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On the tail distribution of the solution to some law equation.

Xinxin CHEN*, Chunhua Ma[†]

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Abstract

We consider a distribution equation which was initially studied by Bertoin [2]:

$$M \stackrel{d}{=} \max\{\tilde{\nu}, \max_{1 \leq k \leq \nu} M_k\}.$$

where $\{M_k\}_{k \geq 1}$ are i.i.d. copies of M and independent of $(\tilde{\nu}, \nu) \in \mathbb{R}_+ \times \mathbb{N}$. We obtain the tail behaviour of the solution of a generalised equation in a different but direct method by considering the joint tail of $(\tilde{\nu}, \nu)$.

1 Introduction to questions

For a random variable $\nu \in \mathbb{N}$, we consider the following equation of distributions on \mathbb{N} :

$$(1.1) \quad M \stackrel{d}{=} \max\{\nu, \max_{1 \leq k \leq \nu} M_k\}$$

where M_k are i.i.d. copies of $M \in \mathbb{N}$ and independent of ν . In fact, for a Galton-Waston tree \mathbb{T} with offspring ν , let

$$M := \sup_{u \in \mathbb{T}} \nu_u$$

be the largest offspring. Clearly M satisfies the equation (1.1).

Note that if $\mathbf{E}[\nu] \leq 1$ then $M < \infty$ a.s. and that if $\mathbf{E}[\nu] > 1$, $\mathbf{P}(M = \infty) = \mathbf{P}(T = \infty)$.

In this paper, we only consider the critical case when $\mathbf{E}[\nu] = 1$. If $\mathbf{E}[\nu] < 1$, the tail distribution of M is of the same order as ν . For two sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, we write $a_n \sim b_n$

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as $n \rightarrow \infty$ if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ and we write $a_n \asymp b_n$ if $K_1 b_n \leq a_n \leq K_2 b_n$ for some positive constants $K_1, K_2 > 0$.

Bertoin [2] considered this equation (1.1) and by use of the link between critical Galton-Watson tree and centred random walk, he proved the following theorem.

Theorem 1.1 (Bertoin). *For M whose distribution satisfies the equation (1.1),*

1. *if $\mathbf{P}(v > n) = n^{-\alpha} \ell(n)$ with $\alpha \in (1, 2)$ and $\ell(\cdot)$ a slowly varying function at infinity, then as $n \rightarrow \infty$,*

$$(1.2) \quad \mathbf{P}(M > n) \sim \frac{C_\alpha}{n},$$

where $C_\alpha \in (0, \infty)$ is a constant which depends only on α ;

2. *if $\sigma^2 := \text{Var}(v) < \infty$, then*

$$(1.3) \quad \mathbf{P}(M > n) \sim \sqrt{\frac{2}{\sigma^2} \mathbf{P}(v > n)}.$$

We will reprove this theorem by direct calculations, using the generating function of v .

More generally, for a random vector (\tilde{v}, v) which takes values in $\mathbb{R}_+ \times \mathbb{N}$ such that $\mathbf{E}[v] = 1$, let us consider the following equation of distribution:

$$(1.4) \quad M \stackrel{d}{=} \max\{\tilde{v}, \max_{1 \leq k \leq v} M_k\}$$

where M_k are i.i.d. copies of $M \in \mathbb{R}_+$ and independent of (\tilde{v}, v) . The distribution of M differs according to the joint distribution of (\tilde{v}, v) . We first consider some special cases in the following.

Theorem 1.2. *For M whose distribution satisfies the equation (1.4), if $\mathbf{E}[v] = 1$ and $\mathbf{E}[v^2] < \infty$, then as $r \rightarrow \infty$,*

$$(1.5) \quad \mathbf{P}(M > r) \sim \sqrt{\frac{2}{\sigma^2} \mathbf{P}(\tilde{v} > r)};$$

where $\sigma^2 = \text{Var}(v)$

For random vector (\tilde{v}, v) where v has infinite variance, we need to consider multivariate regularly varying condition. One can refer to Chapter 6 in [13] for more details.

For a d -dimensional random vector $\mathbf{X} \in \mathbb{R}^d$, its law is regularly varying of index $\alpha \in (0, \infty)$ if for some norm $\|\cdot\|$ on \mathbb{R}^d , there exists a random vector θ on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ such that for any $u \in (0, \infty)$ and as $x \rightarrow \infty$,

$$(1.6) \quad \frac{\mathbf{P}(\|\mathbf{X}\| > ux, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in \cdot)}{\mathbf{P}(\|\mathbf{X}\| > x)} \xrightarrow{\text{weak}} u^{-\alpha} \mathbf{P}(\theta \in \cdot)$$

where the convergence is on weak topology of finite measures, i.e. for $C_b(\mathbb{R}^d)$.

