



# Curves with low Harbourne constants on Kummer and abelian surfaces

Xavier Roulleau

## ► To cite this version:

Xavier Roulleau. Curves with low Harbourne constants on Kummer and abelian surfaces. *Rendiconti del Circolo Matematico di Palermo*, 2018, 67 (3), pp.481-491. 10.1007/s12215-017-0328-4 . hal-02076429

**HAL Id: hal-02076429**

**<https://hal.science/hal-02076429>**

Submitted on 8 Jun 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CURVES WITH LOW HARBOURNE CONSTANTS ON KUMMER AND ABELIAN SURFACES

XAVIER ROULLEAU

ABSTRACT. We construct and study curves with low H-constants on abelian and K3 surfaces. Using the Kummer  $(16_6)$ -configurations on Jacobian surfaces and some  $(16_{10})$ -configurations of curves on  $(1, 3)$ -polarized Abelian surfaces, we obtain examples of configurations of curves of genus  $> 1$  on a generic Jacobian K3 surface with H-constants  $< -4$ .

## 1. INTRODUCTION

The Bounded Negativity Conjecture predicts that for a smooth complex projective surface  $X$  there exists a bound  $b_X$  such that for any reduced curve  $C$  on  $X$  one has

$$C^2 \geq b_X.$$

That conjecture holds in some cases, for instance if  $X$  is an abelian surface, but we do not know whether it remains true if one considers a blow-up of  $X$ . With that question in mind, the H-constants have been introduced in [1]. For a reduced (but not necessarily irreducible) curve  $C$  on a surface  $X$  and  $\mathcal{P} \subset X$  a finite non empty set of points, let  $\pi : \bar{X} \rightarrow X$  be the blowing-up of  $X$  at  $\mathcal{P}$  and let  $\bar{C}$  denotes the strict transform of  $C$  on  $\bar{X}$ . Let us define the number

$$H(C, \mathcal{P}) = \frac{\bar{C}^2}{|\mathcal{P}|},$$

where  $|\mathcal{P}|$  is the order of  $\mathcal{P}$ . We define the Harbourne constant of  $C$  (for short the H-constant) by the formula

$$H(C) = \inf_{\mathcal{P}} H(C, \mathcal{P}) \in \mathbb{R},$$

where  $\mathcal{P} \subset X$  varies among all finite non-empty subsets of  $X$  (note that there is a slight difference with the definition of Hadean constant of a curve given in [1, Remark 2.4], which definition exists only for singular curves; see Remark 4 for the details). Singular curves tend to have low H-constants. It is in general difficult to construct curves having low H-constants, especially if one requires the curve to be irreducible. The (global) Harbourne constant of the surface  $X$  is defined by

$$H_X = \inf_C H(C) \in \mathbb{R} \cup \{-\infty\}$$

---

2000 *Mathematics Subject Classification*. Primary: 14J28 .

where the infimum is taken among all reduced curves  $C \subset X$ . Harbourne constants and their variants are intensively studied (see e.g. [1], [11], [12], [14]); note that the finiteness of  $H_X$  implies the BNC conjecture. Using some elliptic curve configurations in the plane [15], it is known that

$$H_{\mathbb{P}^2} \leq -4,$$

and for any surface  $X$  one has  $H_X \leq H_{\mathbb{P}^2} \leq -4$  (see [14]). However, the curves  $(C_n)_{n \in \mathbb{N}}$  on  $X \neq \mathbb{P}^2$  with H-constant tending to  $-4$  used to prove that  $H_X \leq -4$  are not very explicit and they all satisfy  $H(C_n) > -4$ .

The H-constant is an invariant of the isogeny class of an abelian surface. Using the classical  $(16_6)$  configuration  $R_1$  of 16 genus 2 curves and 16 2-torsion points in a principally polarized abelian surface and a  $(16_{10})$  configuration of 16 smooth genus 4 curves and 16 2-torsion points on a  $(1, 3)$ -polarized abelian surface, plus the dynamic of the multiplication by  $n \in \mathbb{Z}$  map, we construct explicitly some curves with low H-constants on abelian surfaces:

**Theorem 1.** *Let  $A$  be a simple abelian surface. There exists a sequence of curves  $(R_n)_{n \in \mathbb{N}}$  in  $A$  such that  $R_n^2 \rightarrow \infty$  and  $H(R_n) = -4$ .*

*If  $A$  is the Jacobian of a smooth genus 2 curve, the curve  $R_n$  can be chosen either as the union of 16 smooth curves or as an irreducible singular curve.*

It is known that on two particular abelian surfaces with CM there exists a configuration  $C$  of elliptic curves with  $H(C) = -4$ . Moreover for any elliptic curve configuration  $C$  in an abelian surface  $A$ , one always has

$$H(C) \geq -4,$$

with equality if and only if the complement of the singularities of  $C$  is an open ball quotient surface (for these previous results see [14]). Thus elliptic curve configurations with  $H(C) = -4$  are rather special, in particular these configurations are rigid. Indeed to an algebraic family  $(A_t, C_t)_t$  of such surfaces  $A_t$ , each containing a configuration  $C_t$  of elliptic curves with H-constant equals to  $-4$ , such that  $C_t$  varies algebraically with  $A_t$ , one can associate a family of ball quotient surfaces. Since ball quotient surfaces are rigid, the family  $(A_t, C_t)_t$  is trivial and the pairs  $(A_t, C_t)$  are isomorphic.

