



**HAL**  
open science

# On the tail distribution of the solution to some law equation.

Xinxin Chen, Chunhua Ma

► **To cite this version:**

Xinxin Chen, Chunhua Ma. On the tail distribution of the solution to some law equation.. 2019.  
hal-02076851v2

**HAL Id: hal-02076851**

**<https://hal.science/hal-02076851v2>**

Preprint submitted on 28 Mar 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the tail distribution of the solution to some law equation.

Xinxin CHEN\*, Chunhua Ma<sup>†</sup>

March 27, 2019

## Abstract

We consider a distribution equation which was initially studied by Bertoin [2]:

$$M \stackrel{d}{=} \max\{\tilde{\nu}, \max_{1 \leq k \leq \nu} M_k\}.$$

where  $\{M_k\}_{k \geq 1}$  are i.i.d. copies of  $M$  and independent of  $(\tilde{\nu}, \nu) \in \mathbb{R}_+ \times \mathbb{N}$ . We obtain the tail behaviour of the solution of a generalised equation in a different but direct method by considering the joint tail of  $(\tilde{\nu}, \nu)$ .

## 1 Introduction to questions

For a random variable  $\nu \in \mathbb{N}$ , we consider the following equation of distributions on  $\mathbb{N}$ :

$$(1.1) \quad M \stackrel{d}{=} \max\{\nu, \max_{1 \leq k \leq \nu} M_k\}$$

where  $M_k$  are i.i.d. copies of  $M \in \mathbb{N}$  and independent of  $\nu$ . In fact, for a Galton-Waston tree  $\mathbb{T}$  with offspring  $\nu$ , let

$$M := \sup_{u \in \mathbb{T}} \nu_u$$

be the largest offspring. Clearly  $M$  satisfies the equation (1.1).

Note that if  $\mathbf{E}[\nu] \leq 1$  then  $M < \infty$  a.s. and that if  $\mathbf{E}[\nu] > 1$ ,  $\mathbf{P}(M = \infty) = \mathbf{P}(T = \infty)$ .

In this paper, we only consider the critical case when  $\mathbf{E}[\nu] = 1$ . If  $\mathbf{E}[\nu] < 1$ , the tail distribution of  $M$  is of the same order as  $\nu$ . For two sequences  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$ , we write  $a_n \sim b_n$

---

\*Institut Camille Jordan - C.N.R.S. UMR 5208 - Université Claude Bernard Lyon 1 (France).

†School of Mathematical Sciences and LPMC, Nankai University (China).

MSC 2000 60E05 60G70.

*Key words* : distribution equation, multivariate regularly varying

as  $n \rightarrow \infty$  if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$  and we write  $a_n \asymp b_n$  if  $K_1 b_n \leq a_n \leq K_2 b_n$  for some positive constants  $K_1, K_2 > 0$ .

Bertoin [2] considered this equation (1.1) and by use of the link between critical Galton-Watson tree and centred random walk, he proved the following theorem.

**Theorem 1.1** (Bertoin). *For  $M$  whose distribution satisfies the equation (1.1),*

1. *if  $\mathbf{P}(v > n) = n^{-\alpha} \ell(n)$  with  $\alpha \in (1, 2)$  and  $\ell(\cdot)$  a slowly varying function at infinity, then as  $n \rightarrow \infty$ ,*

$$(1.2) \quad \mathbf{P}(M > n) \sim \frac{C_\alpha}{n},$$

where  $C_\alpha \in (0, \infty)$  is a constant which depends only on  $\alpha$ ;

2. *if  $\sigma^2 := \text{Var}(v) < \infty$ , then*

$$(1.3) \quad \mathbf{P}(M > n) \sim \sqrt{\frac{2}{\sigma^2} \mathbf{P}(v > n)}.$$

We will reprove this theorem by direct calculations, using the generating function of  $v$ .

More generally, for a random vector  $(\tilde{v}, v)$  which takes values in  $\mathbb{R}_+ \times \mathbb{N}$  such that  $\mathbf{E}[v] = 1$ , let us consider the following equation of distribution:

$$(1.4) \quad M \stackrel{d}{=} \max\{\tilde{v}, \max_{1 \leq k \leq v} M_k\}$$

where  $M_k$  are i.i.d. copies of  $M \in \mathbb{R}_+$  and independent of  $(\tilde{v}, v)$ . The distribution of  $M$  differs according to the joint distribution of  $(\tilde{v}, v)$ . We first consider some special cases in the following.

**Theorem 1.2.** *For  $M$  whose distribution satisfies the equation (1.4), if  $\mathbf{E}[v] = 1$  and  $\mathbf{E}[v^2] < \infty$ , then as  $r \rightarrow \infty$ ,*

$$(1.5) \quad \mathbf{P}(M > r) \sim \sqrt{\frac{2}{\sigma^2} \mathbf{P}(\tilde{v} > r)};$$

where  $\sigma^2 = \text{Var}(v)$

For random vector  $(\tilde{v}, v)$  where  $v$  has infinite variance, we need to consider multivariate regularly varying condition. One can refer to Chapter 6 in [13] for more details.

