gamma$ . For  $\gamma \in (-1, 0)$ , the risk function  $\ell_\gamma$  weights further the negative local balances than the positive ones.

**Fixed costs** Here, we consider that a power plant generates some fixed costs, including *maintenance* and *investment* costs, whether or not it is producing electricity. In an accounting perspective, the power producers need to assign in their balance book a price to the contingent income generated by their future power plant taking into consideration

<sup>2</sup>The pricing process  $V$  satisfies the replication constraint  $V_T = G_T$ .

<sup>3</sup>Due to the power producers' preferences, we need an almost-quadratic function  $\ell$  weighting more negative arguments. For simplicity, we take the same function  $\ell_\gamma$  as in [GPW18] with negative risk aversion parameter  $\gamma$ .

those costs. For this purpose, they can use the underlying forward market as a mark-to-market way of reducing the contingency of  $G_T = g(X_T)$  by minimizing an integrated measure, similarly to  $R^N[V, \delta]$ , with the asymmetric risk (AR in short) function  $\ell_\gamma$ .

To include the *staggered costs*<sup>4</sup>, we suppose that it depends on the price of forward contract, seen as the best proxy of the future electricity spot price, through a function  $C$ . Then, by taking equidistant trading dates  $t_n = n\Delta t$  with time step  $\Delta t = T/N$ , we consider  $C(X_{t_n})\Delta t$  as the fixed costs associated to the trading period  $[t_n, t_{n+1})$  and subtract it from the local balance  $\mathcal{E}_n$ . In this setting, we perform the asymptotic analysis of the following integrated measure

$$R^N[V, \delta] = \frac{1}{\Delta t} \sum_{n=0}^{N-1} \mathbb{E}[\ell_\gamma(V_{t_{n+1}} - V_{t_n} - \delta_{t_n}(X_{t_{n+1}} - X_{t_n}) - C(X_{t_n})\Delta t)]. \quad (1)$$

**Cost models** Here, we provide two economic points of view to justify the dependence of staggered costs on the future electricity spot price.

First, we use as argument the large usage of the power plant when the electricity spot is expensive. Indeed, an elevated price is usually triggered by a raised demand matching a larger production. This means we have to employ the power plant more frequently. This is the reason why we assume an increasing dependence of staggered costs on the future spot price level. This is typically the case of thermal plants that are mainly used for high demands and high prices.

Second, for some plants, fixed costs related to maintenance are seasonal and greater during the power plant refueling stop. Also, the producer would rather stop its power plant over the period<sup>5</sup> where the demand and the spot price are low. In this sense, the staggered costs and the future spot price are negatively correlated and this is why we consider a decreasing dependence of staggered costs on the future spot price level. This is typically the case of nuclear power plants.

**Nuclear plant example** For an illustration purpose, we estimate fixed costs for a nuclear plant. Following the World Energy Outlook (WEO) 2016 assumptions<sup>6</sup> (for the WEO report, see [IEA16]), we consider

- an *maintenance cost* of \$170 per kW per year;
- an *investment cost* of \$6600 per kW;
- an *exchange rate* at 0.90 €/€ (for 2015 US dollar).

Building a nuclear plant with installed capacity of 1 GW costs \$6.6 billion (or 5.9 billion €). Also, its owner spends \$170 million per year (or 153 million €) in maintenance services.

To soften the financial effort, the owner spread those fixed costs over its power plant lifespan of 20 years. Thus, we obtain the amount of fixed costs per unit (MW)

<sup>4</sup>First, we estimate the total fixed costs related to investment and maintenance services. Then, we spread those costs uniformly over the expected lifespan of the power plant, and then estimate the fixed costs per hour of delivery. By smoothing these fixed costs per hour uniformly over the hedging period, we get the staggered costs.

<sup>5</sup>In France, this occurs in the summer rather than in the winter.

<sup>6</sup>See also the database in the *More Information on...* section and *Investment in power generation* subsection of the World Energy Model website: [www.iea.org/weo/weomodel/](http://www.iea.org/weo/weomodel/).

of installed capacity and per unit of lifetime (hour) equals to 51.40€. For every MW of installed capacity, the owner should pay 33.90€ each hour to refund the investment credit and 17.50€ to maintain its installations.

We considered that interest rates are equal to zero and the investment cost is spread evenly over the power plant lifetime. By looking at the fixed costs generated by a power plant over a 24-hour period, we get 1233.60€ for each MW of installed power. Planning to hedge a one-day production a year in advance leads us to spread that cost over 8760 hours and to get the so-called staggered costs of 0.14€ per unit of installed capacity (MW) per unit of hedging time (hour).

**Contributions** In this paper, we propose an approach for the asymptotic analysis of  $R^N$  in (1) using a nonlinear pricing PDE (partial differential equation). Consider a reference pricing rule  $v$  obtained by the power producer, for example, when no fixed costs are taken into consideration. With this exogenous valuation in hand, the power producer builds his local pricing and hedging rule dependent on  $v$ . In fact, the power producer policy is parameterized by a function  $f$ , modeling a pricing kernel, and is given as follows: the pricing (resp. hedging) rule is (resp. the first derivative of) the solution to the  $f$ -PDE with  $v$  as local terminal condition (see Definition 1). After obtaining the limit  $R[v, f]$  of the integrated measure  $R^N$  (see Theorem 1) given the power producer rule, we minimize the functional  $f \mapsto R[v, f]$  to obtain  $\mathbf{f}$  parameterizing the optimal policies: the pricing (resp. hedging) rule is (resp. the first derivative of) the solution to the  $\mathbf{f}$ -PDE (see (5)) with the payoff function  $g$  as terminal condition.

For the numerical experiments, we take staggered costs depending on the spot price level through convex and concave functions. Also, we compute the pricing and hedging<sup>7</sup> policies associated to the AR valuation in the presence of staggered costs  $C$ . At the end, we are interested to understand some important aspects of this new physical asset valuation approach. How the AR parameter  $\gamma$  impacts the P&L distribution? How does the optimal hedging policy is affected by the staggered costs? As numerical result (see section 3), we observe that  $\gamma$  allows a change in the distribution shape: we are able to distort the P&L into an asymmetric and fat-tailed distribution.

This paper is structured as follows. In section 2, we present the stochastic setting and the optimal pricing PDE. In section 3, we provide two models for the staggered costs and discuss most numerical results. In appendix A, we gather the assumptions and the main theoretical result. In appendix B, we show how to deduce the optimal kernel. Finally, appendix C summarizes important features of the proof of the main result.

## 2 An asymmetric risk valuation with cost management

**Methodology** In view of studying the integrated risk  $R^N$  in the asymptotic regime  $N \rightarrow +\infty$  using the AR function  $\ell_\gamma$ , we follow an approach by PDE valuation. Because the risk function is non quadratic, one expects to deal with nonlinear PDE. This PDE approach is possible if one assumes that the hedging instrument is Markovian, solution to a stochastic differential equation (SDE), which is our standing assumption from now. For simplicity, we assume that the interest rate is zero and that there exists only one risk-neutral measure given by the *probability measure*  $\mathbb{P}$  supporting

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<sup>7</sup>This corresponds to the value/derivative of the numerical solution to the  $\mathbf{f}$ -PDE.

a Brownian motion<sup>8</sup>  $W$ : the hedging instrument we consider is the forward contract  $X := F^T = \{X_t = F_t^T : t \in [0, T]\}$  with (fixed) delivery time  $T$ , which dynamics is given by the (one-factor Gaussian model) SDE<sup>9</sup>

$$dF_t^T = \alpha(t, T)F_t^T dW_t, \quad 0 \leq t \leq T,$$

where the volatility function  $t \in [0, T] \mapsto \alpha(t, T)$  is a deterministic, strictly positive, continuous function<sup>10</sup>. Whenever necessary, we recall that the electricity spot price at the time  $t$  is related to the forward price with same delivery, i.e.,  $F_t^t$ .

As discussed in [Pim18] and [GPW18], the optimal management strategy is designed in two steps. First we consider an exogenous valuation process  $V_t = v(t, X_t)$  through some function  $v$  smooth enough<sup>11</sup>. The value function  $v$  serves as a reference for the power producer to build locally his/her hedging strategy (with asymmetric risk and maintenance cost). So far this is an exogenous value function but later, it will be made endogenous and aligned with the optimal hedging rule.

For the second step, the trick for seeking the optimal hedging strategy is to parameterize the pricing and hedging of the power producer through a function  $f$  representing a pricing/hedging kernel. The value function for the power producer is called  $f$ -PDE: this approach encompasses all nonlinear Markovian pricing rule (see for instance [Cre13, GHL13]). The most basic example is the usual risk-neutral PDE, which corresponds to  $f(\cdot) \equiv 0$ . To formalize the general concept, we state a definition.

