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Uniform controllability of a Stokes problem with a transport term in the zero-diffusion limit

Jon Asier Bárcena-Petisco*[†]

March 23, 2019

0 Abstract and basic information

In this paper we consider a Stokes system with Navier-slip boundary conditions. The main results concern the behaviour of the cost of null controllability with respect to the diffusion coefficient when the control acts in the interior. In particular, we prove in $(0, \pi)^2$ that for a sufficiently large time the cost decays exponentially as the diffusion coefficient vanishes, whereas in $(0, \pi)^3$ we prove that for most of the control domains and for any time $T > 0$ the cost explodes exponentially as the diffusion coefficient vanishes.

Key words: Singular limits, spectral decomposition, Stokes system, transport equation, uniform controllability

AMS subject classification: 35B25, 35P10, 93B05, 93C20

Abbreviated title: A transport-diffusion Stokes control problem

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1 Introduction

1.1 Main results

Throughout the paper we are interested in understanding the control problem:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_d} y + \nabla q = f 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad (Dy \cdot n)_{\text{tg}} = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega. \end{cases} \quad (1.1) \quad \text{eq:sliptrans}$$

As usual, here and throughout the paper, $T > 0$, $\Omega \subset \mathbb{R}^d$ and $\omega \subset \Omega$ are domains, $Q := (0, T) \times \Omega$, $Q_\omega := (0, T) \times \omega$, $\Sigma := (0, T) \times \partial\Omega$, $Du := \frac{1}{2} (\partial_{x_i} u_j + \partial_{x_j} u_i)_{i,j}$ for all $u \in \mathbf{L}^2(\Omega)$ (in the distributional sense), n denotes the normal vector pointing outwards $\partial\Omega$ and $v_{\text{tg}} := v - (v \cdot n)n$. The bold notation is used for denoting vectorial spaces. As for initial data, y^0 , we take it in $\mathcal{H}(\Omega)$, which is defined by:

$$\mathcal{H}(\Omega) := \{u \in \mathbf{L}^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega \text{ and } u \cdot n = 0 \text{ on } \partial\Omega\}.$$

We recall that we can define a normal trace which belongs to $H^{-1/2}(\partial\Omega)$ in the (closed) subspace of $\mathbf{L}^2(\Omega)$ of functions that have a divergence in $L^2(\Omega)$. As for the control force, f , we want it at least in $L^2(Q_\omega)$, and, if possible, of null first component. Finally, throughout the document $\Omega_d := (0, \pi)^d$.

Let us denote the sets of admissible controls as

$$\begin{cases} S_1(y^0) := \{f \in \mathbf{L}^2(Q_\omega) : \Phi_2(y^0, f)(T, \cdot) = 0\}, \\ S_2(y^0) := \{f_2 \in L^2(Q_\omega) : \Phi_2(y^0, (0, f_2))(T, \cdot) = 0\}, \\ S_3(y^0) := \{f \in \mathbf{L}^2(Q_\omega) : \Phi_3(y^0, f)(T, \cdot) = 0\}, \end{cases}$$

for $y^0 \in \mathcal{H}(\Omega_d)$, for $f \in \mathbf{L}^2(Q_\omega)$ and for $\Phi_d(y^0, f)$ the solution of (1.1) in Ω_d . Thanks to [19], we know that S_1 and S_3 are non-empty, whereas we obtain in Section 3.1 that S_2 is non-empty by proving the corresponding observability estimate. As for the cost of the control, we measure it with the usual norms. In fact, we denote:

$$\begin{cases} K_1(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_2) \setminus \{0\}} \inf_{f \in S_1(y^0)} \frac{\|f\|_{\mathbf{L}^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_2)}}, \\ K_2(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_2) \setminus \{0\}} \inf_{f_2 \in S_2(y^0)} \frac{\|f_2\|_{L^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_2)}}, \\ K_3(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_3) \setminus \{0\}} \inf_{f \in S_3(y^0)} \frac{\|f\|_{\mathbf{L}^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_3)}}. \end{cases} \quad (1.2) \quad \text{def:concost}$$

The main results that we prove in the paper are the following ones:

Theorem 1.1. *We have the following results for the control system (1.1):*

1. *Let $\omega \subset \Omega_2$ a domain. Then, there are some $c, C, T_0 > 0$ such that, if $T > T_0$ and $\varepsilon \in (0, 1)$:*

$$K_2(T, \varepsilon, \omega) \leq C e^{-c\varepsilon^{-1}}.$$

t:conexpopp

2. Let $h \in (0, \pi)$ and $\omega \subset (0, \pi) \times (\pi - h, \pi)$ a domain. Then, for any $T \in (0, 2(\pi - h))$, there is $c > 0$ such that, if $\varepsilon \in (0, 1)$:

$$K_1(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

t:conexpdir

3. Let $h \in (0, \pi)$ and $\omega \subset (0, \pi) \times (0, h)$ a domain. Then, for any $T \in (0, \pi - h)$, there is $c > 0$ such that, if $\varepsilon \in (0, 1)$:

$$K_1(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

it:d3bigcon

4. Let $h \in (0, \pi)$ and $\omega \subset (0, \pi)^2 \times (\pi - h, \pi)$ a domain. Then, for any $T > 0$ there is $c > 0$ such that, if $\varepsilon \in (0, 1)$:

$$K_3(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

Remark 1.2. The results in dimension 2 are not surprising as they are similar to those in [10, 21, 15, 25, 26]. The result in dimension 3 (see item 4), however, is surprising because it holds for all $T > 0$ and for a wide range of control domains (in particular, for all the control domains compactly supported in Ω_3). This result can be explained because we have much more freedom to construct eigenfunctions of the adjoint operator (see (1.5) and Section 4 below). Indeed, we can construct in dimension 3 a continuous family of eigenfunctions whose respective eigenvalues do not explode when $\varepsilon \rightarrow 0^+$. The existence of a system with this property is a novelty with respect to the existing literature.

Remark 1.3. The analogous results to the ones stated in Theorem 1.1 and in the lemmas and propositions throughout the paper are true for any rectangle or cuboids and with any velocity parallel to their edges. Indeed, all the operations can be replicated under those assumptions. The reason of restricting to $(0, \pi)^2$ or $(0, \pi)^3$ is to reduce the number of operations so that the reader can focus on the main ideas of the proof.

In order to estimate the cost of the control, we work with the adjoint system:

$$\begin{cases} -\varphi_t - \varepsilon \Delta \varphi - \partial_{x_d} \varphi + \nabla p = 0 & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi \cdot n = 0, \quad (2\varepsilon D\varphi \cdot n + \varphi n_d)_{\text{tg}} = 0 & \text{on } \Sigma, \\ \varphi(T, \cdot) = \varphi^T & \text{on } \Omega. \end{cases} \quad (1.3) \quad \text{eq:stosliptransp}$$

We recall the following classical result:

$$\begin{cases} [K_1(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_2) \setminus \{0\}} \frac{\int_{\Omega} |\varphi(0, \cdot)|^2}{\iint_{Q_\omega} |\varphi|^2}, \\ [K_2(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_2) \setminus \{0\}} \frac{\int_{\Omega} |\varphi(0, \cdot)|^2}{\iint_{Q_\omega} |\varphi_2|^2}, \\ [K_3(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_3) \setminus \{0\}} \frac{\int_{\Omega} |\varphi(0, \cdot)|^2}{\iint_{Q_\omega} |\varphi|^2}. \end{cases} \quad (1.4) \quad \text{eq:Ktepnorda}$$

In order to understand the (energy) solutions of (1.3), we do a spectral decomposition. In particular, we look for solutions of:

$$\begin{cases} -\varepsilon \Delta u - \partial_{x_d} u + \nabla p = \lambda u & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0, \quad (2\varepsilon Du \cdot n + u n_d)_{\text{tg}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.5) \quad \text{eq:stotrans}$$

Remark 1.4. One surprising result that we get with our method is that, as stated in the item 2 of Proposition 2.7 and in Remark 4.2 below, as long as $y^0 \in \mathcal{H}(\Omega_d)$, the gradient of the pressure of the solution of (1.3) is null for any given time.

Throughout the paper we use c and C to denote strictly positive constants, which might be different each time. We denote c for small constants and C for the large ones.

1.2 Historical background

The first control system with a small diffusion and a transport term that was analysed was the heat equation in dimension 1 with Dirichlet boundary conditions in [10]. Afterwards, the same problem but in any dimension and with any speed belonging to $W^{1,+\infty}(\mathbb{R}^+ \times \Omega)$ was studied in [21]. More recently, better approximations of the optimal time in which the cost of the control decays have been given: the upper bound was improved in [15, 25] (in the first one through complex analysis and in the second one by transforming the original equation into the pure heat equation) and the lower bound was improved in [26], through complex analysis and properties of the entire functions. As for similar results, work has been done in the the Burgers equation (see [16]), in the KdV equation (see [17, 18, 3, 4]) and in an artificial advection-diffusion problem (see [6, 7]). As for the Stokes system with small diffusion and a transport term, this is the first time that such a system has been studied, and is, indeed, one of the novelties of the paper.

The study of control problems associated to Stokes systems with Navier-slip boundary condition (see (1.1)) is not new in the literature: the existence of d dimensional controls leading to null controllability was proved in [19], while the existence of $d-1$ dimensional controls has recently been proved in [22]. The need of working in a specific domain to simplify the problem is not new either. For example, recently, in [11], they prove that the Navier-Stokes is (globally) null controllable in any rectangle with boundary controls at two opposing edges and with a phantom force (see also [20] for a weaker result in cuboids).

Finally, the idea of using spectral decomposition is not new in Control Theory (see [9]). Indeed, for the heat and the Stokes context alone, there are many documents which deal with eigenfunctions of the elliptic operator, either for proving the existence of some control (see for instance [23, 24, 5]) or for estimating the cost of the control (see for instance [12]) or for giving negative answer (see for instance [24]) to the existence of a control. As for a system with small diffusion and a transport term, a spectral decomposition indirectly appears in [10, 26], when getting lower bounds for the optimal time T_0 in which the cost of the control decays exponentially with ε . However, as far as the author knows, this is the first time that a spectral decomposition has been used to get the dissipation estimate in a transport-diffusion system.