We observe that for our new examples of curves with  $H(C) = -4$  there is no such links with ball quotient surfaces. Indeed the pairs  $(A, C)$  we give such that  $H(C) = -4$  have deformations.

We then consider the images of the curves  $R_n$  in the associated Kummer surface  $X$  and we obtain:

**Theorem 2.** *Let  $X$  be a Jacobian Kummer surface. For any  $n > 1$ , there are configurations  $C_n$  of curves of genus  $> 1$  such that  $H(C_n) = -4 \frac{n^4}{n^4 - 1} < -4$ .*

The H-constants of curves (and some related variants such as the  $s$ -tuple Harbourne constants) on K3 surfaces have been previously studied, by example in [8] and [12]. In [8], R. Laface and P. Pokora study transversal arrangements  $\mathcal{C}$  of rational curves on K3 surfaces and they give examples of

configurations  $\mathcal{C}$  with a low Harbourne constant. In their examples, one has  $H(\mathcal{C}) \geq -3.777$ , with the exception of two examples on the Schur quartic and the Fermat quartic surfaces, both reaching

$$H(\mathcal{C}) = -8.$$

In the last section, we then turn our attention to irreducible curves with low H-constants in abelian and Kummer surfaces, which are more difficult to obtain, some of which have been recently constructed in [16].

**Acknowledgments.** The author thanks T. Szemberg for sharing his observation that the H-constant of the Kummer configuration is  $-4$  and P. Pokora for a very useful correspondence and his many criticisms. The author moreover thanks the referee for pointing out an error in a previous version of this paper and numerous comments which allowed to improve the exposition of this note

## 2. SMOOTH HYPERELLIPTIC CURVES IN ABELIAN SURFACES AND H-CONSTANTS

**2.1. Preliminaries, Notations.** By [4], an abelian surface  $A$  contains a smooth hyperelliptic curve  $C_0$  of genus  $g$  if and only if it is a generic  $(1, g-1)$ -polarized abelian surface and  $g \in \{2, 3, 4, 5\}$ .

In this section, we study the configurations of curves obtained by translation of these hyperelliptic curves  $C_0$  (of genus 2, 3, 4 or 5) by 2-torsion points and by taking pull-backs by endomorphisms of  $A$ . In the present sub-section, we recall some facts on the computation of the H-constants and some notations.

Let  $C_1, \dots, C_t$  be smooth curves in a smooth surface  $X$  such that the singularities of  $C = \sum_j C_j$  are *ordinary* (i.e. resolved after one blow-up). Let  $\text{Sing}(C)$  be the singularity set of  $C$ ; we suppose that it is non-empty. Let  $f : \bar{X} \rightarrow X$  be the blow-up of  $X$  at  $\text{Sing}(C)$ . For each  $p$  in  $\text{Sing}(C)$ , let  $m_p$  be the multiplicity of  $C$  (we say that such a singularity  $p$  is a  $m_p$ -point) and let  $E_p$  be the exceptional divisor in  $\bar{X}$  above  $p$ . Let us recall the following notation:

$$H(C, \mathcal{P}) = \frac{\bar{C}^2}{|\mathcal{P}|},$$

where  $\bar{C}$  is the strict transform of a curve  $C$  in the blowing-up surface at  $\mathcal{P} \neq \emptyset$ . The following formula is well known:

**Lemma 3.** *Let  $s$  be the cardinal of  $\text{Sing}(C)$ . One has*

$$H(C, \text{Sing}(C)) = \frac{C^2 - \sum_{p \in \mathcal{P}} m_p^2}{s} = \frac{\sum_{j=1}^t C_j^2 - \sum_{p \in \mathcal{P}} m_p}{s}.$$

*Proof.* One can compute  $\bar{C}^2$  in two ways, indeed

$$\bar{C} = f^*C - \sum m_p E_p,$$

thus  $\bar{C}^2 = C^2 - \sum_{p \in \mathcal{P}} m_p^2$  (that formula is valid for any configurations). But  $\bar{C} = \sum_{i=1}^t \bar{C}_i = \sum_{i=1}^t (f^* C_i - \sum_{p \in C_i} E_p)$ , and since the singularities are ordinary, the curves  $\bar{C}_i$  are disjoint, thus

$$\bar{C}^2 = \sum_{i=1}^t \bar{C}_i^2 = \sum_{i=1}^t C_i^2 - \sum_{p \in \text{Sing}(C)} m_p,$$

where we just use the fact that  $\sum_{i=1}^t \sum_{p \in C_i} 1 = \sum_{p \in \text{Sing}(C)} m_p$ .  $\square$

Recall that we define the H-constant of a curve  $C$  by the formula

$$H(C) := \inf_{\mathcal{P}} H(C, \mathcal{P}) \in \mathbb{R},$$

where  $\mathcal{P} \subset X$  varies among all finite non-empty subsets of  $X$ .

