



# COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH RIPPED DENSITY

Raphaël Danchin, Piotr Boguslaw Mucha

## ► To cite this version:

Raphaël Danchin, Piotr Boguslaw Mucha. COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH RIPPED DENSITY. Communications on Pure and Applied Mathematics, 2023, 76 (11), pp.3437-3492. 10.1002/cpa.22116 . hal-02075922v2

**HAL Id: hal-02075922**

**<https://hal.science/hal-02075922v2>**

Submitted on 2 Mar 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH RIPPED DENSITY

RAPHAËL DANCHIN AND PIOTR BOGUSŁAW MUCHA

ABSTRACT. We are concerned with the Cauchy problem for the two-dimensional compressible Navier-Stokes equations supplemented with general  $H^1$  initial velocity and bounded initial density not necessarily strictly positive: it may be the characteristic function of any set, for instance.

In the perfect gas case, we establish global-in-time existence and uniqueness, provided the volume (bulk) viscosity coefficient is large enough. For more general pressure laws (like e.g.  $P = \rho^\gamma$  with  $\gamma > 1$ ), we still get global existence, but uniqueness remains an open question.

As a by-product of our results, we give a rigorous justification of the convergence to the inhomogeneous incompressible Navier-Stokes equations when the bulk viscosity tends to infinity.

In the three-dimensional case, similar results are proved for short time without restriction on the viscosity, and for large time if the initial velocity field is small enough.

## INTRODUCTION

Systems of PDEs coming from classical physics are sources of never-ending challenges for mathematicians. This is the case of Euler and Navier-Stokes systems that are at the basis of fluid mechanics. In that field, a number of new and sometimes unexpected results flourished in the last decade. One can for instance mention the works by C. De Lellis and L. Székelyhidi [16, 17] where a technique based on convex integration is used to construct infinitely many finite energy solutions to the classical incompressible Euler equations. Since their energy may be any nonnegative smooth function of the time variable, those solutions are not physically relevant. For that reason, they are often named wild (or even spam) solutions. Convex integration turned out to be robust enough so as to be adapted to other PDEs for inviscid flows (e.g. the compressible Euler system [4]) and even to models with diffusion like the classical Navier-Stokes system [3].

In accordance with Laplace determinism principle (or, in mathematics, with Hadamard's definition of well-posedness), it is natural to look for conditions on the initial data ensuring uniqueness, global existence and stability by perturbations. However, even for rather simple physical systems, the full answer is not often known. In this regards, one may mention the celebrated Millenium Problem dedicated to the global regularity of solutions to the incompressible Navier-Stokes equations in the three dimensional case<sup>1</sup>. So far, there is no consensus in the community on whether the solutions are unique or not, regular or not, for all time. Positive answer is known for the two space dimensional case, after the work by O.A. Ladyzhenskaya [26] in 1958, that states that weak solutions to the Navier-Stokes equation are unique, stable, and smooth if the data are smooth. Up to some small variations (like e.g. viscous flows with variable density or coupling with a transport equation through a buoyancy term), that case is essentially the only one in classical mathematical fluid mechanics where a complete well-posedness theory is available.

---

<sup>1</sup>See <http://www.claymath.org/millennium-problems>.

In the present paper, we would like to address the global well-posedness issue for the compressible Navier-Stokes system in the barotropic regime, supplemented with general arbitrary large initial data with merely bounded (and nonnegative) density : we have in mind a “ripped” initial density, that is a function that may have nontrivial regions of vacuum, without any extra regularity assumption.

The main achievements here are as follows:

- in the two-dimensional case, for any nonnegative initial density, just bounded, and any initial velocity  $v_0$  in  $H^1$ , if the bulk viscosity  $\nu$  is large enough and  $\|\operatorname{div} v_0\|_{L^2} = \mathcal{O}(\nu^{-1/2})$ , then there exists at least one global solution with uniformly bounded density;
- in the case of a perfect gas, namely if the pressure is positively proportional to the density, then the above solutions are unique (here the bulk viscosity need not be large and  $\operatorname{div} v_0$  need not be small);
- we justify rigorously the (singular) limit of the compressible Navier-Stokes system to the inhomogeneous incompressible one as  $\nu$  tends to infinity (regardless of the fact that the density may vanish and have large variations);
- the above results remain true in the three-dimensional case, provided a suitable smallness condition is prescribed on the whole initial velocity  $v_0$ .

As our goal is to consider as general densities as possible, we do not strive for optimal regularity hypotheses on the initial velocity, and take it in  $H^1$ , for simplicity. Although the density of the solution is only a bounded function, the corresponding velocity has relatively high regularity. However, we do not reach the  $L_{1,loc}(\mathbb{R}_+; C^{0,1})$  regularity so that the classical methods for showing uniqueness fail. Nevertheless, we establish uniqueness in the regime of a perfect gas, and obtain some qualitative results on the regions of vacuum: their growth or decrease is controlled in terms of the data and of the time, they are stable and vacuum cannot appear if the initial density is positive (or cannot disappear if the initial density vanishes on some set with positive measure). In the case where the initial density is the characteristic function of a set, our results provide us with some information on the regularity of the boundary of the support of the density for positive times, even though the flow is not quite Lipschitz.