**Definition 1.** *Let  $f$  be a continuous function. For a terminal time  $\tau > 0$  and a continuous terminal condition  $v(\tau, \cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ , we denote by  $u^\tau : [0, \tau] \times \mathbb{R}^+ \rightarrow \mathbb{R}$  the solution to the  $f$ -PDE*

$$\begin{cases} u_t^\tau + \frac{1}{2}\alpha^2(t, T)x^2 u_{xx}^\tau + f(t, x, u^\tau, u_x^\tau, u_{xx}^\tau) = 0, & \text{for } t < \tau, x \in \mathbb{R}^+, \\ u^\tau(\tau, x) = v(\tau, x). \end{cases} \quad (2)$$

Now, we can proceed to the optimization. Say that there are  $N \in \mathbb{N}$  trading periods of same size on the interval  $[0, T]$ , and thus described by the times  $\{0 = t_0 < t_1 < t_2 < \dots < t_N = T\}$  with  $t_n = n\Delta t$  and  $\Delta t = T/N$ . Given that the maintenance costs for the period  $[t_n, t_{n+1}[$  are  $c(t_n, X_{t_n})\Delta t$ , the local balance of the power producer (using the reference valuation  $v$  and the pricing  $f$ -PDE) is

$$\mathcal{E}_n = \underbrace{v(t_{n+1}, X_{t_{n+1}})}_{u^{t_{n+1}}(t_{n+1}, X_{t_{n+1}})} - u^{t_{n+1}}(t_n, X_{t_n}) - u_x^{t_{n+1}}(t_n, X_{t_n})(X_{t_{n+1}} - X_{t_n}) - c(t_n, X_{t_n})\Delta t.$$

<sup>8</sup>Mathematically speaking, we deal with a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  endowed with the *natural Brownian filtration* completed by  $\mathbb{P}$ -null sets.

<sup>9</sup>The restriction to a one-dimensional Gaussian model is made for sake of simplicity, more general models can be handled as in [Pim18] which describes a framework including the current one.

<sup>10</sup>A standard choice, coherent with the Samuelson effect, is  $\alpha(t, T) = \alpha_0 e^{-\alpha_1(T-t)}$  for some constants  $\alpha_0 > 0$  and  $\alpha_1 \geq 0$  (see Figure 3).

<sup>11</sup>For the reader careful about the precise definition of function spaces, we recall that the set  $C^{1,2}([0, T] \times \mathbb{R})$  denotes the set of functions  $\phi : [0, T] \times \mathbb{R}$  such that the partial derivatives  $\phi_t, \phi_x, \phi_{xx}$  exist and are continuous; furthermore,  $\phi$  is in  $C_{\text{loc, pol}}^{1/2, 1}$  if

$$\|\phi\|_{C_{\text{loc, pol}}^{1/2, 1}} := \sup_{t \neq t' \in [0, T]} \sup_{x \neq x' \in \mathbb{R}} \frac{\|\phi(t, x) - \phi(t', x')\|}{(|t - t'|^{1/2} + |x - x'|)(1 + |x|^\alpha + |x'|^\alpha)} < +\infty$$

for some real  $\alpha \geq 0$ .

The related *conditional risk* (according to the risk function  $\ell_\gamma$ ) is defined by

$$\Delta R_n := \mathbb{E} \left[ \ell_\gamma(\mathcal{E}_n) \middle| \mathcal{F}_{t_n} \right], \quad (3)$$

and the *integrated conditional risk* by

$$R^N := \frac{1}{\Delta t} \sum_{n=0}^{N-1} \mathbb{E}[\Delta R_n].$$

So far, this depends on  $f$  and  $v$ . Then in order to complete the optimization, the determination of the optimal strategy consists in

- a) taking the limit of  $R^N$  as the number  $N$  of trading dates goes to infinity
- b) minimizing the limit over all kernel  $f$ . Denote by  $\mathbf{f}$  the minimizer.

Taking for  $v$  the solution of the  $\mathbf{f}$ -PDE (on the interval  $[0, T]$  with terminal payoff  $g$ ) gives our optimal risk management with an explicit pricing and hedging strategy based on local asymmetric risk. By choosing this pricing rule, the power producer will have a time-consistent way to hedge locally his/her risk, accounting for asymmetric criterion.

**Optimal pricing PDE** Following the analysis in appendix B, we deduce that the optimal kernel  $\mathbf{f}$  is given by

$$\mathbf{f}(t, x, y_1, y_2, y_3) = \alpha^2(t, T)x^2 \left( \mathbf{a}_+ y_3^+ - \mathbf{a}_- y_3^- \right) - c(t, x), \quad (4)$$

for any  $(t, x, y_1, y_2, y_3) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^3$ . The optimal constants  $\mathbf{a}_+$  and  $\mathbf{a}_-$  depends on  $\gamma$  and is given as the solution to the equation in (14). For numerical purposes, we report, in Table 1, the numerical values of  $\mathbf{a}_+$  and  $\mathbf{a}_-$  for a few  $\gamma$ , calculated by a root finding algorithm (using the Mathematica `FindRoot` function).

Table 1: Optimal constants  $\mathbf{a}_+$  and  $\mathbf{a}_-$  as a function of the parameter  $\gamma$

Parameter $\gamma$	Constant $\mathbf{a}_+$	Constant $\mathbf{a}_-$
0.0	0.0000	0.0000
-0.1	-0.0901	0.1044
-0.2	-0.1684	0.2262
-0.3	-0.2366	0.3702
-0.4	-0.2960	0.5434
-0.5	-0.3476	0.7567

We are now in a position to define the pricing and hedging policies, used by the power producer. If  $g$  denotes the future income payoff due to the power plant production, and using the optimal kernel  $\mathbf{f}$  in (4), the pricing and hedging process are given by

$$V_{t_n} = \mathbf{v}(t_n, X_{t_n}), \quad \delta_{t_n} = \mathbf{v}_x(t_n, X_{t_n}),$$

where  $\mathbf{v}$  is the solution to the  $\mathbf{f}$ -PDE

$$\begin{cases} \mathbf{v}_t + \frac{1}{2}\alpha^2(t, T)x^2 \mathbf{v}_{xx} + \mathbf{f}(t, x, \mathbf{v}, \mathbf{v}_x, \mathbf{v}_{xx}) = 0, & \forall t < T, x \in \mathbb{R}^+, \\ \mathbf{v}(T, x) = g(x). \end{cases} \quad (5)$$

Notice that  $v$  depends on the parameter  $\gamma$  and on the staggered cost function  $c(t, x)$ , through the optimal kernel  $\mathbf{f}$ . In the case of symmetric risk, i.e.,  $\gamma = 0$ , note that  $\mathbf{f}(t, x, v, v_x, v_{xx}) = -c(t, x)$  and the PDE becomes linear. In all other cases, the pricing PDE is nonlinear.

### 3 Numerical experiments

In this section, we show the pricing and hedging strategies of the power producer in the presence of fixed costs. We spread those costs over the hedging period to obtain the staggered costs. Then we assume it depends on the forward price according to the following models.

#### 3.1 Staggered cost models

Here, we consider convex-concave models. We first introduce increasing costs as a continuous, differentiable function of the forward price, then we examine decreasing costs described as a piecewise convex function of the forward price.

Let the staggered costs be written as  $C(x) = C_{\text{invest}}(x) + C_{\text{maint}}(x)$  such that

$$C_{\text{invest}}(x) = C_0 \text{ and } C_{\text{maint}}(x) = C_1 h(x/C_2),$$

where  $C_0$ ,  $C_1$  and  $C_2$  are real constants such that  $C_0 \geq 0$ ,  $C_1 > 0$  and  $C_2 > 0$ . Here,  $C_0$  (resp.  $C_1$ ) represents the investment (resp. maintenance) component in the staggered costs (resp. until the price level  $C_2$ ). First, we propose the increasing functions

$$h_0(x) = x^\beta \text{ and } h_1(x) = x^{1/\beta}, x \in \mathbb{R}^+, \quad (6)$$

which are respectively convex and concave for  $\beta > 1$  (see Figure 2a). Then we suggest the decreasing function

$$h_2(x) = e^{-x}, \quad (7)$$

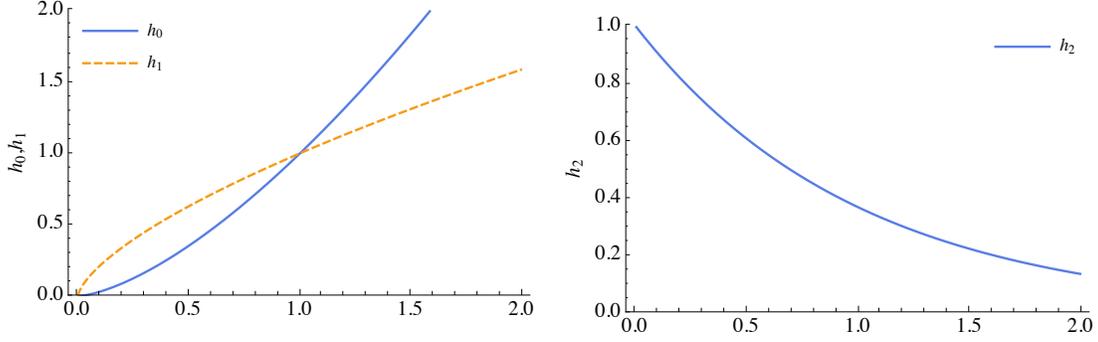
which is continuous, differentiable convex (see Figure 2b). In the following, the real function  $h$  will be either increasing (i.e.,  $h_0$  or  $h_1$ ) or decreasing (i.e.,  $h_2$ ), modeling the dependency on the forward price level.