*Remark 4.* a) If  $C$  is smooth one has  $H(C) = \min(-1, C^2 - 1)$ .  
b) In [1, Remark 2.4] the Hadean constant of a singular curve  $C$  on a surface  $X$  is defined by the formula

$$H_{ad}(C) := \min_{\mathcal{P} \subset \text{Sing}(C), \mathcal{P} \neq \emptyset} H(C, \mathcal{P}).$$

Let  $C$  be an arrangement of  $n \geq 2$  smooth curves intersecting transversally (with at least one intersection point). In [7] is defined and studied the quantity  $H(X, C) := H(C, \text{Sing}(C))$ . An advantage of our definition of H-constant is that it is defined for any curves. Moreover with our definition, it is immediate that the global H-constant of the surface  $X$  satisfies  $H(X) = \inf H(C)$ , where the infimum is taken over reduced curves  $C$  in  $X$ .

Let  $m \in \mathbb{N}^*$  and let  $C \hookrightarrow X$  be a singular curve having singularities of multiplicity  $m$  only (this will be the case for most of the curves in this paper). Let  $s$  be the order of  $\text{Sing}(C)$ .

**Lemma 5.** *One has  $H(C, \text{Sing}(C)) = \frac{C^2}{s} - m^2$ .*

*The H-constant of  $C$  is*

$$H(C) = \min(-1, C^2 - m^2, H(C, \text{Sing}(C))).$$

*Proof.* For integers  $0 \leq a \leq s$ ,  $b \geq 0$ ,  $c \geq 0$  such that  $a + b + c > 0$ , let  $\mathcal{P}_{a,b,c}$  be a set of  $a$   $m$ -points,  $b$  smooth points of  $C$  and  $c$  points in  $X \setminus C$ . Let

$$H_{a,b,c} = H(C, \mathcal{P}_{a,b,c}) = \frac{C^2 - am^2 - b}{a + b + c}.$$

The border cases are  $H_{1,0,0} = C^2 - m^2$ ,  $H_{0,1,0} = C^2 - 1$  and  $H_{0,0,1} = C^2$ . If  $a < \frac{c+C^2}{m^2-1}$  (case which occurs when  $c$  is large) the function  $b \rightarrow H_{a,b,c}$  is decreasing and converging to  $-1$  when  $b \rightarrow \infty$ .

If  $a \geq \frac{c+C^2}{m^2-1}$ , the function  $b \rightarrow H_{a,b,c}$  is increasing, thus if  $a \neq 0$ , one has

$$\inf_{b \geq 0} H_{a,b,c} = H_{a,0,c} = \frac{C^2 - am^2}{a + c},$$

(note that even if  $a < \frac{C^2}{m^2-1}$ , one still has  $\frac{C^2-am^2}{a+c} \geq -1$ ). If  $C^2 - am^2 > 0$ ,  $H_{a,0,c}$  is a decreasing function of  $c$ , with limit 0, otherwise this is an increasing function and the infimum is attained for  $c = 0$ , which gives  $\frac{C^2-am^2}{a}$  (if  $a = 0$ , one gets  $C^2$ ). Then taking the minimum over  $a$ , one obtains the result.  $\square$

Let us recall (see [6]) that for  $a, b, n, m \in \mathbb{N}^*$ , a  $(a_n, b_m)$ -*configuration* is the data of two sets  $A, B$  of order  $a$  and  $b$ , respectively, and a relation  $R \subset A \times B$ , such that  $\forall \alpha \in A, \#\{(\alpha, x) \in R\} = n$  and  $\forall \beta \in B, \#\{(y, \beta) \in R\} = m$ . One has  $an = bm = \#R$ . If  $a = b$  and  $n = m$ , it is called a  $(a_n)$ -*configuration*. If for  $\alpha \neq \alpha'$  in  $A$  the cardinality  $\lambda$  of  $\{(\alpha, x) \in R\} \cap \{(\alpha', x) \in R\}$  does not depend on  $\alpha \neq \alpha'$ , this is called a  $(a_n, b_m)$ -*design* and  $m(n-1) = \lambda(a-1)$ ;  $\lambda$  is called the *type of the design*.

**2.2. Construction of configuration from Genus 2 curves.** Let  $A$  be a principally polarized abelian surface such that the principal polarization  $C_0$  is a smooth genus 2 curve. One can choose an immersion such that  $0 \in A$  is a Weierstrass point of  $C_0$ . The configuration of the 16 translates

$$C_t = t + C_0, t \in A[2]$$

of  $C_0$  by the 2 torsion points of  $A$  is the famous  $(16_6)$  Kummer configuration: there are 6 curves through each point in  $A[2]$ , and each curve contains 6 points in  $A[2]$  (since  $C_t C_{t'} = 2$  for  $t \neq t'$  in  $A[2]$ , it is even a  $(16_6)$ -design of type 2).