In addition, our solutions are physical: total mass and momentum are conserved, and the energy balance is fulfilled for all time. As a consequence, in the case of zero energy initial data (which does not mean that the initial velocity is zero since it may be anything in the regions of vacuum), the only possible solution has null velocity instantaneously.

## 1. THE RESULTS

We are concerned with the following barotropic compressible Navier-Stokes equations in the unit torus  $\mathbb{T}^d$  with  $d = 2, 3$ :

$$(1.1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) - \mu \Delta v - (\lambda + \mu) \nabla \operatorname{div} v + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d. \end{cases}$$

The pressure  $P$  is a given function of the density. The real numbers  $\lambda$  and  $\mu$  designate the bulk and shear viscosity coefficients, respectively, and are assumed to satisfy

$$(1.2) \quad \mu > 0 \quad \text{and} \quad \nu := \lambda + 2\mu > 0.$$

The system is supplemented with the initial data

$$(1.3) \quad v|_{t=0} = v_0, \quad \rho|_{t=0} = \rho_0.$$

It is obvious that the total mass and momentum of smooth enough solutions of (1.1) are conserved through the evolution, namely, for all  $t \geq 0$ ,

$$(1.4) \quad \int_{\mathbb{T}^d} \rho(t, x) dx = \int_{\mathbb{T}^d} \rho_0(x) \quad \text{and} \quad \int_{\mathbb{T}^d} (\rho v)(t, x) dx = \int_{\mathbb{T}^d} (\rho_0 v_0)(x) dx.$$

For expository purposes, we shall always assume that<sup>2</sup>

$$(1.5) \quad \int_{\mathbb{T}^d} \rho_0(x) dx = 1 \quad \text{and} \quad \int_{\mathbb{T}^d} (\rho_0 v_0)(x) dx = 0.$$

Next, if we denote by  $e$  the potential energy of the fluid defined, up to an affine function, by the relation  $\rho e'' = P'$ , and introduce the total energy

$$E(t) := \int_{\mathbb{T}^d} \left( \frac{1}{2} \rho(t, x) |v(t, x)|^2 + e(\rho(t, x)) \right) dx,$$

then (still for smooth enough solutions) the following energy balance holds true:

$$(1.6) \quad E(t) + \int_0^t (\mu \|\nabla \mathcal{P} v(\tau)\|_2^2 + \nu \|\operatorname{div} v(\tau)\|_2^2) d\tau = E_0 := E(0),$$

where  $\mathcal{P}$  denotes the  $L_2$ -projector onto the set of solenoidal vector-fields and  $\|\cdot\|_p$ , the norm in  $L_p(\mathbb{T}^d)$ .

Since the pioneering works by P.-L. Lions in [27] and E. Feireisl in [19] (see also the paper by D. Bresch and P.-E. Jabin [2] that uses recent achievements of the transport theory), it is well understood that in the case of an isentropic pressure law  $P(\rho) = \rho^\gamma$  with  $\gamma > d/2$ , any finite energy initial data generates a global-in-time weak solution to (1.1) satisfying

$$(1.7) \quad E(t) + \int_0^t (\mu \|\nabla \mathcal{P} v(\tau)\|_2^2 + \nu \|\operatorname{div} v(\tau)\|_2^2) d\tau \leq E_0 \quad \text{for all } t \geq 0.$$

However, even in the two-dimensional case, it is not clear that those solutions respect the energy balance (1.6) (just inequality is known), and the regularity and uniqueness issues are widely open. From the viewpoint of the well-posedness theory, those weak solutions are relevant inasmuch as they satisfy the so-called *weak-strong uniqueness* principle : for smooth data, they coincide with the corresponding smooth solution as long as it exists (see [20, 21]).

Regarding the well-posedness issue, there is a number of results in the case of smooth density *bounded away from zero* (some of them like [32] being obtained much before the construction of weak solutions). The general rule is that the solutions are known to exist for small time if the data are large (see e.g. [8, 24, 30, 32]) and for all time if the data are small perturbations of a linearly stable constant state (see [7, 28, 29]). It has been observed by Y. Cho, H.J. Choe and H. Kim [5] that positivity of density may be somewhat relaxed if a suitable compatibility condition involving the initial velocity and high regularity of the density are guaranteed. Let us finally mention that for viscosity coefficients that depend on the density in a very specific way, one can achieve global existence of strong solutions in dimension two, even for large data, if  $\gamma > 3$  (see [33]).

At the end let us mention the work by D. Hoff in [22] devoted to the construction of “intermediate” solutions in between the aforementioned weak solutions and the more regular ones, that may have discontinuous density along some curve ( $d = 2$ ) or surface ( $d = 3$ ).

---

<sup>2</sup>This is not restrictive, as one can rescale the density function and use the Galilean invariance of the system to have those two conditions satisfied.