#### 3.2 Numerical scheme

In the following, we aim to evaluate the power producer rules through the solution  $v$  to the  $\mathbf{f}$ -PDE<sup>12</sup>. To solve this nonlinear PDE, we approximate the second-order derivative by a second-order finite difference evaluated at different times depending whether it belongs or not to the linear part of the equation.

**Centered Time Centered Space method** Here, we reduce the partial differential equation to a system of algebraic equations using a grid  $\{0 = t_0 < t_1 < \dots < t_N = T\}$  in time and  $\{x_0 < x_1 < \dots < x_J\}$  in space defined by the step sizes  $\Delta t$  and  $\Delta x$  (for

<sup>12</sup>Replacing the positive part and negative part by the absolute value in the expression  $\mathbf{f}$  in (4) turns the PDE (5) into the Black-Scholes-Barenblatt (BSB) equation (for more details, see [ALP95]).



(a) Increasing costs  $h_0$  and  $h_1$  for  $\beta = 1.5$ . Around  $x = 1$ , the convex function  $h_0$  presents a steep slope while the concave function  $h_1$  is flatter. (b) Decreasing costs  $h_2$ . At  $x = 0$ , the convex function  $h_2$  and its first derivative take a finite value.

Figure 2: Increasing and decreasing functions modeling the staggered costs.

time and space, respectively). Therefore, we use the backward difference at time  $t_n$  for the linear part and the forward difference at time  $t_{n+1}$  for the kernel part to get

$$\begin{aligned} \frac{\mathbf{v}(t_{n+1}, x_j) - \mathbf{v}(t_n, x_j)}{\Delta t} + \frac{1}{2}\alpha^2(t_n, T)x_j^2 \mathbf{v}_{xx}(t_n, x_j) \\ + \mathbf{f}(t_{n+1}, x_j, \mathbf{v}(t_{n+1}, x_j), \mathbf{v}_x(t_{n+1}, x_j), \mathbf{v}_{xx}(t_{n+1}, x_j)) = 0. \end{aligned}$$

By taking  $u_j^n = \mathbf{v}(t_n, x_j)$ , we use a central space approximation

$$\delta u_j^n := \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}, \quad \delta^2 u_j^n := \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2},$$

to get

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2}\alpha^2(t_n, T)x_j^2 \delta^2 u_j^n + \mathbf{f}(t_{n+1}, x_j, u_j^{n+1}, \delta u_j^{n+1}, \delta^2 u_j^{n+1}) = 0.$$

Finally, replacing the optimal kernel  $\mathbf{f}$  in (4) leads to

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2}\alpha^2(t_n, T)x_j^2 \delta^2 u_j^n + \frac{1}{2}\alpha^2(t_{n+1}, T)x_j^2 (\mathbf{a}_+ - \mathbf{a}_-) |\delta^2 u_j^{n+1}| \\ + \frac{1}{2}\alpha^2(t_{n+1}, T)x_j^2 (\mathbf{a}_+ + \mathbf{a}_-) \delta^2 u_j^{n+1} - c(t_{n+1}, x_j) = 0, \end{aligned}$$

where  $\mathbf{a}_+$  and  $\mathbf{a}_-$  are the real constants depending on  $\gamma$  (see Table 1). By defining the map  $r : (t, x) \mapsto \frac{\Delta t}{2\Delta x^2} \alpha^2(t, T)x^2$  and taking  $r_j^n = r(t_n, x_j)$ , we obtain the following system of algebraic equations:

$$\begin{aligned} -r_j^n u_{j+1}^n + (1 + 2r_j^n) u_j^n - r_j^n u_{j-1}^n = u_j^{n+1} + r_j^{n+1} (\mathbf{a}_+ - \mathbf{a}_-) |u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}| \\ + r_j^{n+1} (\mathbf{a}_+ + \mathbf{a}_-) (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) - c(t_{n+1}, x_j) \Delta t. \end{aligned}$$

In conclusion, we set  $u_j^N = g(x_j)$ , for  $j \in \{0, \dots, J\}$ , and solve the previous system for the vector  $u^n = (u_0^n, \dots, u_J^n)^\top$  backward in time using non-convexity condition at the boundaries:  $u_0^n = 2u_1^n - u_2^n$  and  $u_J^n = 2u_{J-1}^n - u_{J-2}^n$ , for  $n \in \{N-1, \dots, 0\}$  (see [Mey06], for another example of numerical solution to BSB equations). For the volatility function, we take  $\alpha(t, T) = \alpha_0 e^{-\alpha_1(T-t)}$ ,  $0 \leq t \leq T$ , for some constants  $\alpha_0 > 0$  and  $\alpha_1 \geq 0$  (see Figure 3).

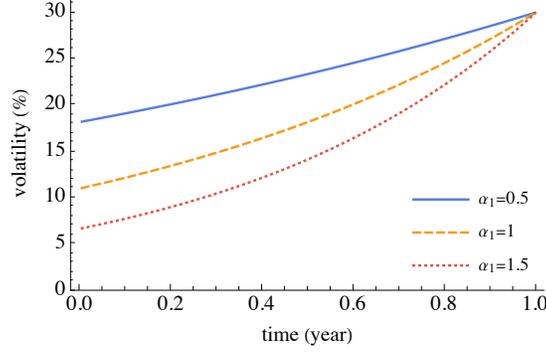


Figure 3: Volatility  $\alpha(t, T)$  increasing in time  $t$  with  $\alpha_0 = 30\%$ . For  $\alpha_1 = 0.5$ , the one-year-to-expiration volatility  $\alpha(0, T)$  is 60.65% of  $\alpha_0$ .

**Transformed optimal pricing PDE** In the analysis of the impact of the staggered costs dependency, it will be useful to have the optimal PDE (5) in terms of the initial forward price. By denoting the initial forward price  $F_0^T$  by  $x_0$ , we consider  $\tilde{v}(t, y) = v(t, y x_0)/x_0$ ,  $\tilde{c}(t, y) = c(t, y x_0)/x_0$  and  $\tilde{g}(y) = g(y x_0)/x_0$ , for any  $x_0 > 0$ . Then, we get the derivatives of  $\tilde{v}$ :

$$\tilde{v}_y(t, y) = v_x(t, y x_0) \text{ and } \tilde{v}_{yy}(t, y) = v_{xx}(t, y x_0)x_0,$$

in terms of the derivatives of  $v$ . Finally, we write the PDE (5) in  $v$  in terms of  $\tilde{v}$ :

$$\begin{cases} \tilde{v}_t + \frac{1}{2}\alpha^2(t, T)y^2 \tilde{v}_{yy} + \tilde{f}(t, y, \tilde{v}, \tilde{v}_y, \tilde{v}_{yy}) = 0, \\ \tilde{v}(T, y) = \tilde{g}(y), \end{cases} \quad (8)$$

where

$$\tilde{f}(t, y, \tilde{v}, \tilde{v}_y, \tilde{v}_{yy}) = \alpha^2(t, T)y^2 \left( \mathbf{a}_+ \tilde{v}_{yy}^+ - \mathbf{a}_- \tilde{v}_{yy}^- \right) - \tilde{c}(t, y).$$

In the following, we solve the transformed PDE to obtain  $\tilde{v}$ , then we get the optimal solution  $v$  through  $v(t, x) = x_0 \tilde{v}(t, x/x_0)$  for  $x \in \mathbb{R}^+$ . Then the pricing and hedging rules at time  $t$  are given by

$$x \mapsto x_0 \tilde{v}(t, x/x_0) \text{ and } x \mapsto \tilde{v}_y(t, x/x_0),$$

where  $\tilde{v}$  is the solution to the transformed  $\tilde{f}$ -PDE (8) and the staggered costs function  $c(t, x) \equiv C(x) = C_0 + C_1 h_i(x/C_2)$ , for  $i \in \{0, 1, 2, 3\}$ , according to section 3.1.

### 3.3 Impact on optimal policies

Here, we analyze the impact of staggered costs on the producer pricing and hedging policy. At the horizon time, power producers decide to turn their power plants on whether or not the electricity spot price is larger than  $K_0$ , representing the costs due to fuel consumption to produce electricity. The future income function  $g$  describing the producer option is equivalent to a call payoff function with strike price  $K_0$  given by  $g(X_T) = (X_T - K_0)^+$  at maturity  $T$ . In this numerical experiment, we consider the following set of parameters:

$x_0$	$K_0$	$T$	$C_0$	$C_1$	$C_2$	$\alpha_0$	$\alpha_1$	$\gamma$
40	10	1	35	15	40	0.3	0.5	-0.5

Given an initial forward price  $x_0$ , we assume that we are interested in getting the optimal policies in two weeks ( $t = 2/52 \approx 0.04$ ). Then, we observe the variation in the hedging position due to the price fluctuation of the forward price. In Figures 4-6, we show the pricing and hedging rule  $x_0 \tilde{v}(t, x/x_0)$  and  $\tilde{v}_y(t, x/x_0)$  at time  $t = 0.04$  for the call payoff  $g$  with  $T = 1$  and  $K_0 = 10$  considering different staggered costs functions. In the plot, we observe three different policies through: (a) an asymmetric risk valuation with staggered costs in full line; (b) a symmetric risk valuation with staggered costs in dashed line; (c) a symmetric risk valuation without staggered costs in dotted line. First we remark that the pricing values given by (a) and (b) are close on all tests. Then on all cases, the pricing values considering staggered costs are obviously lower than those not considering them so that price given by (c) are always lower than prices given by (a) and (b).

