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Adaptive elastic-net selection in a quantile model with diverging number of variable groups

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Abstract

In real applications of the linear model, the explanatory variables are very often naturally grouped, the most common example being the multivariate variance analysis. In the present paper, a quantile model with structure group is considered, the number of groups can diverge with sample size. We introduce and study the adaptive elastic-net group estimator, for improving the parameter estimation accuracy. This method allows automatic selection, with a probability converging to one, of significant groups and further the non zero parameter estimators are asymptotically normal. The convergence rate of the adaptive elastic-net group quantile estimator is also obtained, rate which depends on the number of groups. In order to put the estimation method into practice, an algorithm based on the sub-gradient method is proposed and implemented. The Monte Carlo simulations show that the adaptive elastic-net group quantile estimations are more accurate than other existing group estimations in the literature. Moreover, the numerical study confirms the theoretical results and the usefulness of the proposed estimation method.

Keywords adaptive elastic-net; diverging-dimensional group model; quantile model.

1 Introduction

1.1 Motivation and state of the art literature review

Classically, for a parametric regression model, under assumption that the model errors are i.i.d. random variables with mean zero and finite variance, the model parameters are estimated by Least Square (LS) method. If the model errors don't satisfy the conditions on mean and variance, then the LS method is not appropriate since it can provide inaccurate estimators, i.e. biased and with a large variance. In this case, a very interesting and robust alternative is the quantile estimation method, introduced by Koenker and Bassett (1978). For a complete review on properties of the unpenalized quantile estimators, the reader can see book Koenker (2005), with developments in Koenker et al. (2017). For a linear quantile model with a very large or diverging (with sample size) variable number, it is necessary to automatically detect the significant variables,

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without using hypothesis tests. From where the idea of penalizing the quantile process with LASSO or adaptive LASSO type penalties, introduced by Tibshirani (1996) and Zou (2006), respectively, for the LS loss function. The literature being very large on this topic, we can't give here an exhaustive list, the research on the LASSO method for quantile model being very active in the last decade. Wu and Liu (2009) consider a quantile linear regression, while Zou and Yuan (2008) study a composite quantile regression, the adaptive LASSO penalty being chosen in both works. Many interesting results on the automatic selection have been obtained when the number of quantile model parameters diverges with the sample size. Examples include, Gao and Huanq (2010), Belloni and Chernozhukov (2011), Wang et al. (2012), Zheng et al. (2013), Zheng et al. (2015). However, quite often in practical applications, grouped explanatory variables must be considered in a linear model, an example being the variance analysis model, with the goal of the automatic detection of variable groups that influence the response variable. From where the subject of this paper on the detection of the significant explanatory variable groups in a quantile model when the number of groups depends on the sample size. An adaptive LASSO penalty can be considered, to which one will add another penalty to improve the estimator accuracy. Inspired by Zou and Hastie (2005), where for a linear model with ungrouped explanatory variables was considered, we take the elastic-net as second penalty.

Previously, for a linear model with the number of ungrouped explanatory variables depending on the number n of samples and of order n^c , with $0 \leq c < 1$, Zou and Zhang (2009) show the oracle property for adaptive elastic-net estimator. Recently, always for the LS loss function, Xin et al. (2017) generalize this result for multivariate response variable and considering the model with grouped variables. For a model with ungrouped variables, in fixed number, Slawski et al. (2010) penalize a convex loss function with an elastic-net penalty, while Zeng and Xie (2014) penalize the LS process with the SCAD function and the L_2 norm on the regression coefficients. For a linear model with grouped explanatory variables, the LS loss function is penalized with L_1 and L_2 penalties in Simon et al. (2013), where an algorithm is also proposed in order to fit the model, via accelerated generalized gradient descent and implement this algorithm in the R package *SGL*. It should be noted that, algorithms to estimate a quantile model with penalty of elastic-net type, but not with grouped variables, can be found in the following papers: Slawski (2012) develops solution paths algorithms, while, Yi and Huang (2017) propose an algorithm semismooth Newton coordinate descent, implemented in the R package *hqreg*.

1.2 Contribution of present paper

In the present work, we consider a linear quantile model with grouped explanatory variables, the number of groups can diverge with the sample size. Besides to automatically selecting the groups of significant variables, which is usually done using adaptive LASSO penalty, we propose to improve the accuracy of the parameter estimators. For this, we propose and study the adaptive elastic-net group quantile estimation method. We also show the asymptotic normality for non-zero parameter estimators. For a model with diverging number of variable groups with adaptive elastic-net penalty for a loss function, only the LS loss function has been considered (Xin et al. (2017)). An important contribution of the present paper is to propose an algorithm based on the subgradient method for computing the adaptive elastic-net group quantile estimator of the regression parameters. Using the proposed algorithm, simulations are realized to confirm

the theoretical results and to compare our estimator with existing estimators for a group quantile linear model. Point out that, the simulations confirm that adaptive elastic-net group quantile estimator proposed in the present work is more accurate than the adaptive group LASSO quantile estimator existing in literature. Moreover, for a quantile model with ungrouped explanatory variables, the results obtained by simulations using the proposed algorithm are compatible with those obtained by an existing algorithm.

The remainder of this paper is organized as follows. Section 2 introduces the group quantile model, presents general notations and assumptions. In Section 3, the adaptive elastic-net group quantile estimator is proposed and studied, the convergence rate and oracle property are shown. An algorithm for computing numerically the parameter estimations and a criterion for choosing the tuning parameters are proposed in Section 4. Section 5 presents numerical results on simulations in order to compare our estimation method with existing adaptive quantile methods. All proofs are postponed in Section 6.

2 Model

Let us consider a linear model with g groups of explanatory variables. The number g can depend on n , while, the number of explanatory variables in each group don't depend on n . We suppose, without reducing the generality, that each group contains the same number of variables p . Let us denote by $r_n = gp$. Thus, the following model is considered in the present paper:

$$Y_i = \sum_{j=1}^g \mathbf{X}_{i,j}^t \boldsymbol{\beta}_j + \varepsilon_i = \mathbb{X}_i^t \boldsymbol{\beta}^g + \varepsilon_i, \quad i = 1, \dots, n, \quad (1)$$

with $\boldsymbol{\beta}^g \equiv (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_g) \in \mathbb{R}^{r_n}$ and $\boldsymbol{\beta}_j \in \mathbb{R}^p$ the vector of parameters for the group j , for $j = 1, \dots, g$. For each observation i , the vector $\mathbb{X}_i \in \mathbb{R}^{r_n}$ contains all explanatory variables and $\mathbf{X}_{i,j}$ the explanatory variables of the j th group. The explanatory variables $\mathbb{X}_i = (\mathbf{X}_{i,1}, \dots, \mathbf{X}_{i,g})$, with $\mathbf{X}_{i,j} \in \mathbb{R}^p$, are assumed to be deterministic, for $j = 1, \dots, g$ and $i = 1, \dots, n$. In model (1), Y_i is the response variable, ε_i the model error and $\boldsymbol{\beta}^0 = (\boldsymbol{\beta}_1^0, \dots, \boldsymbol{\beta}_g^0)$ the true value (unknown) of the parameter $\boldsymbol{\beta}^g$. For $p = 1$, the model becomes with ungrouped explanatory variables.

For a fixed quantile index $\tau \in (0, 1)$, we consider the quantile loss function $\rho_\tau(\cdot) : \mathbb{R} \rightarrow (0, \infty)$ defined by $\rho_\tau(u) = u(\tau - \mathbb{1}_{u < 0})$. The associated quantile process for model (1) is:

$$G_n(\boldsymbol{\beta}^g) \equiv \sum_{i=1}^n \rho_\tau(Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^g). \quad (2)$$

The group quantile estimator is by definition:

$$\widetilde{\boldsymbol{\beta}}^g \equiv \arg \min_{\boldsymbol{\beta}^g \in \mathbb{R}^{r_n}} G_n(\boldsymbol{\beta}^g). \quad (3)$$

For the particular case $\tau = 1/2$ we obtain the median regression, for which the quantile process and associated estimator (3), are reduced to the absolute deviation process and to the least abso-

lute deviation estimator, respectively.

For model (1), the purpose of this paper is to automatically detect the significant groups of the explanatory variables, simultaneous with improving the accuracy of the estimators. An adaptive elastic-net method is proposed and studied. The oracle property is usually requested when the parameter number of a model diverges with n . For a group linear model, the oracle property is the satisfaction of the following two properties: the significant groups of explanatory variables are automatically detected with a probability converging to 1 as $n \rightarrow \infty$ and the non-zero parameters are estimated by asymptotically normal estimators. Consider the following set:

$$\mathcal{A}^0 \equiv \{j \in \{1, \dots, g\}; \beta_j^0 \neq \mathbf{0}_p\},$$

which contains the indexes of the significant group of explanatory variables. The set \mathcal{A}^0 is unknown.

2.1 Notation

Give now some general notations. All vectors and matrices are denoted by bold symbols and all vectors are column. For a vector \mathbf{v} , we denote by $\|\mathbf{v}\|_2$ its euclidean norm. We denote also by \mathbf{v}^t its transpose and by $|\mathbf{v}|$ a vector of the same dimension as \mathbf{v} , with the components the absolute values of the components of \mathbf{v} . We use also the notation for the following p -dimensional vectors: $\mathbf{1}_p \equiv (1, \dots, 1)$ and $\mathbf{0}_p \equiv (0, \dots, 0)$. If \mathbf{v} and \mathbf{w} are two vectors of the same dimension, then, for brevity reason, we denote by $\mathbf{v} < \mathbf{w}$ the fact that each component of \mathbf{v} is less than the corresponding component of \mathbf{w} . For a positive definite matrix, we use $\mu_{\min}(\cdot)$ and $\mu_{\max}(\cdot)$ to denote its smallest and largest eigenvalues. When it is not specified, the convergence is for $n \rightarrow \infty$. Throughout the paper, C denotes a positive generic constant not depend on n , which may take a different value in different formula or even in different parts of the same formula. The value of C is not of interest. The notation $\mathbb{1}(\cdot)$ is used for the indicator function. On the other hand, let us consider ε the generic variable for the sequence $(\varepsilon_i)_{1 \leq i \leq n}$. For an index set \mathcal{A} , we denote by $|\mathcal{A}|$ its cardinality and by $\overline{\mathcal{A}}$ its complementary set. We also denote by $\beta_{\mathcal{A}}$ the subvector of β^g containing all subvectors β_j , with $j \in \mathcal{A}$.