For a  $d$ -dimensional random vector  $\mathbf{X} \in \mathbb{R}^d$ , its law is regularly varying of index  $\alpha \in (0, \infty)$  if for some norm  $\|\cdot\|$  on  $\mathbb{R}^d$ , there exists a random vector  $\theta$  on the unit sphere  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$  such that for any  $u \in (0, \infty)$  and as  $x \rightarrow \infty$ ,

$$(1.6) \quad \frac{\mathbf{P}(\|\mathbf{X}\| > ux, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in \cdot)}{\mathbf{P}(\|\mathbf{X}\| > x)} \xrightarrow{\text{weak}} u^{-\alpha} \mathbf{P}(\theta \in \cdot)$$

where the convergence is on weak topology of finite measures, i.e. for  $C_b(\mathbb{R}^d)$ .

The equivalent characterization of multivariate regular variations is as follows. Recall that a measurable function  $V : (0, \infty) \rightarrow (0, \infty)$  is regularly varying of index  $\rho \in \mathbb{R}$  if as  $x \rightarrow \infty$ ,

$$V(xy)/V(x) \rightarrow y^\rho, \forall y \in (0, \infty).$$

A  $d$ -dimensional random vector  $\mathbf{X}$  is then regularly varying of index  $\alpha \in (0, \infty)$  if and only if there exists a regularly varying function  $V$  of index  $-\alpha$  and a nonzero Radon measure  $\mu$  on  $\mathbb{R}^d$  such that, as  $x \rightarrow \infty$ ,

$$(1.7) \quad \frac{1}{V(x)} \mathbf{P}(x^{-1}\mathbf{X} \in \cdot) \xrightarrow{\text{vague}} \mu(\cdot)$$

where vague convergence is for all functions in  $C_K^+(\mathbb{R}^d)$ . Here the measure  $\mu$  is homogeneous of order  $-\alpha$ . A choice for the function  $V$  is that  $V(x) = \mathbf{P}(\|\mathbf{X}\| > x)$  in which case the vague convergence in (1.7) is also weak convergence, and for all  $u \in (0, \infty)$ , by (1.6),

$$\mu(\{x \in \mathbb{R}^d : \|x\| > u, x/\|x\| \in \cdot\}) = u^{-\alpha} \mathbf{P}(\theta \in \cdot).$$

Then the restriction of  $\mu$  on  $\{x : \|x\| > 1\}$  is a probability measure. Hence, for any  $f \in C_b(\mathbb{R}^d)$ ,

$$\mathbf{E}[f(x^{-1}\mathbf{X}) \mid \|\mathbf{X}\| > x] \xrightarrow{x \rightarrow \infty} \int f(y) 1_{\|y\| > 1} \mu(dy).$$

Now we state the following theorem for jointly regularly varying setting.

**Theorem 1.3.** *For  $M$  whose distribution satisfies the equation (1.4) and for  $X = (\tilde{v}, \nu)$ , suppose that  $X$  is regularly varying of index  $\alpha \in (1, 2)$  associated with  $V(x) = \mathbf{P}(\|X\| > x)$  and  $\mu$  given in (1.7) such that*

$$\mu(\{y = (y_1, y_2) \in \mathbb{R}_+^2 : y_1 > 0, y_2 > 1\}) > 0 \text{ and } \mu(\{y = (y_1, y_2) \in \mathbb{R}_+^2 : y_2 = 1 < \|y\|\}) = 0.$$

Suppose moreover that there exist  $c_1, c_2 \in (0, \infty)$  such that

$$\frac{\mathbf{P}(\nu > x)}{\mathbf{P}(\|X\| > x)} \xrightarrow{x \rightarrow \infty} c_1, \text{ and } \frac{\mathbf{P}(\tilde{v} > x)}{\mathbf{P}(\|X\| > x)} \xrightarrow{x \rightarrow \infty} c_2.$$

Then, as  $r \rightarrow \infty$ ,

$$(1.8) \quad \mathbf{P}(M > r) \sim \frac{C_{\alpha, \mu}}{r}$$

where  $C_{\alpha, \mu}$  is the unique positive solution of  $c_1 \alpha \Gamma(-\alpha) x^\alpha - \int_{\mathbb{R}_+^2} e^{-xy_2} 1_{y_1 > 1} \mu(dy) = 0$ .

**Remark:** Instead of the vector  $(\tilde{v}, \nu)$ , if these assumptions hold for the vector  $(\tilde{v}^\gamma, \nu)$  for some  $\gamma > 0$ , these arguments still work for  $M^\gamma$ . In this case, the distribution of  $(\tilde{v}, \nu)$  is called non-standard regularly varying according to [13].