The rest of the paper is organized as follows: in Section 2 we study systems (1.3) and (1.5) for Ω_2 , in Section 3 we study the cost of the control for Ω_2 , in Section 4 we study the control problem (1.1) and

its adjoint system for Ω_3 , and in Section 5 we do some further comments about the techniques and point out some open problems.

2 Spectral decomposition of $\mathcal{H}((0, \pi)^2)$ and some immediate consequences

sec:ana2

In this section we look for solutions of (1.5) in Ω_2 for $d = 2$, and use them to extract information of (1.3) for $d = 2$. In particular, in Section 2.1 we give the proof of a technical result, in Section 2.2 we get some eigenfunctions (solutions of (1.5)) for $d = 2$ which form a total set in $\mathcal{H}(\Omega_2)$, and in Section 2.3 we use those eigenfunctions to extract information about the solutions of (1.3) for $d = 2$.

As for the notation, given any real Banach space $(V, \|\cdot\|)$ and $S \subset V$, we recall that $\text{span}(S)$ denotes the set of all linear combinations of the elements of S . Moreover, we recall that S is a *total set* if $\overline{\text{span}(S)} = V$. Also,

$$l^2 := \left\{ (a_k)_{k \in \mathbb{N}_*} : a_k \in \mathbb{R} \text{ and } \sum_{k \in \mathbb{N}_*} a_k^2 < +\infty \right\}.$$

Besides, for any bounded domain $\Theta \subset \mathbb{R}^d$, we denote:

$$L_0^2(\Theta) := \left\{ g \in L^2(\Theta) : \int_{\Theta} g = 0 \right\}.$$

Moreover, we use $m = (m_1, m_2) \in (\mathbb{N}_*)^2$ to index the eigenfunctions. Finally, we denote the set of functions with separated variables having null normal trace and null divergence as:

$$SV(\Omega_2) := \left\{ \left(\int_0^{x_1} g_1(s) ds g_2(x_2), -g_1(x_1) \int_0^{x_2} g_2(s) ds \right) : g_1, g_2 \in L_0^2(0, \pi) \right\}. \quad (2.1) \quad \text{eq:sepvar}$$

2.1 Some functional analysis results

ec:funcanly

In order to look for eigenfunctions, we look among those which have an easier structure; in particular, among those which belong to $SV(\Omega_2)$. Indeed, by construction, $SV(\Omega_2) \subset \mathcal{H}(\Omega_2)$. Moreover, the interest of $SV(\Omega_2)$ is that it is a total set:

prop:spanSV

Proposition 2.1. *We have:*

$$\overline{\text{span}(SV(\Omega_2))} = \mathcal{H}(\Omega_2).$$

Remark 2.2. This result is not surprising since it is well-known that functions with separated variables form a total set in $L^2(\Omega_2)$.

In order to prove Proposition 2.1, we first need a technical result:

20vec2ortLp

Lemma 2.3. *Let $h \in L^2(\Omega_2)$ a function such that*

$$\int_{\Omega_2} h(x_1, x_2) g_1(x_1) g_2(x_2) = 0 \quad (2.2) \quad \text{hip:htwocoorbis}$$

for all $g_1, g_2 \in L_0^2(0, \pi)$. Then, there are $h_1 \in L_0^2(0, \pi)$ and $h_2 \in L^2(0, \pi)$ such that:

$$h(x_1, x_2) = h_1(x_1) + h_2(x_2) \text{ almost everywhere in } \Omega_2.$$

The fact that the mean of h_1 is null gives us uniqueness and continuity in the decomposition. The proof, up to the author's knowledge, is not written explicitly anywhere, but we can follow the classical scheme of first proving a regular version of the lemma, and then generalizing it to weaker spaces thanks to mollifiers. The proof is not very difficult, so it is omitted for a reason of length, but it is available at [2].

Now we are ready to prove Proposition 2.1:

Proof of proposition 2.1. Since $\overline{\text{span}(SV(\Omega_2))}$ is a closed vectorial subspace of $\mathcal{H}(\Omega_2)$, it suffices to prove that:

$$\overline{\text{span}(SV(\Omega_2))}^\perp = \{0\}. \quad (2.3) \quad \text{eq:totalperp}$$

So, let us consider $u \in \mathcal{H}(\Omega_2)$ such that:

$$\int_{\Omega_2} u \cdot v = 0$$

for any $v \in SV(\Omega_2)$. Recalling (2.1) and with an integration by parts, we have that, for any $g_1, g_2 \in L_0^2(0, \pi)$:

$$\begin{aligned} 0 &= \int_{\Omega_2} u_1(x_1, x_2) \left(\int_0^{x_1} g_1(s) ds \right) g_2(x_2) dx - \int_{\Omega_2} u_2(x_1, x_2) g_1(x_1) \left(\int_0^{x_2} g_2(s) ds \right) dx \\ &= \int_{\Omega_2} \left(- \int_0^{x_1} u_1(s, x_2) ds + \int_0^{x_2} u_2(x_1, s) ds \right) g_1(x_1) g_2(x_2) dx. \end{aligned} \quad (2.4) \quad \text{eq:deru2fg}$$

Thus, using Lemma 2.3, we have that there are $h_1, h_2 \in L^2(0, \pi)$ such that:

$$- \int_0^{x_1} u_1(s, x_2) ds + \int_0^{x_2} u_2(x_1, s) ds = h_1(x_1) + h_2(x_2).$$

If we differentiate with respect to $\partial_{x_1 x_2}$, we have that:

$$0 = -\partial_{x_2} u_1(x_1, x_2) + \partial_{x_1} u_2(x_1, x_2) = \nabla \times u.$$

In addition, since $u \in \mathcal{H}(\Omega_2)$, we also have that $\nabla \cdot u = 0$ and $u \cdot n = 0$. Consequently, since Ω_2 is Lipschitz and simply connected, we have that $u = 0$ (see, for instance, [1, Lemma IV.4.6]). \square

2.2 A total set of $\mathcal{H}(\Omega_2)$ formed by solutions of (1.5)

Throughout this subsection we focus on getting all the solutions of (1.5) (for $d = 2$). In particular, we first get the candidates and then prove that they form a total set:

Proposition 2.4. For all $m = (m_1, m_2) \in (\mathbb{N}_*)^2$ let us set:

$$u_m(x) := \left(\int_0^{x_1} g_{1,m_1}(s) ds g_{2,m_2}(x_2), -g_{1,m_1}(x_1) \int_0^{x_2} g_{2,m_2}(s) ds \right), \quad (2.5) \quad \text{def:umSV}$$

for

$$g_{1,m_1}(s) := \cos(m_1 s), \quad g_{2,m_2}(s) := (2m_2 \varepsilon \cos(m_2 s) - \sin(m_2 s)) e^{-(2\varepsilon)^{-1} s}. \quad (2.6) \quad \text{def:g1m1g2m2}$$

Then, u_m is a solution of the system (1.5) (for $d = 2$) such that

$$\|u_m\|_{\mathbf{L}^2(\Omega)} = \sqrt{\frac{(1 - e^{-\pi\varepsilon^{-1}})m_2^2\pi\varepsilon^3(1 + 4\varepsilon^2(m_1 + m_2^2))}{m_1 + 4m_1m_2^2\varepsilon^2}}, \quad (2.7) \quad \text{def:normcoeftrans}$$

such that its associated pressure only depends on the time variable and such that its associated eigenvalue is:

$$\lambda_m = (m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}. \quad (2.8) \quad \text{eq:eigenval}$$

Remark 2.5. We have that:

$$u_m(x) = e^{-(2\varepsilon)^{-1}x_2} \left(\frac{\sin(m_1x_1)}{m_1} (2m_2\varepsilon \cos(m_2x_2) - \sin(m_2x_2)), -2\varepsilon \cos(m_1x_1) \sin(m_2x_2) \right), \quad (2.9) \quad \text{def:eigenbase}$$

and, using that $m_1, m_2 \geq 1$ and $\varepsilon \in (0, 1)$, the estimate:

$$\|u_m\|_{\mathbf{L}^2(\Omega)} \leq C(m_2\varepsilon + 1). \quad (2.10) \quad \text{est:umbrut}$$

Proof. We can check directly the conclusions of Proposition 2.4 for the functions given in (2.9). In particular, (2.7) has been obtained through a software. However, we show an “intuitive” way to get the u_m , for giving a better insight to the reader. With that purpose, we look for eigenfunctions in $SV(\Omega_2)$ (see (2.1)) such that $g_1, g_2 \in C^\infty([0, \pi])$. We remark that the second boundary condition of (1.5)₃, using that in the square $n_1n_2 = 0$, translates into:

$$\varepsilon(n_1^2 - n_2^2)(\partial_{x_2}u_1 + \partial_{x_1}u_2) - n_2^2u_1 = 0.$$

So, looking at the left, right, bottom and top of the square, if we write the elements of $SV(\Omega_2)$ as in (2.1), we must have:

$$g_1'(0) = g_1'(\pi) = \varepsilon g_2'(0) + g_2(0) = \varepsilon g_2'(\pi) + g_2(\pi) = 0. \quad (2.11) \quad \text{eq:stotransbnDu}$$

If we apply the divergence operator to (1.5)₁ (for $d = 2$), we obtain that $\Delta p = 0$. In particular, we can apply the Laplacian operator to (1.5)₁ to get possible solutions. Indeed, since:

$$\begin{aligned} \Delta u_1 &= g_1'(x_1)g_2(x_2) + \int_0^{x_1} g_1(s)ds g_2''(x_2), \\ \partial_{x_2}\Delta u_1 &= g_1'(x_1)g_2'(x_2) + \int_0^{x_1} g_1(s)ds g_2'''(x_2), \\ \Delta^2 u_1 &= g_1'''(x_1)g_2(x_2) + 2g_1'(x_1)g_2''(x_2) + \int_0^{x_1} g_1(s)ds g_2^{iv}(x_2); \end{aligned}$$

we have that:

$$\begin{aligned} -\varepsilon g_1'''(x_1)g_2(x_2) - 2\varepsilon g_1'(x_1)g_2''(x_2) - \varepsilon \int_0^{x_1} g_1(s)ds g_2^{iv}(x_2) - g_1'(x_1)g_2'(x_2) \\ - \int_0^{x_1} g_1(s)ds g_2'''(x_2) - \lambda g_1'(x_1)g_2(x_2) - \lambda \int_0^{x_1} g_1(s)ds g_2''(x_2) = 0. \end{aligned}$$