For two positive sequences $(a_n), (b_n)$ we denote by $a_n \gg b_n$ the fact that $\lim_{n \rightarrow \infty} a_n/b_n = \infty$. Moreover, for two positive sequences $(a_n), (b_n)$, we write $a_n = O(b_n)$, if $a_n \leq Cb_n$ for some constants $C > 0$ and for n large enough. We also write $a_n = o(b_n)$ if $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$. We also use the following notations for two random variable sequences (U_n) and (V_n) : $V_n = O_{\mathbb{P}}(U_n)$ if, for any $\epsilon > 0$, there exists a constant $C > 0$ such that $\mathbb{P}[U_n/V_n > C] < \epsilon$ for n large enough, $V_n = o_{\mathbb{P}}(U_n)$ when $\lim_{n \rightarrow \infty} \mathbb{P}[U_n/V_n > \epsilon] = 0$ for any $\epsilon > 0$.

2.2 Assumptions

For the errors $(\varepsilon_i)_{1 \leq i \leq n}$ and the design $(\mathbb{X}_i)_{1 \leq i \leq n}$ of model (1), the following general assumptions are considered.

(A1) Random error terms $(\varepsilon_i)_{1 \leq i \leq n}$ are i.i.d. with the distribution function $F : \mathcal{G} \rightarrow [0, 1]$ and f its density function, such that $\mathbb{P}[\varepsilon < 0] = \tau$. The set $\mathcal{G} \subseteq \mathbb{R}$ and $0 \in \mathcal{G}$. The density function f is continuously, strictly positive in a neighbourhood of zero and has a bounded first derivative in the neighbourhood of 0.

(A2) There exist two constants $0 < m_0 \leq M_0 < \infty$, such that $m_0 \leq \mu_{\min}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \leq \mu_{\max}(n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t) \leq M_0$.

(A3) $(g/n)^{1/2} \max_{1 \leq i \leq n} \|\mathbb{X}_i\|_2 \rightarrow 0$, as $n \rightarrow \infty$.

(A4) The number of groups is such that $g = O(n^c)$, with $0 \leq c < 1$.

(A5) Let be $h_0 \equiv \min_{1 \leq j \leq |\mathcal{A}^0|} \|\beta_j^0\|_2$. There exists $M > 0$ such that for $\alpha > (c - 1)/2$ we have $n^{-\alpha} h_0 \geq M$.

Assumption (A1) is standard for a quantile model when the number of parameters can depend on n (see for example Ciuperca (2019), Wu and Liu (2009)), while Assumption (A2), is standard for linear model (see Wu and Liu (2009), Zhang and Xiang (2016)). Assumptions (A3) and (A4) are considered in models with number of parameters depending on n , see Zhang and Xiang (2016), Zou and Zhang (2009), Ciuperca (2019). By assumption (A5), considered also by Zhang and Xiang (2016), we suppose that for significant groups of explanatory variables, the true values of the parameters can depend on n and converge to 0 as $n \rightarrow \infty$, when $\alpha > 0$.

3 Asymptotic properties

In this section we consider that the number g of groups is of order n^c , with the constant c such that $0 \leq c < 1$. If $c = 0$ then the group number is fixed. It is interesting to note that the results stated in this section are original also for a model with ungrouped explanatory variables ($p = 1$). Since the model is well defined now, we can give more precise definition of the oracle property for a parameter estimator. For a model with grouped explanatory variables, an estimator satisfy the oracle property if the *estimation sparsity (consistent selection)* and the *asymptotic normality* properties are satisfied simultaneously. More precisely, the sparsity property (also say, consistency in selection) is when the true non-zero vectors are estimated as non-zero and the null parameter vectors are shrunk directly as a null vector, with a probability converging to 1 as $n \rightarrow \infty$. The asymptotic normality property is when the estimators of the true non-null parameter vectors have a normal asymptotic distribution.

In order to have the oracle property for the estimators but also to improve the accuracy of parameter estimators, we propose a new estimator, inspired by the work of Zou and Zhang (2009), by penalizing the quantile process with an adaptive weighted penalty and with an elastic-net penalty. In Zou and Zhang (2009), the adaptive elastic-net estimator is considered for $p = 1$, $g = n^c$, with $0 \leq c < 1$, for a linear regression with LS loss function.

For model (1), for $\beta^g \in \mathbb{R}^{r_n}$, with $r_n = gp$, let us define the adaptive elastic-net group quantile process by:

$$E_n(\beta^g) \equiv G_n(\beta^g) + \lambda_{1n} \sum_{j=1}^g \tilde{\omega}_{n,j} \|\beta_j\|_2 + \lambda_{2n} \sum_{j=1}^g \|\beta_j\|_2^2, \quad (4)$$

with the process $G_n(\beta^g)$ defined by (2), the weights $\tilde{\omega}_{n;j} \equiv \|\tilde{\beta}_j\|_2^{-\gamma}$, the vector $\tilde{\beta}_j$ being the j -th component of the quantile estimator vector defined by relation (3) and $\gamma > 0$ a positive constant which will be later specified. For the particular case $\lambda_{2n} = 0$, for any $n \in \mathbb{N}$, we obtain the adaptive quantile process, considered in Ciuperca (2019).

The adaptive elastic-net group quantile estimator is defined by:

$$\widehat{\beta}^g = (\widehat{\beta}_1, \dots, \widehat{\beta}_g) = \arg \min_{\beta^g \in \mathbb{R}^{rn}} E_n(\beta^g). \quad (5)$$

In a similar way to the set \mathcal{A}^0 , let us consider the following sets of indexes:

$$\widehat{\mathcal{A}}_n \equiv \{j \in \{1, \dots, g\}; \widehat{\beta}_j \neq \mathbf{0}_p\}.$$

The set $\widehat{\mathcal{A}}_n$ contains the indexes of the non-zero vector estimators. In the present work, the set \mathcal{A}^0 is estimated through the adaptive elastic-net method by the set $\widehat{\mathcal{A}}_n$.

For studying the asymptotic behaviour $\widehat{\beta}^g$ we must first know the asymptotic properties of the group quantile estimator $\widetilde{\beta}^g$ which intervenes in the weights $\tilde{\omega}_{n;j}$. More precisely, we need to know if $\widetilde{\beta}^g$ is consistent and if so, what is its convergence rate. Under assumptions (A1)-(A4), we have by Lemma 1 of Ciuperca (2019) that the convergence rate of the group quantile estimator $\widetilde{\beta}^g$ is of order $(g/n)^{1/2}$: $\|\widetilde{\beta}^g - \beta^0\|_2 = O_{\mathbb{P}}((g/n)^{1/2})$.

A first studied property for the adaptive elastic-net group quantile estimator $\widehat{\beta}^g$ is the convergence rate. In order to find the convergence rate of the $\widehat{\beta}^g$ in the case of g depending on n , we suppose for the tuning parameters $\lambda_{1n}, \lambda_{2n}$, as $n \rightarrow \infty$, that,

$$\lambda_{1n} \rightarrow \infty, \quad \lambda_{1n} n^{(c-1)/2 - \alpha\gamma} \rightarrow 0. \quad (6)$$

$$\lambda_{2n} \rightarrow \infty, \quad \lambda_{2n} n^{(c-1)/2} \rightarrow 0. \quad (7)$$

Condition (6) was also considered by Ciuperca (2019) for studying the asymptotic behaviour of the adaptive group LASSO quantile estimator.

By the following theorem we show that the convergence rate of $\widehat{\beta}^g$ is of order $(g/n)^{1/2}$, which for $c = 0$ (g fixed) becomes $n^{-1/2}$, the convergence rate obtained also in the proof of Remark 3.3, under more appropriate assumptions.

Theorem 3.1. *If $0 \leq c < 1$, under assumptions (A1)-(A5) and conditions (6), (7) for the tuning parameters, we have $\|\widehat{\beta}^g - \beta^0\|_2 = O_{\mathbb{P}}((g/n)^{1/2})$.*

With this result we can now study the oracle property of $\widehat{\beta}^g$. For this, we consider the following additional conditions, as $n \rightarrow \infty$, for the tuning parameters:

$$\lambda_{1n} n^{(1-c)(1+\gamma)/2 - 1} \rightarrow \infty \quad (8)$$

$$\lambda_{1n} |\mathcal{A}^0| n^{-(1+c)/2} \rightarrow 0. \quad (9)$$

$$\lambda_{2n} |\mathcal{A}^0| n^{-(1+c)/2} \rightarrow 0. \quad (10)$$

If in assumption (A5) we consider $\alpha = 0$, that is, the smallest norm of significant groups don't depend on n and if $|\mathcal{A}^0| < \infty$, then condition (6) implies condition (9). If $|\mathcal{A}^0| < \infty$, condition (7) implies condition (10).