**Increasing convex costs** Let us consider the convex staggered costs as  $C(x) = C_0 + C_1 h_0(x/C_2)$  where  $h_0$  is defined in (6), for  $\beta = 1.5$ . Due to these convex costs, the pricing values in case (a) and (b) are concave as shown in Figure 4. Also, we obtain that hedging rules considering costs are below the hedging rules in the case of no cost. Due to the convexity of the cost leading to a decreasing function  $-C'$ , it is natural to get decreasing hedging rules for high values of  $x$  in case (a) and (b) (see analogy with the tangent process method used to estimate the sensibility of a call payoff including similar costs) leading to very different strategies from case (a).

**Increasing concave costs** Here, we consider the concave staggered costs as  $C(x) = C_0 + C_1 h_1(x/C_3)$  where  $h_1$  is defined in (6) by  $\beta = 1.5$ . As shown in Figure 5, for low prices, we need to buy a lot of hedging securities to cover the fixed costs. The concavity of the cost function gives us an increasing to zero function  $-C'$ , such that the hedging rules given by (a) and (b) is below the hedging rule given by (c) but differences between the hedging rules goes to zero as  $x$  increases.

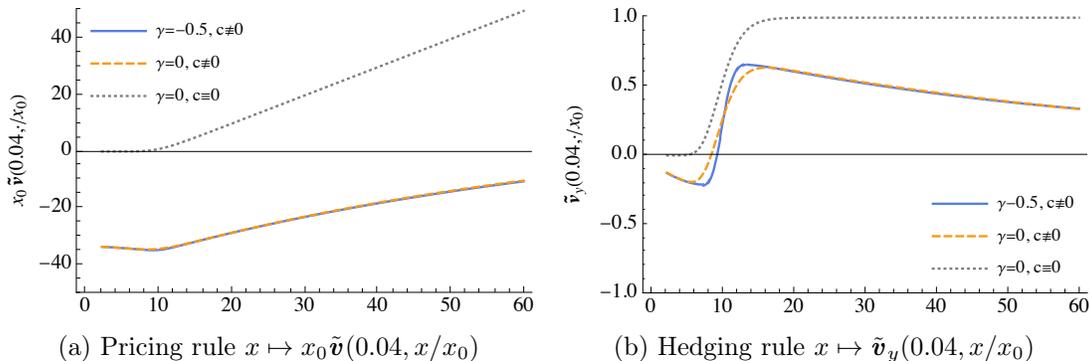


Figure 4: Optimal pricing and hedging rule for a call payoff with  $K_0 = 10$  using an increasing convex function  $h_0$  with  $\beta = 1.5$ .

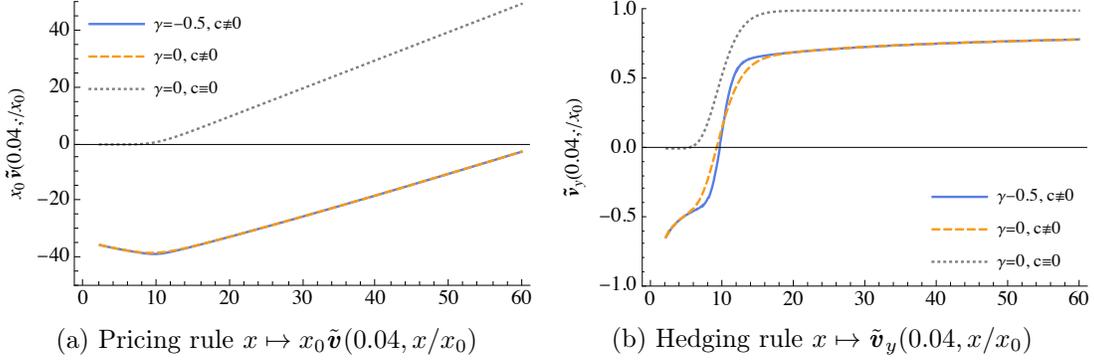


Figure 5: Optimal pricing and hedging rule for a call payoff with  $K_0 = 10$  using an increasing concave function  $h_1$  with  $\beta = 1.5$ .

**Decreasing convex costs** Now let us consider the decreasing staggered costs as  $C(x) = C_0 + C_1 h_2(x/C_3)$ , where  $h_2$  is defined in (7). In Figure 6, we observe the hedging rule considering staggered costs is greater than the hedging rule without staggered costs due to the decreasing property of function  $h_2$ . Again, we get the first derivative  $h_2'$  is almost equal to zero for very large price levels, meaning that the hedging rules will converge to each other.

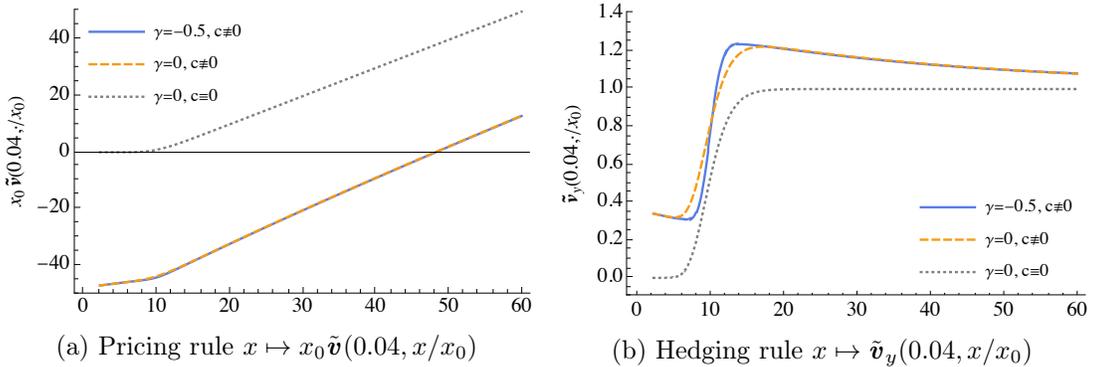


Figure 6: Optimal pricing and hedging rule for a call payoff with  $K_0 = 10$  using a decreasing convex function  $h_2$ , which is continuous differentiable.

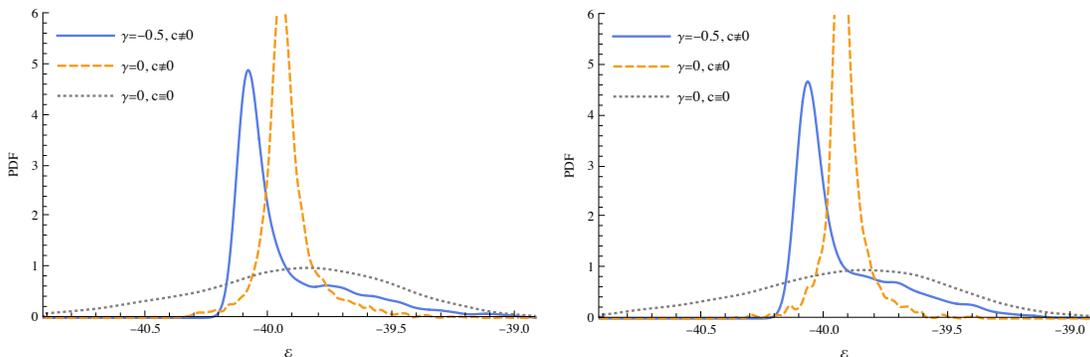
### 3.4 Impact on the P&L distribution

Here, we analyze the impact of the staggered costs on the P&L generated by the producer pricing and hedging policy. We use the contingent income  $G_T$  and the set of parameters provided in section 3.3. Given an initial value  $X_0$  and a staggered costs function  $C$ , we are interested to obtain the P&L distribution  $\mathcal{E} = G_T - \sum_{n=0}^{N-1} \delta(t_n, X_{t_n})(X_{t_{n+1}} - X_{t_n}) - \sum_{n=0}^{N-1} C(X_{t_n})\Delta t$ , produced by a discrete hedging at  $N$  trading dates for different hedging policies  $(t, x) \mapsto \delta(t, x)$ . As hedging policies, we take the optimal hedging rules with staggered costs presented in section 3.3 and we investigate the distortion of the distribution due to the AR parameter  $\gamma$  and the cost function  $c$ . Notice that optimal hedging rule will depend on sample paths of the hedging security  $X$ , which is subject to its initial value  $X_0$ . Then, to obtain different P&L distributions, we choose a value for

$X_0$  such that the optimal hedging rules are distinct at  $X_0$ .

In Figures 7-8 we depict the P&L distribution for different number  $N$  of trading dates. To represent the staggered costs, we use an increasing concave or a decreasing convex models (see section 3.1). By looking Figures 5b-6b, we set  $X_0 = 8$  for the increasing concave model and  $X_0 = 12$  for the decreasing concave model, respectively. In the plot, we compare three different policies: (a) the asymmetric risk valuation with staggered costs in full line; (b) the symmetric risk valuation with staggered costs in dashed line; (c) the symmetric risk valuation without staggered costs in dotted line.