We have that $g_{1,m_1} \in L_0^2(0, \pi)$ (see (2.6)), that g_{1,m_1} satisfies the first two conditions of (2.11), and that g_{1,m_1} is an eigenfunction of the Laplacian; so it is a reasonable function to do an attempt. Under that choice, the functions $g_2 \in L_0^2(0, \pi)$ have to satisfy the two last conditions of (2.11) and the ODE:

$$-\varepsilon m_1^4 g_2 + 2\varepsilon m_1^2 g_2'' - \varepsilon g_2^{iv} + m_1^2 g_2' - g_2''' + \lambda m_1^2 g_2 - \lambda g_2'' = 0. \quad (2.12) \quad \text{eq:ODEf}$$

In order to solve (2.12) we calculate the roots of its characteristic polynomial:

$$-\varepsilon(r^2 - m_1^2)^2 - r(r^2 - m_1^2) - \lambda(r^2 - m_1^2) = 0. \quad (2.13) \quad \text{eq:pchrODEf}$$

Let us focus on the roots of

$$r^2 + \frac{r}{\varepsilon} + \frac{\lambda}{\varepsilon} - m_1^2 = 0, \quad (2.14) \quad \text{eq:pchrlambdaODEf}$$

which are given by:

$$r = \frac{-\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - 4\frac{\lambda}{\varepsilon} + 4m_1^2}}{2}.$$

These roots are complex if λ is sufficiently large with respect to ε^{-1} . We then have that the associated solutions for λ large enough are:

$$g_{2,m_2}(s) := (A \cos(m_2 s) + B \sin(m_2 s)) e^{-(2\varepsilon)^{-1} s}, \quad (2.15) \quad \text{def:g2m2}$$

where $m_2 > 0$ is defined by:

$$m_2 := m_2(\varepsilon, \lambda, m_1) := \sqrt{\frac{\lambda}{\varepsilon} - \frac{1}{4\varepsilon^2} - m_1^2}. \quad (2.16) \quad \text{def:keplambda}$$

Consequently, the g_{2,m_2} that we get are solutions of:

$$-\varepsilon g_2'' - g_2' = (\lambda - \varepsilon m_2^2) g_2. \quad (2.17) \quad \text{eq:ODEg2m2}$$

We are going to see that there are some $g_{2,m_2} \in L_0^2(0, \pi)$ that also satisfy the last two conditions of (2.11). Indeed, we have that:

$$g_{2,m_2}'(s) = \left[\left(-\frac{A}{2\varepsilon} + m_2 B \right) \cos(m_2 s) + \left(-\frac{B}{2\varepsilon} - A m_2 \right) \sin(m_2 s) \right] e^{-(2\varepsilon)^{-1} s}$$

From $g_{2,m_2}(0) = -\varepsilon g_{2,m_2}'(0)$, we obtain that:

$$A = -2m_2 \varepsilon B. \quad (2.18) \quad \text{eq:bnds1}$$

To continue with, from $g_{2,m_2}(\pi) = -\varepsilon g_{2,m_2}'(\pi)$ and (2.18) we have that:

$$-2m_2 \varepsilon B \cos(m_2 \pi) + B \sin(m_2 \pi) = -2m_2 \varepsilon B \cos(m_2 \pi) + \left(\frac{1}{2} - 2m_2^2 \varepsilon^2 \right) B \sin(m_2 \pi).$$

Consequently,

$$B \sin(m_2 \pi) = -4m_2^2 \varepsilon^2 B \sin(m_2 \pi).$$

Hence, either we have $B = 0$, which by (2.18) implies that $g_{2,m_2} = 0$, which just gives the null solution; or $m_2 \in \mathbb{N}_*$. Consequently, the possible values for g_{2,m_2} are (we have taken $B = -1$):

$$g_{2,m_2}(s) = (2m_2 \varepsilon \cos(m_2 s) - \sin(m_2 s)) e^{(2\varepsilon)^{-1} s} = \frac{d \left(2\varepsilon \sin(m_2 s) e^{-(2\varepsilon)^{-1} s} \right)}{ds} : m_2 \in \mathbb{N}_*, \quad (2.19) \quad \text{def:eigenf}$$

which clearly belong to $L_0^2(0, \pi)$. Thus, our candidates are those defined in (2.6) and λ_m satisfies (2.8) because of (2.16).

In order to see that u_m satisfies the first equation of (1.5)₁ (for $d = 2$) with a constant pressure for any given time, we have to use (2.5), that g_{2,m_2} is a solution of (2.17) and that g_{1,m_1} is a solution of $-\varepsilon g''_{1,m_1} = \varepsilon m_1^2 g_{1,m_1}$. Moreover, in order to see that they satisfy the second equation of (1.5)₁ (for $d = 2$), we consider (2.17) and that $\varepsilon g'_{2,m_2}(0) + g_{2,m_2}(0) = 0$, which imply that:

$$(\lambda - \varepsilon m_1^2) \int_0^s g_{2,m_2} = -\varepsilon g'_{2,m_2} - g_{2,m_2} = -\varepsilon \frac{d^2 \left(\int_0^s g_{2,m_2} \right)}{ds^2} - \frac{d \left(\int_0^s g_{2,m_2} \right)}{ds}.$$

□

op:umtotset

Proposition 2.6. *The set $\{u_m\}_{m \in (\mathbb{N}_*)^2}$ is a total set.*

Proof. Thanks to Proposition 2.1, it suffices to prove that:

$$SV(\Omega_2) \subset \overline{\text{span}\{u_m\}}. \quad (2.20)$$

eq:dedniowd

We are going to prove this inclusion constructively; that is, writing an element of $SV(\Omega_2)$ as a limit of linear combinations of the eigenfunctions u_m . We consider a function $u \in SV(\Omega_2)$ (see (2.1)).

Let us recall that $\left\{ \frac{\cos(ks)}{\sqrt{2^{-1}\pi}} \right\}_{k \in \mathbb{N}_*} \cup \{\pi^{-1}\}$ is an orthonormal basis of $L^2(0, \pi)$ (the basis obtained by the diagonalization of the Laplacian with Neumann boundary conditions in $(0, \pi)$). So, since $g_1 \in L^2_0(0, \pi)$, we have that, for some $a_k \in \ell^2$:

$$g_1(s) = \sum_{k \geq 1} a_k \frac{\cos(ks)}{\sqrt{2^{-1}\pi}}. \quad (2.21)$$

eq:decgnmnlpl

That series must be understood as a limit in $L^2(0, \pi)$.

To continue with, we remark that:

$$\left\{ \frac{\gamma_k^\varepsilon}{\sqrt{2^{-1}\pi} \sqrt{1 + 4k^2\varepsilon^2}} \right\}_{k \geq 1} \cup \left\{ \frac{e^{-(2\varepsilon)^{-1}s}}{\sqrt{\varepsilon(1 - e^{-\pi/\varepsilon})}} \right\}, \quad (2.22)$$

base:robin

for

$$\gamma_k^\varepsilon(s) := 2k\varepsilon \cos(ks) - \sin(ks), \quad (2.23)$$

def:gammaepsk

is the orthonormal basis obtained by diagonalization of the Laplacian in $(0, \pi)$ with Robin boundary conditions:

$$\begin{cases} -2\varepsilon g'(0) = g(0), \\ -2\varepsilon g'(\pi) = g(\pi). \end{cases} \quad (2.24)$$

eq:Robbin0pi

Indeed, we have that the Laplacian is self-adjoint in the subspace of $H^2(\Omega_2)$ that satisfies (2.24). Thus, we have, for some $b_k \in \ell^2$:

$$g_2(s) = \left(g_2(s) e^{(2\varepsilon)^{-1}s} \right) e^{-(2\varepsilon)^{-1}s} = \sum_{k \geq 1} b_k \frac{\gamma_k^\varepsilon(s) e^{-(2\varepsilon)^{-1}s}}{\sqrt{2^{-1}\pi} \sqrt{1 + 4k^2\varepsilon^2}}. \quad (2.25)$$

eq:decfrobblpl

We have used that $g_2 \in L^2_0(0, \pi)$ implies that $g_2 e^{(2\varepsilon)^{-1}s}$ is orthogonal to $e^{-(2\varepsilon)^{-1}s}$. Moreover, we have to understand the series as a limit in $L^2(0, \pi)$, not as a pointwise one. In that sense, we can place the exponential into the series because the multiplication of the functions of $L^2(0, \pi)$ by an element of $L^\infty(0, \pi)$ is a continuous linear form.

Next, we recall that the operator \int_0^s is a continuous endomorphism in $L^2(0, \pi)$. Thus, we have that:

$$\int_0^s g_1(s') ds' = \sum_{k \geq 1} a_k \frac{\sin(ks)}{k\sqrt{2^{-1}\pi}}. \quad (2.26) \quad \text{eq:decintgfou}$$

Similarly, we have that:

$$\int_0^s g_2(s') ds' = \sum_{k \geq 1} b_k \frac{2\varepsilon \sin(ks) e^{-(2\varepsilon)^{-1}s}}{\sqrt{2^{-1}\pi} \sqrt{1 + 4k^2\varepsilon^2}}. \quad (2.27) \quad \text{eq:decintffou}$$

The limits in (2.26) and (2.27) are both pointwise and in $L^2(0, \pi)$.