The equivalent characterization of multivariate regular variations is as follows. Recall that a measurable function $V : (0, \infty) \rightarrow (0, \infty)$ is regularly varying of index $\rho \in \mathbb{R}$ if as $x \rightarrow \infty$,

$$V(xy)/V(x) \rightarrow y^\rho, \forall y \in (0, \infty).$$

A d -dimensional random vector \mathbf{X} is then regularly varying of index $\alpha \in (0, \infty)$ if and only if there exists a regularly varying function V of index $-\alpha$ and a nonzero Radon measure μ on \mathbb{R}^d such that, as $x \rightarrow \infty$,

$$(1.7) \quad \frac{1}{V(x)} \mathbf{P}(x^{-1}\mathbf{X} \in \cdot) \xrightarrow{\text{vague}} \mu(\cdot)$$

where vague convergence is for all functions in $C_K^+(\mathbb{R}^d)$. Here the measure μ is homogeneous of order $-\alpha$. A choice for the function V is that $V(x) = \mathbf{P}(\|\mathbf{X}\| > x)$ in which case the vague convergence in (1.7) is also weak convergence, and for all $u \in (0, \infty)$, by (1.6),

$$\mu(\{x \in \mathbb{R}^d : \|x\| > u, x/\|x\| \in \cdot\}) = u^{-\alpha} \mathbf{P}(\theta \in \cdot).$$

Then the restriction of μ on $\{x : \|x\| > 1\}$ is a probability measure. Hence, for any $f \in C_b(\mathbb{R}^d)$,

$$\mathbf{E}[f(x^{-1}\mathbf{X}) \mid \|\mathbf{X}\| > x] \xrightarrow{x \rightarrow \infty} \int f(y) 1_{\|y\| > 1} \mu(dy).$$

Now we state the following theorem for jointly regularly varying setting.

Theorem 1.3. *For M whose distribution satisfies the equation (1.4) and for $X = (\tilde{v}, \nu)$, suppose that X is regularly varying of index $\alpha \in (1, 2)$ associated with $V(x) = \mathbf{P}(\|X\| > x)$ and μ given in (1.7) such that*

$$\mu(\{y = (y_1, y_2) \in \mathbb{R}_+^2 : y_1 > 0, y_2 > 1\}) > 0 \text{ and } \mu(\{y = (y_1, y_2) \in \mathbb{R}_+^2 : y_2 = 1 < \|y\|\}) = 0.$$

Suppose moreover that there exist $c_1, c_2 \in (0, \infty)$ such that

$$\frac{\mathbf{P}(\nu > x)}{\mathbf{P}(\|X\| > x)} \xrightarrow{x \rightarrow \infty} c_1, \text{ and } \frac{\mathbf{P}(\tilde{v} > x)}{\mathbf{P}(\|X\| > x)} \xrightarrow{x \rightarrow \infty} c_2.$$

Then, as $r \rightarrow \infty$,

$$(1.8) \quad \mathbf{P}(M > r) \sim \frac{C_{\alpha, \mu}}{r}$$

where $C_{\alpha, \mu}$ is the unique positive solution of $c_1 \alpha \Gamma(-\alpha) x^\alpha - \int_{\mathbb{R}_+^2} e^{-xy_2} 1_{y_1 > 1} \mu(dy) = 0$.

Remark: Instead of the vector (\tilde{v}, ν) , if these assumptions hold for the vector (\tilde{v}^γ, ν) for some $\gamma > 0$, these arguments still work for M^γ . In this case, the distribution of (\tilde{v}, ν) is called non-standard regularly varying according to [13].

Let us state the following corollary by assuming the joint tail of (\tilde{v}, ν) .

Corollary 1.4. For the random vector (\tilde{v}, ν) with $\mathbf{E}[\nu] = 1$, assume that there exist a non-increasing function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and $\alpha > 0$ such that for any $b \in \mathbb{R}_+$, as $r \rightarrow \infty$,

$$\mathbf{P}(\tilde{v} \geq r, \nu \geq br) \sim \gamma(b)r^{-\alpha},$$

and $\mathbf{P}(\nu \geq r) \sim cr^{-\alpha}$ with some $c \in (0, \infty)$. For random variable M satisfying the equation (1.4),

1. if $\alpha \in (1, 2)$, then

$$(1.9) \quad \mathbf{P}(M > r) \sim \frac{C_{\alpha, \gamma, c}}{r}, \text{ as } r \rightarrow \infty.$$

2. if $\alpha = 2$, then

$$(1.10) \quad \mathbf{P}(M > r) \sim \frac{C_{\alpha, \gamma, c}}{r\sqrt{\log r}}, \text{ as } r \rightarrow \infty.$$

3. if $\alpha > 2$, then

$$(1.11) \quad \mathbf{P}(M > r) \sim \sqrt{\frac{2}{\mathbf{E}[\nu^2] - 1}} \mathbf{P}(\tilde{v} > r), \text{ as } r \rightarrow \infty.$$

Remark: One can refer to [5] for an application and the motivation of this corollary.