We denote by $\mathbb{X}_{i,\mathcal{A}^0}$ the columns $\mathbf{X}_{i,j}$ with $j \in \mathcal{A}^0$. The following theorem states that the adaptive elastic-net group quantile estimator satisfies the oracle property.

Theorem 3.2. *If $0 \leq c < 1$, suppose that assumptions (A1)-(A5) are satisfied and also that the tuning parameters satisfy (6), (7), (8), (9), (10). Then:*

(i) $\mathbb{P} \left[\widehat{\mathcal{A}}_n = \mathcal{A}^0 \right] \rightarrow 1$, for $n \rightarrow \infty$.

(ii) For any vector \mathbf{u} of size $(p|\mathcal{A}^0|)$ such that $\|\mathbf{u}\|_2 = 1$, with notation $\mathbf{\Upsilon}_{n,\mathcal{A}^0} \equiv n^{-1} \sum_{i=1}^n \mathbb{X}_{i,\mathcal{A}^0} \mathbb{X}_{i,\mathcal{A}^0}^t$, we have, $n^{1/2}(\mathbf{u}^t \mathbf{\Upsilon}_{n,\mathcal{A}^0}^{-1} \mathbf{u})^{-1/2} \mathbf{u}^t (\widehat{\boldsymbol{\beta}}^g - \boldsymbol{\beta}^0)_{\mathcal{A}^0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \tau(1 - \tau)f^{-2}(0))$.

More precisely, conditions (6), (7) and (8) are required in the previous theorem for the tuning parameters λ_{1n} , λ_{2n} , for proving the estimator sparsity in Theorem 3.2(i). Conditions (9), (10) are necessary for the asymptotic normality (see the proof of Theorem 3.2 in Section 6). The asymptotic normality of Theorem 3.2(ii) shows that the elastic-net penalty don't affect the asymptotic law of the non-zero parameter estimators. The Gaussian limit distribution being the same as for a quantile estimator, without penalty (see Koenker (2005)). The oracle property obtained in Theorem 3.2 has been shown for other models or other penalties when the number of parameters diverges with the sample size. For a quantile linear model with ungrouped explanatory variables, Zheng et al. (2013), Zheng et al. (2015) and Ciuperca (2019) considered an adaptive LASSO penalty, while Ciuperca (2015) used a seamless L_0 penalty. Wang et al. (2012) shows the sparsity of the quantile estimators with a SCAD and MCP penalty. To the knowledge of the author, the adaptive elastic-net penalty was considered only for a LS loss function by Zou and Zhang (2009) which shows the oracle property for a model with ungrouped variables and by Xin et al. (2017) which shows the starsity property for a model with grouped variables.

Let's take a closer look at the case $c = 0$, in this case the conditions on the design and on the tuning parameters can be simplified. Let us denote $r = r_n$, since it don't depend on n . For the case g fixed, assumption (A3) implies $n^{-1} \max_{1 \leq i \leq n} \mathbb{X}_i^t \mathbb{X}_i \xrightarrow[n \rightarrow \infty]{} 0$. Then, instead of assumptions (A2), (A3) we consider:

(A6) $n^{-1} \max_{1 \leq i \leq n} \mathbb{X}_i^t \mathbb{X}_i \xrightarrow[n \rightarrow \infty]{} 0$ and $n^{-1} \sum_{i=1}^n \mathbb{X}_i \mathbb{X}_i^t \xrightarrow[n \rightarrow \infty]{} \mathbf{\Upsilon}$, with $\mathbf{\Upsilon}$ a $r \times r$ positive definite matrix.

For the penalties, the tuning parameters λ_{1n} , λ_{2n} and the power γ of weight $\tilde{\omega}_{n;j}$ are such that, for $n \rightarrow \infty$:

$$\lambda_{1n} \rightarrow \infty, \quad \lambda_{2n} \rightarrow \infty, \quad n^{-1/2} \lambda_{1n} \rightarrow 0, \quad n^{(\gamma-1)/2} \lambda_{1n} \rightarrow \infty, \quad n^{-1/2} \lambda_{2n} \rightarrow 0. \quad (11)$$

For the tuning parameter λ_{1n} the considered conditions are the same as for the adaptive group LASSO quantile estimator of Ciuperca (2019). By assumption $n^{-1/2} \lambda_{1n} \rightarrow 0$ of relation (11) and $n^{\gamma/2-1} \lambda_{1n} \rightarrow \infty$, as $n \rightarrow \infty$, considered in Remark 3.4 we deduce that $\gamma > 1$.

The following remark shows that for g fixed, only assumptions (A1), (A6) and condition (11) for the tuning parameters are needed for the asymptotic normality of $\widehat{\beta}^g_{\mathcal{A}^0}$. The proof of Remark 3.3 is given in Section 6.

Remark 3.3. *If $c = 0$, under assumptions (A1), (A6) and condition (11), we have $n^{1/2}(\widehat{\beta}^g - \beta^0)_{\mathcal{A}^0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_{p|\mathcal{A}^0}, \tau(1 - \tau)f^{-2}(0)\mathbf{\Upsilon}_{\mathcal{A}^0}^{-1})$, with $(p|\mathcal{A}^0)$ -squared matrix $\mathbf{\Upsilon}_{\mathcal{A}^0}$ defined as the submatrix of $\mathbf{\Upsilon}$ such that with the indexes of rows and columns in \mathcal{A}^0 .*

We also note that Remark 3.3 is a particular case of Theorem 3.2(ii) when $|\mathcal{A}^0|$ is bounded. The following remark gives that the elements of \mathcal{A}^0 and $\widehat{\mathcal{A}}_n$ coincide with a probability converging to one as $n \rightarrow \infty$.

Remark 3.4. *If $c = 0$, under assumptions (A1), (A6) and condition (11) together $n^{\gamma/2-1}\lambda_{1n} \rightarrow \infty$, as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \mathbb{P}[\widehat{\mathcal{A}}_n = \mathcal{A}^0] = 1$.*

The proof of Remark 3.4, based on the Karush-Kuhn-Tucker optimality conditions (given in subsection 4), is omitted since it is similar to that of the proof of Theorem 2 in Ciuperca (2019). Always for a quantile model with grouped explanatory variables, but in finite number, the same variance matrix of the centered normal limit distribution in Theorem 3.4 was obtained by Ciuperca (2019) for adaptive LASSO penalty and by Ciuperca (2017) for adaptive fused LASSO penalty. The oracle property is also true for the adaptive LASSO estimator when the loss function is the LS (see Wang and Leng (2008)).

4 Algorithm

In this section we propose an algorithm, based on the subgradient method, in order to compute the adaptive elastic-net estimator for a group quantile linear model. For this, we will write the Karush-Kuhn-Tucker (KKT) optimality conditions.

For all $j \in \widehat{\mathcal{A}}_n$, the following p equalities hold with probability one,

$$\tau \sum_{i=1}^n \mathbf{X}_{i,j} - \sum_{i=1}^n \mathbf{X}_{i,j} \mathbb{1}_{Y_i < \mathbb{X}_i^t \widehat{\beta}^g} - 2\lambda_{2n} \widehat{\beta}_j = \frac{\lambda_{1n} \widetilde{\omega}_{n;j} \widehat{\beta}_j}{\|\widehat{\beta}_j\|_2}. \quad (12)$$

Moreover, for all $j \notin \widehat{\mathcal{A}}_n$, for all $k = 1, \dots, p$, we have, with probability one, the following inequality

$$\left| \tau \sum_{i=1}^n X_{i,j,k} - \sum_{i=1}^n X_{i,j,k} \mathbb{1}_{Y_i < \mathbb{X}_i^t \widehat{\beta}^g} - 2\lambda_{2n} \widehat{\beta}_{j,k} \right| \leq \lambda_{1n} \widetilde{\omega}_{n;j}. \quad (13)$$

The p inequalities of relation (13), for all $j \notin \widehat{\mathcal{A}}_n$, can be also written:

$$\left| \tau \sum_{i=1}^n \mathbf{X}_{i,j} - \sum_{i=1}^n \mathbf{X}_{i,j} \mathbb{1}_{Y_i < \mathbb{X}_i^t \widehat{\beta}^g} - 2\lambda_{2n} \widehat{\beta}_j \right| \leq \lambda_{1n} \widetilde{\omega}_{n;j} |\mathbf{s}_j|,$$

with \mathbf{s}_j a p -dimensional vector with each component in absolute value less than 1. From this last relation, together relation (12), we consider the following gp gradient equations, for any $j \in \{2, \dots, g\}$:

$$\tau \sum_{i=1}^n \mathbf{X}_{i,j} - \sum_{i=1}^n \mathbf{X}_{i,j} \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}^g} - 2\lambda_{2n} \boldsymbol{\beta}_j + \lambda_{1n} \tilde{\omega}_{n;j} \mathbf{s}_j = \mathbf{0}_p, \quad (14)$$

with \mathbf{s}_j a p -dimensional vector such that,

- if $\|\boldsymbol{\beta}_j\|_2 \neq 0$, then $\mathbf{s}_j = -\frac{\boldsymbol{\beta}_j}{\|\boldsymbol{\beta}_j\|_2}$,
- if $\|\boldsymbol{\beta}_j\|_2 = 0$, then $|\mathbf{s}_j| \leq \mathbf{1}_p$.