We finally recall that the tensor product of two convergent sequences of $L^2(0, \pi)$ converges in $L^2(\Omega_2)$ to the tensor product of their respective limits. Thus, we have that (2.21), (2.25), (2.26) and (2.27) imply:

$$\int_0^{x_1} g_1(s) ds g_2(x_2) = \lim_{k \rightarrow \infty} \sum_{m_1, m_2 \geq 1}^k \frac{2a_{m_1} b_{m_2}}{\pi \sqrt{1 + 4m_2^2\varepsilon^2}} \frac{\sin(m_1 x_1)}{m_1} \gamma_{m_2}^\varepsilon(x_2) e^{-(2\varepsilon)^{-1}x_2},$$

and that

$$-g_1(x_1) \int_0^{x_2} g_2(s) ds = \lim_{k \rightarrow \infty} \sum_{m_1, m_2 \geq 1}^k \frac{2a_{m_1} b_{m_2}}{\pi \sqrt{1 + 4m_2^2\varepsilon^2}} \left(-2\varepsilon \cos(m_1 x_1) \sin(m_2 x_2) e^{-(2\varepsilon)^{-1}x_2} \right).$$

Consequently, we have that:

$$u = \lim_{k \rightarrow \infty} \sum_{m_1, m_2 \geq 1}^k \frac{2a_{m_1} b_{m_2}}{\pi \sqrt{1 + 4m_2^2\varepsilon^2}} u_m,$$

which proves (2.20). □

2.3 Using the spectral decomposition to solve the parabolic equation

Since our operator is not self-adjoint, the total set $\{u_m\}$ given in (2.9) does not have to be orthogonal (and is not). In addition, it is not immediate to write an element of $\mathcal{H}(\Omega_2)$ as a series of elements proportional to those in $\{u_m\}$. Indeed, there are total sets in l^2 (for instance $\{e_1\} \cup \{e_1 + e_k\}_{k \geq 2}$) such that it is not possible to write any element of l^2 (for instance $(k^{-1})_{k \geq 1}$) as a series of elements proportional to those in the total sets. In our particular situation we do not even know if this is possible (probably yes), but we will prove that at least we can express the solutions of (1.3) for $d = 2$ with the help of $\{u_m\}$.

The next step is given any $u \in \text{span}\{u_m\}$, to get its coordinates in our generating system. In order to do so, we are going to get a set of functions $v_m \in \mathbf{L}^2(\Omega_2)$ such that $\{(u_m, v_m)\}$ is a bi-orthogonal system; that is, such that:

$$\int_{\Omega_2} u_m \cdot v_{m'} = 1_{m=m'}. \quad (2.28) \quad \text{eq:coordfunc}$$

Usually, we can get those functions v_m by diagonalization of the adjoint operator, but it is easier to do a systematic search in $\mathbf{L}^2(\Omega_2)$. In that sense we define (recall (2.23)):

$$v_m(x) := \frac{2}{\pi^2} e^{(2\varepsilon)^{-1}x_2} \left(m_1 \sin(m_1 x_1) \frac{\gamma_{m_2}^\varepsilon(x_2)}{1 + 4m_2^2\varepsilon^2}, -\frac{\cos(m_1 x_1) \sin(m_2 x_2)}{2\varepsilon} \right), \quad (2.29) \quad \text{def:vm}$$

We can very easily verify (2.28) because $\{\sqrt{2\pi^{-1}}\sin(ks)\}_{k \in \mathbb{N}_*}$, $\{\sqrt{2\pi^{-1}}\cos(ks)\}_{k \in \mathbb{N}_*} \cup \{\pi^{-1}\}$ and (2.22) are Hilbert basis in $L^2(0, \pi)$. We remark that, for a constant C that does not depend on m_1 , m_2 or ε :

$$\|v_m\|_{\mathbf{L}^2(\Omega_2)} \leq C \left(m_1 + \frac{1}{\varepsilon}\right) e^{(2\varepsilon)^{-1}\pi}. \quad (2.30) \quad \text{bound: vnm}$$

Now we are ready to get the solutions of (1.3) through a spectral decomposition.

Proposition 2.7. *Let $\varphi^T \in \mathcal{H}(\Omega_2)$. Let us denote, for $t \leq T$, $L_\varepsilon^T(t)\varphi^T$ the value of the energy solution of system (1.3) for $d = 2$ and time t . Then, we have that:*

1. For all $t < T$ and $\varepsilon > 0$:

$$L_\varepsilon^T(t)\varphi^T = \sum_{m \in (\mathbb{N}_*)^2} \langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)} \exp\left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}\right)(t - T)\right] u_m. \quad (2.31) \quad \text{eq: solsum}$$

In particular, the series of the right hand-side of (2.31) is well-defined and absolutely convergent in $\mathcal{H}(\Omega_2)$.

2. For all $\varepsilon > 0$, the solution (φ, p) of (1.3) for $d = 2$ and $\varphi^T \in \mathcal{H}(\Omega_2)$ satisfies $\nabla p = 0$.

3. For all $\delta > 0$ there is $C_\delta > 0$ such that for all $T > 0$, $\varphi^T \in \mathcal{H}(\Omega_2)$, $\varepsilon \in (0, 1)$ and $s \leq T - 2\pi - \delta$:

$$\|L_\varepsilon^T(s)\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq C_\delta \exp\left[\frac{s - (T - 2\pi - \delta)}{4\varepsilon}\right] \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}. \quad (2.32) \quad \text{eq: decayexp}$$

Throughout the proof we denote $C_{\varepsilon, t}$ a generic constant which might be different each time and which just depends on ε and t .

Proof. First of all, we have that (2.31) is true if $\varphi^T \in \text{span}\{u_m\}$ as a consequence of Proposition 2.4 (see (2.8)) and (2.28). Moreover, for $t > 0$ and $\varepsilon > 0$ fixed, $L_\varepsilon^T(t)$ is a continuous endomorphism on $\mathcal{H}(\Omega_2)$. Consequently in order to prove item 1, it suffices to prove that the series of the right-hand side of (2.31) is well-defined, absolutely convergent and continuous. We shall prove the three properties at once.

Recalling estimate (2.10), estimate (2.30), and that xe^{-x^2} is bounded in \mathbb{R}^+ , we have that, if $\varepsilon \in (0, 1)$:

$$\begin{aligned} & \sum_{m \in (\mathbb{N}_*)^2} |\langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)}| \exp\left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}\right)(t - T)\right] \|u_m\|_{\mathbf{L}^2(\Omega_2)} \\ & \leq C_{\varepsilon, t} \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \sum_{m \in (\mathbb{N}_*)^2} m_1 m_2 \exp\left((m_1^2 + m_2^2)\varepsilon(t - T)\right) \\ & \leq C_{\varepsilon, t} \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \sum_{m \in (\mathbb{N}_*)^2} \exp\left((m_1^2 + m_2^2)\frac{\varepsilon(t - T)}{2}\right). \end{aligned} \quad (2.33) \quad \text{eq: absconvh}$$

The series in the right-hand side of (2.33) is convergent because, in $(\mathbb{R}^+)^2$:

$$\begin{aligned} & \exp\left((m_1^2 + m_2^2)\frac{\varepsilon(t - T)}{2}\right) 1_{[m_1-1, m_1] \times [m_2-1, m_2]}(x_1, x_2) \\ & \leq \exp\left((x_1^2 + x_2^2)\frac{\varepsilon(t - T)}{2}\right) 1_{[m_1-1, m_1] \times [m_2-1, m_2]}(x_1, x_2), \end{aligned}$$

which implies that:

$$\sum_{m \in (\mathbb{N}_*)^2} \exp\left((m_1^2 + m_2^2) \frac{\varepsilon(t-T)}{2}\right) \leq \int_{(\mathbb{R}^+)^2} \exp\left((x_1^2 + x_2^2) \frac{\varepsilon(t-T)}{2}\right) dx = \frac{2}{\varepsilon(T-t)} \frac{\pi}{4}. \quad (2.34) \quad \text{eq:sumexp}$$

Combining (2.33) and (2.34), we have that:

$$\sum_{m \in (\mathbb{N}_*)^2} |\langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)}| \exp\left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}\right)(t-T)\right] \|u_m\|_{\mathbf{L}^2(\Omega_2)} \leq C_{\varepsilon,t} \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}, \quad (2.35) \quad \text{eq:boundcont}$$

which implies that the series in the right-hand side of (2.31) is absolutely convergent and thus well-defined. Moreover, since the right-hand side of (2.31) is linear, (2.35) implies that the right-hand side of (2.31) is continuous in $\mathcal{H}(\Omega_2)$.

As for item 2, if $\varphi^T \in \text{span}\{u_m\}$, it is a straight consequence of Proposition 2.4. If we consider a general $\varphi^T \in \mathcal{H}(\Omega_2)$, in order to see that the gradient of the pressure is null, we just have to use that $\{u_m\}$ is a total set (see Proposition 2.6) and the continuity of the gradient of the pressure in $H^{-1}(0, T; H^{-1}(\Omega_2))$ with respect to the initial value of (1.3).