Note that Theorem 1.1 is a special case of Theorems 1.3 and 1.2 by taking $\tilde{v} = \nu$. We only need to prove the last two theorems. Let us explain the main idea of proof here, especially for Theorem 1.3 when (\tilde{v}, ν) is regularly varying of index $\alpha \in (1, 2)$.

In fact, let $f(s) := \mathbf{E}[s^\nu]$ be the generating function of ν . Observe from (1.4) that for any $r > 0$,

$$\begin{aligned} \mathbf{P}(M \leq r) &= \mathbf{P}(\tilde{v} \leq r; \max_{1 \leq i \leq \nu} M_k \leq r) \\ &= \mathbf{P}\left(\max_{1 \leq i \leq \nu} M_k \leq r\right) - \mathbf{P}\left(\tilde{v} > r; \max_{1 \leq i \leq \nu} M_k \leq r\right) \\ &= f(1 - \mathbf{P}(M > r)) - \mathbf{E}[(1 - \mathbf{P}(M > r))^\nu; \tilde{v} > r] \end{aligned}$$

where the last equality follows from the independence between (\tilde{v}, ν) and all M_k . Write $x_r := \mathbf{P}(M > r)$ for convenience, we obtain that

$$(1.12) \quad 1 - x_r = f(1 - x_r) - \mathbf{E}[(1 - x_r)^\nu; \tilde{v} > r].$$

Inspired by this equation, we define for any $r > 0$ and for any $x \in [0, r]$:

$$\Phi_r(x) := \left[f\left(1 - \frac{x}{r}\right) - \left(1 - \frac{x}{r}\right)\right] - \mathbf{E}\left[\left(1 - \frac{x}{r}\right)^\nu; \tilde{v} > r\right].$$

Apparently, one sees that $\Phi_r(rx_r) = 0$. On the one hand, we show the tightness of rx_r . On the other hand, for some positive deterministic sequence (γ_r) , we will show that $\gamma_r\Phi_r$ converges to some continuous function Φ uniformly on any compact of $[0, \infty)$ and that the limit function Φ has one unique zero in $[0, \infty)$. This implies that rx_r converges to the unique zero of Φ .

If ν has finite variance, the proof will be similar by changing the rescaling term.

The paper is organised as follows. In Section 2, we study the tail of M when ν has finite variance and prove Theorem 1.2. In Section 3, we prove Theorem 1.3 and Corollary 1.4.

We write $f(x) \asymp g(x)$ as $x \rightarrow x_0$ if $0 < \liminf_{x \rightarrow x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow x_0} \frac{f(x)}{g(x)} < \infty$.

2 Theorem 1.2: (1.4) when $\mathbf{E}[\nu^2] < \infty$

We begin with the simpler case when ν has finite second moment. Note that (1.3) is a special case of (1.5) by taking $\tilde{\nu} = \nu$. So it suffices to prove (1.5) in Theorem 1.2.

Recall that for $x_r = \mathbf{P}(M > r)$, we have the equation (1.12). We first show that $x_r \asymp \sqrt{\mathbf{P}(\tilde{\nu} > r)}$.

For the upper bound, one sees that

$$f(1 - x_r) - (1 - x_r) = \mathbf{E}[(1 - x_r)^\nu; \tilde{\nu} > r] \leq \mathbf{P}(\tilde{\nu} > r).$$

For the generating function f of ν with $f'(1) = 1$, let $b(s)$ be the function on $[0, 1)$ such that

$$(2.1) \quad b(s) = \frac{1}{1 - f(s)} - \frac{1}{f'(1)(1 - s)} = \frac{1}{1 - f(s)} - \frac{1}{1 - s}.$$

Then as proved in Lemma 2.1 of [7], if $\text{Var}(\nu) < \infty$, one has

$$(2.2) \quad 0 \leq b(s) \leq \sigma^2, \forall 0 \leq s < 1, \text{ and } \lim_{s \uparrow 1} b(s) = \sigma^2/2.$$

