

Supplementary Information

1. Derivation of governing equation (2)

The equation governing shear-lag model was derived in [36], it is

$$\frac{E_b^* h_b h}{\mu} \frac{d^2 \bar{\gamma}}{dx_1^2} = \hat{\tau}(\bar{\gamma}), \quad (\text{S1})$$

where E_b^* is the plane strain modulus of the backing and $\hat{\tau}(\bar{\gamma}) \equiv \tau(\bar{\gamma})/\mu$ is the normalized shear stress. The quantity

$$l_{LT} \equiv \sqrt{E_b^* h_b h / \mu} \quad (\text{S2})$$

is defined as the load transfer length. If we normalize distance by

$$\eta = x_1 / l_{LT}, \quad (\text{S3})$$

then (S1) becomes equation (2), i.e.,

$$\frac{d^2 \bar{\gamma}}{d\eta^2} = \hat{\tau}(\bar{\gamma}). \quad (\text{S4})$$

2. Derivation of equation (1)

For small strain linear theory, $\hat{\tau}(\bar{\gamma}) = \mu\gamma / \mu = \gamma$. Therefore, (S4) reduces to $\frac{d^2 \gamma}{d\eta^2} = \gamma$ for the finite bond length subjected to the boundary conditions $\gamma|_{\eta=0} = \gamma_0$, $d\gamma/d\eta|_{\eta=\bar{L}} = 0$. The solution is

$$\gamma = \gamma_0 \frac{\cosh(\bar{L} - \eta)}{\cosh \bar{L}}. \quad (\text{S5})$$

The force is related to γ_0 by

$$F = \mu l_{LT} \gamma_0 \int_0^{\bar{L}} \frac{\cosh(\bar{L} - \eta)}{\cosh \bar{L}} d\eta = -\frac{\mu l_{LT} \gamma_0}{\cosh \bar{L}} \sinh(\bar{L} - \eta) \Big|_0^{\bar{L}} = \frac{\mu l_{LT} \gamma_0}{\cosh \bar{L}} \sinh \bar{L} \Rightarrow \gamma_0 = \frac{F \coth \bar{L}}{\mu l_{LT}} \quad (\text{S6})$$

Using (S5) and (S6), we obtain the equation (1a) in the main text, i.e.,

$$\gamma = \frac{F \coth \bar{L}}{\mu l_{LT}} \frac{\cosh(\bar{L} - \eta)}{\cosh \bar{L}} = \frac{F}{\mu l_{LT}} \frac{\cosh(\bar{L} - \eta)}{\sinh \bar{L}}. \quad (\text{S7})$$

Therefore, the shear stress in the adhesive layer is

$$\tau_{12} = \mu\gamma = \frac{F \coth \bar{L}}{l_{LT}} \frac{\cosh(\bar{L} - \eta)}{\cosh \bar{L}} = \frac{F}{l_{LT}} \frac{\cosh(\bar{L} - \eta)}{\sinh \bar{L}} \quad (\text{S8})$$

The tension in the backing layer is

$$\tau_{11}^B = E_b^* \frac{d(\gamma h)}{d\eta} = -E_b^* h \frac{F}{\mu l_{LT}} \frac{\sinh(\bar{L} - \eta)}{\sinh \bar{L}} \quad (\text{S9})$$

3. Derivation of equation (6a) and (7)

As shown in our previous work [36], the nonlinear ODE (S4) can be reduced to a first order separable equation

$$\frac{d\eta}{d\gamma} = - \left[2 \int_{\gamma_{\min}}^{\gamma} \gamma' f(\gamma') d\gamma' \right]^{-1/2} \Rightarrow \eta = - \int_{\gamma_0}^{\gamma} \left[2 \int_{\gamma_{\min}}^{\gamma'} \gamma'' f(\gamma'') d\gamma'' \right]^{-1/2} d\gamma' \quad (\text{S10})$$

where the integration constant is determined by $\gamma|_{\eta=0} = \gamma_0$. The normalized force can be obtained by integrating the normalized shear stress, i.e.,

$$\bar{F} = \int_0^{\bar{L}} \hat{\tau}(\eta) d\eta = \int_0^{\bar{L}} \frac{d^2\gamma}{d\eta^2} d\eta = \left. \frac{d\gamma}{d\eta} \right|_0^{\bar{L}} = -\gamma'(0) = - \sqrt{2 \int_{\gamma_{\min}}^{\gamma_0} \gamma' f(\gamma') d\gamma'} \quad (\text{S11})$$