Let us state the following corollary by assuming the joint tail of  $(\tilde{v}, \nu)$ .

**Corollary 1.4.** For the random vector  $(\tilde{v}, \nu)$  with  $\mathbf{E}[\nu] = 1$ , assume that there exist a non-increasing function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and  $\alpha > 0$  such that for any  $b \in \mathbb{R}_+$ , as  $r \rightarrow \infty$ ,

$$\mathbf{P}(\tilde{v} \geq r, \nu \geq br) \sim \gamma(b)r^{-\alpha},$$

and  $\mathbf{P}(\nu \geq r) \sim cr^{-\alpha}$  with some  $c \in (0, \infty)$ . For random variable  $M$  satisfying the equation (1.4),

1. if  $\alpha \in (1, 2)$ , then

$$(1.9) \quad \mathbf{P}(M > r) \sim \frac{C_{\alpha, \gamma, c}}{r}, \text{ as } r \rightarrow \infty.$$

2. if  $\alpha = 2$ , then

$$(1.10) \quad \mathbf{P}(M > r) \sim \frac{C_{\alpha, \gamma, c}}{r\sqrt{\log r}}, \text{ as } r \rightarrow \infty.$$

3. if  $\alpha > 2$ , then

$$(1.11) \quad \mathbf{P}(M > r) \sim \sqrt{\frac{2}{\mathbf{E}[\nu^2] - 1}} \mathbf{P}(\tilde{v} > r), \text{ as } r \rightarrow \infty.$$

**Remark:** One can refer to [5] for an application and the motivation of this corollary.

Note that Theorem 1.1 is a special case of Theorems 1.3 and 1.2 by taking  $\tilde{v} = \nu$ . We only need to prove the last two theorems. Let us explain the main idea of proof here, especially for Theorem 1.3 when  $(\tilde{v}, \nu)$  is regularly varying of index  $\alpha \in (1, 2)$ .

In fact, let  $f(s) := \mathbf{E}[s^\nu]$  be the generating function of  $\nu$ . Observe from (1.4) that for any  $r > 0$ ,

$$\begin{aligned} \mathbf{P}(M \leq r) &= \mathbf{P}(\tilde{v} \leq r; \max_{1 \leq i \leq \nu} M_k \leq r) \\ &= \mathbf{P}\left(\max_{1 \leq i \leq \nu} M_k \leq r\right) - \mathbf{P}\left(\tilde{v} > r; \max_{1 \leq i \leq \nu} M_k \leq r\right) \\ &= f(1 - \mathbf{P}(M > r)) - \mathbf{E}[(1 - \mathbf{P}(M > r))^\nu; \tilde{v} > r] \end{aligned}$$

where the last equality follows from the independence between  $(\tilde{v}, \nu)$  and all  $M_k$ . Write  $x_r := \mathbf{P}(M > r)$  for convenience, we obtain that

$$(1.12) \quad 1 - x_r = f(1 - x_r) - \mathbf{E}[(1 - x_r)^\nu; \tilde{v} > r].$$

Inspired by this equation, we define for any  $r > 0$  and for any  $x \in [0, r]$ :

$$\Phi_r(x) := \left[f\left(1 - \frac{x}{r}\right) - \left(1 - \frac{x}{r}\right)\right] - \mathbf{E}\left[\left(1 - \frac{x}{r}\right)^\nu; \tilde{v} > r\right].$$

Apparently, one sees that  $\Phi_r(rx_r) = 0$ . On the one hand, we show the tightness of  $rx_r$ . On the other hand, for some positive deterministic sequence  $(\gamma_r)$ , we will show that  $\gamma_r\Phi_r$  converges to some continuous function  $\Phi$  uniformly on any compact of  $[0, \infty)$  and that the limit function  $\Phi$  has one unique zero in  $[0, \infty)$ . This implies that  $rx_r$  converges to the unique zero of  $\Phi$ .

If  $\nu$  has finite variance, the proof will be similar by changing the rescaling term.

The paper is organised as follows. In Section 2, we study the tail of  $M$  when  $\nu$  has finite variance and prove Theorem 1.2. In Section 3, we prove Theorem 1.3 and Corollary 1.4.

We write  $f(x) \asymp g(x)$  as  $x \rightarrow x_0$  if  $0 < \liminf_{x \rightarrow x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow x_0} \frac{f(x)}{g(x)} < \infty$ .

## 2 Theorem 1.2: (1.4) when $\mathbf{E}[\nu^2] < \infty$

We begin with the simpler case when  $\nu$  has finite second moment. Note that (1.3) is a special case of (1.5) by taking  $\tilde{\nu} = \nu$ . So it suffices to prove (1.5) in Theorem 1.2.

Recall that for  $x_r = \mathbf{P}(M > r)$ , we have the equation (1.12). We first show that  $x_r \asymp \sqrt{\mathbf{P}(\tilde{\nu} > r)}$ .