Finally, let us prove (2.32). Indeed, using (2.10), the triangular inequality, and Cauchy-Schwarz, we have that, for all $\varepsilon \in (0, 1)$:

$$\|L_\varepsilon^T(s) \varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq C \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \sum_{m \in (\mathbb{N}_*)^2} m_2 \|v_m\|_{\mathbf{L}^2(\Omega_2)} \exp\left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}\right)(s-T)\right]. \quad (2.36) \quad \text{eq:estS}$$

Next, using (2.30) and that xe^{-x^2} is bounded in \mathbb{R}^+ , if $s \leq T-2$ and if $\varepsilon \in (0, 1)$, we have that (2.36) turns into:

$$\|L_\varepsilon^T(s) \varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq \frac{C \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}}{\varepsilon^2} \left[\sum_{m \in (\mathbb{N}_*)^2} \exp(-(m_1^2 + m_2^2)\varepsilon) \right] \exp\left(\frac{s-(T-2\pi)}{4\varepsilon}\right). \quad (2.37) \quad \text{est:S(t)almost}$$

Using (2.34) for $t = T-2$, we have that there is $C_\delta > 0$ such that for all $\varepsilon \in (0, 1)$:

$$\frac{1}{\varepsilon^2} \sum_{m \in (\mathbb{N}_*)^2} \exp(-(m_1^2 + m_2^2)\varepsilon) \leq \frac{\pi}{4\varepsilon^3} \leq C_\delta e^{\delta(4\varepsilon)^{-1}}.$$

Consequently, (2.37) implies (2.32). \square

rk:phi2heat *Remark 2.8.* An immediate consequence of Proposition 2.7 is that if $\varphi^T \in \mathcal{H}(\Omega_2)$, φ_2 (the second component of the solution φ of (1.3) for $d = 2$) satisfies:

$$\begin{cases} -z_t - \varepsilon \Delta z - \partial_{x_2} z = 0 & \text{in } Q, \\ \partial_{x_1} z = \partial_n z = 0 & \text{on } \Sigma_{lr}, \\ z = 0 & \text{on } \Sigma_{bt}, \\ z(T, \cdot) = z^T & \text{on } \Omega_2, \end{cases} \quad (2.38) \quad \text{eq:squarephi2}$$

for $z^T = \varphi_2^T$ and for

$$\Sigma_{lr} := (0, T) \times \{0, \pi\} \times (0, \pi), \text{ and } \Sigma_{bt} := (0, T) \times (0, \pi) \times \{0, \pi\}.$$

rk:contin0 *Remark 2.9.* Let $T > 0$ and $\varphi^T \in \mathcal{H}(\Omega_2)$. Using the formula (2.31), we can see that the application

$$\mathbb{R}_*^+ \ni \varepsilon \mapsto \varphi^\varepsilon(0, \cdot) \in \mathcal{H}(\Omega_2)$$

(for φ^ε the solution of (1.3) with $d = 2$ and initial value φ^T) is continuous. Indeed, if $\varepsilon \in (\varepsilon_0/2, 2\varepsilon_0)$, for any $\delta > 0$ we can bound the sum of all the terms of (2.31) except a finite amount by $\delta/3$, and then we can use the continuity on each of the remaining terms.

3 The cost of the control in $(0, \pi)^2$

costsquare In this section we prove the first three items of Theorem 1.1. In the three proofs we use the equivalent definitions of the costs given in (1.4). In addition, for the proof of item 2 we use the eigenfunctions given in (2.9), whereas for the other two proofs we use Remark 2.8 and then inspire in [21]. The three proofs are independent.

In this section we use the notation:

$$\Sigma_{lr} := (0, T) \times \{0, \pi\} \times (0, \pi), \text{ and } \Sigma_{bt} := (0, T) \times (0, \pi) \times \{0, \pi\}.$$

3.1 Proof of item 1 of Theorem 1.1

proofcondeg In order to prove that the cost of the control decays, we first state a Carleman inequality for system (2.38), then use that the divergence is null, and finally apply usual parabolic estimates.

We consider ω_0 an open ball centered at some point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and whose closure is contained in ω . We consider the auxiliary function

$$\eta(x) := (2\pi^2)^{-1} (G(x_1) - (x_2 - \bar{x}_2)^2),$$

for G a regular (positive) function such that $G(0) = G(\pi) = 0$, $G'' < 0$, $G'(\bar{x}_1) = 0$ and $\|G\|_{L^\infty(0, \pi)} \leq \pi^2$. Finally, let us consider the following classical weights, for $\tilde{T} > 0$:

$$\begin{aligned} \alpha(t, x) &:= \frac{e^{4\lambda} - e^{\lambda(\eta(x)+2)}}{t(\tilde{T} - t)}, & \xi(t, x) &:= \frac{e^{\lambda(\eta(x)+2)}}{t(\tilde{T} - t)}, \\ \alpha^*(t) &= \max_{x \in \tilde{\Omega}} \alpha(t, x), & \xi^*(t) &= \min_{x \in \tilde{\Omega}} \xi(t, x). \end{aligned} \tag{3.1}$$

eq:weightssqr

rk:etainf *Remark 3.1.* By the choice of G we have that $\|\eta\|_{L^\infty(\Omega)} \leq 1$, so α is a positive weight. This way of choosing the weight functions is classical (see [14] or [9], for instance).

carlediftr **Proposition 3.2.** *There is a positive constant C such that, for any $\tilde{T} > 0$, for any $z^T \in L^2(\Omega_2)$, the solution z of (2.38) satisfies:*

$$\tau \lambda^2 \iint_Q e^{-2\tau\alpha} \xi |\nabla z|^2 + \tau^3 \lambda^4 \iint_Q e^{-2\tau\alpha} \xi^3 |z|^2 \leq C \tau^3 \lambda^4 \iint_{Q_\omega} e^{-2\tau\alpha} \xi^3 |z|^2, \tag{3.2}$$

car:heattransmix

for any $\varepsilon \in (0, 1)$, for any $\lambda \geq C$ and for any $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$.

The proof of Proposition 3.2 is essentially a combination of the proofs given in [21, Proposition 1] and [13, Lemma 1]. Indeed, we define some auxiliary weights with the help of the following auxiliary function:

$$\eta^-(x) := -(2\pi^2)^{-1} (G(x_1) + (x_2 - \bar{x}_2)^2).$$

Then, we work like in [13, Lemma 1] to deal with the boundary terms of $\Sigma_{l,r}$ and like in [21, Proposition 1] to deal with the boundary terms of $\Sigma_{b,t}$. Since the proof is not original and since it is rather long, we omit it, but it is available at [2].

A straight consequence of Proposition 3.2 and Remark 2.8 is that there is $C > 0$ such that for any $\varphi^T \in \mathbf{L}^2(\Omega)$, if φ denotes the solution of (1.3) (for $d = 2$), for any $\lambda \geq C$, for any $\tau \geq C\varepsilon^{-1}$, and for the weights (3.1) with $\tilde{T} = 1$, we have that:

$$\begin{aligned} \tau\lambda^2 \iint_{(0,1)\times\Omega} e^{-2\tau\alpha\xi} |\nabla\varphi_2(t+T-1, x)|^2 + \tau^3\lambda^4 \iint_{(0,1)\times\Omega} e^{-2\tau\alpha\xi^3} |\varphi_2(t+T-1, x)|^2 \\ \leq C\tau^3\lambda^4 \iint_{(0,1)\times\omega} e^{-2\tau\alpha\xi^3} |\varphi_2(t+T-1, x)|^2. \end{aligned} \quad (3.3) \quad \text{car:u2timelarg}$$

Moreover, since $\varphi_1 = 0$ on $\{0\} \times (0, \pi)$, we have by Poincaré inequality for all $t < T$:

$$\tau\lambda^2 \int_{\Omega} e^{-2\tau\alpha^* \xi^*} |\varphi_1|^2 \leq C\tau\lambda^2 \int_{\Omega} e^{-2\tau\alpha^* \xi^*} |\partial_{x_1}\varphi_1|^2. \quad (3.4) \quad \text{eq:poinu1}$$

Thus, since $\nabla \cdot \varphi = 0$, combining (3.4) with (3.3), we have that, for $\lambda \geq C$, $\tau \geq C\varepsilon^{-1}$:

$$\tau\lambda^2 \iint_{(0,1)\times\Omega} e^{-2\tau\alpha^* \xi^*} |\varphi(t+T-1, x)|^2 \leq C\tau^3\lambda^4 \iint_{(0,1)\times\omega} e^{-2\tau\alpha\xi^3} |\varphi_2(t+T-1, x)|^2. \quad (3.5) \quad \text{car:ulocu2}$$

So, fixing λ large enough, and $\tau = \tau_0\varepsilon^{-1}$ for τ_0 large enough, we have that (3.5) implies that:

$$\|\varphi\|_{\mathbf{L}^2((T-2/3, T-1/3)\times\Omega)} \leq Ce^{C\varepsilon^{-1}} \|\varphi_2\|_{L^2((T-1, T)\times\omega)}. \quad (3.6) \quad \text{est:uL2u2loc}$$

Let $T > 7$, using item 3 of Proposition 2.7 there is $C > 0$ such that for $\delta = 7 - 2\pi - 2/3$, for all $t' \in (T - 2/3, T - 1/3)$ and for $s = 0$, we have that:

$$\|\varphi(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \leq C \exp\left[\frac{7-2/3-t'}{4\varepsilon}\right] \|\varphi(t', \cdot)\|_{\mathbf{L}^2(\Omega)} \leq C \exp\left[\frac{7-T}{4\varepsilon}\right] \|\varphi(t', \cdot)\|_{\mathbf{L}^2(\Omega)}. \quad (3.7) \quad \text{eq:estadju2}$$

So, combining (3.6) and (3.7), we get that, for any $T > 7$:

$$\|\varphi(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \leq C \exp\left[\frac{7-T}{4\varepsilon}\right] \|\varphi\|_{\mathbf{L}^2((T-2/3, T-1/3)\times\Omega)} \leq C \exp\left[\frac{C-T}{4\varepsilon}\right] \|\varphi_2\|_{L^2((T-1, T)\times\omega)}.$$

In particular, if T_0 is sufficiently large, item 1 of Theorem 1.1 is true.