Plugging $s = 1 - x_r$ in (2.1) then using (2.2) gives us that

$$f(1 - x_r) - (1 - x_r) = x_r - \left(\frac{1}{x_r} + b(1 - x_r)\right)^{-1} = \frac{x_r^2 b(1 - x_r)}{1 + x_r b(1 - x_r)} \geq \frac{b(1 - x_r)}{1 + \sigma^2} x_r^2.$$

It follows that

$$x_r^2 \leq \mathbf{P}(\tilde{\nu} > r) \frac{1 + \sigma^2}{b(1 - x_r)}.$$

Note that $\mathbf{P}(M < \infty) = 1$ as $\mathbf{E}[\nu] = 1$. Apparently $x_r \downarrow 0$ as $r \uparrow \infty$. So, $\lim_{r \uparrow \infty} b(1 - x_r) = \sigma^2/2$ by (2.1). Then for r sufficiently large, $b(1 - x_r) \geq \sigma^2/4 > 0$. As a result, for r large enough,

$$x_r \leq 2\sqrt{\frac{1 + \sigma^2}{\sigma^2} \mathbf{P}(\tilde{\nu} > n)}.$$

For the lower bound, as $(1-x)^k \geq 1-kx$ for any $k \in \mathbb{N}$ and $x \in [0, 1]$, we have

$$(2.3) \quad \begin{aligned} f(1-x_r) - (1-x_r) &= \mathbf{E}[(1-x_r)^{\tilde{v}}; \tilde{v} > r] \geq \mathbf{E}[(1-\nu x_r)^{\tilde{v}}; \tilde{v} > r] \\ &= \mathbf{P}(\tilde{v} > r) - x_r \mathbf{E}[\nu; \tilde{v} > r]. \end{aligned}$$

Note that for the generating function f , we have $f'(1) = 1$ and $f''(1) = \sigma^2$. So,

$$f(1-x_r) - (1-x_r) = f(1-x_r) - f(1) + f'(1)x_r \leq \frac{f''(1)}{2}x_r^2.$$

It follows that

$$\frac{\sigma^2}{2}x_r^2 \geq \mathbf{P}(\tilde{v} > r) - x_r \mathbf{E}[\nu; \tilde{v} > r]$$

By Cauchy-Schwartz inequality,

$$(2.4) \quad \mathbf{E}[\nu \mathbf{1}_{\{\tilde{v} > r\}}] \leq \sqrt{\mathbf{E}[\nu^2 \mathbf{1}_{\{\tilde{v} > r\}}] \mathbf{P}(\tilde{v} > r)}.$$

Observe that $\mathbf{E}[\nu^2 \mathbf{1}_{\{\tilde{v} > r\}}] \rightarrow 0$ as $r \rightarrow \infty$. Hence, $\mathbf{E}[\nu \mathbf{1}_{\{\tilde{v} > r\}}] \leq \sqrt{\mathbf{P}(\tilde{v} > r)}$ for r large enough. This yields that

$$\frac{\sigma^2}{2}x_r^2 + x_r \sqrt{\mathbf{P}(\tilde{v} > r)} \geq \mathbf{P}(\tilde{v} > r).$$

Consequently, for all sufficiently large r ,

$$x_r \geq \frac{-\sqrt{\mathbf{P}(\tilde{v} > r)} + \sqrt{1 + 2\sigma^2} \sqrt{\mathbf{P}(\tilde{v} > r)}}{\sigma^2} = \frac{2}{1 + \sqrt{1 + 2\sigma^2}} \sqrt{\mathbf{P}(\tilde{v} > r)}.$$

Therefore, we obtain that

$$x_r \asymp \sqrt{\mathbf{P}(\tilde{v} > n)}.$$

Next, let $\gamma_r := \mathbf{P}(\tilde{v} > r)$. We define for any $r > 0$ and $x \in [0, 1/\gamma_r]$,

$$\Phi_r(x) := f(1-x\sqrt{\gamma_r}) - (1-x\sqrt{\gamma_r}) - \mathbf{E}[(1-x\sqrt{\gamma_r})^{\tilde{v}}; \tilde{v} > r]$$

By (1.12), one has $\Phi_r(\frac{x_r}{\sqrt{\gamma_r}}) = 0$. We are going to show that

$$(2.5) \quad \frac{1}{\gamma_r} \Phi_r(x) \xrightarrow{r \rightarrow \infty} \phi(x) := \frac{\sigma^2}{2}x^2 - 1,$$

uniformly in any compact $K \subset [0, \infty)$. The pointwise convergence is trivial for $x = 0$.