For the upper bound, one sees that

$$f(1 - x_r) - (1 - x_r) = \mathbf{E}[(1 - x_r)^\nu; \tilde{\nu} > r] \leq \mathbf{P}(\tilde{\nu} > r).$$

For the generating function  $f$  of  $\nu$  with  $f'(1) = 1$ , let  $b(s)$  be the function on  $[0, 1)$  such that

$$(2.1) \quad b(s) = \frac{1}{1 - f(s)} - \frac{1}{f'(1)(1 - s)} = \frac{1}{1 - f(s)} - \frac{1}{1 - s}.$$

Then as proved in Lemma 2.1 of [7], if  $\text{Var}(\nu) < \infty$ , one has

$$(2.2) \quad 0 \leq b(s) \leq \sigma^2, \forall 0 \leq s < 1, \text{ and } \lim_{s \uparrow 1} b(s) = \sigma^2/2.$$

Plugging  $s = 1 - x_r$  in (2.1) then using (2.2) gives us that

$$f(1 - x_r) - (1 - x_r) = x_r - \left(\frac{1}{x_r} + b(1 - x_r)\right)^{-1} = \frac{x_r^2 b(1 - x_r)}{1 + x_r b(1 - x_r)} \geq \frac{b(1 - x_r)}{1 + \sigma^2} x_r^2.$$

It follows that

$$x_r^2 \leq \mathbf{P}(\tilde{\nu} > r) \frac{1 + \sigma^2}{b(1 - x_r)}.$$

Note that  $\mathbf{P}(M < \infty) = 1$  as  $\mathbf{E}[\nu] = 1$ . Apparently  $x_r \downarrow 0$  as  $r \uparrow \infty$ . So,  $\lim_{r \uparrow \infty} b(1 - x_r) = \sigma^2/2$  by (2.1). Then for  $r$  sufficiently large,  $b(1 - x_r) \geq \sigma^2/4 > 0$ . As a result, for  $r$  large enough,

$$x_r \leq 2\sqrt{\frac{1 + \sigma^2}{\sigma^2} \mathbf{P}(\tilde{\nu} > n)}.$$

For the lower bound, as  $(1-x)^k \geq 1-kx$  for any  $k \in \mathbb{N}$  and  $x \in [0, 1]$ , we have

$$(2.3) \quad \begin{aligned} f(1-x_r) - (1-x_r) &= \mathbf{E}[(1-x_r)^v; \tilde{v} > r] \geq \mathbf{E}[(1-vx_r); \tilde{v} > r] \\ &= \mathbf{P}(\tilde{v} > r) - x_r \mathbf{E}[v; \tilde{v} > r]. \end{aligned}$$

Note that for the generating function  $f$ , we have  $f'(1) = 1$  and  $f''(1) = \sigma^2$ . So,

$$f(1-x_r) - (1-x_r) = f(1-x_r) - f(1) + f'(1)x_r \leq \frac{f''(1)}{2}x_r^2.$$

It follows that

$$\frac{\sigma^2}{2}x_r^2 \geq \mathbf{P}(\tilde{v} > r) - x_r \mathbf{E}[v; \tilde{v} > r]$$

By Cauchy-Schwartz inequality,

$$(2.4) \quad \mathbf{E}[v\mathbf{1}_{\{\tilde{v} > r\}}] \leq \sqrt{\mathbf{E}[v^2\mathbf{1}_{\{\tilde{v} > r\}}]\mathbf{P}(\tilde{v} > r)}.$$

Observe that  $\mathbf{E}[v^2\mathbf{1}_{\{\tilde{v} > r\}}] \rightarrow 0$  as  $r \rightarrow \infty$ . Hence,  $\mathbf{E}[v\mathbf{1}_{\{\tilde{v} > r\}}] \leq \sqrt{\mathbf{P}(\tilde{v} > r)}$  for  $r$  large enough. This yields that

$$\frac{\sigma^2}{2}x_r^2 + x_r\sqrt{\mathbf{P}(\tilde{v} > r)} \geq \mathbf{P}(\tilde{v} > r).$$

Consequently, for all sufficiently large  $r$ ,

$$x_r \geq \frac{-\sqrt{\mathbf{P}(\tilde{v} > r)} + \sqrt{1+2\sigma^2}\sqrt{\mathbf{P}(\tilde{v} > r)}}{\sigma^2} = \frac{2}{1 + \sqrt{1+2\sigma^2}}\sqrt{\mathbf{P}(\tilde{v} > r)}.$$

Therefore, we obtain that

$$x_r \asymp \sqrt{\mathbf{P}(\tilde{v} > n)}.$$

Next, let  $\gamma_r := \mathbf{P}(\tilde{v} > r)$ . We define for any  $r > 0$  and  $x \in [0, 1/\gamma_r]$ ,

$$\Phi_r(x) := f(1-x\sqrt{\gamma_r}) - (1-x\sqrt{\gamma_r}) - \mathbf{E}[(1-x\sqrt{\gamma_r})^v; \tilde{v} > r]$$

By (1.12), one has  $\Phi_r(\frac{x_r}{\sqrt{\gamma_r}}) = 0$ . We are going to show that

$$(2.5) \quad \frac{1}{\gamma_r}\Phi_r(x) \xrightarrow{r \rightarrow \infty} \phi(x) := \frac{\sigma^2}{2}x^2 - 1,$$

uniformly in any compact  $K \subset [0, \infty)$ . The pointwise convergence is trivial for  $x = 0$ .

