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VALUATIONS AND HENSELIZATION

ANA BELÉN DE FELIPE AND BERNARD TEISSIER

Abstract. We study the extension of valuations centered in a local domain to its henselization. We prove that a valuation ν centered in a local domain R uniquely determines a minimal prime $H(\nu)$ of the henselization R^h of R and an extension of ν centered in $R^h/H(\nu)$, which has the same value group as ν. Our method, which assumes neither that R is noetherian nor that it is integrally closed, is to reduce the problem to the extension of the valuation to a quotient of a standard étale local R-algebra and in that situation to draw valuative consequences from the observation that the Newton–Hensel algorithm for constructing roots of polynomials produces sequences that are always pseudo–convergent in the sense of Ostrowski. We then apply this method to the study of the approximation of elements of the henselization of a valued field by elements of the field and give a characterization of the henselian property of a local domain (R, m_R) in terms of the limits of certain pseudo–convergent sequences of elements of m_R for a valuation centered in it. Another consequence of our work is to establish in full generality a bijective correspondence between the minimal primes of the henselization of a local domain R and the connected components of the Riemann–Zariski space of valuations centered in R.

1. Introduction

Henselization plays an important role in the theory of valued fields, in particular because the valuation of a henselian valued field extends uniquely to any algebraic extension and because maximal valued fields are henselian. Henselization of local rings is also a fundamental tool in algebraic geometry, in particular because of its relation with the implicit function theorem and the detection of local analytic branches of an excellent scheme, but also because of the role it plays in the study of extensions of a valuation of an excellent local domain (R, m_R) to its m_R-adic completion, and thus in some approaches to local uniformization in arbitrary characteristic. In this paper we study the henselization of local domains with a view to applications to algebraic geometry but with methods close to those of the theory of valued fields. More precisely, we study the Newton–Hensel algorithm which is the origin of the henselian property from a valuative viewpoint based on the observation that when applied to a polynomial $F(X) \in R[X]$ defining a standard étale extension this algorithm always produces a pseudo–convergent sequence in the sense of Ostrowski of elements of the maximal ideal of R.

Extension of valuations to the henselization. A valuation on a local domain R is a valuation $\nu\colon K^* \to \Phi$ of the fraction field K of R with values in an abelian totally ordered group Φ such that the *value semigroup* $\nu(R\setminus\{0\})$ of the valuation is contained in $\Phi_{\geq 0}$. One usually adds an element ∞ larger than all elements of Φ and sets $\nu(0) = \infty$. If we denote by $R_\nu \subset K$ the valuation ring of ν, then a valuation ν on R corresponds to an inclusion $R \subseteq R_\nu$. The valuation is *centered in* R if the maximal ideal m_R of R is the ideal of elements with value in $\Phi_{>0} \cup \{\infty\}$. This means that R_ν *dominates* R in the sense that $R \subseteq R_\nu$ and, denoting by m_ν the maximal ideal of R_ν, we have the equality $m_\nu \cap R = m_R$. If $K \hookrightarrow L$ is a field extension, then a valuation $\tilde{\nu}$ of L is an *extension* of ν if $R_{\tilde{\nu}} \cap K = R_\nu$ where $R_{\tilde{\nu}}$ denotes the valuation ring of $\tilde{\nu}$.

The main result of this article is the following generalization of [HOST, Theorem 7.1]:

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Theorem 1. *Let R be a local domain and let R^h be its henselization. If ν is a valuation centered in R, then:*

(1) *There exists a unique prime ideal $H(\nu)$ of R^h lying over the zero ideal of R such that ν extends to a valuation $\tilde{\nu}$ centered in $R^h/H(\nu)$ through the inclusion $R \subset R^h/H(\nu)$. In addition, the ideal $H(\nu)$ is a minimal prime and the extension $\tilde{\nu}$ is unique.*

(2) *With the notation of (1), the valuations ν and $\tilde{\nu}$ have the same value group.*

Remembering that a semivaluation centered in a local ring T and supported at a prime ideal $\mathfrak{p}$ of T means a valuation centered in the local domain $T/\mathfrak{p}$, we can paraphrase this as:
Any valuation centered in a local domain has a unique extension as a semivaluation centered in its henselization, which has the same value group and is supported at a minimal prime.

Recall that the henselization $R \to R^h$ factorizes uniquely local morphisms from R to henselian local rings; it is unique up to unique isomorphism of R-algebras. We fix one.

A large part of Theorem 1 was proved in [HOST] under the additional assumption that R is (quasi)-excellent and thus noetherian. The assumption of quasi–excellence is used in particular to ensure finiteness properties for the normalizations of the ring R and its quotients. We establish the result in full generality and, in contrast to the proof given in [HOST], our proof has the advantage of being constructive.

Motivation. Theorem 1 plays an important role in the classification of extensions of valuations of a quasi-excellent local domain to its completion according to the method of [HOST]. One goal of this classification is to prove the following analogue of the second part of Theorem 1: *There exists an extension of a valuation of a quasi-excellent local domain R to a semivaluation* with the same value group *of its completion $\hat{R}$.* This yet unproven statement seems crucial for proofs of local uniformization in positive characteristic. Such an extension of the valuation allows one to use the advantages of completeness for intermediate steps without losing the algebraicity of the uniformizing modifications in the end. See Conjecture 1.1 of [HOST] which sets in the framework of [HOST] and makes more precise a conjecture of the second author going back to 2003 (see [T1, *proposition* 5.19]).

Theorem 1.(1) is also an essential ingredient in the proof by the first author that, up to homeomorphism in the Zariski topology, the space of valuations centered in a non singular point of an algebraic variety over an algebraically closed field depends only on the dimension (see [F, Theorem 3.5]). In this direction, Theorem 1.(1) allows us to establish in full generality a bijective correspondence between the minimal primes of the henselization of a local domain R and the connected components of the Riemann–Zariski space of valuations centered in R (see Corollary 4.2).

Strategy of the proof. The henselization of a local domain R is the limit of the inductive system of its *Nagata extensions* (also known as standard étale extensions) $S = R[X]/(F(X))_{(m_R,X)}$ where $F(X) = X^n + \cdots + a_{n-1}X - a_n \in R[X]$ is a *Nagata polynomial*: $a_n \in m_R$, $a_{n-1} \notin m_R$. It suffices therefore to study extensions of valuations centered in R to its Nagata extensions. Our approach is to do this from the viewpoint of Ostrowski's pseudo–convergent sequences, which gives a simple description of the minimal prime and an explicit construction of the extended valuation implying immediately that the value group does not change. We give more details in what follows.

Let us keep the notation introduced above. Let S be a Nagata extension of R defined by a Nagata polynomial $F(X) \in R[X]$. Any extension of ν to the algebraic closure $\overline{K}$ of K determines a root $\sigma_\infty \in \overline{K}$ of $F(X)$ that is a limit in the sense of Ostrowski of a pseudo–convergent sequence of elements $\sigma_i \in m_R$, $i \geq 1$, which is attached to $F(X)$ by Newton's method. The minimal polynomial over K of all these distinguished roots σ_∞ coincide, producing a minimal prime $H_S(\nu)$ of S (which is the trace of the minimal prime $H(\nu)$ of R^h). The ideal $H_S(\nu)$ and the extension of ν to a valuation centered in the local domain $S/H_S(\nu)$ are unique, and Theorem 1.(1) follows by passage to the inductive limit.

We prove Theorem 1.(2) by proving that the value group Φ of ν does not change after extension to $S/H_S(\nu)$. To do this, we fix a presentation of the previous quotient as a local R–algebra $R[\sigma_\infty]_{(m_R,\sigma_\infty)} \subset \overline{K}$ and investigate the way in which ν determines the value of the

extended valuation $\tilde{\nu}$ on each element $h(\sigma_\infty)$ with $h(X) \in R[X]$. We describe the behavior of the valuations $\nu(h(\sigma_i))$, $i \geq 1$. Indeed, if these valuations form an eventually constant sequence, then their stationary value is $\tilde{\nu}(h(\sigma_\infty))$; and otherwise, they are cofinal in a certain convex subgroup of Φ. Except in some particular cases (for instance, if the valuation ν is of rank one, in which case the cofinality implies that $h(\sigma_\infty) = 0$), this is not sufficient to obtain the desired result.

In order to compute $\tilde{\nu}(h(\sigma_\infty))$ in general, we present the domain R as the inductive limit of its subdomains A_0 which are essentially of finite type over the prime ring, and for which the restriction ν_0 of ν is of finite rank by Abhyankar's inequality. With the notation of Theorem 1, the local domain $R^h/H(\nu)$ is then the inductive limit of the $A_0^h/H(\nu_0)$ and each $\tilde{\nu}_0$ is obtained by restriction of $\tilde{\nu}$. Hence we can choose A_0 containing the coefficients of $F(X)$ and $h(X)$, and compute $\tilde{\nu}(h(\sigma_\infty))$ as the $\tilde{\nu}_0$–value of $h(\sigma_\infty)$ seen as element of $A_0[\sigma_\infty]_{(m_{A_0},\sigma_\infty)} \subset R[\sigma_\infty]_{(m_R,\sigma_\infty)}$. After an iterative procedure (the main point here is that ν_0 has finite rank), we determine a finite extension of the fraction field of A_0, a unique extension ν_ℓ of ν_0 to this new field, which has the same value group, and a pseudo–convergent sequence $(\chi_i^{(\ell)})_{i\geq 1}$ for ν_ℓ such that $\tilde{\nu}_0(h(\sigma_\infty))$ equals $\nu_\ell(h(\chi_i^{(\ell)}))$ for all i large enough.

Applications of our method. The subdomains of R which are essentially of finite type over the prime ring are excellent, and we could have applied directly this argument to reduce the general case to the case treated in [HOST]. However, we think that our method is much more informative. For example, since it applies to valuation rings we use it to prove the result [Ku1, Theorem 1.1] of F–V. Kuhlmann on the approximation of elements of the henselization of a valued field by elements of the field and a general characterization of the henselian property in terms of pseudo–convergent sequences (see Theorem 4.3 and Proposition 5.2, respectively). We hope that our method can also be used to study the changes in the value *semigroup* which can take place when passing to the henselization, even for regular local rings, as discovered by Cutkosky in [C, Theorem 1.5].

Organization of the paper. In Sections 2 and 3 we prove Theorem 1.(1) and Theorem 1.(2), respectively. We avoid using the approximations of R by its noetherian subrings as much as we can because some general results such as Lemma 3.2 may be of independent interest. The last two sections are applications of our approach. In Subsection 4.1 we study the decomposition into connected components of the Riemann–Zariski space of valuations centered in a local domain, and in Subsection 4.2 we revisit the result of F–V. Kuhlmann mentioned above. In Section 5 we propose a characterization of the henselian property of rings in terms of pseudo–convergent sequences.

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2. Nagata extensions and Newton-Hensel approximations from a valuative viewpoint

In this section we assume that R is a local domain and ν a valuation centered in R, and we study from a valuative viewpoint the process of henselian approximation. We denote by m_R the maximal ideal of R and by K its fraction field.

We start with the following result of J.-P. Lafon:

Proposition 2.1. ([L, Proposition 6]) *Let R be a local ring with maximal ideal m_R and S a local R-algebra with maximal ideal m_S. The following assertions are equivalent:*

(1) *S is a localization of a finite R-algebra and is flat over R, and $S/m_RS = R/m_R = S/m_S$.*
(2) *S is of the form $(R[X]/(F(X)))_{\mathcal{N}}$ where $F(X)$ is a unitary polynomial of the form*

$$X^n + a_1X^{n-1} + \cdots + a_{n-1}X - a_n,$$
$$\text{where } a_i \in R \text{ for } 1 \leq i \leq n \text{ and } a_n \in m_R,\ a_{n-1} \notin m_R,$$

and $\mathcal{N}$ is the maximal ideal of $R[X]/(F(X))$ containing the class x of X modulo $F(X)$, which is the image of the maximal ideal (m_R, X) of $R[X]$.

(3) *S is a localization of a finite R-algebra, and for every local subalgebra R_0 of R essentially of finite type over $\mathbf{Z}$ and containing the coefficients of $F(X)$ the natural map $R_0 \to S_0 = (R_0[X]/(F(X)))_{\mathcal{N}_0}$ induces an isomorphism of the completions.*

Lafon calls such extensions $R \to S$ *Nagata extensions*[1]; they are also called standard étale extensions of R or, assuming that R is noetherian, étale R-algebras quasi-isomorphic to R; see also [EGA, § 18]. For brevity we shall call polynomials $F(X)$ with coefficients in a local ring satisfying the conditions of Proposition 2.1 *Nagata polynomials*[2]. We adopt the convention that the constant term of a Nagata polynomial has a minus sign.

Morphisms of Nagata extensions of R are local morphisms of local R-algebras. A morphism from a Nagata extension S to another one S' exists if and only if there is an element ξ' in the maximal ideal of S' such that $F(\xi') = 0$. There exists at most one such morphism, determined by sending the image $x \in S$ of X to $\xi' \in S'$ and then, by Proposition 2.1, S' is a Nagata extension of S. Lafon proves that Nagata extensions of R form an inductive system and ([L, Théorème 2]) that the henselization R^h of R is the inductive limit of its Nagata extensions. In particular it has the same residue field as R.

Remark 2.2. Keeping the notations of Proposition 2.1, note that if $a_n = 0$, then S is isomorphic to R. The extension is also trivial when $n = 1$. Note also that given any element $\alpha \in m_R$, the polynomial $F_\alpha(X') = F(X' + \alpha) \in R[X']$ with $X' = X - \alpha$ satisfies the same conditions as $F(X)$. Indeed, $F_\alpha(0) = F(\alpha) \in m_R$; and the coefficient of X' in $F_\alpha(X')$ is $F'(\alpha)$, which is not in m_R since $F'(0)$ is not and $\alpha \in m_R$. Moreover, $F_\alpha(X')$ defines the same extension, that is, S is isomorphic to $S_\alpha = (R[X']/(F_\alpha(X')))_{\mathcal{N}'}$. This implies that the Nagata extension defined by the Nagata polynomial $F(X)$ is trivial if and only if $F(X)$ has a zero in the maximal ideal of R.

As a consequence of the following result, we may assume in the definition of a Nagata extension that the polynomial $F(X)$ is irreducible in $R[X]$. An equivalent statement is found in [LM, Chap.13, Proposition 13.15] for the case where R is integrally closed.

Lemma 2.3. *Let R be a local domain and let $F(X) \in R[X]$ be a Nagata polynomial. Let $F(X) = G(X)Q(X)$ be a factorization in $R[X]$, where up to multiplication by a unit of R we write*

$$G(X) = X^s + \cdots + g_{s-1}X + g_s; \qquad Q(X) = X^t + \cdots + q_{t-1}X - q_t.$$

Then, one of the two polynomials $G(X)$, $Q(X)$ must be a Nagata polynomial. It is the factor whose constant term is in m_R. If it is $Q(X)$, then $G(X) \notin (m_R, X)$.

Proof. Let us consider the linear part of $F(X)$ as it is written in Proposition 2.1; we have $-a_n = -q_t g_s \in m_R$, $a_{n-1} = g_s q_{t-1} - g_{s-1} q_t \notin m_R$. Since $a_{n-1} \notin m_R$ it is impossible for both g_s and q_t to be in m_R, but one of them must be since m_R is prime. Let us say that $q_t \in m_R$ and $g_s \notin m_R$ so that $G(X) \notin (m_R, X)$. Then by the second equality we have $q_{t-1} \notin m_R$ so that $Q(X)$ is a Nagata polynomial. □

The next lemma presents Newton's method in the way we are going to use it:

Lemma 2.4. *Let $F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1}X - a_n \in R[X]$ be a Nagata polynomial and note that as an element of $R[X]$, the polynomial $F(X)$ is the same as*

$$F^{(1)}(X_1) = F\left(X_1 - \frac{F(0)}{F'(0)}\right) = F\left(X_1 + \frac{a_n}{a_{n-1}}\right)$$

since $X \mapsto X_1 + \frac{a_n}{a_{n-1}}$ is a change of variable in $R[X]$. Write $F^{(1)}(X_1) = X^n + a_1^{(1)} X^{n-1} + \cdots + a_{n-1}^{(1)} X - a_n^{(1)}$. Then we have:

(1) *The polynomial $F^{(1)}(X_1) \in R[X_1]$ is a Nagata polynomial.*
(2) *The coefficient $a_i^{(1)}$ is congruent to a_i modulo $\frac{a_n}{a_{n-1}}$.*
(3) *$F^{(1)}(0) = -a_n^{(1)} \in a_n^2 R$.*

[1] In [N2], Nagata calls them *quasi-decompositional*.
[2] In [LM] they are called "polynomials satisfying the conditions of the implicit function theorem".