Let now  $n > 0$  be an integer and let  $[n] : A \rightarrow A$  be the multiplication by  $n$  map on  $A$ . For  $t \in A[2]$ , let us define  $D_t = [n]^* C_t$ , in other words

$$D_t = \{x \mid nx \in C_t\} = \{x \mid nx + t \in C_0\}.$$

Since  $[n]$  is étale, the curve  $D_t$  is a smooth curve, thus it is irreducible since its components are the pull back of an ample divisor. By [9, Proposition 2.3.5], since  $C_t$  is symmetric (i.e.  $[-1]^* C_t = C_t$ ), one has  $D_t \sim n^2 C_t$  (in particular  $D_t^2 = 2n^4$ ). The curve

$$W_n = [n]^* \sum_{t \in A[2]} C_t = \sum_{t \in A[2]} D_t$$

has 16 irreducible components and  $16n^4$  ordinary singularities of multiplicity 6 (6-points), which are the torsion points  $A[2n] := \text{Ker } [2n]$ . Each curve  $D_t$  contains  $6n^4$  6-points; the configuration of curves  $D_t$  and singular points of  $W_n$  is a  $(16_{6n^4}, 16n^4_6)$ -configuration. Using Lemma 3, we get:

**Lemma 6.** *One has  $H(W_n, \text{Sing}(W_n)) = -4$ .*

The Harbourne constant  $H_A$  of a surface  $A$  is an invariant of the isogeny class of  $A$  (see [14]). Thus if  $A$  is generic, it is isogeneous to the Jacobian of a smooth genus 2 curve, and we thus obtain the following:

**Proposition 7.** *On a generic abelian surface  $A$ , one has:*

$$H_A \leq -4.$$

Note that when  $A$  is isogeneous to the product of 2 elliptic curves  $E, E'$  (thus non generic in our situation), the H-constant of  $A$  verifies that  $H_A \leq -2$ , and  $H_A \leq -3$  if  $E$  and  $E'$  are isogeneous (see [14]). Moreover, there are two examples of surfaces with CM for which  $H_A \leq -4$ .

*Remark 8.* i) Suppose that  $n$  is odd, then

$$D_t = \{x \mid n(x+t) \in C_0\} = D_0 + t.$$

Moreover, if  $u$  is a 2-torsion point one has  $2u = 0 \Leftrightarrow 2nu = 0$ , thus  $D_0$  and each curve  $D_t$  contains 6 points of 2-torsion.

ii) Suppose that  $n$  is even. Let  $u \in A[2]$  be a 2-torsion point. One has  $u \in D_t \Leftrightarrow nu + t \in C_0 \Leftrightarrow t \in C_0$ . Therefore the 6 curves  $D_t$  with  $t$  in  $A[2] \cap C_0$  contain  $A[2]$ , and the remaining curves do not contain any points from  $A[2]$ .

**2.3. Genus 3 curves.** Let  $A$  be an abelian surface containing a hyperelliptic genus 3 curve  $C_0$  such that 0 is a Weierstrass point. Then the 8 Weierstrass points of  $C_0$  are contained in the set of 2-torsion points of  $A$ . Let  $\mathcal{O}$  be the orbit of  $C_0$  under the action of  $A[2]$  by translation and let  $a$  be the cardinal of  $\mathcal{O}$ . The stabilizer  $S_t$  of  $C_0$  acts as a fix-point free automorphism group of  $C_0$ . Thus considering the possibilities for the genus of  $C_0/S_t$  it is either trivial or an involution, therefore  $a = 16$  or 8. By [5, Remark 1], the curve  $C_0$  is stable by translation by a 2-torsion point, therefore  $a = 8$ . Let  $m$  be the number of curves in  $\mathcal{O}$  through one point in  $A[2]$  (this is well defined because  $A[2]$  acts transitively on itself). The sets of 8 genus 3 curves and  $A[2]$  form a  $(8_8, 16_m)$ -configuration, thus  $m = 4$ . Moreover, since they are translates, two curves  $C, C' \in \mathcal{O}$  satisfy  $CC' = C^2 = 4$ , thus

$$C \sum_{C' \in \mathcal{O}, C' \neq C} C' = 7 \cdot 4.$$

If the singularities of the union of the curves in  $\mathcal{O}$  were only at the points in  $A[2]$  and ordinary, one would have

$$C \sum_{C' \in \mathcal{O}, C' \neq C} C' = 8 \cdot 3.$$

The configuration  $\mathcal{C} = \sum_{C \in \mathcal{O}} C$  contains therefore other singularities than the points in  $A[2]$  or the singularities are non ordinary. It seems less interesting from the point of view of H-constants. Observe that if the singularities at  $A[2]$  are ordinary, one has  $H(\mathcal{C}, A[2]) = -2$ . If there are other singularities, since the configuration is stable by translations by  $A[2]$ , there are at least 16 more singularities.