First we observe the hedging policies given by (c) provide the P&L distribution with greatest variance in all tests, meaning that the hedging policies given by (a) and (b) outperform the usual policy given by (c). In all cases, the P&L distribution for the hedging rule given by (a) is more distorted and asymmetric than the distribution provided by hedging rule given by (b). Due to the asymmetric risk valuation, we observe the P&L distribution for the hedging rule given by (a) presents a thinner left-tail and a fatter right-tail with respect to the distribution given by (b). In comparison with the latter policy, the former policy prevents extreme losses, while allowing more probable moderate losses and disperse gains. Similar numerical results with different asymmetric risk criteria were obtained in [FMW19] using a global criterion and some neural networks methods.

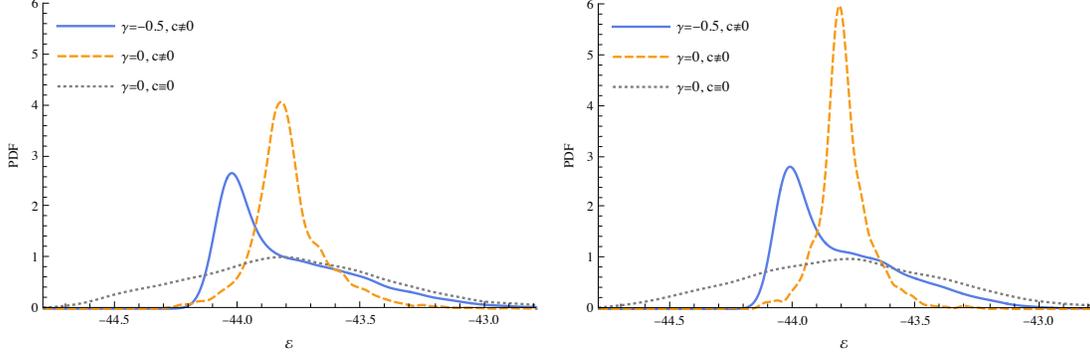


(a)  $N = 24$  trading dates,  $M = 4000$  samples    (b)  $N = 48$  trading dates,  $M = 2000$ , samples

Figure 7: P&L distribution for different hedging rules  $v_x(t, x)$  using an increasing concave function  $h_1$  with  $\beta = 1.5$  and an initial value  $X_0 = 8$ .

## 4 Conclusions and extensions

In this paper, we have described a novel methodology for designing pricing and hedging strategies for future production of power plants with fixed costs. Using an asymmetric risk valuation, we have derived the existence of the asymptotic risk in the case of staggered fixed costs depending on the electricity forward/spot price. Then, we have provided a numerical method to solve the associated PDE and to find the so-called optimal policies. Our model considers state-dependent deterministic staggered costs which can be either a convex or concave function of the spot price level, and we have discussed the behavior of optimal pricing/hedging rules accordingly. In further research, we will consider random maintenance cost function, in order to account for unpredictable



(a)  $N = 24$  trading dates,  $M = 4000$  samples    (b)  $N = 48$  trading dates,  $M = 2000$  samples

Figure 8: P&L distribution for different hedging rules  $v_x(t, x)$  using a decreasing convex function  $h_2$  and an initial value  $X_0 = 12$ .

factors like when a nuclear plant has a breakdown or when the maintenance periods has to be extended.

## A Main asymptotic result

We now turn to the statement of our main results giving the limit of  $R^N$ . Earlier in section 3, we illustrated these results with numerical experiments, with staggered costs depending on the forward price level through convex-concave functions. First we state the set of assumptions to make this asymptotic analysis rigorous, as established in appendix C.

**Assumption 1.** *The volatility function  $t \mapsto \alpha(t, T)$  is strictly positive and continuously differentiable.*

**Assumption 2.** *The reference price  $v: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , its space derivatives  $v_x, v_{xx}: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$  and the kernel  $f: [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are in  $C_{loc, pol}^{1/2, 1}$ . Additionally, the cost function  $c: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$  is in  $C_{loc, pol}^{1/2, 1}$ .*

**Assumption 3.** *For any  $\tau \in (0, T]$ , there is a unique classical solution  $u^\tau: [0, \tau] \times \mathbb{R}^+ \rightarrow \mathbb{R}$  to the PDE (2) with the terminal condition  $u^\tau(\tau, \cdot) = v(\tau, \cdot)$  at the time  $\tau$ . Moreover, partial derivatives  $u_t^\tau, u_x^\tau, u_{xx}^\tau, u_{tx}^\tau, u_{xxx}^\tau$  exist and are in  $C_{loc, pol}^{1/2, 1}$ .*

**Assumption 4.** *The Gamma of the reference price  $v$  does not vanish except on a set of zero-measure:  $v_{xx}(t, X_t) \neq 0$ ,  $dt \otimes d\mathbb{P}$  a.e.*

The next theorem gives the explicit limit of  $R^N$  as  $N \rightarrow +\infty$ . The limit is expressed using an extra Brownian motion  $B = \{B_\theta : \theta \in [0, 1]\}$ , independent of  $W$ : the expectation below are taken w.r.t. both Brownian motions.

**Theorem 1** (Asymptotic risk with staggered costs). *Under the standing assumptions,*

the limit of  $R^N[v, f]$  as  $N \rightarrow \infty$  exists and is given by

$$R[v, f] := \mathbb{E} \left[ \int_0^T \int_0^1 \ell''_\gamma \left( J_1[v](t, X_t) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](t, X_t) \theta \right) \right. \\ \left. \times \left( |J_1[v](t, X_t)|^2 \frac{B_\theta^2}{2} + |J_2[v, f](t, X_t)|^2 \theta - J_1[v](t, X_t) J_2[v, f](t, X_t) \frac{B_\theta^2 - \theta}{2} \right) d\theta dt \right], \quad (9)$$

where the functionals  $J_1[v](t, x)$  and  $J_2[v, f](t, x)$  are given by

$$\begin{cases} J_1[v](t, x) & := |\alpha(t, T)x|^2 v_{xx}(t, x), \\ J_2[v, f](t, x) & := f(t, x, v(t, x), v_x(t, x), v_{xx}(t, x)) + c(t, x). \end{cases} \quad (10)$$

Notice that Theorem 1 is a one-dimensional version of [GPW18, Theorem 2.1], specialized to positive security prices  $X$ , in the presence of staggered costs per unit of time. The proof relies on the same techniques as presented in [GPW18, section 3] and is postponed to appendix C.

## B Optimal kernel

Here we give the explicit expression of  $f$  such that

$$R[v, \mathbf{f}] \leq R[v, f]$$

for any kernel  $f$  as described in the above assumptions. We follow the approach of [GPW18] where  $c \equiv 0$ , by stressing only the main arguments, details are left to the reader.

First, write the asymptotic risk in (9) as a functional

$$R[v, f] = \mathbb{E} \left[ \int_0^T L \left( J_1[v](t, X_t), J_2[v, f](t, X_t) \right) dt \right], \quad (11)$$

in terms of  $L : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by

$$L(x_1, x_2) = \mathbb{E}^B \left[ \int_0^1 \ell''_\gamma \left( x_1 \frac{B_\theta^2 - \theta}{2} - x_2 \theta \right) \left( |x_1|^2 \frac{B_\theta^2}{2} + |x_2|^2 \theta - x_1 x_2 \frac{B_\theta^2 - \theta}{2} \right) d\theta \right], \quad (12)$$

for all  $x_1, x_2 \in \mathbb{R} \times \mathbb{R}$ .

Since  $J_2[v, f](\cdot)$  does depend on  $f$  but  $J_1[v](\cdot)$  does not, we observe that the minimization of (11) over  $f$  is made by minimizing (12) over  $x_2$  as a function of  $x_1$ : denote  $\mathbf{a}(x_1)$  the minimizer. It is established in [GPW18, Section 2.4] that

$$\mathbf{a}(x_1) = \mathbf{a}_+ x_1^+ - \mathbf{a}_- x_1^-, \quad (13)$$

for some real  $\mathbf{a}_+$  and  $\mathbf{a}_-$  (depending on  $\gamma$ )<sup>13</sup>. Denoting by  $\Phi_{\mathcal{N}}$  the cumulative distribution function of the standard normal distribution and  $\phi_{\mathcal{N}} = \Phi'_{\mathcal{N}}$  its density, those constants

<sup>13</sup>Notice that Proposition 2.3 in [GPW18] is stated for  $\gamma$  in  $[0, 1)$ . In fact, a more general condition  $\gamma \in (-1, 1)$  still guarantees that  $\frac{\partial L}{\partial x_2}(1, x_2)$  and  $\frac{\partial L}{\partial x_2}(-1, x_2)$  is strictly increasing in  $x_2$ . By the intermediate value theorem, we get the uniqueness of the value  $\mathbf{a}_+$  and  $\mathbf{a}_-$  defined by  $\frac{\partial L}{\partial x_2}(1, \mathbf{a}_+) = 0$  and  $\frac{\partial L}{\partial x_2}(-1, \mathbf{a}_-)$ , respectively.