(4) *Let $R \to S$ be the Nagata extension defined by $F(X)$. Denoting by x, x_1 the images in S of X, X_1, we have $x_1 \in x^2 S$. In particular, if $\tilde{\nu}$ is any semivaluation on S extending the valuation ν on R, the inequality $\tilde{\nu}(x_1) \geq 2\tilde{\nu}(x)$ holds.*

Proof. The first statement is what was remarked above, and the proof of the next two is a direct computation. The last one follows from the fact that modulo $F(X)$, the element $X_1 = X - \frac{a_n}{a_{n-1}}$ is a multiple of X^2. □

As a consequence, starting from a Nagata polynomial $F(X) \in R[X]$, we can iterate the construction just described to produce:

- A sequence of generators $X_i := X_{i-1} + \frac{F^{(i-1)}(0)}{(F^{(i-1)})'(0)}$ for the polynomial ring $R[X]$, with $X_0 = X$.
- Polynomials $F^{(i)}(X_i) := F^{(i-1)}\left(X_i - \frac{F^{(i-1)}(0)}{(F^{(i-1)})'(0)}\right) \in R[X_i]$, with $F^{(0)}(X) = F(X)$.

Definition 2.5. Let ν be a valuation centered in a local domain R and let $F(X) \in R[X]$ be a Nagata polynomial. Keep the previous notations. We define the following elements of m_R:

$$\delta_k := \frac{a_n^{(k)}}{a_{n-1}^{(k)}} = -\frac{F^{(k)}(0)}{(F^{(k)})'(0)}, \text{ for } k \geq 0.$$

$$\sigma_i := \sum_{k=0}^{i-1} \delta_k, \text{ for } i \geq 1.$$

We say that $(\delta_i)_{i\in\mathbf{N}}$ and $(\sigma_i)_{i\geq 1}$ are the *Newton sequence of values* and the *sequence of partial sums* attached to $F(X)$, respectively.

The polynomials $(F^{(i)}(X_i))_{i\in\mathbf{N}}$ all define the same Nagata extension of R. If at some step $i \geq 0$ we find $F^{(i)}(0) = 0$, this implies that $F(X)$ defines a trivial extension, so we may assume that this does not happen and we shall do so.

By construction, we have $X = X_i + \sigma_i$ and $x_{i+1} = x_i - \delta_i$. We verify by induction that $F^{(i)}(X_i) = F(X_i + \sigma_i)$ for $i \geq 1$. Setting $X_i = 0$ in this identity, we can read the definition of δ_i as given by the equality $F'(\sigma_i)\delta_i = -F(\sigma_i)$. Observe that for all $i \geq 1$, $F(\sigma_i) \neq 0$ because the Nagata extension is not trivial, and $\nu(\delta_i) = \nu(F(\sigma_i))$.

Remark 2.6. Assuming for a moment that R is complete and separated for the m_R-adic topology, Lemma 2.4 tells us in particular that the images in S of the elements X_i converge to $x_\infty = 0$ while the polynomials $F^{(i)}(X_i)$ converge to a polynomial $F^{(\infty)}(X_\infty)$ without constant term because $a_n^{(i)} \in m_R^{2^{i-1}}$. Therefore x_∞ is a root of $F^{(\infty)}(X_\infty)$, which is simple since $a_{n-1}^{(\infty)} \notin m_R$. Since $x_\infty = x - \sum_{k=0}^{\infty} \delta_k$ and $F^{(\infty)}(X_\infty) = F(X)$ this tells us that $\sum_{k=0}^{\infty} \delta_k$ is a simple root of $F(X)$, which is contained in the maximal ideal m_R of R. Since our assumption on $F(X)$ is equivalent to the statement that the image of $F(X)$ in $k[X]$, where $k = R/m_R$, has 0 as a simple root, this is indeed a version of Hensel's lemma.

We stress the fact that by our assumption that the Nagata extension is not trivial we have $\delta_0 \in m_R \setminus \{0\}$ and δ_{i+1} is a non zero multiple of δ_i^2 for any $i \geq 0$, so that we expect to have a root of $F(X)$ which is represented as a sum $\sum_{k=0}^{\infty} \delta_k$ of elements of strictly increasing valuations.

In general, the m_R-adic topology may not be separated, in fact one can have $\bigcap_{n\in\mathbf{N}} m_R^n = m_R$, but the partial sums $(\sigma_i)_{i\geq 1}$ form a pseudo–Cauchy, or pseudo–convergent sequence in the sense of Ostrowski [O, Teil III, § 11] for the valuation ν; see also [K, § 2] and [Ku3, Chapter 8]. We refer to these texts for the following definitions and facts:

A *pseudo–convergent* sequence of elements of a field K endowed with a valuation ν is a family $(y_\tau)_{\tau\in T}$ of elements of K indexed by a well ordered set T without last element, which satisfies the condition that whenever $\tau < \tau' < \tau"$ we have $\nu(y_{\tau'} - y_\tau) < \nu(y_{\tau"} - y_{\tau'})$.

An element $y \in K$ is said to be a *pseudo–limit*, or simply *limit* of this pseudo–convergent sequence if $\nu(y_{\tau'} - y_\tau) \leq \nu(y - y_\tau)$ for $\tau, \tau' \in T$, $\tau < \tau'$. One observes that if $(y_\tau)_{\tau\in T}$ is pseudo–convergent, then for each $\tau \in T$ the value $\nu(y_{\tau'} - y_\tau)$ is independent of $\tau' > \tau$ and can be denoted by ν_τ. Moreover, given $y \in K$, either $\nu(y - y_{\tau'}) > \nu(y - y_\tau)$ whenever $\tau' > \tau$ (in which case y is a limit), or there exists $\tau_0 \in T$ such that $\nu(y - y_{\tau'}) = \nu(y - y_\tau)$ for $\tau' > \tau > \tau_0$.

In other words, the sequence $(\nu(y-y_\tau))_{\tau\in T}$ is either strictly increasing or *eventually constant*. Taking $y=0$ we see that either $\nu(y_{\tau'})>\nu(y_\tau)$ whenever $\tau'>\tau$ or there exists $\tau_0\in T$ such that $\nu(y_{\tau'})=\nu(y_\tau)$ for $\tau'>\tau>\tau_0$. Finally, if y and z are two limits of $(y_\tau)_{\tau\in T}$, we have that for all $\tau\in T$, $\nu(y-z)>\nu_\tau$ since T has no last element.

In this paper, we mostly apply a variant of Ostrowski's method to this particular pseudo–convergent sequence and for a different purpose. From now on we fix an algebraic closure $\overline{K}$ of K and recall that valuations of K extend to $\overline{K}$ (see [B, Ch. VI, § 1, no. 3, Theorem 3]).

Proposition 2.7. *Let $F(X)\in R[X]$ be a Nagata polynomial. Given an extension $\tilde\nu$ of ν to $\overline{K}$, there exists a unique root of $F(X)$ in $\overline{K}$ with positive $\tilde\nu$-value. If we call σ_∞ this root of $F(X)$, then the following also holds:*

(1) *σ_∞ is a limit of the pseudo–convergent sequence $(\sigma_i)_{i\geq 1}$ associated to $F(X)$.*
(2) *For any $z\in\overline{K}\setminus\{\sigma_\infty\}$ such that $F(z)=0$ we have $\tilde\nu(z)=0$.*
(3) *σ_∞ is a simple root of $F(X)$.*

Proof. Write $F(X)=\prod_{j=1}^n X-r_j$ in $\overline{K}[X]$. For all $i\geq 1$, we have $\nu(F(\sigma_i))=\sum_{j=1}^n\tilde\nu(\sigma_i-r_j)$. Hence if none of the r_j is a limit of the pseudo–convergent sequence $(\sigma_i)_{i\geq1}$ then $(\nu(F(\sigma_i)))_{i\geq1}$ is eventually constant. However $\nu(F(\sigma_i))=\nu(\delta_i)$ for all $i\geq1$, so we can assume that r_1 is a limit of $(\sigma_i)_{i\geq1}$. In particular, $\tilde\nu(\sigma_i-r_1)=\nu(\sigma_{i+1}-\sigma_i)=\nu(\delta_i)$ for all $i\geq1$.

For $1\leq j\leq n$, we have $\tilde\nu(r_j)\geq0$ because r_j is integral over R (see [B, Ch. VI, § 1, no.3, Theorem 3]). In addition, $\nu(\sigma_i)=\nu(\delta_0)=\nu(F(0))=\sum_{j=1}^n\tilde\nu(r_j)$ for all $i\geq1$. If $\nu(\sigma_i)>\tilde\nu(r_1)$ for some i, we obtain $\nu(\delta_i)=\tilde\nu(\sigma_i-r_1)=\tilde\nu(r_1)<\nu(\delta_0)$, which gives us a contradiction. We conclude that $\tilde\nu(r_j)=0$ if $j\neq1$ and $\tilde\nu(r_1)=\nu(\delta_0)>0$. □

In what follows, we keep the notation introduced in Proposition 2.7 and an extension $\tilde\nu$ of ν to $\overline{K}$ comes with a distinguished root $\sigma_\infty\in\overline{K}$ of $F(X)$ satisfying $\tilde\nu(\sigma_\infty)=\nu(\delta_0)>0$. In the notation we will omit the dependence of σ_∞ on $\tilde\nu$. Two extensions of ν may choose different roots of $F(X)$ in $\overline{K}$, however their minimal polynomials over K coincide in view of the following:

Corollary 2.8. *In the situation of Proposition 2.7, denote by $F^*(X)$ the minimal polynomial of σ_∞ over K. Then $F^*(X)$ is the only irreducible factor of $F(X)$ in $K[X]$ such that the ν-value of its independent term is positive.*

Proof. The ν-value of the independent term of a polynomial in $K[X]$ is the sum of the $\tilde\nu$-values of all its roots in $\overline{K}$. Use that $\tilde\nu(\sigma_\infty)>0$ and Proposition 2.7.(2) to prove the statement. □

Throughout this section, we denote by $\overline{R}$ the integral closure of R in K. Recall that $\overline{R}\subseteq R_\nu$. The localization $\tilde R=\overline{R}_{m_\nu\cap\overline{R}}$ is an integrally closed local domain dominating R and dominated by R_ν, which is uniquely determined by ν. Note that a Nagata polynomial in $R[X]$ can also be regarded as a Nagata polynomial in $\tilde R[X]$.

Lemma 2.9. *Keeping the same notation, we have the following:*

(1) *The coefficients of $F^*(X)$ belong to $\overline{R}$.*
(2) *The polynomials $F(X)$ and $F^*(X)$ are Nagata polynomials in $\tilde R[X]$ and they define the same Nagata extension of $\tilde R$.*

Proof. Since σ_∞ is integral over R, its minimal polynomial over K belongs to $\overline{R}[X]$ (see [B, Ch. 5, § 1, no.3, Corollary]). Next we prove the second statement.

By Corollary 2.8, $F^*(0)\in m_\nu$. Hence the constant term of $F^*(X)\in\overline{R}[X]\subseteq\tilde R[X]$ is in the maximal ideal of $\tilde R$. The result follows from Lemma 2.3 applied to $\tilde R$ and $F(X)$. The natural epimorphism $\tilde R[X]/(F(X))\to\tilde R[X]/(F^*(X))$ induces an isomorphism of R–algebras from the Nagata extension of $\tilde R$ defined by $F(X)$ to that defined by $F^*(X)$. □

After what we have just seen, the valuation ν determines an irreducible factor $F^*(X)\in\overline{R}[X]$ of $F(X)$ in $K[X]$ with $\nu(F^*(0))=\nu(\delta_0)>0$. Denote by K^* the field $K[X]/(F^*(X))$. Then the natural homomorphism $R[X]/(F(X))\to K^*$ induces a homomorphism of R-algebras

$$E_S(\nu)\colon S=(R[X]/(F(X)))_{\mathcal{N}}\longrightarrow K^*.$$

Indeed, if $F^*(X)$ divides $P(X) \in R[X]$ in $\overline{R}[X]$ (or, equivalently, in $K[X]$) then $P(0) = c\,F^*(0)$ for some $c \in \overline{R}$ and, since ν is centered in R and $F^*(0) \in m_\nu$, we have $P(0) \in m_R$.

Definition 2.10. Let ν be a valuation centered in a local domain R and let S be a Nagata extension of R defined by a Nagata polynomial $F(X) \in R[X]$. We call $H_S(\nu)$ the kernel of the homomorphism $E_S(\nu)$.

Remark 2.11. Let us go back to Lemma 2.9. If $R \to S$ and $\tilde{R} \to \tilde{S}$ are the Nagata extensions defined by $F(X)$, then we have a commutative diagram

$$(\mathrm{D}_1) \qquad \begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[r, "E_S(\nu)"] \arrow[d] & K^* \\ \tilde{R} \arrow[r] & \tilde{S} \arrow[ur, "E_{\tilde{S}}(\nu)"'] & \end{tikzcd}$$

where $S \to \tilde{S}$ is the local ring homomorphism induced by the natural map from $R[X]/(F(X))$ to $\tilde{R}[X]/(F(X))$. By Lemma 2.9.(2), $\tilde{S}$ is also defined by $F^*(X)$. As a consequence, $H_{\tilde{S}}(\nu)$ is the zero ideal and $H_S(\nu)$ is the kernel of $S \to \tilde{S}$. In the case where R is integrally closed, we get that $E_S(\nu)$ is injective and S is a local domain, and therefore R^h is also a local domain.

Observe that the ideal $H_S(\nu)$ depends only on the valuation ν. It has the following properties:

Lemma 2.12. *Let ν be a valuation centered in a local domain R. Then:*

(1) *For any Nagata extension $R \to S$, the ideal $H_S(\nu)$ of S is a minimal prime.*
(2) *Given a map $f\colon S \to S'$ of Nagata extensions of R, we have $f^{-1}(H_{S'}(\nu)) = H_S(\nu)$.*

Proof. Let $F(X) \in R[X]$ be a Nagata polynomial defining the extension $R \to S$. Let $\mathfrak{p}$ be a prime ideal of S such that $\mathfrak{p} \cap R = (0)$. Then $\mathfrak{p}$ is the extension of a prime ideal of $R[X]/(F(X))$ that is contained in the maximal ideal $\mathcal{N}$ and has intersection (0) with R. Since $R \to R[X]/(F(X))$ is an integral extension, we have by the incomparability property that $\mathfrak{p}$ is a minimal prime of S. To prove (1), take $\mathfrak{p} = H_S(\nu)$.

As we saw, a morphism $f\colon S \to S'$ of Nagata extensions is determined by an element $\xi' \in m_{S'}$ such that $F(\xi') = 0$. Then, with the notations of Remark 2.11, the image $\tilde{\xi}'$ of ξ' under the local ring homomorphism $S' \to \tilde{S}'$ determines a map $\tilde{f} : \tilde{S} \to \tilde{S}'$ of Nagata extensions of $\tilde{R}$. We see that mapping the image of X in K^* to the image of $\tilde{\xi}'$ in the field K'^* uniquely defines a map $K^* \to K'^*$ of extensions of K, which has to be injective and and makes the following diagram commute:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "f"'] & \tilde{S} \arrow[r, hook, "E_{\tilde{S}}(\nu)"] \arrow[d, "\tilde{f}"'] & K^* \arrow[d] \\ S' \arrow[r] & \tilde{S}' \arrow[r, hook, "E_{\tilde{S}'}(\nu)"] & K'^* \end{tikzcd}$$

Statement (2) now follows. □

Remarks 2.13. (1) By a direct computation one can check that $H_S(\nu)$ is the extension with respect to the canonical localization homomorphism $R[X]/(F(X)) \to S$ of the prime ideal consisting of the residue classes of the polynomials in $R[X]$ that are divisible by $F^*(X)$ in $K[X]$.
(2) As we have seen in the proof of Lemma 2.12.(1), if $R \to S$ is a Nagata extension, then any prime ideal of S lying over the zero ideal of R is a minimal prime. Conversely, if $\mathfrak{p}$ is a minimal prime ideal of S then $\mathfrak{p} \cap R = (0)$. This follows from the fact that $R \to S$ is flat by using that going–down property holds for flat extensions, see [Ma, (5.D) Theorem 4].