Let us denote by $\mathbb{X}_{i,-j}$ the vector \mathbb{X}_i without subvector $\mathbf{X}_{i,j}$ and $\boldsymbol{\beta}_{-j}^g$ the vector $\boldsymbol{\beta}^g$ without the subvector $\boldsymbol{\beta}_j$. Consequently, if $\|\boldsymbol{\beta}_j\|_2 = 0$ then relation (14) becomes: $\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g}) + \lambda_{1n} \tilde{\omega}_{n;j} \mathbf{s}_j = \mathbf{0}_p$, which implies

$$\mathbf{s}_j = \frac{1}{\lambda_{1n} \tilde{\omega}_{n;j}} \sum_{i=1}^n \mathbf{X}_{i,j} (\mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g} - \tau).$$

Thereby, if $|\mathbf{s}_j| < \mathbf{1}_p$, then $|\sum_{i=1}^n \mathbf{X}_{i,j} (\mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g} - \tau)| < \lambda_{1n} \tilde{\omega}_{n;j} \mathbf{1}_p$. Thus, in the algorithm, we consider when $|\mathbf{s}_j| < \mathbf{1}_p$, that is $|\sum_{i=1}^n \mathbf{X}_{i,j} (\mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g} - \tau)| < \lambda_{1n} \tilde{\omega}_{n;j} \mathbf{1}_p$, that $\boldsymbol{\beta}_j = \mathbf{0}_p$. Otherwise:

$$\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g}) - 2\lambda_{2n} \boldsymbol{\beta}_j = \frac{\lambda_{1n} \tilde{\omega}_{n;j} \boldsymbol{\beta}_j}{\|\boldsymbol{\beta}_j\|_2},$$

from where,

$$\boldsymbol{\beta}_j = \frac{\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g})}{2\lambda_{2n} + \lambda_{1n} \tilde{\omega}_{n;j} \|\boldsymbol{\beta}_j\|_2^{-1}} \quad (15)$$

and considering the euclidean norm we obtain: $\|\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g})\|_2 = 2\lambda_{2n} \|\boldsymbol{\beta}_j\|_2 + \lambda_{1n} \tilde{\omega}_{n;j}$. From this last relation we get:

$$\|\boldsymbol{\beta}_j\|_2 = \frac{\|\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g})\|_2 - \lambda_{1n} \tilde{\omega}_{n;j}}{2\lambda_{2n}}$$

and replacing in relation (15), we obtain:

$$\boldsymbol{\beta}_j = \frac{\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g})}{2\lambda_{2n} + 2\lambda_{1n} \lambda_{2n} \tilde{\omega}_{n;j} (\|\sum_{i=1}^n \mathbf{X}_{i,j} (\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \boldsymbol{\beta}_{-j}^g})\|_2 - \lambda_{1n} \tilde{\omega}_{n;j})^{-1}}.$$

Then, from these relations, we can propose the following algorithm, for λ_{1n} , λ_{2n} , τ , γ fixed.

Algorithm

Step 0

set the initial values of $\beta^{(0)}$ for the coefficient parameters.

Step k

For all $j = 1, \dots, g$ we calculate:

- if $|\sum_{i=1}^n \mathbf{X}_{i,j}(\mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \beta_{-j}^{g(k-1)} - \tau)| < \lambda_{1n} \tilde{\omega}_{n;j} \mathbf{1}_p$, then, $\beta_j^{(k)} = \mathbf{0}_p$,
- otherwise:

$$\beta_j^{(k)} = \frac{\sum_{i=1}^n \mathbf{X}_{i,j}(\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \beta_{-j}^{g(k-1)})}{2\lambda_{2n} + 2\lambda_{1n}\lambda_{2n}\tilde{\omega}_{n;j}(\|\sum_{i=1}^n \mathbf{X}_{i,j}(\tau - \mathbb{1}_{Y_i < \mathbb{X}_{i,-j}^t \beta_{-j}^{g(k-1)})\|_2 - \lambda_{1n}\tilde{\omega}_{n;j})^{-1}}.$$

Stopping the algorithm: The algorithm stops when $\|\beta_j^{(k)} - \beta_j^{(k-1)}\| < \epsilon$, with ϵ a specified precision.

As a starting point, to Step 0, we can take either the group quantile estimator $\beta^{(0)} = \tilde{\beta}^g$, given by (3), or the adaptive group LASSO quantile estimator proposed by Ciuperca (2019). For simulations, we will consider this last estimator.

The reader can find in Tseng (2001) the convergence properties of a block coordinate descent methods applied to minimize a nondifferentiable continuous function.

For choosing the tuning parameters λ_{1n} and λ_{2n} , we can use a criterion of type BIC. For this, let us consider $(S_n)_{n \in \mathbb{N}}$ a deterministic sequence defined by:

- if g is fixed or $g = O(n^c)$ such that $g(\log n)^{-1} = o(1)$, then $S_n = 1$, for any $n \in \mathbb{N}$;
- if $g = O(n^c)$ such that $g(\log n)^{-1} \neq o(1)$, we take (S_n) converging to $+\infty$ such that $g^{-1}(\log n)S_n \xrightarrow[n \rightarrow \infty]{} \infty$ and $n^{-1}(\log n)|\mathcal{A}^0|S_n \xrightarrow[n \rightarrow \infty]{} 0$.

Then, we consider the following criterion :

$$BIC(\lambda_{1n}, \lambda_{2n}) \equiv \log(n^{-1}G_n(\widehat{\beta}^g)) + \frac{\log n}{n}S_n|\widehat{\mathcal{A}}_n| \quad (16)$$

and we choose $\lambda_{1n}, \lambda_{2n}$ that minimize the criterion.

Criterion (16) is of type BIC, introduced by Wang et al. (2012) for a linear model with diverging number of parameters and estimated by penalized LS method. For choosing the tuning parameters, the same type of criterion have been proposed for quantile models with ungrouped variables by Ciuperca (2015) for seamless L_0 penalty, by Zheng et al. (2015) for adaptive LASSO penalty. Taking into account the sparsity of the parameter estimator $\widehat{\beta}^g$ proved in

Theorem 3.2, the choice of sequence (S_n) , is such that it doesn't allow an overfitted or underfitted model. The condition $n^{-1}(\log n)|\mathcal{A}^0|S_n \xrightarrow{n \rightarrow \infty} 0$ avoids an overfitted model and $(\log n)S_n g^{-1} \xrightarrow{n \rightarrow \infty} \infty$ avoids an underfitted model.

Remark 4.1. *In practice, the choice of quantile index can be done as follows. For the explained variable we calculate the standardized values: $\tilde{y}_i = (y_i - \bar{y}_n)/\hat{\sigma}_y$, with $\bar{y}_n, \hat{\sigma}_y$ empirical mean and standard deviation of Y , respectively. Afterwards, we calculate the empirical estimation of τ by $\hat{\tau}_n = n^{-1} \sum_{i=1}^n \mathbb{1}_{\tilde{y}_i < 0}$.*

5 Simulation study

In this section we conduct Monte Carlo simulations in order to evaluate and compare our proposed adaptive elastic-net estimator with existing adaptive quantile estimators. The simulations are based on the algorithm presented in subsection 4.

The considered design is such that:

$$X_{p(j-1)+k} = \frac{Z_j + R_{p(j-1)+k}}{\sqrt{2}}, \quad 1 \leq j \leq g, \quad 1 \leq k \leq p,$$

with Z_j multivariate normal distribution of mean zero and covariance $Cov(Z_{j_1}, Z_{j_2}) = 0.6^{|j_1 - j_2|}$. Moreover, R_1, \dots, R_{p_n} are independent standard normal variables. The considered quantile index is $\tau = 0.5$. In fact, the design is similar to the one considered by Ciuperca (2019) for adaptive group LASSO quantile method and by Wei and Huang (2010) for adaptive group LASSO-LS model. In all simulations, the number of significant groups of explanatory variables was assumed to be four: $|\mathcal{A}^0| = 4$. The number of non-significant groups of explanatory variables will be varied.

In each simulation, for a data set of dimension n , the design and the error distributions are generated. Each simulation is repeated 1000 times.

5.1 Ungrouped variables

In this subsection we consider ungrouped variables. We will compare our estimation method calculated by algorithm in subsection 4, with those obtained by R package *hqreg* for a quantile model with adaptive elastic-net penalty (of Yi and Huang (2017)) and with those obtained by R package *quantreg* for quantile model with adaptive LASSO penalty.

We consider four significant variables, with the true values of the parameters $\beta_0^1 = 0.5, \beta_0^2 = 1, \beta_0^3 = -1, \beta_0^4 = -1.5$. For the adaptive weights in the elastic-net and LASSO penalties, we considered the power $\gamma = 12.25/10$. For the adaptive LASSO method, the considered tuning parameter is $n^{2/5}$ and for adaptive elastic-net estimator in the R package *hqreg*, the considered tuning parameter is $n^{9/20}$. For the adaptive elastic-net quantile method of the present paper we consider the following tuning parameters: $\lambda_{1n} = n^{1-\gamma/2+1/n}$ and $\lambda_{2n} = c_1 n^{2/5}$, with c_1 varied on a value grid such that criterion (16) becomes minimal. For two gaussian errors, the results are given Table 1 where are presented: the median of $|\hat{\mathcal{A}}_n|$, that is the median of the estimated

Table 1: Simulation results for models with ungrouped variables: $p = 1$. Estimation methods: adaptive LASSO quantile (aq), adaptive elastic-net quantile (aEq), adaptive elastic-net group quantile ($aEGq$).