3.2 Proof of item 2 of Theorem 1.1

Let us fix $h \in (0, \pi)$ and a control domain $\omega \subset (0, \pi) \times (\pi - h, \pi)$. We are going to see that, for $T \in (0, 2(\pi - h))$ fixed, the cost of the control is at least exponentially large with respect to ε^{-1} . We recall that

$$u(x) := ce^{-(2\varepsilon)^{-1}x_2} (\sin(x_1)(2\varepsilon \cos(x_2) - \sin(x_2)), -2\varepsilon \cos(x_1) \sin(x_2)),$$

with

$$c := \sqrt{\frac{1 + 4\varepsilon^2}{(1 - e^{-\pi\varepsilon^{-1}})\pi\varepsilon^3(1 + 8\varepsilon^2)}}$$

is a solution of (1.5) for $\lambda_\varepsilon := 2\varepsilon + \frac{1}{4\varepsilon}$ and such that $\|u\|_{\mathbf{L}^2(\Omega)} = 1$ (see Proposition 2.4). In particular, we have that

$$\varphi(t, x) := u(x)e^{-\lambda_\varepsilon(T-t)}$$

is a solution of (1.3) for $d = 2$ and $\varphi^T = u$. On the one hand, we have that:

$$\int_{\Omega} |\varphi(0, \cdot)|^2 = e^{-2\lambda_\varepsilon T}. \quad (3.8) \quad \text{eq:normphiT}$$

On the other hand, we have that, for a fixed $\delta \in (0, \pi - h - T/2)$, since there is $C_\delta > 0$ such that $\varepsilon^{-5/2} \leq C_\delta e^{\delta\varepsilon^{-1}}$ in \mathbb{R}^+ , recalling that $\omega \subset (0, \pi) \times (\pi - h, \pi)$:

$$\iint_{Q_\omega} |\varphi|^2 \leq C_\delta \exp\left[\frac{h - \pi + \delta}{\varepsilon}\right] \int_0^T e^{-2\lambda_\varepsilon(T-t)} dt \leq C_\delta \exp\left[\frac{h - \pi + \delta}{\varepsilon}\right]. \quad (3.9) \quad \text{eq:estinomega}$$

So, combining (1.4), (3.8) and (3.9) we get that:

$$[K_1(T, \varepsilon, \omega)]^2 \geq c_\delta \exp\left[\frac{\pi - h - \delta - 4\varepsilon^2 T - T/2}{\varepsilon}\right],$$

which implies item 2 of Theorem 1.1 by the choice of δ .

3.3 Proof of item 3 of Theorem 1.1

Let $h \in (0, \pi)$, a control domain $\omega \subset (0, \pi) \times (0, h)$ and $T \in (0, \pi - h)$ fixed. In order to prove that the cost of the control blows out exponentially with respect to ε , we shall adapt the proof given in [21, Theorem 1]. Indeed, unlike in [21], φ is a vectorial function, the boundary conditions are not purely Dirichlet, and we cannot pick in (1.3) an initial value φ^T such that $\int_{\Omega} \varphi_2^T > 0$, because, for all $u \in \mathcal{H}(\Omega)$ we have that $\int_{\Omega} u_1 = \int_{\Omega} u_2 = 0$.

To make the proof more clear, we split it with the help of two lemmas:

Lemma 3.3. *Let $h \in (0, \pi)$, $T \in (0, \pi - h)$ and $\delta \in (0, \pi - T - h)$. Let $\varphi^T \in \mathcal{H}(\Omega_2) \cap \mathbf{C}^\infty(\overline{\Omega}_2) \setminus \{0\}$ such that $\text{supp}(\varphi^T) \subset (0, \pi) \times (\pi - \delta, \pi)$ and let φ^ε be the solution of (1.3) for $d = 2$ and initial value φ^T . Then, there is $c > 0$ such that, for all $\varepsilon \in (0, 1)$:*

$$\|\varphi^\varepsilon(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \geq c. \quad (3.10) \quad \text{est:phiTabajo}$$

Remark 3.4. For any $\delta > 0$ there are functions φ^T which satisfy the hypothesis of Lemma 3.3. For example, if $\chi_\delta \in C^\infty([0, \pi])$ is a function supported in $(\pi - \delta, \pi)$ of null mean, we can consider:

$$\varphi^T(x) := \left(\int_0^{x_1} \chi_\delta(s) ds \chi_\delta(x_2), -\chi_\delta(x_1) \int_0^{x_2} \chi_\delta(s) ds \right). \quad (3.11) \quad \text{eq:phi0stratflow}$$

Consequently, neither Lemma 3.3 nor Lemma 3.5 below are empty results.

Proof. We shall prove (3.10) by contradiction. First of all, we remark that, thanks to forward uniqueness of (1.3) (which can be proved by contradiction with backward uniqueness and (2.31)), $\|\varphi^\varepsilon(0, \cdot)\|_{\mathbf{L}^2(\Omega)} > 0$ for all $\varepsilon \in (0, 1]$. In addition, because of Remark 2.9, the only problematic situation is if there is $\varepsilon_k \rightarrow 0$ such that $\|\varphi^{\varepsilon_k}(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \rightarrow 0$. Let us suppose the existence of such sequence ε_k and derive a contradiction.

We have that $\{\varphi_2^\varepsilon : \varepsilon \in (0, 1)\}$ is bounded in $L^2(Q)$ (see Remark 2.8); thus, we can suppose that $\varphi_2^{\varepsilon_k}$ converges weakly in $L^2(Q)$ to some function γ . Let us consider $\psi \in \mathcal{D}([0, T] \times \Omega_2)$. Using Remark 2.8 we have that:

$$0 = \iint_Q \varphi_2^{\varepsilon_k} (\partial_t \psi - \varepsilon_k \Delta \psi + \partial_{x_2} \psi) + \langle \varphi_2^{\varepsilon_k}(0, \cdot), \psi(0, \cdot) \rangle_{L^2(\Omega_2)} - \langle \varphi_2^T, \psi(T, \cdot) \rangle_{L^2(\Omega_2)}. \quad (3.12) \quad \text{eq:vareqpsi}$$

By taking the weak limit of $\varphi_2^{\varepsilon_k}$, the strong limit of $\varepsilon_k \Delta \psi$ (to 0) and the strong limit of $\varphi_2^{\varepsilon_k}(0, \cdot)$ (to 0), we get from (3.12) that:

$$0 = \iint_Q \gamma (\partial_t \psi + \partial_{x_2} \psi) - \langle \varphi_2^T, \psi(T, \cdot) \rangle_{L^2(\Omega_2)}. \quad (3.13) \quad \text{eq:limitPhipsi}$$

We have that equation (3.13) is true in particular for $\psi(t, x) = \varphi_2^T(x_1, x_2 + T - t)$, which belongs to $\mathcal{D}([0, T] \times \Omega)$. Consequently, we get that $\|\varphi_2^T\|_{L^2(\Omega_2)} = 0$. Since $\varphi^T \in \mathcal{H}(\Omega_2) \cap \mathbf{C}^1(\overline{\Omega_2})$ this implies that $\varphi_1^T = 0$, contradicting the fact that $\varphi^T \neq 0$. \square

Lemma 3.5. *Let $h \in (0, \pi)$, $T \in (0, \pi - h)$ and $\delta \in (0, \pi - T - h)$. Then if $\varphi^T \in \mathcal{H}(\Omega_2)$ such that $\text{supp}(\varphi^T) \subset (0, \pi) \times (\pi - \delta, \pi)$ there is $c > 0$ such that for all $\varepsilon \in (0, 1)$:*

$$\iint_{Q_\omega} |\varphi_2^\varepsilon|^2 \leq c e^{-c\varepsilon^{-1}}, \quad (3.14) \quad \text{eq:obsphieps2deca}$$

for φ^ε the solution of (1.3) for $d = 2$ and initial value φ^T .

For the following proof we fix a parameter $\tilde{\delta} > 0$ so that $T \in \left(0, \frac{\pi - h - \delta}{1 + \tilde{\delta}}\right)$.

Proof. First of all, we recall that φ_2^ε satisfies (2.38) for $z^T := \varphi_2^T$. Multiplying (2.38)₁ by

$$2e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + \tilde{\delta}^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon,$$

it is not very difficult to deduce that, for all $t \in [0, T]$:

$$-\frac{d}{dt} \left(\int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + \tilde{\delta}^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2 \right) + 2\varepsilon \int_{\Omega_2} \left| \nabla \left(e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + \tilde{\delta}^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right) \right|^2 = 0.$$

In particular, we have the following Agmon identity:

$$\begin{aligned} & \int_0^T \int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + \tilde{\delta}^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2 \\ & + 2\varepsilon \int_0^T \int_t^T \int_{\Omega_2} \left| \nabla \left(e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + \tilde{\delta}^2)(T - s))\varepsilon^{-1}} \varphi_2^\varepsilon(s, x) \right) \right|^2 dx ds dt = T \int_{\Omega_2} \left| e^{\tilde{\delta}(\pi - x_2)\varepsilon^{-1}} \varphi_2^T \right|^2. \end{aligned} \quad (3.15) \quad \text{eq:agamonsto}$$

On the one hand, using that the support of φ^T is included in $(0, \pi) \times (\pi - \delta, \pi)$, we have that:

$$T \int_{\Omega_2} \left| e^{\tilde{\delta}(\pi - x_2)\varepsilon^{-1}} \varphi_2^T \right|^2 \leq \pi e^{2\tilde{\delta}\delta\varepsilon^{-1}} \int_{\Omega_2} |\varphi_2^T|^2. \quad (3.16) \quad \text{eq:agamonright}$$

On the other hand, recalling that $\omega \subset (0, \pi) \times (0, h)$, we have that:

$$e^{2\tilde{\delta}(\pi - h - T - \tilde{\delta}T)\varepsilon^{-1}} \iint_{Q_\omega} |\varphi_2^\varepsilon|^2 \leq \int_0^T \int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + \tilde{\delta}^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2. \quad (3.17) \quad \text{eq:agamonleft}$$

Combining (3.15), (3.16) and (3.17), we have that:

$$\iint_{Q_\omega} |\varphi_2^\varepsilon|^2 \leq \pi \exp \left[\frac{2\tilde{\delta}(T + \tilde{\delta}T + h + \delta - \pi)}{\varepsilon} \right] \|\varphi_2^T\|_{L^2(\Omega)}^2; \quad (3.18) \quad \text{est:obsexpdecay}$$

that is, we get the decay of the L^2 norm of φ_2 in the control domain. \square

Remark 3.6. So far, combining (1.4), Lemma 3.3, Lemma 3.5 and Remark 3.4, we have that for a given $h \in (0, \pi)$, for all $\omega \subset (0, T) \times (0, h)$, and for all $T \in (0, \pi - h)$, there is $c > 0$ such that for all $\varepsilon \in (0, 1)$:

$$K_2(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

Now we have the tools to end the proof of the item 3 of Theorem 1.1. For that proof, we denote, for $h \in (0, \pi)$ and $T \in (0, \pi - h)$ both fixed parameters:

$$\delta := (\pi - h - T)/3, \quad \omega_1 := (0, \pi) \times (0, h), \quad \text{and} \quad \omega_2 := (0, \pi) \times (0, h + \delta). \quad (3.19)$$

eq:defparamtech

End of the proof of the item 3 of Theorem 1.1. Let φ^ε be the solution of (1.3) for $d = 2$ of initial value (3.11) and with δ as in (3.19). Thanks to (1.4), Lemma 3.3 and Lemma 3.5, the only thing left to prove is that $\iint_{Q_\omega} |\varphi_1^\varepsilon|^2$ decays exponentially. For that purpose, we are going to get an estimate of $\partial_{x_2} \varphi_2^\varepsilon$ in $L^2(Q_{\omega_1})$.