Let us prove that the valuation ν uniquely determines the support of the semivaluation which extends it to the henselization:

Proposition 2.14. *Let ν be a valuation centered in a local domain R and let $R \to S$ be a Nagata extension. If $\mathfrak{p}$ is a prime ideal of S such that $\mathfrak{p} \cap R = (0)$ and ν extends to a valuation centered in $S/\mathfrak{p}$ through the inclusion $R \subset S/\mathfrak{p}$, then $\mathfrak{p} = H_S(\nu)$.*

Proof. Taking into account what we saw in the proof of Lemma 2.12.(1), the ideal $\mathfrak{p}$ corresponds in $R[X]$ to a minimal prime ideal $\mathfrak{q}$ over $(F(X))$ such that $\mathfrak{q} \cap R = (0)$ and $\mathfrak{q} \subseteq (m_R, X)$. Using that $K[X]$ is a principal ideal domain and that $R \to R[X]/(F(X))$ is an integral extension, we see that $\mathfrak{q}$ consists of the polynomials in $R[X]$ that are divisible in $K[X]$ by some irreducible factor $Q(X) \in K[X]$ of $F(X)$. We can write $Q(X) = X^s + q_1 X^{s-1} + \ldots + q_0$ with $q_j \in \overline{R}$ for all j (see [B, Ch. 5, § 1, no.3, Corollary]). Denote by $\bar{x}$ the image of X in $S/\mathfrak{p}$. We have a valuation μ centered in $S/\mathfrak{p}$ which extends ν. It satisfies $\mu(\bar{x}) > 0$ and $\mu(q_j) \geq 0$ for all j. The relation $\bar{x}^s + q_1 \bar{x}^{s-1} + \ldots + q_0 = 0$ implies that $\nu(Q(0)) = \mu(q_0) > 0$. Since the field extension $K \to \mathrm{Frac}(S/\mathfrak{p})$ is algebraic, we can embed $\mathrm{Frac}(S/\mathfrak{p})$ in $\overline{K}$ and extend μ to $\overline{K}$. By Corollary 2.8 we have $Q(X) = F^*(X)$, and therefore $\mathfrak{p} = H_S(\nu)$. □

Given an extension $\tilde{\nu}$ of ν to $\overline{K}$, the evaluation at σ_∞, namely $P(X) \mapsto P(\sigma_\infty)$, induces a K–isomorphism of fields $\pi_{\sigma_\infty} : K^* \to K(\sigma_\infty) \subset \overline{K}$ (recall that $F^*(X)$ is the minimal polynomial of σ_∞ over K). The image of the composition

$$\pi_{\sigma_\infty} \circ E_s(\nu) : S \to K^* \to K(\sigma_\infty)$$

is the local R–subalgebra $R[\sigma_\infty]_{(m_R, \sigma_\infty)}$ of $K(\sigma_\infty)$. Since the ideal is clear from the context, let us denote it simply by $R[\sigma_\infty]_*$. Observe that the quotient $S/H_s(\nu)$ is naturally isomorphic to $R[\sigma_\infty]_*$ and the restriction of $\tilde{\nu}$ to $K(\sigma_\infty)$ is centered in $R[\sigma_\infty]_*$ because $\tilde{\nu}(\sigma_\infty) > 0$ and ν is centered in R. In this way $\tilde{\nu}$ determines a valuation $\tilde{\nu}_S$ centered in $S/H_S(\nu)$ which is an extension of ν. Next we prove the uniqueness of this extension.

Proposition 2.15. *Keep the notation of Proposition 2.14. There is a unique valuation centered in $S/H_S(\nu)$ which extends ν through the inclusion $R \subset S/H_S(\nu)$.*

Proof. Any such extension of ν can be obtained in the way explained above starting from an extension to $\overline{K}$. Therefore it suffices to take two extensions $\tilde{\nu}$ and $\tilde{\nu}'$ of ν to $\overline{K}$ and show that $\tilde{\nu}_S = \tilde{\nu}'_S$. In that situation, by [ZS, Ch. VI, § 7, Corollary 3], there exists a K–automorphism π of $\overline{K}$ such that $\tilde{\nu}' = \tilde{\nu} \circ \pi$. Let σ_∞ and σ'_∞ be the distinguished roots of $F(X)$ in $\overline{K}$ associated to $\tilde{\nu}$ and $\tilde{\nu}'$, respectively (see Proposition 2.7). Since $\tilde{\nu}'(\pi^{-1}(\sigma_\infty)) = \tilde{\nu}(\sigma_\infty) > 0$, the automorphism π must send σ'_∞ to σ_∞. Recall that σ_∞ and σ'_∞ have the same minimal polynomial over K by Corollary 2.8. We have $\pi_{\sigma_\infty} = \pi|_{K(\sigma'_\infty)} \circ \pi_{\sigma'_\infty}$ and $\tilde{\nu}_S = \tilde{\nu}'_S$. □

We are now in position to prove Theorem 1.(1).

Proof of Theorem 1.(1). The henselization R^h is the inductive limit of the Nagata extensions S of R. For every S, denote by $f_S : S \to R^h$ the canonical local ring homomorphism. By Lemma 2.12.(2), the inductive limit $H(\nu) = \varinjlim H_S(\nu)$ is well defined and it is an ideal of R^h such that $f_S^{-1}(H(\nu)) = H_S(\nu)$ for all S. Moreover, it is a minimal prime and lies over the zero ideal of R, because all the $H_S(\nu)$ satisfy this and $\mathfrak{p} = \varinjlim f_S^{-1}(\mathfrak{p})$ for any ideal $\mathfrak{p}$ of R^h. The domain $R^h/H(\nu)$ is the union of the $S/H_S(\nu)$. Since the valuation ν determines uniquely each $H_S(\nu)$ (recall its definition and Proposition 2.14), and according to Proposition 2.15 it extends uniquely to each $S/H_S(\nu)$, the same is true for $R^h/H(\nu)$. □

3. Effective computation of the extended valuation

Let us keep the notation introduced in Section 2. In particular, we have fixed a valuation ν centered in a local domain R and a non trivial Nagata extension $R \to S$ defined by a Nagata polynomial $F(X) \in R[X]$. We present the quotient $S/H_S(\nu)$ in the form $R[\sigma_\infty]_*$ and call $\tilde{\nu}$ the unique valuation centered in $R[\sigma_\infty]_*$ extending ν (see Proposition 2.15 and the paragraph before it).

In this section we show how we can compute the $\tilde{\nu}$-value of an arbitrary non zero element of $R[\sigma_\infty]_*$. Observe that it is enough to study the values of the form $\tilde{\nu}(h(\sigma_\infty))$, where $h(X)$ is a polynomial in $R[X]$ such that $0 \leq \deg h(X) < \deg F^*(X)$. As a direct consequence, we will obtain that the value group of ν coincides with the value group of its extension $\tilde{\nu}$. From this, we deduce that Theorem 1.(2) holds.

3.1. **Initial forms with respect to a valuation.** We begin by recalling some definitions and results that we need for the understanding of the rest of the section.

A valuation ν centered in a local domain R determines a filtration on R indexed by the semigroup of values $\Gamma = \nu(R \setminus \{0\})$. This filtration is defined by the ideals

$$\mathcal{P}_\varphi(R) = \{z \in R \,|\, \nu(z) \geq \varphi\} \text{ and } \mathcal{P}^+_\varphi(R) = \{z \in R \,|\, \nu(z) > \varphi\}.$$

To this filtration is associated a graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\varphi \in \Gamma} \frac{\mathcal{P}_\varphi(R)}{\mathcal{P}^+_\varphi(R)},$$

where each element z of $R \setminus \{0\}$ has a non zero *initial form* $\mathrm{in}_\nu z$, its image in the quotient $\frac{\mathcal{P}_{\nu(z)}(R)}{\mathcal{P}^+_{\nu(z)}(R)}$. By construction, the value of the difference of two elements is larger than the value of each if and only if they have the same initial form.

Let K be the fraction field of R and let Φ be the value group of the valuation ν, that is, $\Phi = \nu(K \setminus \{0\})$. The *rank*, or *height*, of ν is the the *cardinal*[3] of the totally ordered set (for the order opposite to inclusion) of nonzero prime ideals of R_ν, or equivalently of the totally ordered set (for inclusion) of convex subgroups of Φ different from Φ. We refer to [ZS, Chapter VI, Theorem 15] or [B, Chapter VI, § 4, no. 5, Definition 2] for details. If the *rational rank* $\dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ of ν is finite, for example if the ring R is noetherian, (see [ZS, Appendix 2, Proposition 1, Proposition 2]), the rank is finite.

Let Ψ be a proper convex subgroup of Φ, $\Psi \neq (0)$. Let m_Ψ (resp. p_Ψ) be the prime ideal of R_ν (resp. R) corresponding to Ψ, that is,

$$m_\Psi = \{x \in R_\nu \,|\, \nu(x) \notin \Psi\} \text{ and } p_\Psi = m_\Psi \cap R.$$

The valuation ν is composed of a *residual valuation* $\bar{\nu}_\Psi$, whose valuation ring $R_{\bar{\nu}_\Psi}$ is the quotient R_ν/m_Ψ and with values in Ψ, and a valuation ν'_Ψ whose valuation ring is the localization R_{m_Ψ} and with values in Φ/Ψ. With the usual notation, $\nu = \nu'_\Psi \circ \bar{\nu}_\Psi$. For every $x \in R_\nu \setminus m_\Psi$, we have $\bar{\nu}_\Psi(\bar{x}) = \nu(x)$, where $\bar{x}$ denotes the residue class of x in R_ν/m_Ψ. We have an injective local ring map $R/p_\Psi \hookrightarrow R_\nu/m_\Psi$ and the valuation $\bar{\nu}_\Psi$ induces by restriction a valuation centered in R/p_Ψ (with value group contained in Ψ). We denote this valuation also by $\bar{\nu}_\Psi$ and call it the *residual valuation on* R/p_Ψ. We extend its definition to the case of $\Psi = \Phi$ setting $p_\Phi = (0)$ and $\bar{\nu}_\Phi = \nu$.

Let $K \hookrightarrow L$ be an algebraic field extension. For any valuation $\tilde{\nu}$ of L extending ν, by [B, Ch. VI, § 8, no. 1, Lemme 2] the value group $\tilde{\Phi}$ of $\tilde{\nu}$ contains Φ as a subgroup of finite index, so that the map $\tilde{\Psi} \mapsto \tilde{\Psi} \cap \Phi$ is an ordered bijection from the set of convex subgroups of $\tilde{\Phi}$ to the set of convex subgroups of Φ (in general this map is surjective). The inverse map associates to a convex subgroup $\Psi \subset \Phi$ the smallest convex subgroup of $\tilde{\Phi}$ containing Ψ. Each convex subgroup $\tilde{\Psi}$ of $\tilde{\Phi}$ contains $\Psi = \tilde{\Psi} \cap \Phi$ as a subgroup of finite index. In particular, Ψ is cofinal in $\tilde{\Psi}$.

3.2. **Ostrowski's Lemma and initial forms.** Now return to our setting. Let us consider the sequences $(\delta_i)_{i \in \mathbf{N}}$ and $(\sigma_i)_{i \geq 1}$ attached to the Nagata polynomial $F(X) \in R[X]$ (see Definition 2.5). For $i \geq 1$, set

$$\eta_i := \sigma_\infty - \sigma_i \in R[\sigma_\infty]_*.$$

By Proposition 2.7, we have that $\tilde{\nu}(\eta_i) = \nu(\delta_i)$ for all $i \geq 1$.

Let $h(X)$ be a polynomial in $R[X]$ of degree $s \geq 0$. We note that $h(\sigma_i) \in R$ for all $i \geq 1$ and we are going to study the behavior of the $\nu(h(\sigma_i))$ as i increases. Since the Nagata extension is non trivial, the σ_i are all different and thus $h(\sigma_i) \neq 0$ for all i large enough.

Consider the usual expansion $h(X + \alpha) = \sum_{m=0}^{s} h_m(X)\alpha^m$ of $h(X + \alpha)$ as a polynomial in X and α. If the polynomial $h_m(X)$ is not zero, then its degree is $s - m$.

[3]The set of convex subgroups of Φ may not be well ordered; see [B, Exercise 3 to Chap.VI, § 4], where it appears that the set of those convex subgroups of a totally ordered abelian group that are *principal* can realize any totally ordered set. However, the smallest convex subgroup containing a subset of Φ exists as the intersection of such convex subgroups.

Remark 3.1. The maps $\partial_m\colon h(X)\mapsto h_m(X)$ are Hasse–Schmidt derivations satisfying the identities $\partial_m\circ\partial_{m'}=\partial_{m'}\circ\partial_m=\binom{m+m'}{m}\partial_{m+m'}$. Some use the mnemonic notation $\partial_m=\frac{1}{m!}\frac{\partial^m}{\partial X^m}$.

We have the following identities in $K(\sigma_\infty)$:

$$h(\sigma_\infty)=h(\sigma_i)+\sum_{m=1}^{s}h_m(\sigma_i)\eta_i^m, \tag{$\star$}$$

$$h(\sigma_i)=h(\sigma_\infty)+\sum_{m=1}^{s}h_m(\sigma_\infty)(-1)^m\eta_i^m. \tag{$\star\star$}$$

Since $\sigma_{i+1}=\sigma_i+\delta_i$, we also have the identity:

$$h(\sigma_{i+1})=h(\sigma_i)+\sum_{m=1}^{s}h_m(\sigma_i)\delta_i^m. \tag{$\star\star\star$}$$

Lemma 3.2. *The subgroup of the value group Φ of ν generated by the valuations of the δ_i is finitely generated and therefore of finite rational rank.*

Proof. Starting from the equalities $F'(\sigma_i)\delta_i=-F(\sigma_i)=-\sum_{m=1}^n F_m(\sigma_\infty)(-1)^m\eta_i^m$ which follow from ($\star\star$) and applying again the equality ($\star\star$) to the derivative $F'(X)=F_1(X)$, we obtain the following equality, where the $F'_q(X)$ are those polynomials occurring in the expansion $F'(X+\alpha)=F'(X)+\sum_{q=1}^{n-1}F'_q(X)\alpha^q$:

$$F'(\sigma_\infty)\eta_{i+1}=\sum_{m=2}^{n}F_m(\sigma_\infty)(-1)^m\eta_i^m+\sum_{q=1}^{n-1}F'_q(\sigma_\infty)(-1)^q\eta_i^q\delta_i.$$

By construction, we have the identity $F'_q(X)=(q+1)F_{q+1}(X)$ for $1\le q\le n-1$, so that the previous equality can be rewritten as

$$F'(\sigma_\infty)\eta_{i+1}=\sum_{m=2}^{n}(\eta_i-m\delta_i)F_m(\sigma_\infty)(-1)^m\eta_i^{m-1}.$$

For $2\le m\le n$, we have $\tilde\nu(\eta_i-m\delta_i)=\tilde\nu(\eta_{i+1}-(m-1)\delta_i)=\nu(\delta_i)$. Since the sequence $(\nu(\delta_i))_{i\in\mathbf{N}}$ is strictly increasing, the $\tilde\nu$-values of any two terms of the sum of the right hand side are different for all large enough i. Therefore, remembering that $\tilde\nu(F'(\sigma_\infty))=0$ and that none of the δ_i or η_i is zero (because the extension $R\to S$ is non trivial), this equality shows that $\tilde\nu(\eta_{i+1})=\nu(\delta_{i+1})$ is, for large i, in the semigroup generated by the ν-values of finitely many δ_i and the $\tilde\nu$-value of the $F_m(\sigma_\infty)$, $m=2,\dots,n$. Thus, the subgroup of Φ generated by the $\nu(\delta_i)$ is contained in a finitely generated subgroup of the value group $\tilde\Phi$ of $\tilde\nu$ and so is finitely generated. □

Remark 3.3. The fact that the group generated by the $\nu(\delta_i)$ is of finite rational rank also follows from Abhyankar's inequality since this group is contained in the value group of the restriction of ν to a subring of R which is essentially of finite type over the prime ring and contains the coefficients of $F(X)$. Lemma 3.2 gives a stronger result.