n	g	ε	median true $\neq 0$			median true = 0			sd ($\beta^0 - \hat{\beta}$)			mean $ Y - \hat{Y} $		
			aq	aEq	aEGq	aq	aEq	aEGq	aq	aEq	aaEGq	aq	aEq	aEGq
30	5	$\mathcal{N}(0, 3)$	2	3	2	1	0	1	1.33	1.47	0.96	2.25	2.18	2.61
		$\mathcal{N}(0, 1)$	3	4	3	1	1	1	1.15	1.27	0.97	0.86	0.75	1.15
60	5	$\mathcal{N}(0, 3)$	3	3	2	1	1	1	1.28	1.30	1.20	2.33	2.30	2.50
		$\mathcal{N}(0, 1)$	4	4	3	1	1	1	1.26	1.28	1.29	0.79	0.78	0.90
100	5	$\mathcal{N}(0, 3)$	3	3	3	1	1	1	1.24	1.22	1.07	2.37	2.39	2.55
		$\mathcal{N}(0, 1)$	4	4	4	1	1	1	1.26	1.25	1.29	0.80	0.82	0.84
100	20	$\mathcal{N}(0, 3)$	3	3	3	13	14	15	0.66	0.64	0.56	2.27	2.35	2.46
200	40	$\mathcal{N}(0, 3)$	4	3	3	33	36	35	0.46	0.42	0.50	2.30	2.44	2.39
200	60	$\mathcal{N}(0, 3)$	4	3	3	50	55	54	0.37	0.33	0.41	0.75	0.82	0.80

number of groups with significant explanatory variables by three adapted penalized method estimation: adaptive LASSO quantile (aq), adaptive elastic-net quantile (aEq) of Yi and Huang (2017) and adaptive elastic-net group quantile ($aEGq$) method proposed in the present paper, subsection 4. We also give in Table 1 the median of $|\hat{\mathcal{A}}_n|$ of the estimated number of non-significant groups by the three estimation methods. Always for the three methods, we calculate the standard-deviation of the parameter estimations and the mean of the absolute value of $Y - \hat{Y}$, with \hat{Y} the corresponding model prediction of Y . From the significant and non significant group identification point of view, the results are similar by the three estimation methods. The aq and aEq methods provide parameter estimations with the same standard-deviation, while by the $aEGq$ method, the parameter estimations are more precise. For the prevision of the response variable Y , for small n , the prevision by $aEGq$ method is slightly worse, but when n increases, we obtain the same precisions by $aEGq$ and aEq methods.

5.2 Grouped variables

For two error distributions, standard Normal $\mathcal{N}(0, 1)$ and Cauchy $\mathcal{C}(0, 1)$, we compare the results of the proposed adaptive elastic-net method with those of Ciuperca (2019) by adaptive group LASSO quantile method. The tuning parameters are for the our method: $\lambda_{1n} = c_2 c (\sigma + c) g n^{((1-c)/2 + 1 - (1-c)(1+\gamma)/2)/2}$, $\lambda_{2n} = c_3 c (\sigma + c) n^{1/2 - c/2 - 1/n}$, with $c = \log(g) / \log(n)$ and $\gamma = \max(1.225, 2c/(1-c) + 2/n)$, where c_2, c_3 are positive constants, varied on a value grid such that $\lambda_{1n}, \lambda_{2n}$ make criterion (16) minimal. We took $S_n = 1$. We first consider that each group contains $p = 2$ explanatory variables, with $|\mathcal{A}^0| = 4$, more precisely, $\beta_1^0 = (0.5, 1)$, $\beta_2^0 = (1, 1)$, $\beta_3^0 = (-1, 0)$, $\beta_4^0 = (-1.5, 1)$. In Table 2, we give the same indicators as in the case of the ungrouped variables for two penalized estimation methods: adaptive group LASSO quantile (aGq) method and adaptive elastic-net group quantile ($aEGq$) method (4), for this last the algorithm presented in subsection 4 being used. On the explanatory variables, the both methods give similar results in the detection of the true significant groups of and of the true non significant groups, but the $aEGq$ estimations are more accurate. To better exemplify this last finding, we give in Table 3 the medians of the parameter estimations.

Table 2: Simulation results for models with grouped variables: $p = 2$. Estimation methods: adaptive group LASSO quantile (aGq), adaptive elastic-net group quantile ($aEGq$).

n	g	ε	true param $\neq 0$				true param = 0				sd($\beta^0 - \hat{\beta}$)		mean $Y - \hat{Y}$	
			median		mean		median		mean		aGq	aEGq	aGq	aEGq
			aGq	aEGq	aGq	aEGq	aGq	aEGq	aGq	aEGq	aGq	aEGq	aGq	aEGq
50	5	$\mathcal{N}(0, 1)$	4	4	3.9	3.99	0	1	0.47	0.85	1.01	0.99	1.03	1.14
		$\mathcal{C}(0, 1)$	4	4	3.5	3.9	1	1	0.51	0.64	0.98	0.93	7.50	7.59
100	10	$\mathcal{N}(0, 1)$	4	3	3.6	2.88	5	6	4.9	6	0.67	0.62	1.35	1.82
		$\mathcal{C}(0, 1)$	3	3	2.8	2.4	6	6	5.29	6	0.65	0.61	6.53	6.84

Table 3: Median of $\hat{\beta}_{A^0}$, for models with grouped variables: $p = 2$. Comparison adaptive elastic-net group quantile ($aEGq$) estimations with adaptive group LASSO quantile (aGq) estimations.

n	g	ε	method	$\beta_1^0 = (0.5, 1)$	$\beta_2^0 = (1, 1)$	$\beta_3^0 = (-1, 0)$	$\beta_4^0 = (-1.5, 1)$
50	5	$\mathcal{N}(0, 1)$	aGq	(0.34, 0.84)	(0.67, 0.70)	(-0.37, 0)	(-0.82, 0.19)
			aEGq	(0.60, 0.74)	(0.74, 0.74)	(-0.51, -0.20)	(-0.61, 0.16)
		$\mathcal{C}(0, 1)$	aGq	(0.22, 0.75)	(0.45, 0.52)	(-0.05, 0)	(-0.41, 0)
			aEGq	(0.45, 0.55)	(0.55, 0.55)	(-0.26, -0.06)	(-0.40, 0.13)
100	10	$\mathcal{N}(0, 1)$	aGq	(0.22, 0.72)	(0.55, 0.57)	(-0.09, 0)	(-0.48, 0)
			aEGq	(0.19, 0.24)	(0.29, 0.30)	(-0.51, -0.20)	(-0.61, 0.16)
		$\mathcal{C}(0, 1)$	aGq	(0.07, 0.53)	(0.33, 0.38)	(0, 0)	(-0.06, 0)
			aEGq	(0.13, 0.17)	(0.20, 0.20)	(0, 0)	(-0.10, 0.03)

To conclude our numerical study, we consider that each group of explanatory variables contains $p = 5$ variables. The first four groups are significant, with $\beta_1^0 = (0.5, 1, 1.5, 1, 0.5)$, $\beta_2^0 = (1, 1, 1, 1, 1)$, $\beta_3^0 = (-1, 0, 1, 2, 1.5)$, $\beta_4^0 = (-1.5, 1, 0.5, 0.5, 0.5)$, other groups are not significant. The design is generated as for the case $p = 2$. The results are presented in Table 4. The finding made for groups of two variables remain true: the $aEGq$ method provides more accurate parameter estimations than those obtained by aGq method.

The simulations were performed on a computer with CPU 1.90 GHz and 4 GB RAM. Execution time for 1000 Monte Carlo replications for $n = 50$, $g = 5$ and for a given value of λ_{1n} , λ_{2n} in criterion (16) is 2.4 minutes, a single Monte Carlo replication taking 0.99 seconds.

6 Proofs

In the proof of Theorem 3.3 we will use the following theorem.

Theorem 6.1. (Geyer (1996)). *If \mathcal{F}_n and \mathcal{F} are two random lower semicontinuous convex functions from \mathbb{R}^p to $\mathbb{R} \cup \{\infty\}$ such that $\mathcal{F}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{F}$ and \mathcal{F} has a unique minimizer, then*

$$\arg \min_{\mathbf{u} \in \mathbb{R}^p} \mathcal{F}_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \arg \min_{\mathbf{u} \in \mathbb{R}^p} \mathcal{F}(\mathbf{u}).$$

Table 4: Simulation results for models with grouped variables: $p = 5$. Comparison adaptive elastic-net group quantile ($aEGq$) estimations with adaptive group LASSO quantile (aGq) estimations.

n	g	ε	true param $\neq 0$				true param = 0				$sd(\beta^0 - \hat{\beta})$		$mean Y - \hat{Y} $	
			median		mean		median		mean		aGq	aEGq	aGq	aEGq
			aGq	aEGq	aGq	aEGq	aGq	aEGq	aGq	aEGq	aGq	aEGq	aGq	aEGq
50	5	$\mathcal{N}(0, 1)$	4	4	3.8	4	1	1	0.58	0.84	1.02	0.77	1.71	4.20
		$\mathcal{C}(0, 1)$	4	4	3.6	3.9	1	1	0.74	0.55	1.04	0.81	6.98	7.25
100	10	$\mathcal{N}(0, 1)$	4	4	3.85	4	6	6	5.33	6	0.78	0.65	1.69	3.09
		$\mathcal{C}(0, 1)$	4	4	3.6	3.9	6	6	5.6	5.8	0.77	0.63	7.37	8.42
200	20	$\mathcal{N}(0, 1)$	4	4	3.8	4	15	16	15.2	16	0.57	0.47	1.85	3.66
		$\mathcal{C}(0, 1)$	4	4	3.96	3.96	16	16	15.5	16	0.57	0.50	44.3	45.1

In the proofs, for all $x, y \in \mathbb{R}$, the following identity on ρ_τ will be used:

$$\rho_\tau(x - y) - \rho_\tau(x) = y(\mathbb{1}_{x < 0} - \tau) + \int_0^\tau (\mathbb{1}_{x \leq v} - \mathbf{u}_{x \leq 0}) dv. \quad (17)$$

Proof of Theorem 3.1 By definition of adaptive elastic-net group quantile estimator, we have that

$$\hat{\beta}^g = \arg \min_{\beta^g \in \mathbb{R}^{r_n}} n^{-1} [E_n(\beta^g) - E_n(\beta^0)].$$