$\mathbf{a}_+$  and  $\mathbf{a}_-$  are respectively the unique solutions to

$$\begin{aligned} (1 + \gamma^2) \mathbf{a}_+ + \gamma T(\mathbf{a}_+) &= 0, \\ (1 + \gamma^2) \mathbf{a}_- - \gamma T(\mathbf{a}_-) &= 0, \end{aligned} \tag{14}$$

with

$$T(a) = 2a + 8a \Phi_{\mathcal{N}}(-\sqrt{2a+1}) \mathbf{1}_{2a+1>0} - 4\sqrt{2a+1} \phi_{\mathcal{N}}(\sqrt{2a+1}) \mathbf{1}_{2a+1>0}.$$

In view of (9)-(11)-(12)-(13), we easily deduce that the optimal kernel  $\mathbf{f}$  is given by

$$\mathbf{f}(t, x, y_1, y_2, y_3) = \mathbf{a}(\alpha^2(t, T)x^2 y_3) - c(t, x) = \alpha^2(t, T)x^2(\mathbf{a}_+ y_3^+ - \mathbf{a}_- y_3^-) - c(t, x),$$

for any  $t, x, y_1, y_2, y_3 \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ . Observe that the above  $\mathbf{f}$  satisfies Assumption 2. Besides, we depict, in Figure 9, the global minimizer  $\mathbf{a}$  of  $\min_{x_2 \in \mathbb{R}} L(x_1, x_2)$  in function of  $x_1$ .

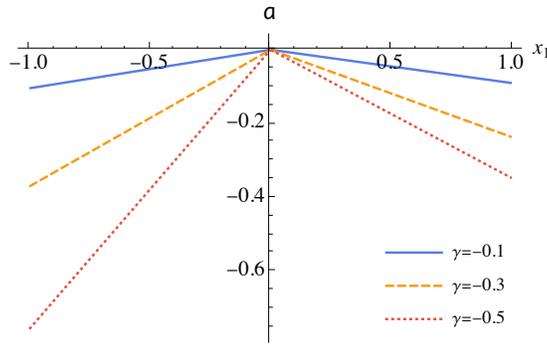


Figure 9: Global minimizer  $\mathbf{a}$  as function of  $x_1$  for different values of  $\gamma$ . For  $\gamma \in (-1, 0)$ , the map  $x_1 \mapsto \mathbf{a}(x_1)$  is negative and piecewise linear.

## C Existence of the asymptotic risk

In the following, we discuss how Assumptions 1-4 lead to an analysis similar to that given in [GPW18]. First, we have that the map  $t \mapsto \alpha(t, T)$  is globally bounded, and that its derivative  $t \mapsto \alpha_t(t, T)$  is locally bounded because of Assumption 1. Therefore, the diffusion function  $(t, x) \mapsto \sigma(t, x) = \alpha(t, T)x$  is Lipschitz in space and time, and it has linear growth in space uniformly in time, which is in  $C_{\text{loc, pol}}^{1/2, 1}$ . Next, Assumption 2 is equivalent to Assumptions 2.2-2.3 in [GPW18]. Moreover, it implies that the map

$$(t, x) \mapsto J_2[v, f](t, x) = f(t, x, v(t, x), v_x(t, x), v_{xx}(t, x)) + c(t, x)$$

is in  $C_{\text{loc, pol}}^{1/2, 1}$ . Now, any estimation or convergence result valid for

$$(t, x) \mapsto J_0[v](t, x) = f(t, x, v(t, x), v_x(t, x), v_{xx}(t, x)),$$

which is  $C_{\text{loc, pol}}^{1/2, 1}$ , is also valid for  $J_2[v, f](\cdot)$ . Also, the expectation w.r.t. the distribution of  $B$ , or  $W$ , or both, is denoted by  $\mathbb{E}^B$ , or  $\mathbb{E}^W$ , or  $\mathbb{E}^{W \otimes B}$ .

Now, we review some important aspects of the computation of the asymptotic risk. First of all, we consider a time-space scaling of the hedging security dynamics in order to study the conditional risk  $\Delta R_n$  (see (3)) on each interval  $[t_n, t_{n+1}]$ . Then, we obtain an expansion of  $\Delta R_n$  in terms of the time step  $\Delta t$ . With expansion in hands, we aggregate the expectation of those conditional  $\Delta R_n$  and take the limit, after dividing by  $\Delta t$ .

**Scaling and conditioning aspect** As explained in section 2, we deal with one hedging instrument  $X$ , which is the forward contract with delivery time  $T$ , satisfying the following SDE

$$dX_t = \alpha(t, T)X_t dW_t, \quad 0 \leq t \leq T, \quad (15)$$

where  $W$  is a standard Brownian motion.

In view of the small-time approximations, we define a version  $X_\theta^{\Delta t}$  of the solution  $X$  of the SDE (15)

$$X_\theta^{\Delta t} = x + \Delta t^{1/2} \int_0^\theta \alpha(t_n + \theta' \Delta t, T) X_{\theta'}^{\Delta t} dB_{\theta'}, \quad 0 \leq \theta \leq 1, \quad (16)$$

where  $B = \{B_\theta : \theta \in [0, 1]\}$  is an extra standard Brownian motion independent from  $W$ . Depending on the trading dates,  $X^{\Delta t}$  is also dependent on the number  $N$  of time steps. Also, it was supposed the original probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is large enough to contain an additional Brownian motion  $B$ .

**Stochastic expansion aspect** In view of expanding the conditional risk  $\Delta R_n$

$$\mathbb{E} \left[ \ell_\gamma \left( u^{t_{n+1}}(t_{n+1}, X_{t_{n+1}}) - u^{t_{n+1}}(t_n, X_{t_n}) - u_x^{t_{n+1}}(t_n, X_{t_n})(X_{t_{n+1}} - X_{t_n}) - c(t_n, X_{t_n})\Delta t \right) \middle| \mathcal{F}_{t_n} \right],$$

for any  $n \in \{0, \dots, N-1\}$ , we consider the stochastic process  $\mathcal{E}_\theta^{\Delta t}$  defined by

$$\mathcal{E}_\theta^{\Delta t} = u^{t_{n+1}}(t_n + \theta \Delta t, X_\theta^{\Delta t}) - u^{t_{n+1}}(t_n, x) - u_x^{t_{n+1}}(t_n, x)(X_\theta^{\Delta t} - x) - c(t, x)\theta \Delta t, \quad (17)$$

where  $X_\theta^{\Delta t}$  is the solution to the SDE (16) starting from  $x \in \mathbb{R}^+$ . Using the fact that the processes  $\{X_{t_n + \theta \Delta t}^{t_n, x} : \theta \in [0, 1]\}$  and  $\{X_\theta^{\Delta t} : \theta \in [0, 1]\}$  have the same distribution, we rewrite  $\Delta R_n$  as a continuous function in terms of  $X^{t_n, x}$  and  $X^{\Delta t}$  at  $\theta = 0$ , respectively. Indeed, by setting

$$P^{\Delta t}(t_n, x) = \frac{1}{(\Delta t)^2} \mathbb{E}[\ell_\gamma(\mathcal{E}_1^{\Delta t})] = \frac{1}{(\Delta t)^2} \mathbb{E} \left[ \ell_\gamma \left( u^{t_{n+1}}(t_{n+1}, X_1^{\Delta t}) - u^{t_{n+1}}(t_n, x) - u_x^{t_{n+1}}(t_n, x)(X_1^{\Delta t} - x) - c(t, x)\Delta t \right) \right], \quad (18)$$

we obtain

$$\Delta R_n = (\Delta t)^2 P^{\Delta t}(t_n, X_{t_n}) = \mathbb{E}[\ell_\gamma(\mathcal{E}_1^{\Delta t})]_{x=X_{t_n}}. \quad (19)$$

In the following, we consider several constants  $K_{n, N}(x)$  depending polynomially on  $x$ , uniformly in the interval  $[t_n, t_{n+1}]$  and in the number  $N$  of time steps. Whenever we note  $K_{n, N}(x) \in K_{\text{pol}}$ , we mean that, for some real  $\alpha > 0$ , it holds

$$\sup_{N \geq 1} \sup_{0 \leq n \leq N-1} \sup_{x \in \mathbb{R}^+} \frac{|K_{n, N}(x)|}{1 + |x|^\alpha} < +\infty.$$

In the following proposition, we state the stochastic expansion of  $P^{\Delta t}$  at point  $(t_n, x)$  in terms of the time step  $\Delta t$  in the presence of a cost function  $c(t, x)$  proportional to  $\Delta t$ .

**Proposition 1.** *Suppose that assumptions of Theorem 1 hold. Let  $t_n = n\Delta t$ ,  $x \in \mathbb{R}$ ,  $X_\theta^{\Delta t}$  given in (16),  $\mathcal{E}_\theta^{\Delta t}$  given in (17) and  $P^{\Delta t}(t_n, x)$  be defined in (18). Then, it holds*

$$\begin{aligned} P^{\Delta t}(t_n, x) &= \mathbb{E}^B \left[ \int_0^1 \ell''_\gamma \left( J_1[u^{t_{n+1}}](t_n, x) \frac{B_\theta^2 - \theta}{2} - J_2[u^{t_{n+1}}, f](t_n, x)\theta + R_\theta^{\Delta t}(t_n, x) \right) \right. \\ &\times \left. \left( |J_1[u^{t_{n+1}}](t_n, x)|^2 \frac{B_\theta^2}{2} + |J_2[u^{t_{n+1}}, f](t_n, x)|^2 \theta - J_1[u^{t_{n+1}}](t_n, x) J_2[u^{t_{n+1}}, f](t_n, x) \frac{B_\theta^2 - \theta}{2} \right) d\theta \right] \\ &+ K_{n,N}(x)\Delta t^{1/2}, \end{aligned}$$

where

$$R_\theta^{\Delta t}(t_n, x) = \frac{\mathcal{E}_\theta^{\Delta t}}{\Delta t} - J_1[u^{t_{n+1}}](t_n, x) \frac{B_\theta^2 - \theta}{2} + J_2[u^{t_{n+1}}, f](t_n, x)\theta,$$

for some constant  $K_{n,N}(x) \in K_{pol}$ .