Since the sequence $(\nu(\delta_i))_{i\in\mathbf{N}}$ is strictly increasing, for large i there are no two terms of the sum in ($\star\star$) with the same $\tilde\nu$-value. However we do not see immediately that there exists an index i_0 such that their $\tilde\nu$-values remain in the same order for $i\ge i_0$. The purpose of the next proposition is to establish this, at least for groups of finite rank (see Remark 3.6 below).

Proposition 3.4. (Ostrowski-Kaplansky; see [O, Teil III, statement IV, p.371 ff.] and [K, Lemma 4]) *Let Φ be a totally ordered abelian group of finite rank. Let $\beta_1,\dots,\beta_s\in\Phi$ and distinct integers $t_1,\dots,t_s\in\mathbf{N}\setminus\{0\}$ be given. Let $(\gamma_\tau)_{\tau\in T}$ be a strictly increasing family of elements of Φ indexed by a well ordered set T without last element. There exist an element $\iota\in T$ and a permutation $(k_1,\dots,k_s)$ of $(1,\dots,s)$ such that for all $\tau\ge\iota$ we have the inequalities*

$$\beta_{k_1}+t_{k_1}\gamma_\tau<\beta_{k_2}+t_{k_2}\gamma_\tau<\dots<\beta_{k_s}+t_{k_s}\gamma_\tau.$$

Proof. It suffices to prove the following statement, due to Ostrowski: There exist a $\iota \in T$ and a $k \in \{1, \dots, s\}$ such that for all $\tau \geq \iota$ we have $\beta_k + t_k\gamma_\tau < \beta_j + t_j\gamma_\tau$ for all $j \neq k$, and then repeat the argument with $\{1, \dots, s\} \setminus \{k\}$ and take the largest ι obtained at the end.

We use induction on the rank h of Φ. If $h = 1$, we may assume that Φ is an ordered subgroup of $\mathbf{R}$. Let M be the the upper bound in $\mathbf{R}$ of the set $(\gamma_\tau)_{\tau\in T}$, with $M = \infty$ if the set is not bounded. Consider the equations

$$\beta_\ell + t_\ell x = \beta_j + t_j x \ \text{ for all pairs } (\ell, j) \text{ with } \ell \neq j.$$

Since the t_i are all different, their solutions form a finite set. We denote by A the largest one which is less than M, and set $A = 0$ if there is no such solution which is greater or equal than zero. For $A < x < M$ the values of the functions $\phi_j = \beta_j + t_j x$ remain distinct. Denote by γ_η the smallest member of the family (γ_τ) which is greater than A and let k be the integer such that $\phi_k(\gamma_\eta) < \phi_j(\gamma_\eta)$ for $j \neq k$. Since the functions $\phi_k(x) - \phi_j(x)$ have no zero between γ_η and M the orders of their values are preserved, so that $\beta_k + t_k\gamma_\tau < \beta_j + t_j\gamma_\tau$ for $\tau \geq \eta$, and the value M is never reached since T has no last element and the family $(\gamma_\tau)_{\tau\in T}$ is strictly increasing. This proves the proposition for $h = 1$. (This is Ostrowski's original proof).

Assume now that the result is true for all ranks less than h, where h is the rank of Φ. Let $\Psi_1 \subset \Phi$ be the largest convex subgroup and let us denote by $\pi\colon \Phi \to \Phi/\Psi_1$ the natural map. If the family $(\pi(\gamma_\tau))_{\tau\in T}$ is strictly increasing at least for large τ, we apply the previous argument to the rank one group Φ/Ψ_1 and obtain the result. Otherwise there exist $\theta \in T$ and $\overline{\delta} \in \Phi/\Psi_1$ such that $\pi(\gamma_\tau) = \overline{\delta}$ for $\tau \geq \theta$. We consider the minimum value of the $\pi(\beta_i) + t_i\overline{\delta}$ for $i = 1, \dots, m$ and denote it by $\overline{\zeta}$. If this minimum value is attained for a single index k our lemma is proved. Otherwise, denoting by $J \subset \{1, \dots, s\}$ the set of indices i such that $\pi(\beta_i) + t_i\overline{\delta} = \overline{\zeta}$, we choose an element δ in $\pi^{-1}(\overline{\delta})$ which is smaller than all the $\gamma_\tau \in \pi^{-1}(\overline{\delta})$. Then, we choose an element $\zeta \in \pi^{-1}(\overline{\zeta})$ which is smaller than the $\beta_i + t_i\delta$ for all indices $i \in J$. Applying the induction hypothesis in Ψ_1 with the $\beta'_i = \beta_i + t_i\delta - \zeta$ for $i \in J$ and the $\gamma'_\tau = \gamma_\tau - \delta$, and with $T' = \{\tau \in T \mid \tau \geq \theta\}$, proves the proposition. □

Corollary 3.5. *In the situation of Proposition 3.4, if the subgroup of Φ generated by the elements γ_τ is of finite rational rank, then the proposition is valid for this family $(\gamma_\tau)_{\tau\in T}$.*

Proof. Apply Proposition 3.4 to the subgroup generated by $\beta_1, \dots, \beta_s, (\gamma_\tau)_{\tau\in T}$; it is of finite rational rank, hence of finite rank. □

Thus, in view of Lemma 3.2, we do not need in what follows to assume that the value group Φ of ν is of finite rank to use Proposition 3.4 taking as family $(\gamma_\tau)_{\tau\in T}$ the sequence $(\nu(\delta_i))_{i\in\mathbf{N}}$.

Remark 3.6. Another proof of Proposition 3.4, which does not assume Φ to be of finite rank, is given by F.-V. Kuhlmann in [Ku3, Lemma 8.8], which is unfortunately not yet published. With this proof, we do not need Lemma 3.2 and Corollary 3.5 to use Proposition 3.4.

We can now use Proposition 3.4 to prove the

Proposition 3.7. *Let $h(X) \in R[X]$ be a polynomial of degree $s > 0$. Keep the notations of the identity ($\star\star$) above. There exist $i_0 \in \mathbf{N}$ and $k \in \{1, \dots, s\}$ such that for $i \geq i_0$ we have*

$$\mathrm{in}_{\tilde{\nu}}(h(\sigma_\infty) - h(\sigma_i)) = -\mathrm{in}_{\tilde{\nu}}(h_k(\sigma_\infty)(-1)^k\eta_i^k).$$

In particular, $\tilde{\nu}(h(\sigma_\infty) - h(\sigma_i)) = \tilde{\nu}(h_k(\sigma_\infty)(-1)^k\eta_i^k)$ for $i \geq i_0$ and $h(\sigma_\infty)$ is a limit for the valuation $\tilde{\nu}$ of the pseudo–convergent sequence $(h(\sigma_i))_{i\geq i_0}$.

Proof. The polynomial $h_s(X)$ is a nonzero constant polynomial. If $s = 1$ then the first statement is trivial. In the general case it is enough to apply Proposition 3.4 to the $\beta_m = \tilde{\nu}(h_m(\sigma_\infty))$ in the value group $\tilde{\Phi}$ of $\tilde{\nu}$, with $t_m = m$, $\gamma_i = \nu(\delta_i)$, and $T = \mathbf{N}$, recalling that $\tilde{\nu}(\eta_i) = \nu(\delta_i)$ and $\nu(\delta_{i+1}) \geq 2\nu(\delta_i)$.

The second statement follows directly from the first. In order to prove the last part of the proposition it is enough to observe that if $h(\sigma_\infty) \in K(\sigma_\infty)$ is such that $\tilde{\nu}(h(\sigma_\infty) - h(\sigma_i))$ is less than $\tilde{\nu}(h(\sigma_\infty) - h(\sigma_j))$ for $i_0 \leq i < j$, then $(h(\sigma_i))_{i\geq i_0}$ is necessarily a pseudo–convergent sequence for $\tilde{\nu}$ having $h(\sigma_\infty)$ as a limit. Since $\tilde{\nu}(\eta_i) < \tilde{\nu}(\eta_j)$ for $i < j$, this ends the proof. □

Remarks 3.8. (1) Proposition 3.7 is essentially a slightly more precise version of a result of Ostrowski (see [O, part III, statement III, p.371 ff.], [K, Lemma 5] and [Ku3, Chapter 8]) to the effect that the values taken by a polynomial with coefficients in R on a pseudo–convergent sequence of elements of R (in this case the σ_i) form themselves a pseudo–convergent sequence and therefore their valuations are eventually either constant or strictly increasing. Ostrowski's result, proved for rank one valuations in [O], stated for arbitrary rank in [K] and proved in [Ku3], is more general in that it applies to all pseudo–convergent sequences.
(2) We shall see below in Proposition 4.4 that there exist an index i_0, an element $a \in R_\nu$ and an integer e with $0 \leq e \leq s$ such that for $i \geq i_0$ we have $\mathrm{in}_\nu h(\sigma_i) = \mathrm{in}_\nu(a\delta_i^e)$.

Corollary 3.9. *Keep the notations of Proposition 3.7. If the sequence $(\nu(h(\sigma_i)))_{i\geq 1}$ is eventually constant, then $\mathrm{in}_{\tilde\nu}(h(\sigma_\infty)) = \mathrm{in}_\nu(h(\sigma_i))$ for all i large enough. Otherwise, for all i large enough we have $\tilde\nu(h(\sigma_\infty)) > \nu(h(\sigma_{i+1})) > \nu(h(\sigma_i))$ and $\mathrm{in}_\nu(h(\sigma_i)) = -\mathrm{in}_{\tilde\nu}(h_k(\sigma_\infty)(-1)^k\eta_i^k)$.*

Proof. Since $(h(\sigma_i))_{i\geq i_0}$ is pseudo–convergent, then either there exists $i_1 \geq i_0$ such that the sequence $(\nu(h(\sigma_i)))_{i\geq i_1}$ is constant or $\nu(h(\sigma_j)) > \nu(h(\sigma_i))$ whenever $j > i \geq i_0$.

Assume that we are in the first case and call $\varphi = \nu(h(\sigma_i))$, $i \geq i_1$. Since $(\tilde\nu(\eta_i))_{i\geq 1}$ is strictly increasing, for all i large enough, φ is different from $\tilde\nu(h_k(\sigma_\infty)(-1)^k\eta_i^k)$. By the same argument, $\tilde\nu(h(\sigma_\infty)) \neq \tilde\nu(h_k(\sigma_\infty)(-1)^k\eta_i^k)$ for any sufficiently large i. Then, for large i we must have $\tilde\nu(h(\sigma_\infty)) = \nu(h(\sigma_i)) < \tilde\nu(h(\sigma_\infty) - h(\sigma_i)) = \tilde\nu(h_k(\sigma_\infty)(-1)^k\eta_i^k)$ and, as a consequence, $\mathrm{in}_{\tilde\nu}(h(\sigma_\infty)) = \mathrm{in}_\nu(h(\sigma_i))$.

If $\nu(h(\sigma_j)) > \nu(h(\sigma_i))$ for $j > i \geq i_0$, then $y = 0$ is a limit for $\tilde\nu$ of the pseudo–convergent sequence $(h(\sigma_i))_{i\geq i_0}$. If $h(\sigma_\infty) = 0$ then the result is clear because $h(\sigma_i) \neq 0$ for large i, so assume that $h(\sigma_\infty) \neq 0$. From the fact that $h(\sigma_\infty)$ is also a limit we deduce that

$$\tilde\nu(h(\sigma_\infty)) = \tilde\nu(h(\sigma_\infty) - y) > \nu(h(\sigma_j) - h(\sigma_i)) = \nu(h(\sigma_i))$$

for $j > i \geq i_0$. Hence for $i \geq i_0$, $\mathrm{in}_{\tilde\nu}(h(\sigma_\infty) - h(\sigma_i)) = -\mathrm{in}_\nu(h(\sigma_i))$, which coincides with $\mathrm{in}_{\tilde\nu}(h_k(\sigma_\infty)(-1)^k\eta_i^k)$. □

3.3. **The convex subgroups associated to a Nagata polynomial.** In this subsection we denote by Φ the value group of ν. We make the first steps towards the computation of $\tilde\nu$. Our main tools are Lemma 3.14, Corollary 3.9, and two convex subgroups of Φ that we associate to a Nagata polynomial, one of them invariant after any change of variable $X \mapsto X' + \alpha$ in $R[X]$ with $\alpha \in m_R$.

Definition 3.10. Let ν be a valuation centered in a local domain R and let Φ be its value group. Let $F(X) \in R[X]$ be a Nagata polynomial such that $\delta_i \neq 0$ for all $i \geq 0$, where $(\delta_i)_{i\in\mathbf{N}}$ is the Newton sequence of values attached to it. The *convex subgroup Ψ_F of Φ associated to $F(X)$* is the smallest convex subgroup of Φ containing all the $\nu(\delta_i)$.

In particular the subgroup Ψ_F is defined for Nagata polynomials giving rise to a non trivial Nagata extension. We have $\Psi_F \neq (0)$ because all the δ_i belong to m_R. We now come back to the behavior of the δ_i with the following two observations:

Lemma 3.11. *Let $\tilde\Phi$ be the value group of $\tilde\nu$ and let $\widetilde{\Psi_F}$ be the smallest convex subgroup of $\tilde\Phi$ containing Ψ_F. The $\nu(\delta_i)$ are cofinal in Ψ_F and therefore in $\widetilde{\Psi_F}$.*

Proof. We have seen in Subsection 3.1 that Ψ_F is cofinal in $\widetilde{\Psi_F}$. Next we prove that $(\nu(\delta_i))_{i\in\mathbf{N}}$ is cofinal in Ψ_F.

Observe that Ψ_F is the smallest convex subgroup of Φ which contains the subgroup $\Phi' \subset \Phi$ generated by the $\nu(\delta_i)$. Let us show that Φ' is cofinal in Ψ_F. It suffices to deal with the positive semigroups of the groups in sight. If there exists $\varphi \in \Psi_{F,>0}$ such that for any $\zeta \in \Phi'_{>0}$ we have $\zeta < \varphi$, we see that the elements $\{\theta \in \Phi_{\geq 0} \,|\, \forall n \in \mathbf{N},\ n\theta < \varphi\}$ form a semigroup: given θ, θ' with this property we may assume that $\theta \leq \theta'$ and then $n(\theta + \theta') \leq 2n\theta' < \varphi$ for all $n \in \mathbf{N}$. This semigroup contains $\Phi'_{\geq 0}$, is the positive part of a convex subgroup of Φ and does not contain φ. This contradicts the minimality of Ψ_F. Thus, Φ' is cofinal in Ψ_F and to finish the proof it suffices to show that the $\nu(\delta_i)$ are cofinal in Φ', which has finite rank by Lemma 3.2.

Let Ψ'_1 be the largest proper convex subgroup of Φ' (it exists because Φ' has finite rank). By construction it cannot contain all the $\nu(\delta_i)$. We know from Proposition 2.4 that $\nu(\delta_{i+1}) \geq 2\nu(\delta_i)$.