**2.4. Construction of configurations from Genus 4 curves.** Traynard in [17], almost one century later Barth, Nieto in [3], and Naruki in [10] constructed  $(16_{10})$  configurations of lines lying on a 3-dimensional family of quartic K3 surfaces  $X$  in  $\mathbb{P}^3$ : there exist two sets  $\mathcal{C}, \mathcal{C}'$  of 16 disjoint lines in  $X$  such that each line in  $\mathcal{C}$  meets exactly 10 lines in  $\mathcal{C}'$ , and vice versa.

By the famous results of Nikulin characterizing Kummer surfaces, there exists a double cover  $\pi : \tilde{A} \rightarrow X$  branched over  $\mathcal{C}$ . That cover contains 16  $(-1)$ -curve over  $\pi^{-1}\mathcal{C}$ . The contraction  $\mu : \tilde{A} \rightarrow A$  of these 16 exceptional divisors is an abelian surface and the image of these 16 curves is the set  $A[2]$  of two torsion points of  $A$ .

We denote by  $C_1, \dots, C_{16}$  the 16 smooth curves images by  $\mu_*\pi^*$  of the 16 disjoint lines in  $\mathcal{C}'$ . By [3, Section 6], the 16 curves  $C_1, \dots, C_{16}$  are translates of each other by the action by the group  $A[2]$  of 2-torsion points; the argument is that if  $C'_i$  is a translate of  $C_i$  by a 2-torsion point, then  $\pi_*\mu^*C'_i$  is a line in the quartic  $X$ , but a such a generic quartic has exactly 32 lines.

**Proposition 9.** *The curves  $C_1, \dots, C_{16}$  in  $A$  are smooth of genus 4. The 16 2-torsion points  $A[2]$  and these 16 curves form a  $(16_{10})$ -design of type 6: 10 curves though one point in  $A[2]$ , a curve contains 10 points in  $A[2]$  and two curves meet at 6 points in  $A[2]$ .*

*The H-constant of that configuration  $\sum C_i$  is  $H = -4$ .*

*Proof.* The 10 intersection points between the lines in  $\mathcal{C}$  and  $\mathcal{C}'$  are transverse, therefore by the Riemann-Hurwitz Theorem, the genus of the 16 irreducible components of  $\pi^*\mathcal{C}'$  is 4. The intersections of the 16 components in  $\mu_*\pi^*\mathcal{C}'$  are transverse (since  $\pi^*\mathcal{C}'$  is a union of disjoint curves) and that intersection holds over points in  $A[2]$  (which is the image of the exceptional divisors of  $\tilde{A}$ ).

Since the curves in  $\mathcal{C}$  and  $\mathcal{C}'$  form a  $(16_{10})$  configuration, the 16 curves  $C_1, \dots, C_{16}$  and the 2 torsion points in  $A$  have the described  $(16_{10})$  configuration.

Since the strict transform in  $\tilde{A}$  of the curves  $C_i \neq C_j$  are two disjoint curves, the 6 intersection points of  $C_i \neq C_j$  are 2-torsion points, the configuration is therefore a  $(16_{10})$ -design of type 6.

It is then immediate to compute the H-constant of  $\mathcal{C} = C_1 + \dots + C_{16}$ .  $\square$

*Remark 10.* Since the 16 curves are the orbit of a curve by the group  $A[2]$  of torsion points, one can change the notations and define  $C_t = C_0 + t$  for  $t \in A[2]$ , for a chosen curve  $C_0$  containing 0. As in sub-section 2.2, let us define  $D_t = [n]^*C_t$ ; this is a smooth curve. It is then immediate to check that the  $H$ -constant of the curve  $W_n = \sum D_t$  equals  $-4$ . We will use these configurations of curves in Section 3.

**2.5. Genus 5 curves.** By [4], a generic  $(1, 4)$ -polarized abelian surface contains a smooth genus 5 curve  $C$  which is hyperelliptic, the set of Weierstrass points in  $C$  is 12 2-torsion points, and  $C$  is stable by a sub-group of  $A[2]$  isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^2$ . Thus the orbit of  $C$  by the translations by elements of  $A[2]$  is the union of 4 genus 5 curves.

The intersection of two of these curves equals  $C^2 = 2g - 2 = 8$ . Since each of these two curves contains 12 points in  $A[2]$ , the intersections are transverse and are on 8 points in  $A[2]$ . The 4 curves and the 16 2-torsion



points form a  $(4_{12}, 16_3)$  configuration. The  $H$ -constant of that configuration is  $H = \frac{4 \cdot 8 - 16 \cdot 3}{16} = -1$ .

### 3. CONFIGURATIONS OF CURVES WITH LOW H-CONSTANT IN KUMMER SURFACES

In this Section, we study the images in the Kummer surface  $\text{Km}(A)$  of the various curve configurations studied in Section 2 in abelian surfaces  $A$ .