We treat $f(1-x\sqrt{\gamma_r}) - (1-x\sqrt{\gamma_r})$ and $\mathbf{E}[(1-x\sqrt{\gamma_r})^{\tilde{v}}; \tilde{v} > r]$ separately. First, for any $x \in (0, \infty)$, by (2.1) and (2.2),

$$n^2 [f(1-\frac{x}{n}) - (1-\frac{x}{n})] = \frac{b(1-\frac{x}{n})}{1+\frac{x}{n}b(1-\frac{x}{n})} x^2 \rightarrow \frac{\sigma^2}{2} x^2,$$

as $n \rightarrow \infty$. As $f(1 - \frac{x}{n}) - (1 - \frac{x}{n})$ is monotone for x , Dini's theorem shows that this convergence is uniform in any compact $K \subset [0, \infty)$. Note that $\gamma_r \downarrow 0$ as $r \uparrow \infty$. Replacing n by $1/\sqrt{\gamma_r}$, we get

$$\frac{1}{\gamma_r} [f(1 - x\sqrt{\gamma_r}) - (1 - x\sqrt{\gamma_r})] \rightarrow \frac{\sigma^2}{2} x^2.$$

uniformly for x in a compact $K \subset [0, \infty)$. It remains to show that for any $x \geq 0$

$$(2.6) \quad \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] \sim \gamma_r.$$

It is immediate that

$$\mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] \leq \mathbf{P}(\tilde{\nu} > r) = \gamma_r.$$

Similarly as (2.3) and (2.4),

$$\begin{aligned} \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] &\geq \mathbf{P}(\tilde{\nu} > r) - x\sqrt{\gamma_r} \mathbf{E}[v; \tilde{\nu} > r] \\ &\geq \gamma_r - x\gamma_r \sqrt{\mathbf{E}[v^2 \mathbf{1}_{\{\tilde{\nu} > r\}}]}. \end{aligned}$$

This implies that

$$\left| \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] - \gamma_r \right| \leq x\gamma_r \sqrt{\mathbf{E}[v^2 \mathbf{1}_{\{\tilde{\nu} > r\}}]}.$$

where $\lim_{r \rightarrow \infty} \gamma_r \sqrt{\mathbf{E}[v^2 \mathbf{1}_{\{\tilde{\nu} > r\}}]} = 0$. Moreover, we also have the uniform convergence

$$\left| \mathbf{E} [(1 - x\sqrt{\gamma_r})^\nu; \tilde{\nu} > r] - \gamma_r \right| \rightarrow 0$$

in any compact set on \mathbb{R}_+ . We hence deduce that

$$\frac{1}{\gamma_r} \Phi_r(x) \rightarrow \phi(x).$$

uniformly in any compact set on \mathbb{R}_+ .

Now let us prove the convergence of $\frac{x_r}{\sqrt{\gamma_r}}$ by showing that any convergent subsequence converges towards the same limit. In fact, note that for any subsequence of $\{x_r\}$ such that as $k \rightarrow \infty$,

$$\frac{x_{r_k}}{\sqrt{\gamma_{r_k}}} \rightarrow x^* \in \mathbb{R}_+,$$

the uniform convergence (2.5) and the continuity of ϕ yield that

$$\lim_{k \rightarrow \infty} \frac{1}{\gamma_{r_k}} \Phi_{r_k} \left(\frac{x_{r_k}}{\sqrt{\gamma_{r_k}}} \right) = \phi(x^*).$$

By (1.12),

$$\phi(x^*) = 0.$$

So, $x^* = \sqrt{2/\sigma^2}$ and

$$x_r \sim \sqrt{\frac{2}{\sigma^2} \mathbf{P}(\tilde{\nu} > r)}.$$

3 Theorem 1.3: (1.4) in the jointly regularly varying case

In this section, we study the tail of M given in the equation

$$M \stackrel{d}{=} \max\{\tilde{\nu}, \max_{1 \leq k \leq \nu} M_k\},$$

where $X = (\tilde{\nu}, \nu)$ is independent of $M_k, k \geq 1$, and has multivariate regularly varying tail. For $x_r = \mathbf{P}(M > r)$ with $r > 0$, we first show that

$$x_r \asymp \frac{1}{r}, \text{ as } r \rightarrow \infty.$$

According to the assumptions of Theorem 1.3, we can write $V(r) = r^{-\alpha}\ell(r)$ for some $\alpha \in (1, 2)$ and ℓ slowly varying function, then

$$(3.1) \quad \mathbf{P}(\nu > r) = r^{-\alpha}\ell_1(r), \text{ and } \mathbf{P}(\tilde{\nu} > r) = r^{-\alpha}\ell_2(r)$$

where ℓ_1 and ℓ_2 are two slowly varying function at infinity such that $\ell_1(r) \sim c_1\ell(r)$ and $\ell_2(r) \sim c_2\ell(r)$ as $r \rightarrow \infty$.