In particular, the normalized force is the slope of the shear strain at the origin. Note that the solution works for a wide range of hyperelastic models.

4. Numerical method to find shear strain

A simple way of finding the strain is to solve the differential equation S4 directly; by writing it as two first-order ODEs with $y_1 = \gamma$, $y_2 = \gamma'$:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= f(y_1) y_1 \end{aligned} \quad (\text{S12})$$

where prime stands for the derivative with respect to η . The boundary conditions are $y_2 = 0$ at $\eta = L/l_{LT}$ and $y_2 = -\bar{F}$ at $\eta = 0$, (see (S11) above). (S12) is solved using the BVP function in MATLAB.

5. FE model

The FE model is shown schematically in Fig S1, and implemented in a commercial FEM software, ABAQUS. The adhesive is modeled by a three term Yeoh's material. The backing is modeled as a linear elastic solid with Young's modulus. Material parameters can be found in [36]. In the simulation, the overlap length L is finite and varies from $0.5l_{LT}$ to $5l_{LT}$, and the overhanging length is $20h$. The initial thickness of the backing h_b is taken to be $2h$, which is a typical value [5].

The displacements on the adhesive/substrate interface are fixed to be zero. The constant force F is mimicked by a uniform horizontal traction t acting on the left end of the backing. To balance the accuracy and efficiency of the computation, we choose a very fine mesh near the origin and right edge in the adhesive, while the element size in between increases and is about $0.2h$ in the middle of the bond length. Hybrid plane strain elements CPE4H and CPE3H are used to handle the incompressibility of the adhesive. Our convergence test shows that further refinement of mesh does not affect the FE results (except the first few elements near the origin where there is a singularity).

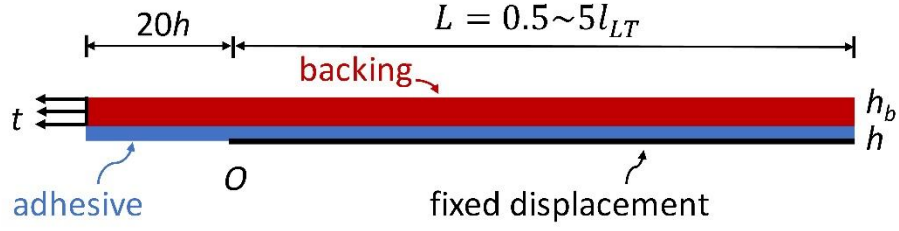


Fig. S1. Schematic of finite element model. The red and blue filled rectangles represent the backing and adhesive layer respectively, and the rigid substrate is modeled by the fixed displacement boundary condition.

6. Energy release rate for more general hyperelasticity models

We extend our three-term Yeoh's model to any hyperelastic solids for which the strain energy density W only depends on I_1 , where I_1 is the trace of the right Cauchy-Green tensor. In our previous work [36], we have shown that for large shear,

$$\tau_{11} \approx 2 \frac{dW}{dI_1} \gamma^2, \tau_{12} \approx 2 \frac{dW}{dI_1} \gamma, \tau_{22} \approx 0 \quad (\text{S13a-c})$$

Using the relation $I_1 - 3 \approx \gamma^2$, we have $W(I_1) = \hat{W}(\gamma^2)$. We next use (16) to obtain the energy release rate

$$\Gamma = \frac{F^2}{2E_b^* h_b} + h \hat{W}(\gamma_{\min}^2) \quad (\text{S14})$$