**3.1. The genus 2 case.** We keep the notations and hypothesis of subsection 2.2. In particular,  $A$  is the Jacobian of a genus 2 curve. Let  $\mu : \tilde{A} \rightarrow A$  be the blow-up of  $A$  at the 16 2-torsion points. We denote by  $\bar{D}$  the strict transform in  $\tilde{A}$  of a curve  $D \hookrightarrow A$ . Let  $\pi : \tilde{A} \rightarrow X$  be the quotient map by the automorphism  $[-1]$ . Since on  $A$  one has  $[-1]^*C_t = [-1]^*(t + C_0) = C_t$ , one obtains

$$[-1]^*D_t = D_t$$

and the map  $\bar{D}_t \rightarrow D'_t = \pi(\bar{D}_t)$  has degree 2, thus  $D_t'^2 = \frac{1}{2}(\bar{D}_t)^2$ .

**Proposition 11.** *Let be  $n > 1$ . The configuration  $\mathcal{D}$  of the 16 curves  $D'_t$  with  $t \in A[2]$  in the Kummer surface  $X$  has Harbourne constant*

$$H\left(\sum_{t \in A[2]} D'_t\right) = -4 \frac{n^4}{n^4 - 1}.$$

*Proof.* If  $n$  is even, a curve  $D_t$  contains 16 or 0 points of 2 torsion depending if  $t \in C_0$  or not (thus there are 10 curves without points of 2 torsion, and 6 with). If  $n$  is odd, each curve  $D_t$  contains 6 points of 2-torsion and then one has:

$$D_t'^2 = \frac{1}{2}(2n^4 - 6) = n^4 - 3.$$

If  $n$  is even, one has:

$$D_t'^2 = \frac{1}{2}(2n^4 - 16) = n^4 - 8 \text{ or } D_t'^2 = n^4,$$

according if  $t \in A[2]$  is in  $C_0$  or not. The configuration  $\mathcal{D}$  contains

$$\frac{1}{2}(16n^4 - 16) = 8(n^4 - 1)$$

6-points and no other singularities. If  $n$  is even, then the configuration has 10 curves with self-intersection  $n^4$  and 6 curves with self-intersection  $n^4 - 8$ . Thus if  $n$  is even one has

$$H(\mathcal{D}) = \frac{10n^4 + 6(n^4 - 8) - 8(n^4 - 1)6}{8(n^4 - 1)} = -4 \frac{n^4}{n^4 - 1} \sim -4,$$

which for  $n = 2$  gives  $H = -64/15 \simeq -4.2\bar{6}$ .

If  $n$  is odd, one has 16 curves with self-intersection  $n^4 - 3$ , and we get the same formula:

$$H(\mathcal{D}) = \frac{\sum D_t'^2 - 8(n^4 - 1)6}{8(n^4 - 1)} = \frac{16(n^4 - 3) - 8(n^4 - 1)6}{8(n^4 - 1)} = -4 \frac{n^4}{n^4 - 1}.$$

□

*Remark 12.* a) The H-constants of the various configurations are  $< -4$ .  
 b) For  $n = 1$ , the H-constant is  $-2$ .

**3.2. The genus 4 case.** Let us consider the configuration  $(16_{10})$  considered in sub-section 2.4 of 16 genus 4 curves  $C_t$ ,  $t \in A[2]$  in a generic  $(1, 3)$ -polarized abelian surface  $A$ . Let  $X = \text{Km}(A)$  be the Kummer surface associated to  $A$ . Let  $\mu : \tilde{A} \rightarrow A$  the blow-up at the points in  $A[2]$ , and  $\pi : \tilde{A} \rightarrow X$  be the quotient map. Let us consider as in Remark 10 the 16 curves  $D_t = [n]^*C_t$ ,  $t \in A[2]$  in  $A$ . Let be  $\bar{D}_t$  the strict transform in  $\tilde{A}$  of  $D_t$  and  $D'_t = \pi(\bar{D}_t)$ .

**Proposition 13.** *For  $n > 1$ , the configuration  $\mathcal{C} = \sum_{t \in A[2]} D'_t$  in the Kummer surface  $X$  has Harbourne constant*

$$H(\mathcal{C}) = -4 \frac{n^4}{n^4 - 1}.$$

*Proof.* The involution  $[-1] : A \rightarrow A$  fixes the set  $A[2]$  and stabilizes the configuration  $\mathcal{C} = \sum_{t \in A[2]} C_t$ , since a curve  $C_t$  in  $\mathcal{C}$  is determined by the 2-torsion points it contains,  $[-1]$  stabilizes each curve  $C_t$ ,  $t \in A[2]$ , and thus also one has  $[-1]^*D_t = D_t$ . Thus the restriction  $\bar{D}_t \rightarrow D'_t$  of  $\pi$  has degree 2. Numerically, one has  $D_t = n^2 C_t$  and  $C_t^2 = 6$ .