We treat  $f(1-x\sqrt{\gamma_r}) - (1-x\sqrt{\gamma_r})$  and  $\mathbf{E}[(1-x\sqrt{\gamma_r})^v; \tilde{v} > r]$  separately. First, for any  $x \in (0, \infty)$ , by (2.1) and (2.2),

$$n^2[f(1-\frac{x}{n}) - (1-\frac{x}{n})] = \frac{b(1-\frac{x}{n})}{1+\frac{x}{n}b(1-\frac{x}{n})}x^2 \rightarrow \frac{\sigma^2}{2}x^2,$$

as  $n \rightarrow \infty$ . As  $f(1 - \frac{x}{n}) - (1 - \frac{x}{n})$  is monotone for  $x$ , Dini's theorem shows that this convergence is uniform in any compact  $K \subset [0, \infty)$ . Note that  $\gamma_r \downarrow 0$  as  $r \uparrow \infty$ . Replacing  $n$  by  $1/\sqrt{\gamma_r}$ , we get

$$\frac{1}{\gamma_r} [f(1 - x\sqrt{\gamma_r}) - (1 - x\sqrt{\gamma_r})] \rightarrow \frac{\sigma^2}{2} x^2.$$

uniformly for  $x$  in a compact  $K \subset [0, \infty)$ . It remains to show that for any  $x \geq 0$

$$(2.6) \quad \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] \sim \gamma_r.$$

It is immediate that

$$\mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] \leq \mathbf{P}(\tilde{\nu} > r) = \gamma_r.$$

Similarly as (2.3) and (2.4),

$$\begin{aligned} \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] &\geq \mathbf{P}(\tilde{\nu} > r) - x\sqrt{\gamma_r} \mathbf{E}[v; \tilde{\nu} > r] \\ &\geq \gamma_r - x\gamma_r \sqrt{\mathbf{E}[v^2 \mathbf{1}_{\{\tilde{\nu} > r\}}]}. \end{aligned}$$

This implies that

$$\left| \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] - \gamma_r \right| \leq x\gamma_r \sqrt{\mathbf{E}[v^2 \mathbf{1}_{\{\tilde{\nu} > r\}}]}.$$

where  $\lim_{r \rightarrow \infty} \gamma_r \sqrt{\mathbf{E}[v^2 \mathbf{1}_{\{\tilde{\nu} > r\}}]} = 0$ . Moreover, we also have the uniform convergence

$$\left| \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] - \gamma_r \right| \rightarrow 0$$

in any compact set on  $\mathbb{R}_+$ . We hence deduce that

$$\frac{1}{\gamma_r} \Phi_r(x) \rightarrow \phi(x).$$

uniformly in any compact set on  $\mathbb{R}_+$ .

Now let us prove the convergence of  $\frac{x_r}{\sqrt{\gamma_r}}$  by showing that any convergent subsequence converges towards the same limit. In fact, note that for any subsequence of  $\{x_r\}$  such that as  $k \rightarrow \infty$ ,

$$\frac{x_{r_k}}{\sqrt{\gamma_{r_k}}} \rightarrow x^* \in \mathbb{R}_+,$$

the uniform convergence (2.5) and the continuity of  $\phi$  yield that

$$\lim_{k \rightarrow \infty} \frac{1}{\gamma_{r_k}} \Phi_{r_k} \left( \frac{x_{r_k}}{\sqrt{\gamma_{r_k}}} \right) = \phi(x^*).$$

By (1.12),

$$\phi(x^*) = 0.$$

So,  $x^* = \sqrt{2/\sigma^2}$  and

$$x_r \sim \sqrt{\frac{2}{\sigma^2} \mathbf{P}(\tilde{\nu} > r)}.$$



### 3 Theorem 1.3: (1.4) in the jointly regularly varying case

In this section, we study the tail of  $M$  given in the equation

$$M \stackrel{d}{=} \max\{\tilde{\nu}, \max_{1 \leq k \leq \nu} M_k\},$$

where  $X = (\tilde{\nu}, \nu)$  is independent of  $M_k, k \geq 1$ , and has multivariate regularly varying tail. For  $x_r = \mathbf{P}(M > r)$  with  $r > 0$ , we first show that