Let us fix $\theta(s)$ a positive regular scalar cut-off function supported in $(-\infty, h + \delta)$ and such that $\theta = 1$ in $(-\infty, h]$. We define:

$$\psi^\varepsilon(t, x) := \theta(x_2) \varphi_2^\varepsilon(t, x). \quad (3.20)$$

def:zepschi

Recalling the support of φ^T (see (3.11)), we have that ψ^ε satisfies:

$$\begin{cases} -\partial_t \psi^\varepsilon - \varepsilon \Delta \psi^\varepsilon - \partial_{x_2} \psi^\varepsilon = \varepsilon \theta''(x_2) \varphi_2^\varepsilon - 2\varepsilon \partial_{x_2} (\theta'(x_2) \varphi_2^\varepsilon) - \theta'(x_2) \varphi_2^\varepsilon & \text{in } Q, \\ \psi^\varepsilon = 0 & \text{on } \Sigma_{b,t}, \\ \partial_{x_1} \psi^\varepsilon = 0 & \text{on } \Sigma_{l,r}, \\ \psi^\varepsilon(T, \cdot) = 0 & \text{on } \Omega_2. \end{cases} \quad (3.21)$$

eq:zepsheat

In particular, if we multiply (3.21) by ψ^ε and we integrate by parts, it is not difficult to deduce that:

$$\iint_Q |\nabla \psi^\varepsilon|^2 \leq \frac{C}{\varepsilon} \iint_{Q_{\omega_2}} |\varphi_2^\varepsilon|^2. \quad (3.22)$$

est:nablazeps

Using Lemma 3.5 for $h' = h + \delta$, $T' = T$ and $\delta' = \delta$ and using that for all $\tilde{\delta} > 0$ there is $C_{\tilde{\delta}} > 0$ such that $\varepsilon^{-1} \leq C_{\tilde{\delta}} e^{\tilde{\delta} \varepsilon^{-1}}$ in \mathbb{R}^+ , we get from (3.22) that:

$$\iint_{Q_{\omega_1}} |\partial_{x_2} \varphi_2^\varepsilon|^2 \leq Ce^{-c\varepsilon^{-1}}.$$

In order to conclude, using that the divergence of φ^ε is null, we get that:

$$\iint_{Q_{\omega_1}} |\partial_{x_1} \varphi_1^\varepsilon|^2 \leq Ce^{-c\varepsilon^{-1}}.$$

In addition, since φ_1^ε is null on $x_1 = 0$, we have, thanks to Poincaré inequality:

$$\iint_{Q_\omega} |\varphi_1^\varepsilon|^2 \leq \iint_{Q_{\omega_1}} |\varphi_1^\varepsilon|^2 \leq Ce^{-c\varepsilon^{-1}}. \quad (3.23)$$

eq:petitphi1

So, combining (1.4), (3.10), (3.14) and (3.23) we end the proof of the item 3 of Theorem 1.1. \square

4 The control problem (1.1) in $(0, \pi)^3$

sec:dim3

In this section we first give some solutions of (1.5) in Ω_3 for $d = 3$ and then, we prove item 4 of Theorem 1.1.

4.1 Brief study of the spectral decomposition of $\mathcal{H}((0, \pi)^3)$

In order to do the spectral decomposition, we consider the set of separated variables with null divergence, null normal trace, and a null component:

$$\begin{aligned} SV(\Omega_3) &:= SV_1(\Omega_3) \cup SV_2(\Omega_3) \cup SV_3(\Omega_3) := \\ &\left\{ \left(0, g_1(x_1) \int_0^{x_2} g_2(s) ds g_3(x_3), -g_1(x_1) g_2(x_2) \int_0^{x_3} g_3(s) ds \right) : g_1 \in L^2(0, \pi); g_2, g_3 \in L_0^2(0, \pi) \right\} \\ &\cup \left\{ \left(-\int_0^{x_1} g_1(s) ds g_2(x_2) g_3(x_3), 0, g_1(x_1) g_2(x_2) \int_0^{x_3} g_3(s) ds \right) : g_2 \in L^2(0, \pi); g_1, g_3 \in L_0^2(0, \pi) \right\} \\ &\cup \left\{ \left(\int_0^{x_1} g_1(s) ds g_2(x_2) g_3(x_3), -g_1(x_1) \int_0^{x_2} g_2(s) ds g_3(x_3), 0 \right) : g_3 \in L^2(0, \pi); g_1, g_2 \in L_0^2(0, \pi) \right\}. \end{aligned}$$

Remark 4.1. Unlike in Ω_2 , now the mean of one of the three auxiliary functions is not necessarily 0.

In order to look for eigenfunctions in $SV(\Omega_3)$, we remark that if g_1, g_2, g_3 are regular functions, the boundary conditions of (1.3) translates for $d = 3$ into:

$$g_1'(0) = g_1'(\pi) = g_2'(0) = g_2'(\pi) = \varepsilon g_3'(0) + g_3(0) = \varepsilon g_3'(\pi) + g_3(\pi) = 0 \quad (4.1)$$

eq:boundslipSV3

Let us focus on $SV_3(\Omega_3)$. We can again consider for $m_1, m_2 \in \mathbb{N}_*$, $g_{1,m_1}(s) := \cos(m_1 s)$ and $g_{2,m_2}(s) := \cos(m_2 s)$. As for g_3 , we can look at the solutions of:

$$g'' + \frac{g'}{\varepsilon} + \left(\frac{\lambda}{\varepsilon} - m_1^2 - m_2^2 \right) g = 0,$$

an equation that can be obtained as in the proof of Proposition 2.4. Since the mean of g_3 is not necessarily 0 and since $\int_0^s g_3(s') ds'$ does not have to be an eigenfunction of $-\varepsilon \Delta + \partial_{x_3}$ (because the third component of an element of $SV_3(\Omega_3)$ is null), we can consider small values of λ . In particular, we get, for $m_1, m_2 \in \mathbb{N}_*$, the following eigenfunctions:

$$\left(\frac{1}{m_1} \sin(m_1 x_1) \cos(m_2 x_2) e^{-\varepsilon^{-1} x_3}, -\frac{1}{m_2} \cos(m_1 x_1) \sin(m_2 x_2) e^{-\varepsilon^{-1} x_3}, 0 \right). \quad (4.2)$$

eq:probeeigenvec3

We remark that their associated eigenvalues are:

$$\lambda_{m_1, m_2}^\varepsilon := \varepsilon(m_1^2 + m_2^2), \quad (4.3)$$

eq:probeigenval3

and that their associated pressure terms are constant for almost every time.

rk:gen3

Remark 4.2. We can get for dimension 3 similar results to those of Proposition 2.1, Proposition 2.4 and of Proposition 2.6, with the only difference that some eigenvalues do not explode with ε . In that sense, we can get similar results as items 1 and 2 of Proposition 2.7, but we cannot get a general dissipation result as the one in item 3 of Proposition 2.7.

4.2 Proof of item 4 of Theorem 1.1

For this proof we consider the function

$$u(x) := \frac{2e^{-\varepsilon^{-1}x_3}}{\pi\sqrt{\varepsilon(1-e^{-2\pi\varepsilon^{-1}})}} (\sin(x_1) \cos(x_2), -\cos(x_1) \sin(x_2), 0),$$

which by (4.2) and (4.3) is a solution of (1.5) for $d = 3$, $\lambda = 2\varepsilon$ and a pressure which is constant for almost every time. Moreover, we can easily check that $\|u\|_{\mathbf{L}^2(\Omega)} = 1$ and that $\varphi(t, x) := u(x)e^{-2\varepsilon(T-t)}$ is a solution of (1.3) for $d = 3$.

On the one hand, we have that:

$$\int_{\Omega_3} |\varphi(0, \cdot)|^2 = e^{-4\varepsilon T}. \quad (4.4) \quad \text{eq:normphiT3}$$

On the other hand, we have that, for all $\delta > 0$, since $\varepsilon^{-1/2} \leq C_\delta e^{\delta\varepsilon^{-1}}$ in \mathbb{R}^+ , recalling that $\omega \subset (0, \pi)^2 \times (\pi - h, \pi)$:

$$\iint_{Q_\omega} |\varphi|^2 \leq TC_\delta \exp\left[\frac{2(-\pi + h + \delta)}{\varepsilon}\right]. \quad (4.5) \quad \text{eq:estinomega3}$$

So, if we combine (4.4) and (4.5), we have that (remember (1.4) and that $\varepsilon \in (0, 1)$):

$$[K_3(T, \varepsilon, \omega)]^2 \geq \frac{\int_{\Omega} |\varphi(0, \cdot)|^2}{\iint_{Q_\omega} |\varphi|^2} \geq c_\delta \frac{e^{-4T}}{T} \exp\left[\frac{2(\pi - h - \delta)}{\varepsilon}\right];$$

that is, for T fixed the cost of the control explodes exponentially when $\varepsilon \rightarrow 0^+$.

5 Further comments and open problems

openproblem

- **About the boundary control.** The case where we have a control on the boundary of (1.1) (as proposed in [8]) instead of in the interior remains an open problem.