The images of the $\nu(\delta_i)$ in the rank one group Φ'/Ψ'_1 satisfy the same inequality and therefore are cofinal in this archimedian ordered group. This implies that the $\nu(\delta_i)$ are cofinal in Φ'. $\square$

Lemma 3.12. *There exists $i_0 \geq 0$ such that Ψ_F is the smallest convex subgroup of Φ containing $\nu(\delta_{i_0})$.*

Proof. By Lemma 3.2, the subgroup Φ' of Φ generated by the $\nu(\delta_i)$ is finitely generated, so there exists $i_0 \geq 0$ such that $\Phi' = \mathbf{Z}\nu(\delta_0) + \ldots + \mathbf{Z}\nu(\delta_{i_0})$. Therefore Ψ_F is the smallest convex subgroup of Φ that contains the set $\{\nu(\delta_i)\}_{i=0}^{i_0}$. To finish the proof, use that $\nu(\delta_i) < \nu(\delta_{i+1})$. $\square$

Remark 3.13. With the notations of Lemma 3.11, if $\Psi_F = \Phi$ then $\widetilde{\Psi_F} = \tilde{\Phi}$.

Let $\bar{\nu}_{\Psi_F}$ be the residual valuation on R/p_{Ψ_F} (see Subsection 3.1). We call L_F the fraction field of R/p_{Ψ_F} and we fix an algebraic closure $\overline{L}_F$ of L_F. Given $a \in R$, we denote by $\bar{a}$ the residue class of a modulo p_{Ψ_F}.

The image $\overline{F}(X) \in R/p_{\Psi_F}[X]$ of the Nagata polynomial $F(X) \in R[X]$ is again a Nagata polynomial. In addition, the Newton sequence of values and the sequence of partial sums attached to $\overline{F}(X)$ are $(\bar{\delta}_i)_{i\in\mathbf{N}}$ and $(\bar{\sigma}_i)_{i\geq 1}$, respectively (that is, they are obtained from the sequences attached to $F(X)$ by reduction modulo p_{Ψ_F}). By construction, $\bar{\delta}_i$ is different from zero and $\nu(\delta_i) = \bar{\nu}_{\Psi_F}(\bar{\delta}_i)$ for all $i \geq 0$, and the convex subgroup $\Psi_{\overline{F}}$ associated to $\overline{F}(X)$ is the whole group of the valuation $\bar{\nu}_{\Psi_F}$.

Hence we have a Nagata extension $R/p_{\Psi_F} \to S_F = (R/p_{\Psi_F}[X]/(\overline{F}(X)))_{\mathcal{N}_F}$, where $\mathcal{N}_F$ corresponds to the maximal ideal $(m_R/p_{\Psi_F}, X)$ of $R/p_{\Psi_F}[X]$; and a valuation $\bar{\nu}_{\Psi_F}$, in the sequel called $\bar{\nu}_F$, which is centered in the local domain R/p_{Ψ_F}. According to what we saw in Section 2, the valuation $\bar{\nu}_F$ determines a minimal prime ideal $H_{S_F}(\bar{\nu}_F)$ of S_F with the property that $\bar{\nu}_F$ extends uniquely to a valuation $\tilde{\bar{\nu}}_F$ centered in $S_F/H_{S_F}(\bar{\nu}_F)$ through the inclusion $R/p_{\Psi_F} \subset S_F/H_{S_F}(\bar{\nu}_F)$. We present the quotient $S_F/H_{S_F}(\bar{\nu}_F)$ in the form $R/p_{\Psi_F}[\bar{\sigma}_\infty]_*$ where $\bar{\sigma}_\infty \in \overline{L}_F$ satisfies $\overline{F}(\bar{\sigma}_\infty) = 0$ and $\tilde{\bar{\nu}}_F(\bar{\sigma}_\infty) > 0$.

We shall need the following lemma, in which we keep these notations. Observe that in the case where $\Psi_F = \Phi$ it also holds (one has $R/p_{\Psi_F} = R$, $S_F = S$, and $\bar{\nu}_F = \nu$).

Lemma 3.14. *Let $h(X) \in R[X]$ be such that its image $\bar{h}(X)$ in $R/p_{\Psi_F}[X]$ is not zero. Then for all i large enough we have $\nu(h(\sigma_i)) \in \Psi_F$, and the following are equivalent:*

(1) *The sequence $(\nu(h(\sigma_i)))_{i\geq 1}$ is not eventually constant.*
(2) *The image of $\bar{h}(X)$ in S_F belongs to $H_{S_F}(\bar{\nu}_F)$.*

Proof. This is automatically true if $\bar{h}(X)$ is a non zero constant polynomial, so assume that $\deg \bar{h}(X) > 0$. The elements $\bar{h}(\bar{\sigma}_i) \in R/p_{\Psi_F}$ are the images of the $h(\sigma_i)$ under the natural epimorphism $R \to R/p_{\Psi_F}$, and they cannot be zero for infinitely many values of i, so that $h(\sigma_i) \notin p_{\Psi_F}$, which means that $\nu(h(\sigma_i)) = \bar{\nu}_F(\bar{h}(\bar{\sigma}_i)) \in \Psi_F$, at least for large i. This equality proves the first part of the result and will be implicitly used in what follows.

Let us now prove that (1) and (2) are equivalent. Recall that (2) holds if and only if $\bar{h}(\bar{\sigma}_\infty) = 0$ in $L_F(\bar{\sigma}_\infty)$. In view of Corollary 3.9, if $\nu(h(\sigma_i))$ is constant for large i, its value is the $\tilde{\bar{\nu}}_F$-value of the element $\bar{h}(\bar{\sigma}_\infty)$ and is in Ψ_F, therefore $\bar{h}(\bar{\sigma}_\infty) \neq 0$. If $\nu(h(\sigma_i))$ is not constant for large i, by Corollary 3.9 there exists k such that $1 \leq k \leq \deg \bar{h}(X)$ and

$$\tilde{\bar{\nu}}_F(\bar{h}(\bar{\sigma}_\infty)) > \bar{\nu}_F(\bar{h}(\bar{\sigma}_i)) = \tilde{\bar{\nu}}_F(\bar{h}_k(\bar{\sigma}_\infty)(-1)^k\bar{\eta}_i^k) = \tilde{\bar{\nu}}_F(\bar{h}_k(\bar{\sigma}_\infty)) + k\,\bar{\nu}_F(\bar{\delta}_i)$$

for all i large enough, where $\bar{\eta}_i = \bar{\sigma}_\infty - \bar{\sigma}_i$ and $\bar{h}_k(X) \in R/p_{\Psi_F}[X]$. These inequalities and Lemma 3.11 imply that the $\bar{\nu}_F(\bar{h}(\bar{\sigma}_i))$ are cofinal in the value group of $\tilde{\bar{\nu}}_F$. Thus, $\bar{h}(\bar{\sigma}_\infty) = 0$. $\square$

Remark 3.15. We know how to compute the $\tilde{\nu}$-value of any non zero element of $K(\sigma_\infty)$ once we know to calculate $\tilde{\nu}(h(\sigma_\infty))$ for $h(X) \in R[X]$ such that $0 < \deg h(X) < \deg F^*(X)$. Let us apply Lemma 3.14 and Corollary 3.9 to such a polynomial. When $\Psi_F = \Phi$, these results say that $\tilde{\nu}(h(\sigma_\infty)) = \nu(h(\sigma_i))$ for all i large enough. If $\Psi_F \subsetneq \Phi$, then we would reach the same conclusion if could guarantee that $\bar{h}(X) \in R/p_{\Psi_F}[X]$ is a non zero polynomial whose degree is less that the degree of the minimal polynomial of $\bar{\sigma}_\infty$ over L_F. This condition on the degree is satisfied if $F(X)$ becomes irreducible, more precisely, when its image $\overline{F}(X) \in R/p_{\Psi_F}[X]$ is irreducible in $L_F[X]$. Observe that $\deg \bar{h}(X) \leq \deg h(X) < \deg F^*(X) \leq \deg F(X) = \deg \overline{F}(X)$.

As explained in Remark 2.2, given $\alpha \in m_R$, $F_\alpha(X') := F(X'+\alpha) \in R[X']$ is a Nagata polynomial defining a Nagata extension S_α of R that is isomorphic to S. Moreover, since $\nu(F^*(\alpha)) > 0$ we have $F^*_\alpha(X') = F^*(X'+\alpha)$, and therefore $S/H_S(\nu)$ is isomorphic to $S_\alpha/H_{S_\alpha}(\nu)$.

So we may first make a change of variable $X' = X - \alpha$ with $\alpha \in m_R$, and then construct the Newton sequence of values $(\delta_i^{(\alpha)})_{i\in\mathbf{N}}$ and the sequence of partial sums $(\sigma_i^{(\alpha)})_{i\geq 1}$ attached to the polynomial $F_\alpha(X')$ by iterating Newton's method.

Since the Nagata extension $R \to S$ is non trivial, $F(a) \neq 0$ for all $a \in m_R$, and the set $\{\nu(F(a)) \,|\, a \in m_R\}$ is contained in Φ. This subset of the value group does not depend on the choice of the variable: for all $\alpha \in m_R$, it equals the set $\{\nu(F_\alpha(a)) \,|\, a \in m_R\} \subseteq \Phi$.

Definition 3.16. Let ν be a valuation centered in a local domain R and let Φ be its value group. Let $F(X) \in R[X]$ be a Nagata polynomial defining a non trivial Nagata extension of R. The *intrinsic convex subgroup* Ψ *of* Φ *associated to* $F(X)$ is the smallest convex subgroup of Φ containing all the $\nu(F(a))$ with $a \in m_R$.

Given $\alpha \in m_R$, in Definition 3.10 we associated to the polynomial $F_\alpha(X') \in R[X']$ a convex subgroup Ψ_{F_α} of Φ. Next we explain the relationship between the intrinsic convex subgroup Ψ and these Ψ_{F_α}.

Lemma 3.17. *With the notations of this subsection, we have:*

(1) $\Psi_{F_\alpha} \subseteq \Psi$.
(2) $\Psi_{F_\alpha} = \Psi$ *if and only if* $\overline{F_\alpha}(X') \in R/p_{\Psi_{F_\alpha}}[X']$ *defines a non trivial Nagata extension of the local domain* $R/p_{\Psi_{F_\alpha}}$.

Proof. Statement (1) follows from the identities $\nu(\delta_0^{(\alpha)}) = \nu(F(\alpha))$ and $\nu(\delta_i^{(\alpha)}) = \nu(F(\sigma_i^{(\alpha)}+\alpha))$ for $i \geq 1$. Having a strict inclusion $\Psi_{F_\alpha} \subsetneq \Psi$ is equivalent to the existence of $a \in m_R$ such that $F_\alpha(a) \in p_{\Psi_{F_\alpha}}$. In turn, this is equivalent to saying that the polynomial $\overline{F_\alpha}(X') \in R/p_{\Psi_{F_\alpha}}[X']$ has a root in $m_R/p_{\Psi_{F_\alpha}}$, which means that it defines a trivial Nagata extension of $R/p_{\Psi_{F_\alpha}}$. □

Definition 3.18. We say that $X' = X - \alpha$, where $\alpha \in m_R$, is a *good variable* if the following conditions are satisfied:

(1) $\Psi_{F_\alpha} = \Psi$.
(2) Ψ_{F_α} is the smallest convex subgroup of Φ containing $\nu(\delta_0^{(\alpha)})$.

In the finite rank case, we can assume that our Nagata extension is of the form S_α with $\alpha \in m_R$ defining a good variable:

Lemma 3.19. *If the rank of* ν *is finite then there exists a good variable.*

Proof. We first concentrate on (1) in Definition 3.18. If it is satisfied for $\alpha = 0$ then we are done. Assume that this is not the case. Then we can choose $a_1 \in m_R$ such that $F(a_1) \in p_{\Psi_F}$. Set $\alpha_1 = a_1$ and consider $F_{\alpha_1}(X') = F(X'+a_1)$. Since $\nu(\delta_0^{(\alpha_1)}) = \nu(F(a_1)) \notin \Psi_F$ and the convex subgroups of Φ are totally ordered by inclusion, it follows that $\Psi_F \subsetneq \Psi_{F_{\alpha_1}}$. If $\Psi_{F_{\alpha_1}} = \Phi$, then α_1 satisfies (1) and we stop. Otherwise we pick $a_2 \in m_R$ such that $F_{\alpha_1}(a_2)$ belongs to $p_{\Psi_{F_{\alpha_1}}}$ and set $\alpha_2 = \alpha_1 + a_2$. Now consider the polynomial $F_{\alpha_2}(X') = F(X'+\alpha_2)$. We have $\nu(\delta_0^{(\alpha_2)}) = \nu(F_{\alpha_1}(a_2)) \notin \Psi_{F_{\alpha_1}}$ and, by the same argument as before, we get a chain $\Psi_F \subsetneq \Psi_{F_{\alpha_1}} \subsetneq \Psi_{F_{\alpha_2}} \subseteq \Phi$. After a finite number of iterations this process has to stop because our assumption on its rank. But it cannot stop unless $\Psi_{F_\alpha} = \Psi$.

Suppose that $\Psi_F = \Psi$ but (2) in Definition 3.18 is not satisfied. By Lemma 3.12, there exists $i_0 > 0$ such that Ψ_F is the smallest convex subgroup of Φ containing $\nu(\delta_{i_0})$. Then $X' = X - \sigma_{i_0}$ is a good variable. □

Remark 3.20. The assumption on the rank of the valuation is needed to guarantee that condition (1) in Definition 3.18 can be achieved.

3.4. **End of the proof of Theorem 1.** Finally, we prove the main result of this section and obtain Theorem 1.(2) as a corollary of it.

Proposition 3.21. *Let ν be a valuation centered in a local domain R with value group Φ. Let $R \to S$ be a Nagata extension and let $\tilde{\nu}$ be the unique valuation centered in $S/H_S(\nu)$ extending ν through the inclusion $R \subset S/H_S(\nu)$. Then, the value group of $\tilde{\nu}$ is Φ.*

Proof. Let $F(X)$ be a Nagata polynomial of degree n defining the extension $R \to S$, which we assume to be non trivial.

Reduction to the case where R is integrally closed: Keep the notation introduced before Lemma 2.9. In view of the following commutative diagram (see (D_1) in Remark 2.11),

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, hook] & S/H_S(\nu) \arrow[r, "\simeq"] \arrow[d, hook] & R[\sigma_\infty]_* \arrow[r, hook] \arrow[d, hook] & K(\sigma_\infty) \\
\tilde{R} \arrow[r, hook] & \tilde{S} \arrow[r, "\simeq"] & \tilde{R}[\sigma_\infty]_* \arrow[ur, hook] &
\end{tikzcd}$$

where all the vertical maps are local ring homomorphisms, it suffices to show the result for the Nagata extension $\tilde{R} \to \tilde{S}$ and the same valuations ν and $\tilde{\nu}$.

From now on, R is an integrally closed local domain. By Lemma 2.9 and our assumption on R, we can assume that $F(X) = F^*(X)$, which simplifies the notation in what follows. Any element of the value group of $\tilde{\nu}$ is of the form $\tilde{\nu}(a\, h(\sigma_\infty))$ where $a \in R \setminus \{0\}$ and $h(X)$ is a polynomial in $R[X]$ such that $0 \leq \deg h(X) < n$. Therefore it is enough to prove that $\tilde{\nu}(h(\sigma_\infty))$ belongs to Φ. If its degree is zero, then $\tilde{\nu}(h(\sigma_\infty)) = \nu(h(\sigma_\infty)) \in \Phi$; thus we may assume that $\deg h(X) > 0$.

Reduction to case where ν is of finite rank: Let P_0 be the prime ring in R, that is, $\mathbf{Z}/p\mathbf{Z}$ if R is of characteristic p, and $\mathbf{Z}$ otherwise. The ring R is the inductive limit of its P_0–subalgebras that are essentially of finite type. Since R is integrally closed, one may restrict the inductive system to the integrally closed local subalgebras of R that are essentially of finite type over P_0. This is because both $\mathbf{Z}$ and $\mathbf{F}_p$ are universally japanese (see [LM, Ch. 14, no.1]) so that the integral closure of R_0 in its field of fractions is noetherian and its localization at the center of the valuation is again noetherian. Let us consider a subalgebra A_0 of R of this type and containing the coefficients of the polynomial $F(X)$. We call $K_0 \subseteq K$ the fraction field of A_0.