$$x_r \asymp \frac{1}{r}, \text{ as } r \rightarrow \infty.$$

According to the assumptions of Theorem 1.3, we can write  $V(r) = r^{-\alpha}\ell(r)$  for some  $\alpha \in (1, 2)$  and  $\ell$  slowly varying function, then

$$(3.1) \quad \mathbf{P}(\nu > r) = r^{-\alpha}\ell_1(r), \text{ and } \mathbf{P}(\tilde{\nu} > r) = r^{-\alpha}\ell_2(r)$$

where  $\ell_1$  and  $\ell_2$  are two slowly varying function at infinity such that  $\ell_1(r) \sim c_1\ell(r)$  and  $\ell_2(r) \sim c_2\ell(r)$  as  $r \rightarrow \infty$ .

We first show that  $x_r = O(\frac{1}{r})$ . In fact, by (1.12), one has

$$(3.2) \quad x_r = 1 - f(1 - x_r) + \mathbf{E}[(1 - x_r)^\nu; \tilde{\nu} > r] \leq 1 - f(1 - x_r) + \mathbf{P}(\tilde{\nu} > r).$$

Note that  $f$  is the generating function of  $\nu$ , thus,

$$\begin{aligned} f(1 - x_r) &= \mathbf{E}[(1 - x_r)^\nu] \geq \mathbf{E}[(1 - x_r)^\nu; \nu \leq r] \\ &\geq \mathbf{E}[(1 - \nu x_r); \nu \leq r] \\ &= \mathbf{P}(\nu \leq r) - x_r \mathbf{E}[\nu; \nu \leq r]. \end{aligned}$$

Plugging it into (3.2) yields that

$$\begin{aligned} x_r &\leq 1 - \mathbf{P}(\nu \leq r) + x_r \mathbf{E}[\nu; \nu \leq r] + \mathbf{P}(\tilde{\nu} > r) \\ &= \mathbf{P}(\nu > r) + x_r \mathbf{E}[\nu; \nu \leq r] + \mathbf{P}(\tilde{\nu} > r). \end{aligned}$$

Recall that  $1 = \mathbf{E}[\nu]$ , so

$$x_r(1 - \mathbf{E}[\nu; \nu \leq r]) = x_r \mathbf{E}[\nu; \nu > r] \leq \mathbf{P}(\nu > r) + \mathbf{P}(\tilde{\nu} > r).$$

By (3.1),  $\mathbf{P}(\tilde{\nu} > r) \sim \frac{c_2}{c_1}\mathbf{P}(\nu > r)$ . Moreover, the tail of  $\nu$  is regularly varying of index  $\alpha \in (1, 2)$ , then Karamata's theorem implies that  $\mathbf{E}[\nu; \nu > r] \sim \frac{\alpha}{\alpha-1}\mathbf{P}(\nu > r)r$ . Consequently,  $x_r = O(\frac{1}{r})$ .

Apparently,  $0 \leq x_r r \leq C < \infty$  for some constant  $C > 0$ . This means the tightness of  $\{rx_r\}_{r>0}$ . Now we turn to the uniform convergence of  $\Phi_r$  defined as follows:

$$(3.3) \quad \Phi_r(x) = f(1 - \frac{x}{r}) - (1 - \frac{x}{r}) - \mathbf{E}[(1 - \frac{x}{r})^\nu; \tilde{\nu} > r], \forall r > x \geq 0.$$

Clearly,  $\Phi_r(0) \sim -c_2 r^{-\alpha} \ell(r)$ .

On the one hand, by Bingham *et al.* [4, Theorem 8.1.6; P 333], as  $\mathbf{P}(v > r) = r^{-\alpha} \ell_1(r)$ , we get that for  $x > 0$ ,

$$(3.4) \quad f\left(1 - \frac{x}{r}\right) - \left(1 - \frac{x}{r}\right) \sim \alpha \Gamma(-\alpha) \left(\frac{x}{r}\right)^\alpha \ell_1\left(\frac{r}{x}\right) \sim c_1 \alpha \Gamma(-\alpha) x^\alpha V(r), \quad r \rightarrow \infty,$$

On the other hand, we study the convergence of  $\frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r]$  by use of the multivariate regularly varying tail of  $X = (\tilde{v}, v)$ . Note that

$$\begin{aligned} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] &= \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r \mid \|X\| > r] \\ &\leq \mathbf{E}[e^{-x \frac{v}{r}} \mathbf{1}_{\{\tilde{v} > r\}} \mid \|X\| > r] \end{aligned}$$

For the lower bound, note that for any  $\varepsilon \in (0, 1)$  and  $x > 0$ , there exists  $r(x, \varepsilon) > 0$  such that for all  $r \geq r(x, \varepsilon)$ , one has  $(1 - \frac{x}{r}) \geq e^{-(1+\varepsilon)\frac{x}{r}}$ . Therefore,

$$\frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] \geq \mathbf{E}[e^{-(1+\varepsilon)x \frac{v}{r}} \mathbf{1}_{\{\tilde{v} > r\}} \mid \|X\| > r].$$