We first show that $x_r = O(\frac{1}{r})$. In fact, by (1.12), one has

$$(3.2) \quad x_r = 1 - f(1 - x_r) + \mathbf{E}[(1 - x_r)^\nu; \tilde{\nu} > r] \leq 1 - f(1 - x_r) + \mathbf{P}(\tilde{\nu} > r).$$

Note that f is the generating function of ν , thus,

$$\begin{aligned} f(1 - x_r) &= \mathbf{E}[(1 - x_r)^\nu] \geq \mathbf{E}[(1 - x_r)^\nu; \nu \leq r] \\ &\geq \mathbf{E}[(1 - \nu x_r); \nu \leq r] \\ &= \mathbf{P}(\nu \leq r) - x_r \mathbf{E}[\nu; \nu \leq r]. \end{aligned}$$

Plugging it into (3.2) yields that

$$\begin{aligned} x_r &\leq 1 - \mathbf{P}(\nu \leq r) + x_r \mathbf{E}[\nu; \nu \leq r] + \mathbf{P}(\tilde{\nu} > r) \\ &= \mathbf{P}(\nu > r) + x_r \mathbf{E}[\nu; \nu \leq r] + \mathbf{P}(\tilde{\nu} > r). \end{aligned}$$

Recall that $1 = \mathbf{E}[\nu]$, so

$$x_r(1 - \mathbf{E}[\nu; \nu \leq r]) = x_r \mathbf{E}[\nu; \nu > r] \leq \mathbf{P}(\nu > r) + \mathbf{P}(\tilde{\nu} > r).$$

By (3.1), $\mathbf{P}(\tilde{\nu} > r) \sim \frac{c_2}{c_1}\mathbf{P}(\nu > r)$. Moreover, the tail of ν is regularly varying of index $\alpha \in (1, 2)$, then Karamata's theorem implies that $\mathbf{E}[\nu; \nu > r] \sim \frac{\alpha}{\alpha-1}\mathbf{P}(\nu > r)r$. Consequently, $x_r = O(\frac{1}{r})$.

Apparently, $0 \leq x_r r \leq C < \infty$ for some constant $C > 0$. This means the tightness of $\{rx_r\}_{r>0}$. Now we turn to the uniform convergence of Φ_r defined as follows:

$$(3.3) \quad \Phi_r(x) = f(1 - \frac{x}{r}) - (1 - \frac{x}{r}) - \mathbf{E}[(1 - \frac{x}{r})^\nu; \tilde{\nu} > r], \forall r > x \geq 0.$$

Clearly, $\Phi_r(0) \sim -c_2 r^{-\alpha} \ell(r)$.

On the one hand, by Bingham *et al.* [4, Theorem 8.1.6; P 333], as $\mathbf{P}(v > r) = r^{-\alpha} \ell_1(r)$, we get that for $x > 0$,

$$(3.4) \quad f\left(1 - \frac{x}{r}\right) - \left(1 - \frac{x}{r}\right) \sim \alpha \Gamma(-\alpha) \left(\frac{x}{r}\right)^\alpha \ell_1\left(\frac{r}{x}\right) \sim c_1 \alpha \Gamma(-\alpha) x^\alpha V(r), \quad r \rightarrow \infty,$$

On the other hand, we study the convergence of $\frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r]$ by use of the multivariate regularly varying tail of $X = (\tilde{v}, v)$. Note that

$$\begin{aligned} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] &= \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r \mid \|X\| > r] \\ &\leq \mathbf{E}[e^{-x \frac{v}{r}} \mathbf{1}_{\{\tilde{v} > r\}} \mid \|X\| > r] \end{aligned}$$

For the lower bound, note that for any $\varepsilon \in (0, 1)$ and $x > 0$, there exists $r(x, \varepsilon) > 0$ such that for all $r \geq r(x, \varepsilon)$, one has $(1 - \frac{x}{r}) \geq e^{-(1+\varepsilon)\frac{x}{r}}$. Therefore,

$$\frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] \geq \mathbf{E}[e^{-(1+\varepsilon)x \frac{v}{r}} \mathbf{1}_{\{\tilde{v} > r\}} \mid \|X\| > r].$$