Since  $D_t$  as a set is  $\{x \in A \mid nx + t \in C_0\}$ , a point  $t' \in A[2]$  is in  $D_t$  if and only if  $nt' + t \in C_0$ . Thus if  $n$  is even, the curve  $D_t$  contains 16 or 0 points of 2 torsion depending if  $t \in C_0$  or not (thus there are 6 curves without points of 2 torsion, and 10 with), moreover one has:

$$D_t'^2 = \frac{1}{2}(\bar{D}_t)^2 = \frac{1}{2}(6n^4 - 16) = 3n^4 - 8 \text{ or } D_t'^2 = 3n^4,$$

according if  $t \in A[2]$  is in  $C_0$  or not. If  $n$  is odd, each curve  $D_t$  contains 10 points of 2-torsion and

$$D_t'^2 = \frac{1}{2}(6n^4 - 10) = 3n^4 - 5.$$

The configuration  $\mathcal{C} = \sum D'_t$  has  $\frac{1}{2}(16n^4 - 16) = 8(n^4 - 1)$  10-points and no other singularities. If  $n$  is even, then the configuration contains 6 curves with self-intersection  $3n^4$  and 10 curves with self-intersection  $3n^4 - 8$ , thus

$$H(\mathcal{C}) = \frac{6 \cdot 3n^4 + 10(3 \cdot n^4 - 8) - 8(n^4 - 1)10}{8(n^4 - 1)} = -4 \frac{n^4}{n^4 - 1} \sim -4.$$

If  $n$  is odd, one has 16 curves with self-intersection  $3n^4 - 5$ , and one gets the same formula. □

*Remark 14.* The multiplication by  $n$  map  $[n]$  on  $A$  induces a rational map  $[n] : X \dashrightarrow X$ . The configurations  $\sum D_t$  we are describing are the pull back by  $[n]$  of a configuration in  $X = \text{Km}(A)$  of 16 disjoint rational curves.

#### 4. IRREDUCIBLE CURVES WITH LOW H-CONSTANT IN ABELIAN AND KUMMER SURFACES.

Obtaining irreducible curves with low Harbourne constant is in general a difficult problem. Let  $k > 0$  be an integer. In [16], we prove that in a generic abelian surface polarized by  $M$  with  $M^2 = k(k+1)$  there exists a hyperelliptic curve  $\Gamma_k$  numerically equivalent to  $4M$  such that  $\Gamma_k$  has a unique singularity of multiplicity  $4k+2$ . Thus:

**Proposition 15.** *The H-constant of  $\Gamma_k$  is*

$$H(\Gamma_k) = \Gamma_k^2 - (4k+2)^2 = -4.$$

Let us study the case  $k = 1$  and define  $T_1 = \Gamma_1$ . This is a curve of geometric genus 2 in an abelian surface  $A$  with one 6-point singularity, which we can suppose in 0. In [13] such a curve  $T_1$  is constructed:  $A$  is the Jacobian of a genus 2 curve  $C_0$  and  $T_1$  is the image by the multiplication by 2 map of  $C_0$  in  $A = J(C_0)$ . The self-intersection of  $T_1$  is  $T_1^2 = 32$  and the singularity of  $T_1$  is a 6-point. Let  $n \in \mathbb{N}$  be odd. The curve  $T_n = [n]^*T_1 \sim n^2T_1$  has 6-points singularities at each points of  $A[n]$ , the set of  $n$ -torsion points of  $A$ . Let  $[n]$  be the multiplication by  $n$  map. The following diagram of curve configurations in  $A$  is commutative

$$\begin{array}{ccc} \sum_{t \in A[2]} D_t & \xrightarrow{[2]} & T_n \\ \downarrow [n] & & \downarrow [n] \\ \sum_{t \in A[2]} C_t & \xrightarrow{[2]} & T_1 \end{array}$$

Since the multiplication by 2 map  $[2]$  has degree 16 and the curves  $D_t$  are permuted by translations by elements of  $A[2]$ , the map  $D_t \xrightarrow{[2]} T_n$  is birational, thus  $T_n$  is irreducible. Its singularities are 6-points over each  $n$ -torsion points.

**Theorem 16.** *Let  $n \in \mathbb{N}$  be odd. The Harbourne constant of the irreducible curve  $T_n \hookrightarrow A$  is :*

$$H(T_n) = \frac{32n^4 - 36n^4}{n^4} = -4.$$

Let  $\bar{T}_n$  be the strict transform of  $T_n$  under the blowing-up map  $\tilde{A} \rightarrow A$  at the points from  $A[2]$ . Since  $n$  is odd, among points in  $A[2]$ ,  $T_n$  contains only 0, thus  $\bar{T}_n^2 = 32n^4 - 36$ . Moreover  $[-1]^*T_n = T_n$  (it can be seen using the map  $C^{(2)} \rightarrow A$  that  $[-1]^*T_1 = T_1$ , and therefore  $[-1]^*T_n = T_n$ ). The image of  $\bar{T}_n$  on the Kummer surface  $X = \tilde{A}/[-1]$  is an irreducible curve  $W_n$  with  $\frac{1}{2}(n^4 - 1)$  6-points if  $n$  is odd. The map

$$\bar{T}_n \rightarrow W_n$$

has degree 2 and

$$W_n^2 = 16n^4 - 18,$$

thus

**Proposition 17.** *Let  $n \in \mathbb{N}$  be odd. The  $H$ -constant of the irreducible curve  $W_n$  in the Kummer surface  $X$  is*