The multivariate regularly tail of the random vector  $X$  implies the weak convergence of  $\mathbf{P}((\frac{\tilde{v}}{r}, \frac{v}{r}) \in \cdot \mid \|X\| > r)$  towards  $\mu_+(\cdot) := \mu(\cdot \cap \{y = (y_1, y_2) \in \mathbb{R}_+^2 : \|y\| > 1\})$ . Moreover, for  $A := (1, \infty) \times \mathbb{R}_+$ ,  $\mu_+(\partial A) = 0$  by the assumption. As a consequence, for any  $x > 0$ ,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\mathbf{P}(\tilde{v} > r)}{V(r)} &= \mu_+(A) = c_2 \\ \liminf_{r \rightarrow \infty} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] &\geq \int e^{-(1+\varepsilon)xy_2} \mathbf{1}_{\{y_1 > 1\}} \mu_+(dy) \\ \limsup_{r \rightarrow \infty} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] &\leq \int e^{-xy_2} \mathbf{1}_{\{y_1 > 1\}} \mu_+(dy) \end{aligned}$$

Letting  $\varepsilon > 0$  yields that

$$(3.5) \quad \lim_{r \rightarrow \infty} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] = \int e^{-xy_2} \mathbf{1}_{\{y_1 > 1\}} \mu_+(dy).$$

Note that  $f(1 - \frac{x}{r}) - (1 - \frac{x}{r})$  is non-decreasing in  $x$  and  $\frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r]$  is non-increasing in  $x$ . So, by Dini's theorem, both the convergences (3.4) and (3.5) are uniform on any compact in  $[0, \infty)$ . Going back to (3.3), one obtains that

$$\frac{1}{V(r)} \Phi_r(x) \xrightarrow{r \rightarrow \infty} \Phi(x) := c_1 \alpha \Gamma(-\alpha) x^\alpha - \int e^{-xy_1} \mathbf{1}_{\{y_2 > 1\}} \mu_+(dy),$$

uniformly on any compact in  $[0, \infty)$ . Note that by dominated convergence theorem,  $\Phi$  is continuous for  $x \geq 0$ . Moreover,  $\Phi$  is strict increasing on  $\mathbb{R}_+$ . The zero  $C_{\alpha, \mu} > 0$  of  $\Phi$  exists and is unique because

$$\Phi(0) = -c_2 < 0 \text{ and } \lim_{x \rightarrow \infty} \Phi(x) = \infty.$$

The equation (1.12) means that  $\Phi_r(rx_r) = 0$ . So we conclude that  $rx_r \rightarrow C_{\alpha, \mu}$ .

## 4 Proof of Corollary 1.4

It is clear that when  $\alpha > 2$ , (1.11) follows from Theorem 1.2 and that when  $\alpha \in (1, 2)$ , (1.9) follows from Theorem 1.3. We only need to prove (1.10) for  $\alpha = 2$ .

First note that  $\mathbf{P}(v \geq r) \sim cr^{-\alpha}$  and  $\mathbf{P}(\tilde{v} \geq r) \sim \gamma(0)r^{-\alpha}$ . By Theorem 1.5 in [6], one has for  $r$  sufficiently large,

$$p_r := \mathbf{P}(M \geq r) \asymp \frac{1}{r\sqrt{\log r}}.$$

For  $r \geq e$  and  $x \in [0, r\sqrt{\log r}]$ , define

$$\Phi_r(x) = f\left(1 - \frac{x}{r\sqrt{\log r}}\right) - \left(1 - \frac{x}{r\sqrt{\log r}}\right) - \mathbf{E} \left[ \left(1 - \frac{x}{r\sqrt{\log r}}\right)^v ; \tilde{v} \geq r \right].$$

Similarly, one has for any  $x \geq 0$ , as  $r \rightarrow \infty$ ,

$$f\left(1 - \frac{x}{r\sqrt{\log r}}\right) - \left(1 - \frac{x}{r\sqrt{\log r}}\right) \sim C_2 \left(\frac{x}{r\sqrt{\log r}}\right)^2 \log\left(\frac{r\sqrt{\log r}}{x}\right) \sim C_2 \frac{x^2}{r^2}.$$

For any  $\varepsilon \in (0, 1)$  and  $r \gg 1$ ,

$$\mathbf{E} \left[ e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] \leq \mathbf{E} \left[ \left(1 - \frac{x}{r\sqrt{\log r}}\right)^v ; \tilde{v} \geq r \right] \leq \mathbf{P}(\tilde{v} \geq r)$$