The valuation ν induces a valuation ν_0 centered in A_0 whose valuation ring is $R_{\nu_0} = R_\nu \cap K_0$. Since A_0 is essentially of finite type over P_0, the value group Φ_0 of ν_0 has finite rational rank. Indeed, this rank is bounded by the transcendence degree of K_0 over the fraction field of P_0 by Abhyankar's inequality (see [B, Ch. VI, § 10, no.3, Cor.1]). Therefore Φ_0 has finite rank.

The polynomial $F(X) \in A_0[X]$ is a Nagata polynomial and, if S_0 is the Nagata extension of A_0 defined by $F(X)$ and $\tilde{\nu}_0$ is the extension of ν_0 to S_0 (note that $H_{S_0}(\nu_0) = (0)$, see Remark 2.11), we have that $R_{\tilde{\nu}_0}$ corresponds to $R_{\tilde{\nu}} \cap K_0(\sigma_\infty)$ and a commutative diagram as follows:

$$\begin{tikzcd}
A_0 \arrow[r, hook] \arrow[d, hook] & S_0 \arrow[r, "\simeq"] \arrow[d, hook] & A_0[\sigma_\infty]_* \arrow[r, hook] \arrow[d, hook] & K_0(\sigma_\infty) \arrow[d, hook] \\
R \arrow[r, hook] & S \arrow[r, "\simeq"] & R[\sigma_\infty]_* \arrow[r, hook] & K(\sigma_\infty)
\end{tikzcd}$$

where all the vertical maps are local ring homomorphisms. Since given a finite number of polynomials in $R[X]$ one may assume that A_0 contains the coefficients of all the polynomials, we see that it is enough to prove the result in the case where the local domain is integrally closed and the rank of the valuation is finite.

Proof in the case where R is integrally closed and ν is of finite rank: According to Lemma 3.14 and Corollary 3.9, if $\Psi_F = \Phi$ then $\tilde{\nu}(h(\sigma_\infty)) = \nu(h(\sigma_i))$ for all i large enough, and this ends the proof. Next we treat the case $\Psi_F \subsetneq \Phi$.

The polynomial $F(X)$ defines non trivial Nagata extensions of R and R_ν that sit in a natural commutative diagram of local ring homomorphisms:

$$(D_2) \qquad \begin{tikzcd}
R \arrow[r, hook] \arrow[d, hook] & S = R[\sigma_\infty]_* \arrow[r, hook] \arrow[d, hook] & R_{\tilde{\nu}} \\
R_\nu \arrow[r, hook] & R_\nu[\sigma_\infty]_* \arrow[ur, hook] &
\end{tikzcd}$$

Every finitely generated ideal of R_ν is principal and generated by an element of its set of generators. Therefore $h(X)$ can be written in $R_\nu[X]$ as $h(X) = h_t H(X)$ with $h_t \in R_\nu \setminus \{0\}$ and $H(X) \in R_\nu[X]$ having a coefficient equal to one.

Let $\bar\nu := \bar\nu_{\Psi_F}$ be the residual valuation (corresponding to Ψ_F) with which ν is composed and let L be the fraction field of its valuation ring $R_{\bar\nu} = R_\nu/m_{\Psi_F}$. Fix an algebraic closure of L and write the Nagata extension defined by the image $\overline{F}(X)$ of $F(X)$ in $R_{\bar\nu}[X]$ as $R_{\bar\nu} \hookrightarrow R_{\bar\nu}[\bar\sigma_\infty]_*$, where $\overline{F}(\bar\sigma_\infty) = 0$.

Let us first assume that $\overline{F}(X) \in R_{\bar\nu}[X]$ is irreducible. Since $R_{\bar\nu}$ is integrally closed (recall that any valuation ring has this property, see [B, Ch. VI, § 1, no. 3, Corollary 1]) and the polynomial is monic, this is equivalent to being irreducible in $L[X]$. Therefore $\overline{F}(X)$ is the minimal polynomial of $\bar\sigma_\infty$ over L. Then we finish the proof as follows:

On the one hand, the image $\overline{H}(X)$ of $H(X)$ in $R_{\bar\nu}[X]$ is not zero because at least one of its coefficients equals 1. On the other hand, we have $\deg \overline{H}(X) \le \deg h(X) < n = \deg \overline{F}(X)$ and thus $\overline{H}(\bar\sigma_\infty) \neq 0$. Applying Lemma 3.14 and Corollary 3.9 (to R_ν, $F(X)$, and $H(X)$) we conclude that $\tilde\nu(H(\sigma_\infty)) = \nu(H(\sigma_i))$ for all i large enough. This fact and the relation between $h(X)$ and $H(X)$ show that, for large i, $\tilde\nu(h(\sigma_\infty)) = \nu(h(\sigma_i))$.

Next we address the proof in the case where $\overline{F}(X)$ is a multiple of the minimal polynomial of $\bar\sigma_\infty$ over L. This minimal polynomial has coefficients in $R_{\bar\nu}$ by Lemma 2.9.(1). Let us write it as the image $\overline{Q}(X)$ in $R_{\bar\nu}[X]$ of a monic polynomial $Q(X) \in R_\nu[X]$ with $\deg Q(X) = \deg \overline{Q}(X) < n$. We have $\bar\nu(\overline{Q}(0)) > 0$ and an equality $\overline{F}(X) = \overline{G}(X)\overline{Q}(X)$ in $R_{\bar\nu}[X]$. Hence by Lemma 2.3, $\overline{Q}(X)$ is a Nagata polynomial and $\overline{G}(0)$ does not belong to the maximal ideal $m_{\bar\nu} = m_\nu/m_{\Psi_F}$ of $R_{\bar\nu}$. It is straightforward that $Q(X) \in R_\nu[X]$ is an irreducible Nagata polynomial.

Let $(\epsilon_i)_{i\in\mathbf{N}}$ and $(\tau_i)_{i\ge1}$ be the Newton sequence of values and the sequence of partial sums attached to $Q(X)$. We can extend $\tilde\nu$ to the algebraic closure $\overline{K}$ of K and therefore consider the distinguished root $\tau_\infty^{(0)} \in \overline{K}$ of $Q(X)$ satisfying $\tilde\nu(\tau_\infty^{(0)}) = \nu(\epsilon_0) > 0$. Note that $Q(X)$ is the minimal polynomial of $\tau_\infty^{(0)}$. In addition, by Euclidean division (in the rings $R_\nu[X]$ and $R_{\bar\nu}[X]$),

$$F(X) = G_1(X)Q(X) + B(X) \tag{1}$$

with $\overline{G}_1(X) = \overline{G}(X)$ and $\overline{B}(X) = 0$ in $R_{\bar\nu}[X]$, and thus $G_1(0) \notin m_\nu$ and $B(X) \in m_{\Psi_F} R_\nu[X]$.

Since ν has finite rank, by Lemma 3.19 we can assume that X is a good coordinate. Under this assumption we have $\Psi_F = \Psi$, and hence $\overline{Q}(X)$ and $Q(X)$ both define non trivial Nagata extensions. Indeed, $\overline{Q}(X)$ defines the same Nagata extension as $\overline{F}(X)$ and, if $Q(a) = 0$ for some $a \in m_R$, then $\nu(F(a)) = \nu(B(a)) \notin \Psi_F$, which is a contradiction.

Let Ψ_Q (resp. Ψ') be the convex subgroup (resp. the intrinsic convex subgroup) of Φ associated to $Q(X)$. Given $a \in m_R$, evaluating in a the expression (1) above, and taking into account that $\nu(G_1(a)) = 0$ and that $\nu(B(a))$ is greater than any element in Ψ_F, we obtain that $\nu(F(a)) = \nu(Q(a))$. On the one hand, by definition of Ψ' and Lemma 3.17.(1), we have $\Psi_Q \subseteq \Psi' = \Psi = \Psi_F$. On the other hand, taking $a = 0$ gives that $\nu(\delta_0) = \nu(\epsilon_0)$, and since Ψ_F is the smallest convex subgroup containing $\nu(\delta_0)$, we get $\Psi_F \subseteq \Psi_Q$. This shows that $\Psi_Q = \Psi' = \Psi = \Psi_F$.

We write the Nagata extension defined by $Q(X) \in R_\nu[X]$ as $R_\nu \hookrightarrow R_1 = R_\nu[\tau_\infty^{(0)}]_* \subset K(\tau_\infty^{(0)})$. Let ν_1 be the valuation centered in R_1 which extends ν and denote by R_{ν_1} its valuation ring. Note that R_{ν_1} coincides with $R_{\tilde\nu} \cap K(\tau_\infty^{(0)})$. Moreover, the value group of ν_1 is the group Φ. Indeed, we have $\bar\nu_{\Psi_Q} = \bar\nu_F$ and $\overline{Q}(X) \in R_{\bar\nu}[X]$ irreducible, and we have seen before that the result is true in this case. The key point is that $\Psi_Q = \Psi_F$.

Let us now consider the polynomial $F_1(X_1) = F(X_1 + \tau_\infty^{(0)}) \in R_1[X_1]$. It is still a Nagata polynomial of degree n and vanishes at $\sigma_\infty^{(1)} = \sigma_\infty - \tau_\infty^{(0)}$. In fact, $F(X)$ is a Nagata polynomial in $R_1[X]$ and $\tau_\infty^{(0)} \in m_{R_1}$, so with the notation of Subsection 3.3, $F_1(X_1)$ is the polynomial $F_\alpha(X_1)$ for $\alpha = \tau_\infty^{(0)}$, $X_1 = X - \alpha$. In addition, $\tilde\nu$ takes a positive value on $\sigma_\infty^{(1)}$ because $\tilde\nu(\sigma_\infty)$ and $\tilde\nu(\tau_\infty^{(0)})$ are both positive (more precisely, we have $\tilde\nu(\sigma_\infty) = \nu(\delta_0) = \nu(\epsilon_0) = \tilde\nu(\tau_\infty^{(0)}) \in \Psi_{F,>0}$). Since R_1 is an integrally closed local domain (it is a Nagata extension of an integrally closed domain, see [L, Proposition 7]), the polynomial $F_1(X_1)$ determines the Nagata extension

$$R_1 \hookrightarrow S_1 = R_1[\sigma_\infty^{(1)}]_* \subset K(\sigma_\infty, \tau_\infty^{(0)}),$$

which contains σ_∞. The extension $R_\nu[\sigma_\infty]_* \hookrightarrow S_1$ therefore is a Nagata extension and, setting $h_1(X_1) = h(X_1 + \tau_\infty^{(0)})$, the element $h(\sigma_\infty)$ is mapped to $h_1(\sigma_\infty^{(1)})$ which is non zero because $\deg h(X) < n$ and $h(\sigma_\infty)$ and $h(\sigma_\infty^{(1)} + \tau_\infty^{(0)})$ have the same image in the henselization of R_ν. The valuation $\widetilde{\nu_1}$ centered in S_1 extending ν_1 has valuation ring $R_{\widetilde{\nu_1}} = R_{\tilde{\nu}} \cap K(\sigma_\infty, \tau_\infty^{(0)})$.

Figure 1 might help the reader to visualize this construction: we have a commutative diagram of local ring homomorphisms, where the last horizontal arrow corresponds to the Nagata extension defined by $F_1(X_1)$ seen as Nagata polynomial in $R_{\nu_1}[X_1]$.

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, hook] & S = R[\sigma_\infty]_* \arrow[d, hook] \arrow[dr, hook] & \\
R_\nu \arrow[r, hook] \arrow[d, hook] & R_\nu[\sigma_\infty]_* \arrow[r, hook] \arrow[d, hook] & R_{\tilde{\nu}} \arrow[d, hook] \\
R_1 = R_\nu[\tau_\infty^{(0)}]_* \arrow[r, hook] \arrow[d, hook] & S_1 = R_1[\sigma_\infty^{(1)}]_* \arrow[r, hook] \arrow[d, hook] & R_{\widetilde{\nu_1}} \\
R_{\nu_1} \arrow[r, hook] & R_{\nu_1}[\sigma_\infty^{(1)}]_* \arrow[ur, hook] &
\end{tikzcd}$$

FIGURE 1. Commutative diagram constructed after one step.

Let $(\delta_i^{(1)})_{i\in\mathbf{N}}$ and $(\sigma_i^{(1)})_{i\geq 1}$ be the Newton sequence of values and the sequence of partial sums attached to $F_1(X_1)$, respectively. Let Ψ_{F_1} be the convex subgroup of Φ associated to $F_1(X_1)$. Since $\nu_1(\delta_0^{(1)}) = \nu_1(F_1(0)) = \nu_1(F(\tau_\infty^{(0)})) = \nu_1(B(\tau_\infty^{(0)})) \notin \Psi_F$ and the set of convex subgroups of Φ is totally ordered by inclusion, we have $\Psi_F \subsetneq \Psi_{F_1} \subseteq \Phi$. Note that

$$\nu_1(\delta_0^{(1)}) = \tilde{\nu}(\sigma_\infty^{(1)}) > \tilde{\nu}(\sigma_\infty) = \tilde{\nu}(\tau_\infty^{(0)}).$$

If $\Psi_{F_1} = \Phi$ then Lemma 3.14 and Corollary 3.9 imply that,

$$\tilde{\nu}(h(\sigma_\infty)) = \widetilde{\nu_1}(h_1(\sigma_\infty^{(1)})) = \nu_1(h_1(\sigma_i^{(1)})) = \nu_1(h(\sigma_i^{(1)} + \tau_\infty^{(0)})) \text{ for all } i \text{ large enough.} \tag{2}$$

Assume that $\Psi_{F_1} \subsetneq \Phi$. Let $\overline{\nu_1} := \overline{\nu_1}_{\Psi_{F_1}}$ be the residual valuation (corresponding to Ψ_{F_1}) with which ν_1 is composed. Suppose that the image $\overline{F_1}(X_1) \in R_{\overline{\nu_1}}[X_1]$ of $F_1(X_1)$ is irreducible (which in turn implies that $F_1(X_1)$ is the minimal polynomial of $\sigma_\infty^{(1)}$ over $K(\tau_\infty^{(0)})$). Then, writing $h_1(X_1) = h_{1,t_1}H_1(X_1)$ in $R_{\nu_1}[X_1]$, where h_{1,t_1} is a non zero constant and $H_1(X_1)$ has a coefficient equal to one, and repeating the same arguments as before, we conclude that the statement (2) above also holds in this case. If $\overline{F_1}(X_1)$ is reducible, we lift its Nagata factor $\overline{Q_1}(X_1)$ to an irreducible Nagata polynomial $Q_1(X_1) \in R_{\nu_1}[X_1]$, which has a unique root of positive $\tilde{\nu}$-value $\tau_\infty^{(1)} \in \overline{K}$ and repeat the construction.

Since ν_1 has finite rank, we can assume that X_1 is a good variable. We produce a Nagata extension $R_{\nu_1} \hookrightarrow R_2 = R_{\nu_1}[\tau_\infty^{(1)}]_* \subset K(\tau_\infty^{(0)}, \tau_\infty^{(1)})$ such that again the valuation extends uniquely to a valuation ν_2 centered in R_2 and the value group does not change (so it is equal to Φ). The Nagata polynomial $F_2(X_2) = F_1(X_2 + \tau_\infty^{(1)}) \in R_2[X_2]$ determines a Nagata extension

$$R_2 \hookrightarrow S_2 = R_2[\sigma_\infty^{(2)}]_* \subset K(\sigma_\infty, \tau_\infty^{(0)}, \tau_\infty^{(1)})$$

associated to the root of positive $\tilde{\nu}$-value $\sigma_\infty^{(2)} = \sigma_\infty^{(1)} - \tau_\infty^{(1)} = \sigma_\infty - \sum_{k=0}^{1} \tau_\infty^{(k)}$, and which contains $R_{\nu_1}[\sigma_\infty^{(1)}]_*$. We call $\widetilde{\nu_2}$ the valuation centered in S_2 extending ν_2, $R_{\widetilde{\nu_2}} = R_{\tilde{\nu}} \cap K(\sigma_\infty, \tau_\infty^{(0)}, \tau_\infty^{(1)})$. The element $h(\sigma_\infty)$ is written in S_2 as the non zero element $h_2(\sigma_\infty^{(2)}) = h(\sigma_\infty^{(2)} + \sum_{k=0}^{1} \tau_\infty^{(k)})$. In addition, we have $\Psi_F \subsetneq \Psi_{F_1} \subsetneq \Psi_{F_2} \subseteq \Phi$ and $\nu_2(\delta_0^{(2)}) = \tilde{\nu}(\sigma_\infty^{(2)}) > \tilde{\nu}(\sigma_\infty^{(1)}) = \tilde{\nu}(\tau_\infty^{(1)})$.