For $\mathbf{u} \in \mathbb{R}^{r_n}$, with $\|\mathbf{u}\|_2 = 1$, $B > 0$ a constant, in order to show the theorem, we study: $n^{-1} [E_n(\beta^0 + B(g/n)^{1/2} \mathbf{u}) - E_n(\beta^0)]$ which is, taking into account condition (6) that $\lambda_{1n} n^{(c-1)/2 - \alpha\gamma} \xrightarrow[n \rightarrow \infty]{} 0$, using also the proof of Theorem 3 of Ciuperca (2019), strictly bigger that

$$\begin{aligned} & B^2 f(0) g n^{-1} \left(n^{-1} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u} \right) (1 + o_{\mathbb{P}}(1)) - B O_{\mathbb{P}}(g n^{-1}) \\ & + \lambda_{2n} n^{-1} \sum_{j=1}^g [\|\beta_j^0 + g n^{-1} B \mathbf{u}_j\|_2^2 - \|\beta_j^0\|_2^2], \end{aligned} \quad (18)$$

for B large enough. On the other hand, using the triangular inequality, we have

$$\begin{aligned} \frac{\lambda_{2n}}{n} \sum_{j=1}^g [\|\beta_j^0 + \frac{g}{n} B \mathbf{u}_j\|_2^2 - \|\beta_j^0\|_2^2] & \leq \frac{\lambda_{2n}}{n} \sum_{j=1}^g \left[\frac{g}{n} B^2 \|\mathbf{u}_j\|_2^2 + 2 \|\beta_j^0\|_2 \sqrt{\frac{g}{n}} B \|\mathbf{u}_j\|_2 \right] \\ & \simeq \lambda_{2n} \left[\left(\frac{g}{n} \right)^2 B^2 + \left(\frac{g}{n} \right)^{3/2} B \right], \end{aligned}$$

which is, using condition (7), of order $o(B^2 f(0) g n^{-1} (n^{-1} \sum_{i=1}^n \mathbf{u}^t \mathbb{X}_i \mathbb{X}_i^t \mathbf{u}) + B O_{\mathbb{P}}(g n^{-1}))$. Then, taking into account relation (18), together with Assumption (A6), we obtain that for all $\epsilon > 0$, there exists B_ϵ large enough such that for any n large enough:

$$\mathbb{P} \left[\inf_{\substack{\mathbf{u} \in \mathbb{R}^{r_n} \\ \|\mathbf{u}\|_2 = 1}} E_n(\beta^0 + B_\epsilon \sqrt{\frac{g}{n}} \mathbf{u}) > E_n(\beta^0) \right] > 1 - \epsilon$$

and the theorem follows. ■

Proof of Theorem 3.2

The proof follows the same general lines as the proof of Theorem 4 in Ciuperca (2019) Consequently, some calculation details are omitted.

(i) Let be the following two sets of parameters $\mathcal{V}_g(\beta^0) \equiv \{\beta^g \in \mathbb{R}^{r_n}; \|\beta^g - \beta^0\|_2 \leq B(g/n)^{1/2}\}$ and $\mathcal{W}_n \equiv \{\beta^g \in \mathcal{V}_g(\beta^0); \|\beta_{\mathcal{A}^0} - \beta^0\|_2 > 0\}$. By Theorem 3.1, the estimator $\widehat{\beta}^g$ belongs to $\mathcal{V}_g(\beta^0)$ with a probability converging to 1 as $n \rightarrow \infty$ and for B large enough. For proving the theorem, we will first show that

$$\lim_{n \rightarrow \infty} \mathbb{P}[\widehat{\beta}^g \in \mathcal{W}_n] = 0. \quad (19)$$

For this, we consider the parameter vector $\beta^g = (\beta_{\mathcal{A}^0}, \beta_{\overline{\mathcal{A}^0}}) \in \mathcal{W}_n$. Let be also another parameter vector $\beta^{(1)} = (\beta_{\mathcal{A}^0}^{(1)}, \beta_{\overline{\mathcal{A}^0}}^{(1)}) \in \mathcal{V}_g(\beta^0)$, such that $\beta_{\mathcal{A}^0}^{(1)} = \beta_{\mathcal{A}^0}$ and $\beta_{\overline{\mathcal{A}^0}}^{(1)} = \mathbf{0}_{r_n - p|\mathcal{A}^0|}$. For proving relation (19), let us consider the following difference:

$$\begin{aligned} D_n(\beta^g, \beta^{(1)}) &\equiv n^{-1}E_n(\beta^g) - n^{-1}E_n(\beta^{(1)}) \\ &= \frac{1}{n} \sum_{i=1}^n [\rho_\tau(Y_i - \mathbb{X}_i^t \beta^g) - \rho_\tau(Y_i - \mathbb{X}_i^t \beta^{(1)})] \\ &\quad + \frac{\lambda_{1n}}{n} \sum_{j=|\mathcal{A}^0|+1}^g \tilde{\omega}_{n;j} \|\beta_j\|_2 + \frac{\lambda_{2n}}{n} \sum_{j=|\mathcal{A}^0|+1}^g \|\beta_j\|_2^2. \end{aligned} \quad (20)$$

Using identity (17), we have:

$$\begin{aligned} \sum_{i=1}^n [\rho_\tau(Y_i - \mathbb{X}_i^t \beta^g) - \rho_\tau(Y_i - \mathbb{X}_i^t \beta^{(1)})] &= \sum_{i=1}^n (\beta^g - \beta^{(1)})^t \mathbb{X}_i [\mathbb{1}_{Y_i - \mathbb{X}_i^t \beta^{(1)} \leq 0} - \tau] \\ &\quad + \sum_{i=1}^n \int_0^{\mathbb{X}_i^t (\beta^g - \beta^{(1)})} [\mathbb{1}_{Y_i - \mathbb{X}_i^t \beta^{(1)} \leq v} - \mathbb{1}_{Y_i - \mathbb{X}_i^t \beta^{(1)} \leq 0}] dv \\ &\equiv T_{1n} + T_{2n}. \end{aligned}$$

For T_{1n} , using assumption (A3), since $\beta^g \in \mathcal{V}_g(\beta^0)$ and together with that f, f' are bounded in a neighbourhood of 0 by assumption (A1), we have, taking also into account assumption (A2), that, $\mathbb{E}[T_{1n}] = \sum_{i=1}^n (\beta^g - \beta^{(1)})^t \mathbb{X}_i [F(\mathbb{X}_i^t (\beta^{(1)} - \beta^0)) - F(0)] = O(n \|\beta^g - \beta^{(1)}\|_2 \|\beta^{(1)} - \beta^0\|_2) = O(n \|\beta^g - \beta^{(1)}\|_2^2)$. By similar arguments, using assumptions (A1), (A2), (A3), we have for the variance $\text{Var}[T_{1n}] \leq \mathbb{E}[T_{1n}^2] = \sum_{i=1}^n ((\beta^g - \beta^{(1)})^t \mathbb{X}_i)^2 = O(n \|\beta^g - \beta^{(1)}\|_2^2)$. Then, by the Law of Large Numbers, we deduce,

$$T_{1n} = Cn \|\beta^g - \beta^{(1)}\|_2^2 (1 + o_{\mathbb{P}}(1)) \quad (21)$$

For $T_{2n} = \sum_{i=1}^n \int_0^{\mathbb{X}_i^t (\beta^g - \beta^{(1)})} [\mathbb{1}_{\varepsilon_i \leq v + \mathbb{X}_i^t (\beta^{(1)} - \beta^0)} - \mathbb{1}_{\varepsilon_i \leq \mathbb{X}_i^t (\beta^{(1)} - \beta^0)}] dv$, since $\beta^g \in \mathcal{V}_g(\beta^0)$, we have, by a Taylor expansion, that,

$$\mathbb{E}[T_{2n}] = \sum_{i=1}^n \int_0^{\mathbb{X}_i^t (\beta^g - \beta^{(1)})} [F(v + \mathbb{X}_i^t (\beta^{(1)} - \beta^0)) - F(\mathbb{X}_i^t (\beta^{(1)} - \beta^0))] dv.$$

Since the derivative f' is bounded in a neighbourhood of 0, taking into account Assumption (A3), we have:

$$\mathbb{E}[T_{2n}] = O\left(\sum_{i=1}^n (\mathbb{X}_i^t(\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}))^2\right) = O(n\|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2^2). \quad (22)$$

For the variance of T_{2n} , since the errors ε_i are independent, we have:

$$\begin{aligned} \text{Var}[T_{2n}] &= \sum_{i=1}^n \text{Var}\left[\int_0^{\mathbb{X}_i^t(\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)})} [\mathbb{1}_{\varepsilon_i \leq v + \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)} - \mathbb{1}_{\varepsilon_i \leq \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)}] dv\right] \\ &= \sum_{i=1}^n \mathbb{E}\left[\int_0^{\mathbb{X}_i^t(\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)})} ([\mathbb{1}_{\varepsilon_i \leq v + \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)} - \mathbb{1}_{\varepsilon_i \leq \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)}] \right. \\ &\quad \left. - [F(v + \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)) - F(\mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0))]) dv\right]^2 \\ &\leq \sum_{i=1}^n \mathbb{E}\left[\left|\int_0^{\mathbb{X}_i^t(\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)})} ([\mathbb{1}_{\varepsilon_i \leq v + \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)} - \mathbb{1}_{\varepsilon_i \leq \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)}] \right. \right. \\ &\quad \left. \left. - [F(v + \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)) - F(\mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0))]) dv\right|\right] 2|\mathbb{X}_i^t(\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)})| \\ &\leq 2 \sum_{i=1}^n \int_0^{\mathbb{X}_i^t(\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)})} [F(v + \mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0)) - F(\mathbb{X}_i^t(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^0))] dv \\ &\quad \cdot 2 \max_{1 \leq l \leq n} \|\mathbb{X}_l\|_2 \|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2. \end{aligned}$$