$$H(W_n) = \frac{-4n^4}{n^4 - 1}.$$

*In particular, for  $n = 3$  one has  $H(W_3) = -\frac{81}{20}$ .*

## 5. SOME REMARKS ON $H$ -CONSTANTS OF ABELIAN SURFACES

Let  $\phi : C \hookrightarrow A$  be an irreducible curve of geometric genus  $g$  in an abelian surface  $A$ . Let  $m_p = m_p(C)$  be the multiplicity of  $C$  at a point  $p$ . One has

$$C^2 = 2g - 2 + 2\gamma,$$

where

$$\gamma \geq \sum_p \frac{1}{2} m_p(m_p - 1),$$

with equality if all singularities are ordinary. Thus

$$H(C, \text{Sing}(C)) = \frac{1}{\#\text{Sing}(C)} (C^2 - \sum_{p \in \text{Sing}(C)} m_p^2) \geq \frac{1}{\#\text{Sing}(C)} (2g - 2 - \sum_{p \in \text{Sing}(C)} m_p)$$

with equality if all singularities are ordinary. From the previous construction in Section 2, one can ask the following

**Problem 18.** Does there exist an abelian surface containing a curve  $C$  of geometric genus  $g$  such that

$$-4 > \frac{1}{\#\text{Sing}(C)} (2g - 2 - \sum_{p \in \text{Sing}(C)} m_p) ?$$

We were not able to find any example of such a curve. If the answer is no, it would imply that the bounded negativity conjecture holds true for surfaces which are blow-ups of abelian surfaces.

## REFERENCES

- [1] Bauer T., Di Rocco S., Harbourne B., Huizenga J., Lundman A., Pokora P., Szemberg T., Bounded negativity and arrangement of lines, *Int. Math. Res. Notices.* (2015) 2015 (19): 9456–9471
- [2] Bauer T., Harbourne B., Knutsen A.L.; Küronya A., Müller-Stach S., Roulleau X., Szemberg T., Negative curves on algebraic surfaces, *Duke Math. J.* 162 (2013), 1877–1894.
- [3] Barth W., Nieto I., abelian surfaces of type  $(1, 3)$  and quartic surfaces with 16 skew lines, *J. Algebraic Geom.* 3 (1994), no. 2, 173–222.
- [4] Borówka P., Ortega A., Hyperelliptic curves on  $(1, 4)$  polarised abelian surfaces, *arXiv:1708.01270*
- [5] Borówka P., Sankaran G., Hyperelliptic genus 4 curves on abelian surfaces, *Proc. Amer. Math. Soc.* 145 (2017), 5023–5034
- [6] Dolgachev I., Abstract configurations in Algebraic Geometry, *The Fano Conference*, 423–462, Univ. Torino, Turin, 2004.
- [7] Laface R., Pokora P., Local negativity of surfaces with non-negative Kodaira dimension and transversal configurations of curves, *arXiv:1602.05418*

- [8] Laface R., Pokora P., On the local negativity of surfaces with numerically trivial canonical class, to appear in *Rend. Lin. Mat. e App.*
- [9] Birkenhake C., Lange H., *Complex abelian varieties*, Second edition. *Grund. der Math. Wiss.*, 302. Springer-Verlag, Berlin, 2004. xii+635 pp.
- [10] Naruki I., On smooth quartic embedding of Kummer surfaces, *Proc. Japan Acad.*, 67, Ser. A (1991), pp. 223–225
- [11] Pokora P., Tutaj-Gasińska H., Harbourne constants and conic configurations on the projective plane, *Math. Nachr.* 289 (2016), no. 7, 888–894.
- [12] Pokora P., Harbourne constants and arrangements of lines on smooth hypersurfaces in  $\mathbb{P}_{\mathbb{C}}^3$ , *Taiwanese J. Math.* 20 (2016), no. 1, 25–31
- [13] Polizzi F., Rito C., Roulleau X., A pair of rigid surfaces with  $p_g = q = 2$  and  $K^2 = 8$  whose universal cover is not the bidisk, preprint
- [14] Roulleau X., Bounded negativity, Miyaoka-Sakai inequality and elliptic curve configurations, *Int. Math. Res. Notices* (2017) 2017 (8): 2480–2496
- [15] Roulleau X., Urzúa G., Chern slopes of simply connected complex surfaces of general type are dense in  $[2, 3]$ , *Annals of Math.* 182 (2015), 287–306
- [16] Roulleau X., Sarti A., Construction of Nikulin configurations on some Kummer surfaces and applications, *arXiv* 1711.05968
- [17] Traynard E., Sur les fonctions thêta de deux variables et les surfaces Hyperelliptiques, *Ann. Sci. École Norm. Sup.* (3) 24 (1907), 77–177.

Xavier Roulleau,  
 Aix-Marseille Université, CNRS, Centrale Marseille,  
 I2M UMR 7373,  
 13453 Marseille, France  
 Xavier.Roulleau@univ-amu.fr

URL: <https://old.i2m.univ-amu.fr/~roulleau.x/>