We now ask whether $\Psi_{F_2} = \Phi$, or $\Psi_{F_2} \subsetneq \Phi$ and we get an irreducible polynomial after reducing the coefficients of $F_2(X_2)$ modulo the prime ideal of R_{ν_2} corresponding to Ψ_{F_2}. If one of these conditions holds then, for all i large enough,

$$\tilde{\nu}(h(\sigma_\infty)) = \widetilde{\nu_2}(h_2(\sigma_\infty^{(2)})) = \nu_2(h_2(\sigma_i^{(2)})) = \nu_2(h(\sigma_i^{(2)} + \sum_{k=0}^{1} \tau_\infty^{(k)});$$

otherwise, we repeat the construction.

As we iterate this construction, the value groups of the valuations $\{\nu_k\}_{k\geq 1}$ that we create remain equal to Φ which has a bounded rank, and the convex subgroups $\{\Psi_{F_k}\}_{k\geq 1}$ that we determine grow strictly, so this process has to stop after finitely many steps, say $\ell \geq 1$ steps. This proves that $\tilde{\nu}(h(\sigma_\infty))$ is the value of the image $h_\ell(\sigma_\infty^{(\ell)}) = h(\sigma_\infty^{(\ell)} + \sum_{k=0}^{\ell-1} \tau_\infty^{(k)})$ of $h(\sigma_\infty)$ in a finite extension of K, for a uniquely defined extension $\tilde{\nu}_\ell$ of ν; and it is also the ν_ℓ-value for large i of $h(\sigma_i^{(\ell)} + \sum_{k=0}^{\ell-1} \tau_\infty^{(k)})$. The value group is preserved since by construction the value group of ν_ℓ is the same as that of ν. □

The proof of Theorem 1.(2) is now straightforward.

Proof of Theorem 1.(2). Since the valuation ν extends uniquely to each $S/H_S(\nu)$ without changing the value group by Proposition 3.21, the same is true for $R^h/H(\nu) = \bigcup_S S/H_S(\nu)$. □

Remark 3.22. If it was infinite, the sequence $\tau_\infty^{(0)} + \tau_\infty^{(1)} + \tau_\infty^{(2)} + \cdots + \tau_\infty^{(k)}$ built in the proof of Proposition 3.21 would be a pseudo–convergent sequence of elements of the maximal ideal of R_ν^h for the extension $\tilde{\nu}$ of ν, which would have the property that the smallest convex subgroup containing $\tilde{\nu}(\tau_\infty^{(k+1)})$ is strictly larger than the one containing $\tilde{\nu}(\tau_\infty^{(k)})$. Let us say that such a sequence is *pseudo–convergent in scales*. What we use here is that by Abhyankar's inequality (see [B, Ch. VI, § 10, no. 3, Corollary 1]) there can be no infinite such sequence in a field of bounded transcendence degree over a valued field with a value group of bounded rational rank, in our case the prime field with the trivial valuation. By construction, σ_∞ is in fact represented in S_ℓ by a finite sum $\sigma_\infty = \tau_\infty^{(0)} + \tau_\infty^{(1)} + \tau_\infty^{(2)} + \cdots + \tau_\infty^{(\ell-1)} + \sigma_\infty^{(\ell)}$.
Here and at the end of this paper we have only used the fact that the value group has finite rank. A class of rings which have this property for any valuation and are not all noetherian are the rings of *finite valuative dimension* in the sense of Jaffard in [J, Ch. IV].

We may summarize the conclusion of the preceding discussion as follows:

Definition 3.23. Let ν be a valuation centered in a local domain R and let $F(X) \in R[X]$ be a Nagata polynomial. Let Ψ_F be the convex subgroup of the value group of ν attached to $F(X)$ as in Definition 3.10 and let p_{Ψ_F} be the corresponding prime ideal of R. We say that $F(X)$ is *ν-residually irreducible* if the image $\overline{F}(X) \in R/p_{\Psi_F}[X]$ of $F(X)$ is irreducible in $L_F[X]$, where L_F denotes the fraction field of R/p_{Ψ_F}.

This irreducibility implies that $\overline{F}(X)$ defines a non trivial Nagata extension of R/p_{Ψ_F} and by Lemma 3.17 that Ψ_F is the intrinsic convex subgroup of the Nagata extension defined by $F(X)$. Remembering the discussion in Remark 3.15 and the factorization $h(X) = h_t H(X)$ used in the proof of Proposition 3.21, it also implies that for any polynomial $h(X) \in R[X]$ such that $0 \leq \deg h(X) < \deg F^*(X)$, we have for large i the equality $\mathrm{in}_\nu(h(\sigma_i)) = \mathrm{in}_{\tilde{\nu}}(h(\sigma_\infty))$.

The algorithm described above has the following consequence:

Proposition 3.24. *Let ν be a valuation of finite rank centered in an integrally closed local domain R. Given a Nagata extension $R \hookrightarrow R[\sigma_\infty]_*$ corresponding to a Nagata polynomial $F(X) \in R[X]$, there exist a local domain R' dominating R, dominated by the henselization of R_ν and containing σ_∞, to which the valuation ν extends uniquely to a valuation centered in R' and with the same value group, and an element $a' \in R'$ such that the Nagata polynomial $F'(X) = F(X + a') \in R'[X]$ is $\tilde{\nu}$-residually irreducible.*

4. Applications

4.1. **Connected components of the Riemann–Zariski space.** In this subsection we apply Theorem 1.(1) in the study of the Riemann–Zariski space $\mathrm{RZ}(R)$ of valuations centered in a local domain R.

We start with the following result which, in a slightly different formulation, is classical (see [Ku2, Theorem 5.14]). We give a proof in the spirit of this paper:

Corollary 4.1. *The henselization of a valuation ring is a valuation ring with the same value group.*

Proof. A valuation ring R_ν is integrally closed, so the henselization R_ν^h of R_ν is a local domain (and there is no minimal prime to consider, see Remark 2.11) and it is integrally closed (see [LM, Theorem 13.12] or [EGA, Theorem 18.6.9]). As such, according to [B, Ch. VI, § 1, no. 3, Theorem 3], R_ν^h is the intersection of all the valuation rings of K^h which dominate it, where K^h is the fraction field of R_ν^h. By the uniqueness of the extension of the valuation ν (Theorem 1.(1)), among all the valuation rings of K^h which dominate R_ν there is only one which also dominates R_ν^h, so R_ν^h is this valuation ring. The fact that the value group is the same follows from Theorem 1.(2). □

Let R be a local domain and let $\mathrm{RZ}(R)$ be the space of valuations centered in R. If K is the fraction field of R, then $\mathrm{RZ}(R)$ consist of the set of all valuation rings of K which dominate R endowed with the Zariski topology (see [ZS, Ch. VI, § 17]). This topology is obtained by taking as a basis of open sets the subsets $U(A)$, whose elements are the valuation rings of K dominating R and containing A, where A ranges over the family of all finite subsets of K.

Corollary 4.2. *Let R be a local domain and let $\{H_\iota\}_{\iota\in I}$ be the set of minimal primes of R^h. Let $\varphi : \mathrm{RZ}(R) \to \bigsqcup_{\iota\in I} \mathrm{RZ}(R^h/H_\iota)$ be the map which to a valuation ring $R_\nu \in \mathrm{RZ}(R)$ associates the minimal prime $H(\nu)$ of R^h and the valuation ring $R_{\tilde\nu} \in \mathrm{RZ}(R^h/H(\nu))$ of the extension $\tilde\nu$ of ν to $R^h/H(\nu)$. Then, the map φ satisfies the following:*

(1) *It is a homeomorphism.*
(2) *It induces a bijection between the set of connected components of $\mathrm{RZ}(R)$ and $\{H_\iota\}_{\iota\in I}$.*

Proof. Let K be the fraction field of R and let $\overline{R}$ be the the integral closure of R in K. Any maximal ideal of $\overline{R}$ is of the form $m_\nu \cap \overline{R}$ for some valuation ring R_ν of K dominating R (see [B, Ch. VI, § 1, no. 3, Theorem 3]). The ideal $H(\nu)$ of R^h associated to ν then appears as the kernel of the canonical map $R^h \to \tilde{R}^h$, where $\tilde{R} = \overline{R}_{m_\nu\cap\overline{R}}$. Indeed, taking inductive limits of Nagata extensions in the commutative diagram obtained by combination of (D_1) and the diagram (D_2) for $\tilde{R}$, yields that $R^h \to R_\nu^h$ can be written as the injection $\tilde{R}^h \hookrightarrow R_\nu^h$ composed with $R^h \to \tilde{R}^h$. This defines a map from the set of maximal ideals of $\overline{R}$ to $\{H_\iota\}_{\iota\in I}$. We now define its inverse map.

As in Remarks 2.13.(2), the fact that R^h is flat over R implies that for each $\iota \in I$ we have $H_\iota \cap R = (0)$. Since the natural composed map $p_\iota : R \to R^h \to R^h/H_\iota$ is injective, it induces an injection $\overline{R} \hookrightarrow \overline{R^h/H_\iota}$, where again the bar means integral closure. The maximal ideal of the domain $\overline{R^h/H_\iota}$, which is local by [R, Ch. IX, Corollaire 1], induces a maximal ideal, say $\overline{m}_\iota$, of $\overline{R}$. We associate to H_ι this $\overline{m}_\iota$. Denoting by $\overline{R}_{\overline{m}_\iota}$ the localization of $\overline{R}$ at $\overline{m}_\iota$, we have injections $R \hookrightarrow \overline{R}_{\overline{m}_\iota} \hookrightarrow \overline{R^h/H_\iota}$ and a commutative diagram of local ring maps considering these maps and $R^h/H_\iota \hookrightarrow \overline{R^h/H_\iota}$ composed with p_ι. Since the integral closure of a henselian local domain is a henselian local domain as an inductive limit of finite algebras (see [EGA, Ch. IV, § 18, Theorem 18.5.11 and Proposition 18.6.14]), $\overline{R^h/H_\iota}$ is henselian and by the universal property of henselization the natural map $R^h \to \overline{R^h/H_\iota}$ factors uniquely through the map $(\overline{R}_{\overline{m}_\iota})^h \to \overline{R^h/H_\iota}$. This map is injective because its kernel should have intersection zero with $\overline{R}_{\overline{m}_\iota}$ and therefore by [B, Ch. V, § 2, no. 1, Corollary 1] should be a minimal prime of the domain $(\overline{R}_{\overline{m}_\iota})^h$. This shows that H_ι is indeed the kernel of the map $R^h \to (\overline{R}_{\overline{m}_\iota})^h$. Observe also that we have local ring maps $R^h/H_\iota \hookrightarrow (\overline{R}_{\overline{m}_\iota})^h \hookrightarrow \overline{R^h/H_\iota}$. Hence if $R_{\tilde\mu} \in \mathrm{RZ}(R^h/H_\iota)$, then $R_{\tilde\mu}$ dominates $\overline{R^h/H_\iota}$ and the valuation ring $R_\mu = R_{\tilde\mu} \cap K$ of K belongs to $\mathrm{RZ}(\overline{R}_{\overline{m}_\iota})$ and $m_\mu \cap \overline{R} = \overline{m}_\iota$ (since $\overline{m}_\iota \overline{R}_{\overline{m}_\iota} \cap \overline{R} = \overline{m}_\iota$).

Thus, we have established a bijection between $\{H_\iota\}_{\iota\in I}$ and the set of maximal ideals of $\overline{R}$, and in what follows we write this last set as $\{\overline{m}_\iota\}_{\iota\in I}$ with $\overline{m}_\iota$ corresponding to H_ι.

As a set $\mathrm{RZ}(R)$ is the disjoint union of the family of subsets $\{\mathrm{RZ}(\overline{R}_{\overline{m}_\iota})\}_{\iota\in I}$. To prove that they are homeomorphic we observe that the Zariski topology on any $\mathrm{RZ}(\overline{R}_{\overline{m}_\iota})$ coincides with the topology induced by the topology of $\mathrm{RZ}(R)$. In addition, the local integral domains $\overline{R}_{\overline{m}_\iota}$ are unibranch and we can apply [Tm, Theorem 2.4.2] which ensures the connectedness of the spaces $\mathrm{RZ}(\overline{R}_{\overline{m}_\iota})$. So, in order to finish the proof, it suffices to take $\iota \in I$ and show that the bijective map φ_ι from $\mathrm{RZ}(\overline{R}_{\overline{m}_\iota})$ to $\mathrm{RZ}(R^h/H_\iota)$ induced by φ is a homeomorphism.

On the one hand, the henselization $(\overline{R}_{\overline{m}_\iota})^h$ is a local domain (because $\overline{R}_{\overline{m}_\iota}$ is an integrally closed local domain, see Remark 2.11), and therefore by [F, Proposition 3.4], the map from

$\mathrm{RZ}(\overline{R}_{\overline{m}_\iota})$ to $\mathrm{RZ}((\overline{R}_{\overline{m}_\iota})^h)$ which sends a valuation to its unique extension is a homeomorphism. Note that in [F, Proposition 3.4] the noetherianity and excellence assumptions on the local ring are only needed to show that the previous map is well defined and bijective. On the other hand, as we have seen above, we have local ring maps $R^h/H_\iota \hookrightarrow (\overline{R}_{\overline{m}_\iota})^h \hookrightarrow \overline{R^h/H_\iota}$ and hence $\mathrm{RZ}((\overline{R}_{\overline{m}_\iota})^h)$ coincides with $\mathrm{RZ}(R^h/H_\iota)$. These facts imply that φ_ι is a homeomorphism. □

4.2. **Approximation of Henselian elements.** In this subsection, instead of considering extensions of all the valuations centered in a local domain to its henselization, we study the extension of the valuation of a valuation ring to its henselization; we revisit a result of Franz-Viktor Kuhlmann in [Ku1, Theorem 1.1]. This result concerns the approximation of elements of the henselization $(K^h, \tilde{\nu})$ of a valued field (K, ν) by elements of K and we can state it as follows since we know by Corollary 4.1 that $R^h_\nu = R_{\tilde{\nu}}$ and the value groups are equal:

Theorem 4.3. (Kuhlmann) *Let K be a field endowed with a valuation ν determined by the valuation ring R_ν and let Φ be the value group of ν. Let K^h be the field of fractions of the henselization $R^h_\nu = R_{\tilde{\nu}}$ of R_ν. For every element $z \in K^h \setminus K$ there exist a convex subgroup Ψ of Φ and an element $\varphi \in \Phi$ such that $\varphi + \Psi$ is cofinal in the ordered set*

$$\tilde{\nu}(z - K) = \{\tilde{\nu}(z - c) \,|\, c \in K\} \subset \Phi.$$

Before giving the proof of this result, let us come back to the nature of the growth of the $\nu(h(\sigma_i))$, which is also a consequence of Corollary 3.9, but here we see directly that the coefficient of $\mathrm{in}_\nu \delta_i^e$ is the initial form of an element of R_ν.