The multivariate regularly tail of the random vector X implies the weak convergence of $\mathbf{P}((\frac{\tilde{v}}{r}, \frac{v}{r}) \in \cdot \mid \|X\| > r)$ towards $\mu_+(\cdot) := \mu(\cdot \cap \{y = (y_1, y_2) \in \mathbb{R}_+^2 : \|y\| > 1\})$. Moreover, for $A := (1, \infty) \times \mathbb{R}_+$, $\mu_+(\partial A) = 0$ by the assumption. As a consequence, for any $x > 0$,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\mathbf{P}(\tilde{v} > r)}{V(r)} &= \mu_+(A) = c_2 \\ \liminf_{r \rightarrow \infty} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] &\geq \int e^{-(1+\varepsilon)xy_2} \mathbf{1}_{\{y_1 > 1\}} \mu_+(dy) \\ \limsup_{r \rightarrow \infty} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] &\leq \int e^{-xy_2} \mathbf{1}_{\{y_1 > 1\}} \mu_+(dy) \end{aligned}$$

Letting $\varepsilon > 0$ yields that

$$(3.5) \quad \lim_{r \rightarrow \infty} \frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r] = \int e^{-xy_2} \mathbf{1}_{\{y_1 > 1\}} \mu_+(dy).$$

Note that $f(1 - \frac{x}{r}) - (1 - \frac{x}{r})$ is non-decreasing in x and $\frac{1}{V(r)} \mathbf{E}[(1 - \frac{x}{r})^v; \tilde{v} > r]$ is non-increasing in x . So, by Dini's theorem, both the convergences (3.4) and (3.5) are uniform on any compact in $[0, \infty)$. Going back to (3.3), one obtains that

$$\frac{1}{V(r)} \Phi_r(x) \xrightarrow{r \rightarrow \infty} \Phi(x) := c_1 \alpha \Gamma(-\alpha) x^\alpha - \int e^{-xy_1} \mathbf{1}_{\{y_2 > 1\}} \mu_+(dy),$$

uniformly on any compact in $[0, \infty)$. Note that by dominated convergence theorem, Φ is continuous for $x \geq 0$. Moreover, Φ is strict increasing on \mathbb{R}_+ . The zero $C_{\alpha, \mu} > 0$ of Φ exists and is unique because

$$\Phi(0) = -c_2 < 0 \text{ and } \lim_{x \rightarrow \infty} \Phi(x) = \infty.$$

The equation (1.12) means that $\Phi_r(rx_r) = 0$. So we conclude that $rx_r \rightarrow C_{\alpha, \mu}$.

4 Proof of Corollary 1.4

It is clear that when $\alpha > 2$, (1.11) follows from Theorem 1.2 and that when $\alpha \in (1, 2)$, (1.9) follows from Theorem 1.3. We only need to prove (1.10) for $\alpha = 2$.

First note that $\mathbf{P}(v \geq r) \sim cr^{-\alpha}$ and $\mathbf{P}(\tilde{v} \geq r) \sim \gamma(0)r^{-\alpha}$. By Theorem 1.5 in [6], one has for r sufficiently large,

$$p_r := \mathbf{P}(M \geq r) \asymp \frac{1}{r\sqrt{\log r}}.$$

For $r \geq e$ and $x \in [0, r\sqrt{\log r}]$, define

$$\Phi_r(x) = f\left(1 - \frac{x}{r\sqrt{\log r}}\right) - \left(1 - \frac{x}{r\sqrt{\log r}}\right) - \mathbf{E} \left[\left(1 - \frac{x}{r\sqrt{\log r}}\right)^v ; \tilde{v} \geq r \right].$$

Similarly, one has for any $x \geq 0$, as $r \rightarrow \infty$,

$$f\left(1 - \frac{x}{r\sqrt{\log r}}\right) - \left(1 - \frac{x}{r\sqrt{\log r}}\right) \sim C_2 \left(\frac{x}{r\sqrt{\log r}}\right)^2 \log\left(\frac{r\sqrt{\log r}}{x}\right) \sim C_2 \frac{x^2}{r^2}.$$

For any $\varepsilon \in (0, 1)$ and $r \gg 1$,

$$\mathbf{E} \left[e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] \leq \mathbf{E} \left[\left(1 - \frac{x}{r\sqrt{\log r}}\right)^v ; \tilde{v} \geq r \right] \leq \mathbf{P}(\tilde{v} \geq r)$$

Apparently $r^2 \mathbf{P}(\tilde{v} \geq r) \rightarrow c$ as $r \rightarrow \infty$. For the lower bound, we see that for any $x > 0$,

$$\begin{aligned} r^2 \mathbf{E} \left[1 - e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] &= \int_0^\infty e^{-u} r^2 \mathbf{P}(\tilde{v} \geq r, v \geq r \frac{u}{\varepsilon x}) du \\ &\leq \int_0^\infty e^{-u} du \left(r^2 \mathbf{P}(\tilde{v} \geq r) \right) \end{aligned}$$