Apparently  $r^2 \mathbf{P}(\tilde{v} \geq r) \rightarrow c$  as  $r \rightarrow \infty$ . For the lower bound, we see that for any  $x > 0$ ,

$$\begin{aligned} r^2 \mathbf{E} \left[ 1 - e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] &= \int_0^\infty e^{-u} r^2 \mathbf{P}(\tilde{v} \geq r, v \geq r \frac{u}{\varepsilon x}) du \\ &\leq \int_0^\infty e^{-u} du \left( r^2 \mathbf{P}(\tilde{v} \geq r) \right) \end{aligned}$$

As  $\mathbf{P}(\tilde{v} \geq r) \sim cr^{-\alpha}$  and  $\mathbf{P}(\tilde{v} \geq r, v \geq br) \sim \gamma(b)r^{-\alpha}$ , by dominated convergence theorem,

$$\lim_{r \rightarrow \infty} r^2 \mathbf{E} \left[ 1 - e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] = \int_0^\infty \gamma\left(\frac{u}{\varepsilon x}\right) e^{-u} du$$

Here one can show that  $\gamma(b) \downarrow 0$  as  $b \uparrow \infty$ . In fact, for any  $b > 0$ , we have

$$r^2 \mathbf{P}(\tilde{v} \geq r, v \geq br) \leq r^2 \mathbf{P}(v \geq br).$$

Taking limit shows that  $\gamma(b) \leq cb^{-\alpha}$ . Consequently,

$$\lim_{r \rightarrow \infty} r^2 \mathbf{E} \left[ e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] = \int_0^\infty [\gamma(0) - \gamma\left(\frac{u}{\varepsilon x}\right)] e^{-u} du \geq \int_0^\infty [\gamma(0) - \frac{c\varepsilon^2 x^2}{u^2} \wedge \gamma(0)] e^{-u} du.$$

Note that by dominated convergence theorem,

$$\int_0^\infty \frac{c\varepsilon^2 x^2}{u^2} \wedge \gamma(0)e^{-u} du \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Therefore,

$$\begin{aligned} \gamma(0) + o_\varepsilon(1) &\leq \liminf_{r \rightarrow \infty} r^2 \mathbf{E} \left[ \left( 1 - \frac{x}{r\sqrt{\log r}} \right)^\nu ; \tilde{\nu} \geq r \right] \\ &\leq \limsup_{r \rightarrow \infty} r^2 \mathbf{E} \left[ \left( 1 - \frac{x}{r\sqrt{\log r}} \right)^\nu ; \tilde{\nu} \geq r \right] \leq \gamma(0) \end{aligned}$$

We end up with

$$r^2 \Phi_r(x) \rightarrow \phi(x) := C_2 x^2 - \gamma(0).$$

One can check the uniform convergence in any compact set on  $\mathbb{R}_+$  by Dini's theorem. We hence conclude that  $r\sqrt{\log r} p_r$  converges to  $\sqrt{\frac{\gamma(0)}{C_2}}$ .

## References

- [1] Addario-Berry, L., Devroye, L. and Janson, S. (2013): Sub-Gaussian tail bounds for the width and height of conditioned Galton-Watson trees. *Ann. Probab.* **41**, 1072-1087.
- [2] Bertoin, J. (2011): On the maximal offspring in a critical branching process with infinite variance. *J. Appl. Probab.* **48**, 576-582.
- [3] Borovkov, K. A. and Vatutin. V. A. (1996): On distribution tails and expectations of maxima in critical branching processes. *J. Appl. Probab.* **33**, 614-622.
- [4] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987): *Regular Variation*. Cambridge University Press, Cambridge (UK).
- [5] Chen, X. and de Raphélis, L. (2019+) The most visited edges of randomly biased random walks on a supercritical Galton-Watson tree (I). *In preparation*.
- [6] Chen, D., de Raphélis, L. and Hu, Y. (2018): Favorite sites of randomly biased random walks on a supercritical Galton-Watson tree. *Stoc. Proc. Appl.* **128**, 1525-1557.
- [7] Geiger, J. and Kersting, G. (2000): The survival probability of a critical branching process in a random environment. *Theory Probab. Appl.* **45**, 517-525.
- [8] Li, Z.H. (2006): A limit theorem for discrete Galton-Watson branching processes with immigration. *Journal of Applied Probability.* **43**, 289-295.
- [9] Kortchemski, I. (2017): Sub-exponential tail bounds for conditioned stable Bienaymé-Galton-Watson trees. *Probability Theory and Related Fields.* **168**, 1-40.

- [10] Lindvall, T. (1976): On the maximum of a branching process. *Scandinavian Journal of Statistics*. **3**, 209-214.
- [11] Ma, C. (2009): A limit theorem of two-type Galton-Watson branching processes with immigration. *Statistics and probability letters*. **79**, 1710-1716.
- [12] Resnick, S.I. (1987): *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.
- [13] Resnick, S.I. (2009): *Heavy-Tail Phenomena: Probability and Statistical Modelling*. Springer, New York.