Proposition 4.4. *With the notations of Proposition 3.7, given a Nagata extension of the valued local domain R and a polynomial $h(X) \in R[X]$, there exist $a \in R_\nu$ and $e \in \mathbf{N}$, $0 \le e \le \deg h(X)$ such that for all i large enough we have the equality $\mathrm{in}_\nu h(\sigma_i) = \mathrm{in}_\nu(a\delta_i^e)$.*

Proof. Considering $h(X)$ as a polynomial in $R_\nu[X]$ we see that it suffices to prove the result when one of the coefficients of $h(X)$ is equal to one, so we assume this. We use the notation of the proof of Proposition 3.21. In particular, let $Q(X) \in R_\nu[X]$ denote the lifting of the minimal polynomial of $\bar{\sigma}_\infty$ over L and let us write the $Q(X)$-adic expansion of $h(X)$ as

$$h(X) = A_r(X)Q(X)^r + A_{r-1}(X)Q(X)^{r-1} + \cdots + A_0(X) \ \text{ with } \deg A_j(X) < \deg Q(X).$$

We have a similar expression after passing to the quotient by m_{Ψ_F}. Let J be the non–empty set consisting of those j such that $0 \le j \le r$ and $\overline{A}_j(X) \neq 0$. For all $j \in J$, the condition $\deg \overline{A}_j(X) < \deg \overline{Q}(X)$ implies that there exists $\varphi_j \in \Psi_F$ such that $\nu(A_j(\sigma_i)) = \varphi_j$ for all i large enough (see Lemma 3.14), and thus we can find $b_j \in R_\nu$ such that $\mathrm{in}_\nu A_j(\sigma_i) = \mathrm{in}_\nu b_j$ for large i (see Corollary 3.9). Evaluating in σ_i the identity (1) of the proof of Proposition 3.21, we get $\mathrm{in}_\nu Q(\sigma_i) = \mathrm{in}_\nu(-a_{n-1}G_1(0)^{-1}\delta_i)$ for all $i \ge 1$. Since $\nu(A_j(\sigma_i)Q(\sigma_i)^j) \notin \Psi_F$ if $j \notin J$, the result follows from Lemma 3.2, Corollary 3.5 and Proposition 3.4 with $\gamma_i = \nu(Q(\sigma_i))$, and for all $j \in J$, $\beta_j = \varphi_j$ and $t_j = j$. The case where $(\nu(h(\sigma_i)))_{i \ge 1}$ is eventually constant corresponds to $e = 0$ and then we have $a \in R$. □

Remark 4.5. Since R_ν and its henselization $R_{\tilde{\nu}}$ have the same residue field, another way to state that ν and $\tilde{\nu}$ have the same value group is to say (see [T1, Proposition 4.1]) that the natural graded injection $\mathrm{gr}_\nu R_\nu \hookrightarrow \mathrm{gr}_{\tilde{\nu}} R_{\tilde{\nu}}$ is an equality. It is therefore not surprising that the initial form of $h_k(\sigma_\infty)$ in Proposition 3.7 appears as the initial form of an element of R_ν.

Proof of Theorem 4.3. Let us first assume that z lies in $R_{\tilde{\nu}}$. Using the fact that R_ν and $R_{\tilde{\nu}}$ have the same residue field, by removing an element of R_ν of value zero, we can exclude the case where $\tilde{\nu}(z) = 0$ and assume $\tilde{\nu}(z) > 0$. Then z lies in the maximal ideal of a Nagata extension $R_0[\sigma_\infty]_* \subset R_{\tilde{\nu}}$ of a normal local domain $R_0 \subset R_\nu$ essentially of finite type over the prime ring. We call ν_0 the restriction of ν to the fraction field K_0 of R_0 and Φ_0 its value group. Let $F(X) \in R_0[X]$ be an irreducible Nagata polynomial defining $R_0[\sigma_\infty]_*$ and let $(\delta_i)_{i \in \mathbf{N}}$ be its Newton sequence of values. After Proposition 3.24, since R_0 is noetherian and Φ_0 of finite rank, we may assume that σ_∞ is a limit of the pseudo–convergent sequence $(\sigma_i)_{i \ge 1}$ associated to $F[X]$ in such a way that for any polynomial $P(X) \in R_0[X]$ with $0 \le \deg P(X) < \deg F(X)$, we have that $\mathrm{in}_{\tilde{\nu}} P(\sigma_\infty) = \mathrm{in}_\nu P(\sigma_i)$ for large i.

We write $z = \frac{h(\sigma_\infty)}{q(\sigma_\infty)}$ in $R_0[\sigma_\infty]_*$, where $h(X), q(X) \in R_0[X]$ are polynomials of degree less than $\deg F(X)$, $h(0) \in m_{R_0}$, and $q(0) \notin m_{R_0}$. The polynomial

$$H(X) = h(\sigma_\infty)q(X) - h(X)q(\sigma_\infty) \in R_0[\sigma_\infty]_*[X]$$

satisfies the equation $H(\sigma_\infty) = 0$. It is of positive degree since otherwise it would be identically zero and $\frac{h(X)}{q(X)}$ would be constant. By the same argument as in the proof of Proposition 3.7, the $\tilde{\nu}$-value of $H(\sigma_i)$ is, for large i, of the form $\tilde{\nu}(H_k(\sigma_\infty)\delta_i^k)$ with $k \geq 1$. In addition, for all i large enough we have $\nu(q(\sigma_i)) = \tilde{\nu}(q(\sigma_\infty)) = 0$ and the $\tilde{\nu}$-value of $H(\sigma_i) = h(\sigma_\infty)q(\sigma_i) - h(\sigma_i)q(\sigma_\infty)$ coincides with that of $\frac{h(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h(\sigma_i)}{q(\sigma_i)}$. Thus, by Lemma 3.11 there exists $i_0 \geq 1$ such that

$$\left\{\tilde{\nu}\left(\frac{h(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h(\sigma_i)}{q(\sigma_i)}\right) \mid i \geq i_0\right\} \text{ is cofinal in } \varphi + \Psi,$$

where $\varphi = \tilde{\nu}(H_k(\sigma_\infty)) \in \Phi_{\geq 0}$ and Ψ is the smallest convex subgroup of Φ containing all the $\nu(\delta_i)$. Note that since the δ_i associated to $F(X)$ are in R_0, the smallest convex subgroup of Φ_0 containing the $\nu(\delta_i)$ is the intersection with Φ_0 of the convex subgroup Ψ_F of Φ associated to the polynomial $F(X)$ seen as a Nagata polynomial in $R_\nu[X]$ and is cofinal in it by Lemma 3.11. Next we prove that $\varphi + \Psi$ is cofinal in $\tilde{\nu}(z - K)$.

Let us first verify the inclusion $\varphi + \Psi \subset \tilde{\nu}(z - K)$. Given $\psi \in \Psi$ and an element $c \in K$ such that $\nu(c) = \varphi + \psi$, the cofinality we verified above implies that there exists $i \geq i_0$ such that $\tilde{\nu}\left(\frac{h(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h(\sigma_i)}{q(\sigma_i)}\right) > \varphi + \psi$. Therefore we also have $\varphi + \psi = \tilde{\nu}\left(\frac{h(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h(\sigma_i)}{q(\sigma_i)} + c\right)$, which is an element of $\tilde{\nu}(z - K)$.

Now let us prove that for any $c \in K$ there exists $\psi \in \Psi$ such that $\varphi + \psi > \tilde{\nu}(z - c)$. We may assume that $c \in R_\nu$ since otherwise $\tilde{\nu}(z - c) = \tilde{\nu}(c) < 0$ and the result is clear. Then, enlarging the local ring R_0 if necesssary, we can also assume that $c \in R_0$. So let us consider the polynomial $h_{(c)}(X) = h(X) - cq(X) \in R_0[X]$ and write $z - c$ as $\frac{h_{(c)}(\sigma_\infty)}{q(\sigma_\infty)}$ in $R_0[\sigma_\infty]_*$. Since both $h_{(c)}(X)$ and $q(X)$ are nonzero polynomials of degree less than $\deg F(X)$, we have by Corollary 3.9 or Proposition 4.4 the equality

$$\mathrm{in}_{\tilde{\nu}}(h_{(c)}(\sigma_\infty)q(\sigma_i)) = \mathrm{in}_{\tilde{\nu}}(h_{(c)}(\sigma_i)q(\sigma_\infty)) \text{ for all } i \text{ large enough,}$$

which implies the inequality $\tilde{\nu}\left(\frac{h_{(c)}(\sigma_\infty)}{q(\sigma_\infty)}\right) < \tilde{\nu}\left(\frac{h_{(c)}(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h_{(c)}(\sigma_i)}{q(\sigma_i)}\right)$. But $\frac{h_{(c)}(\sigma_\infty)}{q(\sigma_\infty)} = z - c$ and $\frac{h_{(c)}(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h_{(c)}(\sigma_i)}{q(\sigma_i)} = \frac{h(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h(\sigma_i)}{q(\sigma_i)}$ for large i, so that we can find $i_1 \geq i_0$ such that

$$\tilde{\nu}(z - c) < \tilde{\nu}\left(\frac{h(\sigma_\infty)}{q(\sigma_\infty)} - \frac{h(\sigma_i)}{q(\sigma_i)}\right) = \varphi + \nu(\delta_i^k) \text{ for all } i \geq i_1.$$

This inequality gives the result we want in this case.

If $z \in K^h \setminus R_{\tilde{\nu}}$, using the fact that the value groups of ν and $\tilde{\nu}$ are the same, we choose $d \in m_\nu$ such that $dz \in m_{\tilde{\nu}}$, apply to dz the argument we have just seen and use the fact that $\tilde{\nu}(dz - K) = \tilde{\nu}(z - K) + \nu(d)$. Replacing the element φ associated to dz as above by $\varphi - \nu(d)$ gives the result. □

5. Etale type and the henselian property

In this section we relate in greater generality Nagata polynomials with certain pseudo–convergent sequences and obtain a valuative characterization of the henselian property.

After Ostrowski and Kaplansky, one says that a pseudo–convergent sequence $(y_\tau)_{\tau \in T}$ of elements of a valued field (K, ν) is of *algebraic type* if there exist polynomials $h(X) \in K[X]$ such that $(\nu(h(y_\tau)))_{\tau \in T}$ is not eventually constant. We propose the following, where as usual $\tau + 1$ designates the successor of τ in the well ordered set T:

Definition 5.1. Let ν be a valuation centered in a local domain R. A pseudo–convergent sequence $(y_\tau)_{\tau \in T}$ of elements of the maximal ideal m_R of R is *of étale type* if there exist polynomials $h(X) \in R[X]$ such that one has the equality $\nu(h(y_\tau)) = \nu(y_{\tau+1} - y_\tau)$ for $\tau \geq \tau_0 \in T$, where τ_0 may depend on the polynomial $h(X)$.

Note that if the values of the y_τ are not eventually constant, then $\nu(y_{\tau+1}) > \nu(y_\tau)$ for large τ and then $h(X) = X - a$ with $a \in R$ such that $\nu(a) > \nu(y_\tau)$ for all $\tau \in T$ is such a polynomial. The element $a \in R$ is a limit of $(y_\tau)_{\tau\in T}$. By the argument given in Lemma 3.11, a pseudo-convergent sequence such that the $\nu(y_\tau)$ are not eventually constant is tested as being of étale type by a linear polynomial $X - a$ if $\nu(a)$ is not in the smallest convex subgroup containing the $\nu(y_\tau)$.

It is a classical result (see [K, Theorem 4], [Ku3, Theorem 8.19]) that a valued field is maximal (has no non trivial immediate extension, which means a valued extension with the same value group and the same residue field) if and only if all pseudo–convergent sequences in the field have a limit in the field. Since henselization is an immediate extension by Corollary 4.1, maximal valued fields have a henselian valuation ring. We give a somewhat more precise and more general result in the following valuative criterion for the henselian property:

Proposition 5.2. *Let R be a local domain with maximal ideal m_R, and let ν a valuation of finite rank centered in R. The local domain R is henselian if and only if every pseudo–convergent sequence of elements of m_R which is of étale type has a limit in m_R.*

Proof. A local ring in which every Nagata polynomial has a root in the maximal ideal is henselian; see [L, Lemma, p.94]. This and the fact that the henselization is the inductive limit of Nagata extensions imply that the local domain R is henselian if and only if every Nagata polynomial in $R[X]$ has a root in m_R.

Let us assume that every pseudo–convergent sequence of elements of m_R which is of étale type has a limit in m_R. We proceed by contradiction and suppose that there exists a Nagata polynomial $F(X) \in R[X]$ of degree n such that $F(a) \neq 0$ for all $a \in m_R$. By Lemma 3.19, we may assume that X is a good variable. Then the polynomial $F(X)$ comes with the pseudo–convergent sequence of partial sums $(\sigma_i)_{i\geq 1}$ and the associated convex subgroup Ψ_F of the value group of ν, which equals the intrinsic convex subgroup Ψ. Our assumption implies that $(\sigma_i)_{i\geq 1}$ has a limit $y \in m_R$. We have $F(y) = F(\sigma_i) + \sum_{m=1}^n F_m(\sigma_i)(y - \sigma_i)^m$ as in the identity $(\star)$ of Section 3, and since $\nu(y - \sigma_i) \geq \nu(\delta_i)$ and $\nu(F(\sigma_i)) = \nu(\delta_i)$, we get that $\nu(F(y)) \geq \nu(\delta_i)$ for all i. Since $\nu(F(y)) \in \Psi = \Psi_F$ and the $\nu(\delta_i)$ are strictly increasing and cofinal in Ψ_F, this gives us a contradiction.

Let us now assume that R is henselian and let $(y_\tau)_{\tau\in T}$ be a pseudo–convergent sequence of étale type of elements of m_R. Let $h(X) \in R[X]$ be a polynomial verifying the equality $\nu(h(y_\tau)) = \nu(y_{\tau+1} - y_\tau)$ for large τ. Suppose that $h(X)$ is, up to multiplication by an invertible element of R, a Nagata polynomial. Then $h(X)$ has a root $y \in m_R$ since R is henselian. We can write $h(X) = (X - y)G(X)$ in $R[X]$ and by Lemma 2.3, since $X - y$ is a Nagata polynomial, we have that $G(X) \notin (m_R, X)$ so that $\nu(h(y_\tau)) = \nu(y_{\tau+1} - y_\tau) = \nu(y - y_\tau)$ for large τ, which shows that y is a limit of $(y_\tau)_{\tau\in T}$. Therefore, in order to end the proof, it is sufficient to show that $h(X)$ is a Nagata polynomial up to multiplication by a unit of R.

Writing $h(X) = b_0X^s + \cdots + b_{s-1}X + b_s = b_0\prod_{j=1}^s(X - r_j)$ in a splitting field of $h(X)$, after extension of ν we see that our assumption immediately implies that $\nu(b_0) = 0$ and exactly one of the r_j is a limit of $(y_\tau)_{\tau\in T}$ while the values of the other $y_\tau - r_j$ are zero. So after multiplication by an invertible element of R we may assume that $h(X)$ is unitary. We have the identity $h(y_{\tau+1}) - h(y_\tau) = \sum_{m=1}^s h_m(y_\tau)(y_{\tau+1} - y_\tau)^m$ as in the identity $(\star\star\star)$ of Section 3. Since $\nu(h(y_{\tau+1})) > \nu(h(y_\tau))$ at least for large τ, the previous equality then implies the equality

$$\begin{aligned}\mathrm{in}_\nu h(y_\tau) &= -\mathrm{in}_\nu\big(\sum_{m=1}^s h_m(y_\tau)(y_{\tau+1} - y_\tau)^m\big)\\ &= -\mathrm{in}_\nu\big((y_{\tau+1} - y_\tau)(h_1(y_\tau) + \sum_{m=2}^s h_m(y_\tau)(y_{\tau+1} - y_\tau)^{m-1})\big),\end{aligned}$$

so that $h(y_\tau)$ can have the same value as $y_{\tau+1} - y_\tau$ only if $h_1(y_\tau)$ is invertible, which in turn implies, since $y_\tau \in m_R$, that the coefficient b_{s-1} of X in $h(X)$ is invertible in R. Finally, we get that $h(0) = b_s \in m_R$ since $y_\tau \in m_R$ for all τ. This shows that $h(X)$ is a Nagata polynomial. □

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