As $\mathbf{P}(\tilde{v} \geq r) \sim cr^{-\alpha}$ and $\mathbf{P}(\tilde{v} \geq r, v \geq br) \sim \gamma(b)r^{-\alpha}$, by dominated convergence theorem,

$$\lim_{r \rightarrow \infty} r^2 \mathbf{E} \left[1 - e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] = \int_0^\infty \gamma\left(\frac{u}{\varepsilon x}\right) e^{-u} du$$

Here one can show that $\gamma(b) \downarrow 0$ as $b \uparrow \infty$. In fact, for any $b > 0$, we have

$$r^2 \mathbf{P}(\tilde{v} \geq r, v \geq br) \leq r^2 \mathbf{P}(v \geq br).$$

Taking limit shows that $\gamma(b) \leq cb^{-\alpha}$. Consequently,

$$\lim_{r \rightarrow \infty} r^2 \mathbf{E} \left[e^{-\varepsilon \frac{xv}{r}} ; \tilde{v} \geq r \right] = \int_0^\infty [\gamma(0) - \gamma\left(\frac{u}{\varepsilon x}\right)] e^{-u} du \geq \int_0^\infty [\gamma(0) - \frac{c\varepsilon^2 x^2}{u^2} \wedge \gamma(0)] e^{-u} du.$$

Note that by dominated convergence theorem,

$$\int_0^\infty \frac{c\varepsilon^2 x^2}{u^2} \wedge \gamma(0)e^{-u} du \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Therefore,

$$\begin{aligned} \gamma(0) + o_\varepsilon(1) &\leq \liminf_{r \rightarrow \infty} r^2 \mathbf{E} \left[\left(1 - \frac{x}{r\sqrt{\log r}} \right)^\nu ; \tilde{\nu} \geq r \right] \\ &\leq \limsup_{r \rightarrow \infty} r^2 \mathbf{E} \left[\left(1 - \frac{x}{r\sqrt{\log r}} \right)^\nu ; \tilde{\nu} \geq r \right] \leq \gamma(0) \end{aligned}$$

We end up with

$$r^2 \Phi_r(x) \rightarrow \phi(x) := C_2 x^2 - \gamma(0).$$

One can check the uniform convergence in any compact set on \mathbb{R}_+ by Dini's theorem. We hence conclude that $r\sqrt{\log r} p_r$ converges to $\sqrt{\frac{\gamma(0)}{C_2}}$.

References

- [1] Addario-Berry, L., Devroye, L. and Janson, S. (2013): Sub-Gaussian tail bounds for the width and height of conditioned Galton-Watson trees. *Ann. Probab.* **41**, 1072-1087.
- [2] Bertoin, J. (2011): On the maximal offspring in a critical branching process with infinite variance. *J. Appl. Probab.* **48**, 576-582.
- [3] Borovkov, K. A. and Vatutin, V. A. (1996): On distribution tails and expectations of maxima in critical branching processes. *J. Appl. Probab.* **33**, 614-622.
- [4] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987): *Regular Variation*. Cambridge University Press, Cambridge (UK).
- [5] Chen, X. and de Raphélis, L. (2019+) The most visited edges of randomly biased random walks on a supercritical Galton-Watson tree (I). *In preparation*.
- [6] Chen, D., de Raphélis, L. and Hu, Y. (2018): Favorite sites of randomly biased random walks on a supercritical Galton-Watson tree. *Stoc. Proc. Appl.* **128**, 1525-1557.
- [7] Geiger, J. and Kersting, G. (2000): The survival probability of a critical branching process in a random environment. *Theory Probab. Appl.* **45**, 517-525.
- [8] Li, Z.H. (2006): A limit theorem for discrete Galton-Watson branching processes with immigration. *Journal of Applied Probability.* **43**, 289-295.
- [9] Kortchemski, I. (2017): Sub-exponential tail bounds for conditioned stable Bienaymé-Galton-Watson trees. *Probability Theory and Related Fields.* **168**, 1-40.

- [10] Lindvall, T. (1976): On the maximum of a branching process. *Scandinavian Journal of Statistics*. **3**, 209-214.
- [11] Ma, C. (2009): A limit theorem of two-type Galton-Watson branching processes with immigration. *Statistics and probability letters*. **79**, 1710-1716.
- [12] Resnick, S.I. (1987): *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.
- [13] Resnick, S.I. (2009): *Heavy-Tail Phenomena: Probability and Statistical Modelling*. Springer, New York.