*Proof.* Following the same argument as in [GPW18, Proposition 3.1], we use Ito's lemma in  $u^{t_{n+1}}$  and its gradient  $u_x^{t_{n+1}}$  to obtain the dynamics of  $\mathcal{E}_\theta^{\Delta t}$ . Then, we apply to the Ito-Tanaka formula to  $\ell_\gamma(\mathcal{E}_\theta^{\Delta t})$  in order to obtain the announced result. Indeed, replacing  $X_\theta^{\Delta t}$  in (16) into  $\mathcal{E}_\theta^{\Delta t}$  in (17) leads to

$$\begin{aligned} \mathcal{E}_\theta^{\Delta t} &= u^{t_{n+1}}(t_n + \theta\Delta t, X_\theta^{\Delta t}) - u^{t_{n+1}}(t_n, x) - c(t, x)\theta\Delta t \\ &\quad - \Delta t^{1/2} \int_0^\theta u_x^{t_{n+1}}(t_n, x) \alpha(t_n + \theta'\Delta t, T) X_{\theta'}^{\Delta t} dB_{\theta'}. \end{aligned}$$

Applying Ito's formula to  $u^{t_{n+1}}(t_n + \theta\Delta t, X_\theta^{\Delta t})$ , we obtain

$$\begin{aligned} \mathcal{E}_\theta^{\Delta t} &= -\Delta t \int_0^\theta c(t, x) d\theta' + \Delta t \int_0^\theta \left( u_t^{t_{n+1}} + \frac{1}{2} |\alpha(\cdot, T) \cdot|^2 u_{xx}^{t_{n+1}} \right) (t_n + \theta'\Delta t, X_{\theta'}^{\Delta t}) d\theta' \\ &\quad + \Delta t^{1/2} \int_0^\theta \left( u_x^{t_{n+1}}(t_n + \theta'\Delta t, X_{\theta'}^{\Delta t}) - u_x^{t_{n+1}}(t_n, x) \right) \alpha(t_n + \theta'\Delta t, T) X_{\theta'}^{\Delta t} dB_{\theta'}. \end{aligned} \tag{20}$$

Now we replace the PDE (2) satisfied by the function  $u^{t_{n+1}}$  in (20). Again applying Ito's formula to the gradient  $u_x^{t_{n+1}}(t_n + \theta\Delta t, X_\theta^{\Delta t})$  in (20), we get

$$\begin{aligned} \mathcal{E}_\theta^{\Delta t} &= \Delta t |\alpha(t_n, T)x|^2 u_{xx}^{t_{n+1}}(t_n, x) \frac{B_\theta^2 - \theta}{2} + \Delta t R_\theta^{\Delta t}(t_n, x) \\ &\quad - \Delta t \left( f(t_n, x, u^{t_{n+1}}(t_n, x), u_x^{t_{n+1}}(t_n, x), u_{xx}^{t_{n+1}}(t_n, x)) + c(t, x) \right) \theta, \end{aligned} \tag{21}$$

where the remainder process  $R_\theta^{\Delta t}$  is given by

$$\begin{aligned}
R_\theta^{\Delta t}(t_n, x) &= \int_0^\theta \left( u_t^{t_{n+1}}(t_n + \theta' \Delta t, X_{\theta'}^{\Delta t}) - u_t^{t_{n+1}}(t_n, x) \right) d\theta' \\
&+ \frac{1}{2} \int_0^\theta \left( |\alpha(t_n + \theta' \Delta t, T) X_{\theta'}^{\Delta t}|^2 u_{xx}^{t_{n+1}}(t_n + \theta' \Delta t, X_{\theta'}^{\Delta t}) - |\alpha(t_n, T) x|^2 u_{xx}^{t_{n+1}}(t_n, x) \right) d\theta' \\
&+ \int_0^\theta \left( \frac{u_x^{t_{n+1}}(t_n + \theta' \Delta t, X_{\theta'}^{\Delta t}) - u_x^{t_{n+1}}(t_n, x)}{\Delta t^{1/2}} |\alpha(t_n, T) X_{\theta'}^{\Delta t}|^2 - u_{xx}^{t_{n+1}}(t_n, x) |\alpha(t_n, T) x|^2 B_{\theta'} \right) dB_{\theta'}.
\end{aligned}$$

Due to the result in [GPW18, Lemma 3.5],  $R_\theta^{\Delta t}$  is small in the following sense:  $\sup_{\theta \in [0,1]} \mathbb{E}^B |R_\theta^{\Delta t}(t_n, x)|^2 \leq K_{n,N}(x) \Delta t$ , for some constant  $K_{n,N}(x) \in K_{\text{pol}}$ .

Thanks to the relation  $\ell''_\gamma(y)y = \ell'_\gamma(y)$  for all  $y \in \mathbb{R}$ , we get by applying the Ito-Tanaka formula:  $\ell_\gamma(\mathcal{E}_\theta^{\Delta t}) = \int_0^\theta \ell''_\gamma(\mathcal{E}_{\theta'}^{\Delta t})(\mathcal{E}_{\theta'}^{\Delta t} d\mathcal{E}_{\theta'}^{\Delta t} + \frac{1}{2} d\langle \mathcal{E}^{\Delta t} \rangle_{\theta'})$ . After replacing the expression of  $\mathcal{E}^{\Delta t}$  in (21), we take the expectation to obtain

$$\begin{aligned}
\mathbb{E}^B[\ell_\gamma(\mathcal{E}_1^{\Delta t})] &\stackrel{(19)}{=} \Delta t^2 P^{\Delta t}(t_n, x) \\
&= \Delta t^2 \mathbb{E}^B \left[ \int_0^1 \ell''_\gamma \left( J_1[u^{t_{n+1}}](t_n, x) \frac{B_\theta^2 - \theta}{2} \Delta t - J_2[u^{t_{n+1}}, f](t_n, x) \theta \Delta t + R_\theta^{\Delta t}(t_n, x) \Delta t \right) \right. \\
&\quad \times \left( |J_1[u^{t_{n+1}}](t_n, x)|^2 \frac{B_\theta^2}{2} + |J_2[u^{t_{n+1}}, f](t_n, x)|^2 \theta \right. \\
&\quad \left. \left. - J_1[u^{t_{n+1}}](t_n, x) J_2[u^{t_{n+1}}, f](t_n, x) \frac{B_\theta^2 - \theta}{2} \right) d\theta \right] + K_{n,N}(x) \Delta t^{5/2},
\end{aligned}$$

where  $J_2[u^{t_{n+1}}, f](\cdot)$  and  $J_1[u^{t_{n+1}}](\cdot)$  are given in (10).

We omit the study of the terms in the remainder, because they were already treated in the proof of [GPW18, Proposition 3.1]. Using the relation  $\ell''_\gamma(\varepsilon y) = \ell''_\gamma(y)$  for any  $\varepsilon > 0$ , we obtain the announced result.  $\square$

**Aggregating aspect** From the relation between  $P^{\Delta t}(t_n, X_{t_n})$  and  $\Delta R_n$  in (19), we write  $R^N[v, f]$  using the expansion of  $P^{\Delta t}(t, x)$  obtained in Proposition 1:

$$\begin{aligned}
R^N[v, f] &= \frac{1}{\Delta t} \sum_{n=0}^{N-1} \mathbb{E}[\Delta R_n] = \sum_{n=0}^{N-1} \mathbb{E} \left[ P^{\Delta t}(t_n, X_{t_n}) \right] \Delta t \\
&= \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_0^1 \ell''_\gamma \left( J_1[u^{t_{n+1}}](t_n, X_{t_n}) \frac{B_\theta^2 - \theta}{2} - J_2[u^{t_{n+1}}, f](t_n, X_{t_n}) \theta + R_\theta^{\Delta t}(t_n, X_{t_n}) \right) \right. \\
&\quad \times \left( |J_1[u^{t_{n+1}}](t_n, X_{t_n})|^2 \frac{B_\theta^2}{2} + |J_2[u^{t_{n+1}}, f](t_n, X_{t_n})|^2 \theta \right. \\
&\quad \left. \left. - J_1[u^{t_{n+1}}](t_n, X_{t_n}) J_2[u^{t_{n+1}}, f](t_n, X_{t_n}) \frac{B_\theta^2 - \theta}{2} \right) d\theta \right] \Delta t + \sum_{n=0}^{N-1} \mathbb{E} [K_{n,N}(X_{t_n}) \Delta t^{3/2}].
\end{aligned} \tag{22}$$