- **The spectral method in the transport-diffusion elemental equation.** Let us consider the control problem studied in [10, 21]:

$$\begin{cases} y_t + \mathcal{L}^\varepsilon y = 1_\omega f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega, \end{cases} \quad (5.1) \quad \text{eq:diftranscongen}$$

for $\mathcal{L}^\varepsilon := -\varepsilon\Delta + \partial_{x_d}$, for Ω any open regular domain. In order to prove the dissipation result for the adjoint system, we can use a spectral decomposition. In particular, if $\{w_m\}$ is the spectral basis when we diagonalize the Dirichlet Laplacian in Ω , of eigenvalues λ_m , for $\tilde{w}_m := w_m e^{(2\varepsilon)^{-1}x_d}$ we have that $\{\tilde{w}_m\}$ is a linearly independent total set such that $\mathcal{L}^\varepsilon \tilde{w}_m = (\varepsilon\lambda_m + (4\varepsilon)^{-1})\tilde{w}_m$, and such that each \tilde{w}_m satisfies Dirichlet boundary conditions. Consequently, we can replicate the procedure of Section 2.1 and get the dissipation result, with an accurate result (see (2.32)), which looking at the eigenvalues clearly is the optimal one. As for the optimal dissipation rate, there is another proof in [10, Lemma 4] which uses a representation theorem and a comparison theorem.

• **The control problem (1.1) when we change the boundary conditions.** We can attempt for $\Omega = (0, \pi)^2$ and $d = 2$ to replicate this method to get similar results with a boundary condition of the type:

$$(-2\varepsilon Du \cdot n + \gamma n_2 u)_{\text{tg}} = 0, \quad (5.2)$$

eq:slipthetaR

for $\gamma \in \mathbb{R}$ (it is specially interesting the case $\gamma = 1/2$, when the operator is self-adjoint). Even if it actually produces eigenfunctions, it does it by taking into account the other two roots of (2.13); that is, $r = \pm m_1$. In particular, we get eigenfunctions with g_{2, m_2} of the type (see (2.15)):

$$A \cos(m_2 \pi s) e^{(2\varepsilon)^{-1} s} + B \sin(m_2 \pi s) e^{(2\varepsilon)^{-1} s} + C e^{m_1 s} + D e^{-m_1 s},$$

and with m_2 not necessarily in \mathbb{N}_* (and not necessarily easy to calculate explicitly). The root $r = m_1$ does not depend on ε or, more astonishingly, on λ , because:

$$u(x_1, x_2) := (\sin(m_1 x_1) e^{m_1 x_2}, -\cos(m_1 x_1) e^{m_1 x_2}) = \nabla \left(\frac{-\cos(m_1 x_1) e^{m_1 x_2}}{m_1} \right).$$

Therefore, that term is an eigenfunction independent of m_2 or ε because of the help of the pressure term. A similar thing happens with the term of $D e^{-m_1 s}$. All this, of course, adds some difficulties in proving that they form a total set (if they actually do), difficulties which we have been unable to overcome. Therefore, getting the eigenfunctions of the Stokes system with a boundary condition of the type (5.2) (together with $u \cdot n = 0$) can be considered for future work.

As for the control problem (1.1) with Dirichlet boundary conditions, replicating the method of this paper has several problems. For instance, in dimension 2, using the notation of (2.1), we now have the boundary conditions:

$$g_1(0) = g_1(\pi) = g_2(0) = g_2(\pi) = 0.$$

So, the analogous choice for g_{1, m_1} would be $\sin(m_1 s)$ (see (2.6)₁); but, for m_1 odd it does not belong to $L_0^2(0, \pi)$. Moreover, for any $m_1 \in \mathbb{N}_*$, $\int_0^s \sin(m_1 s') ds' = m_1^{-1} (1 - \cos(m_1 s))$, which is not an eigenfunction of the Laplacian. Consequently, it is very likely that another method must be used to get the cost of the control.

• **The control problem (1.1) in other domains.** It is clear that working in $(0, \pi)^2$ or $(0, \pi)^3$ is really helpful, not only for the dissipation, but for the cost of the control as well. Indeed, we have used several times in Section 3 that the solutions of (1.3) are solutions of the heat equation. Consequently, it would be interesting to know what results can be obtained in other domains.

• **The cost of the control in dimension 3 if ω is near $x_3 = 0$.** It is an open question if $K_3(T, \varepsilon, \omega)$ (see (1.2)) also explodes with ε for large times T if there is $\bar{x} \in (0, \pi)^2 \times \{0\}$ and $\delta > 0$ such that $B(\bar{x}, \delta) \cap \{x_3 > 0\} \subset \omega$.

• **The optimal T_0 in which the cost of the control decays with ε .** The optimal T_0 in which the cost of the control decays with ε is an open problem (see item 1 of Theorem 1.1). Thanks to items 2

and 3 of Theorem 1.1 we can get a lower bound. Moreover, we could have got an upper bound by doing all the operations of the proof of the Carleman (Proposition 3.2) explicitly as in [10], but it would be far from optimal and so is not worth the effort. A more subtle technique would be to transform the problem in a fast-control problem as in [25]; but, on the one hand, when we multiply by a weight which depends on a spatial variable, the fact that the divergence is null is lost; and, on the other hand, the fact that fast controls have a cost of $e^{CT^{-1}}$ (without specifying the C) in the Stokes system is just a recent result (see [5]). In addition, this problem is probably more difficult than getting the optimal T_0 for (5.1), which is still an open problem, even in dimension 1.

• **Relation of the cost of control with a force of $d - 1$ components and with a force of d components.** An interesting question is to see if the fact that $K_1(T, \varepsilon, \omega)$ decays or explodes with ε is equivalent to the fact that $K_2(T, \varepsilon, \omega)$ decays or explodes with ε (see (1.2) for the definitions).

References

- [1] F. Boyer and P. Fabrie. *Mathematical Tool for the Study of the Incompressible Navier-Stokes Equations and Related Models*. Springer, first edition, 2013.
- [2] J. A. Bárcena-Petisco. Some control and existence problems in fluid mechanics, PhD Thesis. 2020.
- [3] N. Carreño and S. Guerrero. On the non-uniform null controllability of a linear KdV equation. *Asymptotic Anal.*, 94(1-2):33–69, 2015.
- [4] N. Carreño and S. Guerrero. Uniform null controllability of a linear KdV equation using two controls. *J. Math. Anal. Appl.*, 457(1):922–943, 2018.
- [5] F. W. Chaves-Silva and G. Lebeau. Spectral inequality and optimal cost of controllability for the Stokes system. *ESAIM: CoCV*, 22(4):1137–1162, 2016.
- [6] P. Cornilleau and S. Guerrero. Controllability and observability of an artificial advection–diffusion problem. *Math. Control Signal*, 24(3):265–294, 2012.
- [7] P. Cornilleau and S. Guerrero. On the cost of null-control of an artificial advection-diffusion problem. *ESAIM: CoCV*, 19(4):1209–1224, 2013.
- [8] J.-M. Coron. On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM: CoCV*, 1:35–75, 1996.
- [9] J.-M. Coron. *Control and Nonlinearity*. Number 136. American Mathematical Soc., 2007.
- [10] J.-M. Coron and S. Guerrero. Singular optimal control: a linear 1-D parabolic–hyperbolic example. *Asymptotic Anal.*, 44(3, 4):237–257, 2005.
- [11] J.-M. Coron, F. Marbach, F. Sueur, and P. Zhang. Controllability of the Navier-Stokes equation in a rectangle with a little help of a distributed phantom force. *arXiv preprint arXiv:1801.01860*, 2018.

- [12] S. Ervedoza and E. Zuazua. Sharp observability estimates for heat equations. *Arch. Ration. Mech. An.*, 202(3):975–1017, 2011.
- [13] E. Fernández-Cara, M. González-Burgos, S. Guerrero, and J.-P. Puel. Null controllability of the heat equation with boundary Fourier conditions: the linear case. *ESAIM: COCV*, 12(3):442–465, 2006.
- [14] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of Evolution Equations*. Number 34. Seoul National University, 1996.
- [15] O. Glass. A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit. *J. Funct. Anal.*, 258(3):852–868, 2010.
- [16] O. Glass and S. Guerrero. On the uniform controllability of the Burgers equation. *SIAM J. Control Optim.*, 46(4):1211–1238, 2007.
- [17] O. Glass and S. Guerrero. Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. *Asymptotic Anal.*, 60(1-2):61–100, 2008.
- [18] O. Glass and S. Guerrero. Uniform controllability of a transport equation in zero diffusion–dispersion limit. *Math. Mod. Meth. Appl. S.*, 19(09):1567–1601, 2009.
- [19] S. Guerrero. Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions. *ESAIM: CoCV*, 12(3):484–544, 2006.
- [20] S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. A result concerning the global approximate controllability of the Navier–Stokes system in dimension 3. *J. Math. Pure. et Appl.*, 98(6):689–709, 2012.
- [21] S. Guerrero and G. Lebeau. Singular optimal control for a transport-diffusion equation. *Commun. Part. Diff. Eq.*, 32(12):1813–1836, 2007.
- [22] S. Guerrero and C. Montoya. Local null controllability of the N-dimensional Navier–Stokes system with nonlinear Navier-slip boundary conditions and $N - 1$ scalar controls. *J. Math. Pure. et Appl.*, 113:37–69, 2018.
- [23] G. Lebeau and L. Robbiano. Contrôle exact de l’équation de la chaleur. *Commun. Part. Diff. Eq.*, 20(1-2):335–356, 1995.
- [24] J.-L. Lions and E. Zuazua. A generique uniqueness result for the Stokes system and its control theoretical consequences. *Partial differential equations and applications: Collected Papers in Honor of Carlo Pucci*, 177:221–235, 1996.
- [25] P. Lissy. A link between the cost of fast controls for the 1-D heat equation and the uniform controllability of a 1-D transport-diffusion equation. *CR Acad. Sci. Paris Ser. I*, 2012.
- [26] P. Lissy. Explicit lower bounds for the cost of fast controls for some 1-D parabolic or dispersive equations, and a new lower bound concerning the uniform controllability of the 1-D transport–diffusion equation. *J. Differ. Equations*, 259(10):5331–5352, 2015.