Taking into account assumptions (A1)-(A3) we obtain: $\text{Var}[T_{2n}] = o(\mathbb{E}[T_{2n}])$. Hence, taking also into account relation (22), by Bienaymé-Tchebychev inequality we obtain: $T_{2n} = Cn\|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2^2(1 + o_{\mathbb{P}}(1))$. Then, together with relation (21), we obtained that, $T_{1n} + T_{2n} = Cn\|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2^2(1 + o_{\mathbb{P}}(1))$. Thus, returning to relation (20), we get,

$$D_n(\boldsymbol{\beta}^g, \boldsymbol{\beta}^{(1)}) = C\|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2^2(1 + o_{\mathbb{P}}(1)) + \frac{\lambda_{1n}}{n} \sum_{j=|\mathcal{A}^0|+1}^g \left(\frac{g}{n}\right)^{(1-\gamma)/2} + \frac{\lambda_{2n}}{n} \sum_{j=|\mathcal{A}^0|+1}^g \frac{g}{n}. \quad (23)$$

From where,

$$\begin{aligned} \frac{D_n(\boldsymbol{\beta}^g, \boldsymbol{\beta}^{(1)})}{\|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2} &\geq C\|\boldsymbol{\beta}^g - \boldsymbol{\beta}^{(1)}\|_2(1 + o_{\mathbb{P}}(1)) + \frac{\lambda_{1n}}{n} O_{\mathbb{P}}\left(\left(\frac{g}{n}\right)^{-\gamma/2}\right) + \frac{\lambda_{2n}}{n} \frac{g}{n} \\ &= O_{\mathbb{P}}\left(\left(\frac{g}{n}\right)^{1/2} + \frac{\lambda_{1n}}{n} \left(\frac{g}{n}\right)^{-\gamma/2} + \frac{\lambda_{2n}}{n} \frac{g}{n}\right) \\ &\geq O_{\mathbb{P}}\left(\frac{\lambda_{1n}}{n} \left(\frac{g}{n}\right)^{-\gamma/2}\right) = O_{\mathbb{P}}(\lambda_{1n} n^{\gamma(1-c)/2-1}). \end{aligned} \quad (24)$$

We have similarly to relation (23) that $D_n(\beta^0, \beta^{(1)}) = C\|(\beta^0 - \beta^{(1)})_{\mathcal{A}^0}\|_2^2(1 + o_{\mathbb{P}}(1))$. From where,

$$\frac{D_n(\beta^0, \beta^{(1)})}{\|\beta^0 - \beta^{(1)}\|_2} = O_{\mathbb{P}}((g/n)^{1/2}) = O_{\mathbb{P}}(n^{(c-1)/2}). \quad (25)$$

Then, taking into account condition (8), we obtain for relation (24,) by (25),

$$\frac{D_n(\beta^g, \beta^{(1)})}{\|\beta^g - \beta^{(1)}\|_2} \gg \frac{D_n(\beta^0, \beta^{(1)})}{\|\beta^0 - \beta^{(1)}\|_2},$$

which implies relation (19). Relation (19) involves, with a probability converging to 1, as $n \rightarrow \infty$, that, for $\|\beta^g - \beta^0\|_2 \leq C(g/n)^{1/2}$, we have,

$$E_n(\beta_{\mathcal{A}^0}^g, \mathbf{0}_{r_n-p|\mathcal{A}^0}|) = \min_{\|\beta_{\mathcal{A}^0}^g\|_2 \leq C\sqrt{\frac{g}{n}}} E_n(\beta_{\mathcal{A}^0}^g, \beta_{\mathcal{A}^0}^g). \quad (26)$$

On the other hand, by Assumption (A5) and Theorem 3.1 we have: $\lim_{n \rightarrow \infty} \mathbb{P}[\min_{j \in \mathcal{A}^0} \|\widehat{\beta}_j^g\|_2 > 0] = 1$. This relation together with relation (26) imply claim (i).

(ii) By claim (i), we have with a probability converging to one as $n \rightarrow \infty$, that $\widehat{\beta}^g$ is under the form $\beta^0 + (gn^{-1})^{1/2}\delta$, with the r_n -dimensional vector $\delta = (\delta_{\mathcal{A}^0}, \mathbf{0}_{r_n-p|\mathcal{A}^0}|)$, $\delta_{\mathcal{A}^0}$ being a $(p|\mathcal{A}^0|)$ -dimensional vector such that $\|\delta_{\mathcal{A}^0}\|_2 < C$. We consider the following difference:

$$\begin{aligned} E_n(\beta^0 + \sqrt{\frac{g}{n}}\delta) - E_n(\beta^0) &= \frac{1}{n} \sum_{i=1}^n [\rho_{\tau}(Y_i - \mathbb{X}_i^t(\beta^0 + \sqrt{\frac{g}{n}}\delta)) - \rho_{\tau}(\varepsilon_i)] \\ &+ \frac{\lambda_{1n}}{n} \sum_{j=1}^{|\mathcal{A}^0|} \widetilde{\omega}_{n;j} [\|\beta_j^0 + \sqrt{\frac{g}{n}}\delta_j\|_2 - \|\beta_j^0\|_2] + \frac{\lambda_{2n}}{n} \sum_{j=1}^{|\mathcal{A}^0|} [\|\beta_j^0 + \sqrt{\frac{g}{n}}\delta_j\|_2^2 - \|\beta_j^0\|_2^2]. \end{aligned} \quad (27)$$

For relation (27), we denote the following two sums $\mathcal{P}_1 \equiv n^{-1}\lambda_{1n} \sum_{j=1}^{|\mathcal{A}^0|} \widetilde{\omega}_{n;j} [\|\beta_j^0 + (g/n)^{1/2}\delta_j\|_2 - \|\beta_j^0\|_2]$ and $\mathcal{P}_2 \equiv n^{-1}\lambda_{2n} \sum_{j=1}^{|\mathcal{A}^0|} [\|\beta_j^0 + (g/n)^{1/2}\delta_j\|_2^2 - \|\beta_j^0\|_2^2]$.

Since $\|\widehat{\beta}^g - \beta^0\|_2 = O_{\mathbb{P}}((g/n)^{1/2})$, we have $\mathcal{P}_1 = O_{\mathbb{P}}(n^{-1}\lambda_{1n}|\mathcal{A}^0|(gn^{-1})^{1/2})$. Then, using condition (9), we obtain:

$$\mathcal{P}_1/(g/n) = O_{\mathbb{P}}(n^{-1}\lambda_{1n}|\mathcal{A}^0|(gn^{-1})^{-1/2}) = O_{\mathbb{P}}(\lambda_{1n}|\mathcal{A}^0|n^{-(1+c)/2}) = o_{\mathbb{P}}(1). \quad (28)$$

For the absolute value of \mathcal{P}_2 we have by Cauchy-Schwarz inequality, that

$$\begin{aligned} |\mathcal{P}_2| &\leq \frac{\lambda_{2n}}{n} \sum_{j=1}^{|\mathcal{A}^0|} \left(\frac{g}{n} \|\delta_j\|_2^2 + 2\sqrt{\frac{g}{n}} \|\delta_j\|_2 \|\beta_j^0\|_2 \right) \\ &= O_{\mathbb{P}}\left(\frac{\lambda_{2n}}{n} \sum_{j=1}^{|\mathcal{A}^0|} (g/n)^{1/2} \right) = O_{\mathbb{P}}(|\mathcal{A}^0|\lambda_{2n}n^{-1+(c-1)/2}). \end{aligned}$$

Then, using condition (10), we have

$$\frac{\mathcal{P}_2}{g/n} = O_{\mathbb{P}}(\lambda_{2n}|\mathcal{A}^0|n^{-(1+c)/2}) = o_{\mathbb{P}}(1). \quad (29)$$

For the first term of the right-hand side of (27), using the proof of Theorem 4 of Ciuperca (2019), we have:

$$\begin{aligned} n^{-1} \sum_{i=1}^n [\rho_\tau(Y_i - \mathbb{X}_i^t(\boldsymbol{\beta}^0 + \sqrt{\frac{g}{n}}\boldsymbol{\delta})) - \rho_\tau(\varepsilon_i)] &= \frac{1}{n} \sqrt{\frac{g}{n}} \sum_{i=1}^n \mathbb{X}_{i,\mathcal{A}^0}^t \boldsymbol{\delta}_{\mathcal{A}^0} [\mathbb{1}_{\varepsilon_i < 0} - \tau] \\ &\quad + \frac{f(0)}{2} \frac{g}{n} \boldsymbol{\delta}_{\mathcal{A}^0}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}^0} \boldsymbol{\delta}_{\mathcal{A}^0} (1 + o_{\mathbb{P}}(1)). \end{aligned} \quad (30)$$

Comparing (30) with (28) and (29) we deduce that minimizing (27) amounts to minimize (30), which have that minimizer:

$$\sqrt{\frac{g}{n}} \boldsymbol{\delta}_{\mathcal{A}^0} = -\frac{1}{n} \frac{1}{f(0)} \boldsymbol{\Upsilon}_{n,\mathcal{A}^0}^{-1} \left(\sum_{i=1}^n \mathbb{X}_{i,\mathcal{A}^0} (\mathbb{1}_{\varepsilon_i \leq 0} - \tau) \right).$$