In the previous expression, we substitute the solution  $u^{t_{n+1}}(t_n, \cdot)$  by its terminal condition  $v(t_{n+1}, \cdot)$ . This is equivalent to replace  $J_2[u^{t_{n+1}}, f](t_n, \cdot)$  and  $J_1[u^{t_{n+1}}](t_n, \cdot)$  by  $J_2[v, f](t_{n+1}, \cdot)$  and  $J_1[v](t_{n+1}, \cdot)$ . Then, we rewrite  $R^N[v, f]$  in (22) as follows

$$\begin{aligned}
R^N[v, f] &= \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_0^1 \ell''_\gamma \left( J_1[v](t_{n+1}, X_{t_n}) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](t_{n+1}, X_{t_n}) \theta + \bar{R}_\theta^{\Delta t}(t_n, X_{t_n}) \right) \right. \\
&\times \left( |J_1[v](t_{n+1}, X_{t_n})|^2 \frac{B_\theta^2}{2} + |J_2[v, f](t_{n+1}, X_{t_n})|^2 \theta \right. \\
&\left. \left. - J_1[v](t_{n+1}, X_{t_n}) J_2[v, f](t_{n+1}, X_{t_n}) \frac{B_\theta^2 - \theta}{2} \right) d\theta \right] \Delta t + \sum_{n=0}^{N-1} \mathbb{E} [(\bar{C}^{\Delta t}(t_n, X_{t_n}) + K_{n,N}(X_{t_n}) \Delta t^{3/2})], \tag{23}
\end{aligned}$$

where

$$\begin{aligned}
\bar{R}_\theta^{\Delta t}(t_n, x) &:= R_\theta^{\Delta t}(t_n, x) + (J_1[u^{t_{n+1}}](t_n, x) - J_1[v](t_{n+1}, x)) \frac{B_\theta^2 - \theta}{2} \\
&\quad - (J_2[u^{t_{n+1}}, f](t_n, x) - J_2[v, f](t_{n+1}, x)) \theta
\end{aligned}$$

and

$$\begin{aligned}
\bar{C}^{\Delta t}(t_n, x) &:= \mathbb{E}^B \left[ \int_0^1 \ell''_\gamma \left( J_1[u^{t_{n+1}}](t_n, x) \frac{B_\theta^2 - \theta}{2} - J_2[u^{t_{n+1}}, f](t_n, x) \theta + R_\theta^{\Delta t}(t_n, x) \right) \right. \\
&\times \left\{ (|J_1[u^{t_{n+1}}](t_n, x)|^2 - |J_1[v](t_{n+1}, x)|^2) \frac{B_\theta^2}{2} + (|J_2[u^{t_{n+1}}, f](t_n, x)|^2 - |J_2[v, f](t_{n+1}, x)|^2) \theta \right. \\
&\left. \left. - (J_1[u^{t_{n+1}}](t_n, x) J_2[u^{t_{n+1}}, f](t_n, X_{t_n}) - J_1[v](t_{n+1}, x) J_2[v, f](t_{n+1}, x)) \frac{B_\theta^2 - \theta}{2} \right\} d\theta \right].
\end{aligned}$$

Noticed that the terms  $\bar{R}_\theta^{\Delta t}(t_n, X_{t_n})$  and  $\bar{C}^{\Delta t}(t_n, X_{t_n})$  need to be estimated in order to pass to limit  $N \rightarrow \infty$  in the expression (23). From the result in [GPW18, Proposition 3.3], the previous terms converge to zero in the following sense:

- $\sup_{0 \leq n \leq N-1} \mathbb{E} |\bar{C}^{\Delta t}(t_n, X_{t_n})| \leq K \Delta t^{1/2}$ , for some constant  $K > 0$ .
- $\sup_{0 \leq n \leq N-1} \sup_{\theta \in [0,1]} |\bar{R}_\theta^{\Delta t}(t_n, X_{t_n})| \xrightarrow[N \rightarrow \infty]{} 0$ ,  $d\mathbb{P}^W \otimes d\mathbb{P}^B$ -almost surely.

Now we write the integrated conditional risk  $R^N$  in (23) as the expectation of a double integrable. By setting  $\varphi_t^N := \sup\{t_n : t \geq t_n\}$  and  $\bar{\varphi}_t^N := \inf\{t_n : t < t_n\}$ , we obtain from equation (22)

$$\begin{aligned}
R^N[v, f] &= \mathbb{E} \left[ \int_0^T \int_0^1 \ell''_\gamma \left( J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N}) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N}) \theta + \bar{R}_\theta^{\Delta t}(\bar{\varphi}_t^N, X_{\varphi_t^N}) \right) \right. \\
&\times \left\{ |J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N})|^2 \frac{B_\theta^2}{2} + |J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N})|^2 \theta \right. \\
&\left. \left. - J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N}) J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N}) \frac{B_\theta^2 - \theta}{2} \right\} d\theta dt \right] + \sum_{n=0}^{N-1} \mathbb{E} [\bar{C}^{\Delta t}(t_n, X_{t_n}) + K_{n,N}(X_{t_n}) \Delta t^{3/2}]. \tag{24}
\end{aligned}$$

When the number of trading dates  $N \rightarrow +\infty$ , the last term of  $R^N[v, f]$  in (24) goes to zero, due to the estimates in [GPW18, Lemma 3.3] and [GPW18, Proposition 3.1]. On the other hand, the limit of the first term in (24) is achieved by applying the dominated convergence theorem, as we now explain.

**Step 1.** Because of the continuity of the volatility  $\alpha(t, T)$  in time  $t$ , the reference price  $v$  and its derivatives  $v_x, v_{xx}$  in time  $t$  and in space  $x$ , the kernel  $f$ , the cost function  $c$ , and the path-continuity of  $X$ , we get

$$\begin{aligned} J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N}) &\xrightarrow{N \rightarrow \infty} J_1[v](t, X_t) \\ J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N}) &\xrightarrow{N \rightarrow \infty} J_2[v, f](t, X_t), \end{aligned}$$

$d\mathbb{P}^W$ -a.s., for all  $t \in [0, T]$ . Moreover, the result of item (c) of [GPW18, Lemma 3.3] states:  $\sup_{0 \leq n \leq N-1} \sup_{\theta \in [0, 1]} |\bar{R}_\theta^{\Delta t}(t_n, X_{t_n})| \xrightarrow{N \rightarrow \infty} 0$ ,  $d\mathbb{P}^W \otimes d\mathbb{P}^B$ -a.s.. Then, we get

$$\begin{aligned} &J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N}) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N})\theta + \bar{R}_\theta^{\Delta t}(\bar{\varphi}_t^N, X_{\varphi_t^N}) \\ &\xrightarrow{N \rightarrow \infty} J_1[v](t, X_t) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](t, X_t)\theta \end{aligned}$$

and

$$\begin{aligned} &|J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N})|^2\theta + |J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N})|^2 \frac{B_\theta^2}{2} - J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N})J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N}) \frac{B_\theta^2 - \theta}{2} \\ &\xrightarrow{N \rightarrow \infty} |J_2[v, f](t, X_t)|^2\theta + |J_1[v](t, X_t)|^2 \frac{B_\theta^2}{2} - J_2[v, f](t, X_t)J_1[v](t, X_t) \frac{B_\theta^2 - \theta}{2}, \end{aligned}$$

$d\mathbb{P}^W \otimes d\mathbb{P}^B$ -a.s., for all  $(\theta, t) \in [0, 1] \times [0, T]$ .

**Step 2.** Because of the second derivative  $\ell''_\gamma$  is discontinuous at 0 but the set

$$\mathcal{A} := \left\{ (\omega, t, \theta) \in \Omega \times [0, T] \times [0, 1] : J_1[v](t, X_t(\omega)) \frac{B_\theta(\omega)^2 - \theta}{2} - J_2[v, f](t, X_t(\omega))\theta = 0 \right\}$$

has measure zero (owing to Assumption 4 and  $X_t > 0$ ), it holds

$$\begin{aligned} &\ell''_\gamma \left( J_1[v](\bar{\varphi}_t^N, X_{\varphi_t^N}) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](\bar{\varphi}_t^N, X_{\varphi_t^N})\theta + \bar{R}_\theta^{\Delta t}(\bar{\varphi}_t^N, X_{\varphi_t^N}) \right) \\ &\xrightarrow{N \rightarrow \infty} \ell''_\gamma \left( J_1[v](t, X_t) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](t, X_t)\theta \right), \end{aligned}$$

$d\mathbb{P}^W \otimes d\mathbb{P}^B \otimes dt \otimes d\theta$ -a.s.

**Step 3.** Because of the boundedness of  $\ell''_\gamma$ , the polynomial growth of  $v, v_x, v_{xx}$ , and  $c$  in space  $x$ , we have that the integrand in (24) is bounded by  $C(1 + \sup_{t \in [0, T]} |X_t| + |B_\theta|)^\alpha$ , for some positive constants  $C$  and  $\alpha$ . Finally, we take the limit of the first term of  $R^N[v, f]$  in (24) to get

$$\begin{aligned} R[v, f] &= \mathbb{E} \left[ \int_0^T \int_0^1 \ell''_\gamma \left( J_1[v](t, X_t) \frac{B_\theta^2 - \theta}{2} - J_2[v, f](t, X_t)\theta \right) \right. \\ &\quad \left. \times \left( |J_1[v](t, X_t)|^2 \frac{B_\theta^2}{2} + J_2[v, f](t, X_t)|^2\theta - J_1[v](t, X_t) J_2[v, f](t, X_t) \frac{B_\theta^2 - \theta}{2} \right) d\theta dt \right], \end{aligned}$$

by dominated convergence theorem, which concludes the proof of Theorem 1.

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