On the other hand, by Central Limit Theorem (CLT) for independent random variable sequences, we have

$$\sqrt{n} f(0) \frac{\mathbf{u}^t (\widehat{\boldsymbol{\beta}}_{\mathcal{A}^0}^g - \boldsymbol{\beta}_{\mathcal{A}^0}^0)}{\sqrt{\tau(1-\tau) (\mathbf{u}^t \boldsymbol{\Upsilon}_{n,\mathcal{A}^0}^{-1} \mathbf{u})}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

Since, $\widehat{\boldsymbol{\beta}}_{\mathcal{A}^0}^g - \boldsymbol{\beta}_{\mathcal{A}^0}^0 = (g/n)^{1/2} \boldsymbol{\delta}_{\mathcal{A}^0}$, claim (ii) follows. \blacksquare

Proof of Remark 3.3 Let us consider the vector $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_g) \equiv \sqrt{n}(\boldsymbol{\beta}^g - \boldsymbol{\beta}^0) \in \mathbb{R}^r$. Then $Y_i - \mathbb{X}_i^t \boldsymbol{\beta}^g = -n^{-1/2} \mathbb{X}_i^t \mathbf{u} + \varepsilon_i$. We consider also the following random process:

$$\begin{aligned} L_n(\mathbf{u}) &\equiv \sum_{i=1}^n [\rho_\tau(\varepsilon_i - n^{-1/2} \mathbb{X}_i^t \mathbf{u}) - \rho_\tau(\varepsilon_i)] + \lambda_{1n} \sum_{j=1}^g \tilde{\omega}_{n;j} [\|\boldsymbol{\beta}_j^0 + n^{-1/2} \mathbf{u}_j\|_2 - \|\boldsymbol{\beta}_j^0\|_2] \\ &\quad + \lambda_{2n} \sum_{j=1}^g [\|\boldsymbol{\beta}_j^0 + n^{-1/2} \mathbf{u}_j\|_2^2 - \|\boldsymbol{\beta}_j^0\|_2^2] \\ &= [\mathbf{z}_n^t \mathbf{u} + B_n(\mathbf{u})] + \lambda_{1n} \sum_{j=1}^g \tilde{\omega}_{n;j} [\|\boldsymbol{\beta}_j^0 + n^{-1/2} \mathbf{u}_j\|_2 - \|\boldsymbol{\beta}_j^0\|_2] \frac{\sqrt{n}}{\sqrt{n}} \\ &\quad + \lambda_{2n} \sum_{j=1}^g [\|\boldsymbol{\beta}_j^0 + n^{-1/2} \mathbf{u}_j\|_2^2 - \|\boldsymbol{\beta}_j^0\|_2^2] \\ &\equiv S_{1n} + S_{2n} + S_{3n}, \end{aligned} \quad (31)$$

with $\mathbf{z}_n \equiv n^{-1/2} \sum_{i=1}^n \mathbb{X}_i ((1-\tau)\mathbb{1}_{\varepsilon_i < 0} - \tau\mathbb{1}_{\varepsilon_i \geq 0})$ and $B_n(\mathbf{u}) \equiv \sum_{i=1}^n \int_0^{n^{-1/2} \mathbb{X}_i^t \mathbf{u}} [\mathbb{1}_{\varepsilon_i < v} - \mathbb{1}_{\varepsilon_i < 0}] dv$. As in the proof of Theorem 1 of Ciuperca (2019) we have:

$$S_{2n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \sum_{j=1}^g W(\boldsymbol{\beta}_j^0, \mathbf{u}), \quad \text{with } W(\boldsymbol{\beta}_j^0, \mathbf{u}) \equiv \begin{cases} 0, & \text{if } \boldsymbol{\beta}_j^0 \neq \mathbf{0}_p, \\ 0, & \text{if } \boldsymbol{\beta}_j^0 = \mathbf{0}_p \text{ and } \mathbf{u}_j = \mathbf{0}_j, \\ \infty, & \text{if } \boldsymbol{\beta}_j^0 = \mathbf{0}_p \text{ and } \mathbf{u}_j \neq \mathbf{0}_j. \end{cases} \quad (32)$$

By Cauchy-Schwarz inequality, we have: $\|\mathbf{x} - \mathbf{y}\|_2^2 - \|\mathbf{x}\|_2^2 \leq \|\mathbf{y}\|_2^2 + 2\|\mathbf{x}\|_2\|\mathbf{y}\|_2$. Taking $\mathbf{x} = \beta_j^0$, $\mathbf{y} = n^{-1/2}\mathbf{u}_j$, using also condition (11), we obtain for S_{3n} :

$$|S_{3n}| \leq \lambda_{2n} \sum_{j=1}^g [n^{-1}\|\mathbf{u}_j\|_2^2 + 2\|\beta_j^0\|_2 n^{-1/2}\|\mathbf{u}_j\|_2] = C\lambda_{2n}n^{-1/2}(1 + o(1)) \xrightarrow[n \rightarrow \infty]{} 0. \quad (33)$$

For S_{1n} , using Assumptions (A1), (A6), we have, by the CLT, that, $\mathbf{z}_n^t \mathbf{u} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathbf{z}^t \mathbf{u}$, with \mathbf{z} a r -dimensional random vector of law $\mathcal{N}(\mathbf{0}_r, \tau(1 - \tau)\mathbf{\Upsilon})$. Taking into account assumption (A1) that the density f has a bounded derivative in the neighborhood of 0, we have

$$\mathbb{E}[B_n(\mathbf{u})] \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}f(0)\mathbf{u}^t \mathbf{\Upsilon} \mathbf{u}. \quad (34)$$

On the other hand,

$$\begin{aligned} \text{Var}[B_n(\mathbf{u})] &= \sum_{i=1}^n \mathbb{E} \left[\int_0^{n^{-1/2}\mathbb{X}_i^t \mathbf{u}} ((\mathbb{1}_{\varepsilon_i < v} - \mathbb{1}_{\varepsilon_i < 0}) - (F(v) - F(0))) dv \right]^2 \\ &\leq \sum_{i=1}^n \mathbb{E} \left[\int_0^{n^{-1/2}\mathbb{X}_i^t \mathbf{u}} |(\mathbb{1}_{\varepsilon_i < v} - \mathbb{1}_{\varepsilon_i < 0}) - (F(v) - F(0))| dv \right] \cdot 2n^{-1/2} |\mathbb{X}_i^t \mathbf{u}| \\ &\leq 2n^{-1/2} \max_{1 \leq i \leq n} |\mathbb{X}_i^t \mathbf{u}| \sum_{i=1}^n \mathbb{E} \left[\int_0^{n^{-1/2}\mathbb{X}_i^t \mathbf{u}} (|\mathbb{1}_{\varepsilon_i < v} - \mathbb{1}_{\varepsilon_i < 0}| + |F(v) - F(0)|) dv \right] \\ &= 4n^{-1/2} \max_{1 \leq i \leq n} |\mathbb{X}_i^t \mathbf{u}| \sum_{i=1}^n \int_0^{n^{-1/2}\mathbb{X}_i^t \mathbf{u}} (F(v) - F(0)) dv \\ &= 4\mathbb{E}[B_n(\mathbf{u})]n^{-1/2} \max_{1 \leq i \leq n} |\mathbb{X}_i^t \mathbf{u}| \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

by relation (34) and Assumption (A6). Then, taking into account relation (34), we obtain by Bienaymé-Tchebichev inequality that $B_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2^{-1}f(0)\mathbf{u}^t \mathbf{\Upsilon} \mathbf{u}$.

Thus, taking into account relations (31), (32), (33) and the fact that $\mathbf{z}_n^t \mathbf{u} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathbf{z}^t \mathbf{u}$, together with $B_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 2^{-1}f(0)\mathbf{u}^t \mathbf{\Upsilon} \mathbf{u}$, we have:

$$L_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathbf{z}^t \mathbf{u} + \frac{1}{2}f(0)\mathbf{u}^t \mathbf{\Upsilon} \mathbf{u} + \sum_{j=1}^g W(\beta_j^0, \mathbf{u}). \quad (35)$$

Since $L_n(\mathbf{u})$ and the right-hand side of relation (35) are convex in \mathbf{u} , we have by Theorem 6.1 that:

$$\arg \min_{\mathbf{u} \in \mathbb{R}^r} L_n(\mathbf{u}) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \arg \min_{\mathbf{u} \in \mathbb{R}^r} \left(\mathbf{z}^t \mathbf{u} + \frac{1}{2}f(0)\mathbf{u}^t \mathbf{\Upsilon} \mathbf{u} + \sum_{j=1}^g W(\beta_j^0, \mathbf{u}) \right). \quad (36)$$

On the other hand, we have $\widehat{\mathbf{u}}^n \equiv \arg \min_{\mathbf{u} \in \mathbb{R}^r} L_n(\mathbf{u}) \stackrel{\text{denoted}}{=} (\widehat{\mathbf{u}}_{1n}, \widehat{\mathbf{u}}_{2n})$, with $\widehat{\mathbf{u}}_{1n}$ the first $(p|\mathcal{A}^0|)$ elements of $\widehat{\mathbf{u}}^n$ and $\widehat{\mathbf{u}}_{2n}$ the next $(r - p|\mathcal{A}^0|)$ elements of $\widehat{\mathbf{u}}^n$. Taking into account relations (36), (35) and (32) we obtain that $\widehat{\mathbf{u}}_{2n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbf{0}_{r-p|\mathcal{A}^0|}$ and $\widehat{\mathbf{u}}_{1n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_{p|\mathcal{A}^0|}, \tau(1 - \tau)f^{-2}(0)\mathbf{\Upsilon}_{\mathcal{A}^0}^{-1})$. The proof of the theorem is thus complete. \blacksquare

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