We here want to provide the reader with a complete global-in-time existence theory in the case where the initial velocity is in  $H^1(\mathbb{T}^d)$  and the initial density is just bounded. In the two dimensional case, we shall achieve our goal provided that  $\nu^{1/2}\|\operatorname{div} v_0\|_2 \leq K$  for some given  $K > 0$  and that  $\nu$  is large enough (the assumption on  $\operatorname{div} v_0$  comes up naturally when defining a suitable energy functional that controls the  $H^1$  regularity). A remarkable feature of our result is that, even though the density is rough and need not be positive, one can exhibit some parabolic gain of regularity for the velocity, which entails that both  $\operatorname{div} v$  and  $\operatorname{curl} v$  are almost in  $L_{2,loc}(\mathbb{R}_+; L_\infty)$ . Although this does not quite imply that the full gradient of  $v$  is in  $L_{1,loc}(\mathbb{R}_+; L_\infty)$ , we will get uniqueness in the case where  $P(\rho) = \rho$ .

Let us first state our global existence result in the two-dimensional case.

**Theorem 1.1.** *Assume that the pressure law is  $P(\rho) = a\rho^\gamma$  for some  $a > 0$  and  $\gamma \geq 1$ . Fix some  $K > 0$ , and consider any vector field  $v_0$  in  $H^1(\mathbb{T}^2)$  satisfying  $\|\operatorname{div} v_0\|_2 \leq K\nu^{-1/2}$  and nonnegative bounded function  $\rho_0$  fulfilling (1.5).*

*There exists a positive number  $\nu_0$  depending only on  $K, \gamma, \mu, E_0, \|\nabla v_0\|_2$  and  $\|\rho_0\|_\infty$  such that if  $\nu \geq \nu_0$  then System (1.1) admits a global-in-time solution  $(\rho, v)$  fulfilling the conservations laws (1.4), the energy balance (1.6),*

$$\rho \in L_\infty(\mathbb{R}_+ \times \mathbb{T}^2) \cap \mathcal{C}(\mathbb{R}_+; L_p(\mathbb{T}^2)), \quad p < \infty \quad \text{and} \quad \sqrt{\rho} v \in \mathcal{C}(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$

*In addition, we have, denoting  $\dot{v} := v_t + v \cdot \nabla v$  and  $G := \nu \operatorname{div} v - P$ ,*

$$(1.8) \quad \begin{aligned} v &\in L_\infty(\mathbb{R}_+; H^1(\mathbb{T}^2)), \quad (\nabla^2 \mathcal{P}v, \nabla G, \sqrt{\rho} \dot{v}) \in L_2(\mathbb{R}_+ \times \mathbb{T}^2), \\ \sqrt{\rho t} \dot{v} &\in L_{\infty,loc}(\mathbb{R}_+; L_2(\mathbb{T}^2)), \quad \sqrt{t} \nabla \dot{v} \in L_{2,loc}(\mathbb{R}_+; L_2(\mathbb{T}^2)), \end{aligned}$$

*and both  $\operatorname{div} v$  and  $\operatorname{curl} v$  are in  $L_{r,loc}(\mathbb{R}_+; L_\infty(\mathbb{T}^2))$  for all  $r < 2$ .*

**Remark 1.1.** *For simplicity, we focussed on the physically relevant case where the pressure function  $P$  is given by  $P(\rho) = a\rho^\gamma$  for some  $\gamma \geq 1$  and  $a > 0$ . However, the above theorem remains true whenever:*

$$(1.9) \quad P \text{ is a } C^1 \text{ nonnegative function on } \mathbb{R}_+ \text{ such that } \rho \mapsto \rho^{-1}P(\rho) \text{ is nondecreasing.}$$

In the case of a linear pressure law, our existence result is supplemented with uniqueness.

**Theorem 1.2.** *Assume  $P(\rho) = a\rho$  for some  $a > 0$ . Then, for any  $T > 0$ , any nonnegative  $\rho_0$  in  $L_\infty(\mathbb{T}^2)$  and  $v_0$  in  $H^1(\mathbb{T}^2)$ , and any viscosity coefficients  $(\lambda, \mu)$ , there exists at most one solution to System (1.1) supplemented with data  $(\rho_0, v_0)$  on  $[0, T] \times \mathbb{T}^2$ , with the regularity given in Theorem 1.1 (restricted to interval  $[0, T]$ ).*

Since the norms of the solution constructed in Theorem 1.1 may be bounded uniformly with respect to  $\nu$ , one gets almost for free the all-time convergence when  $\nu$  tends to  $+\infty$  to the following inhomogeneous incompressible Navier-Stokes equations:

$$(1.10) \quad \begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^2, \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) - \mu \Delta v + \nabla \Pi = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^2, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^2. \end{cases}$$

Let us give the statement in the case of a fixed initial data (for simplicity):

**Theorem 1.3.** *Fix some data  $(\rho_0, v_0)$  in  $L_\infty(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$  satisfying  $\operatorname{div} v_0 = 0$  and  $\rho_0 \geq 0$ , and denote by  $(\rho^\nu, v^\nu)$  the corresponding global solution of (1.1) of Theorem 1.1 for  $\nu \geq \nu_0$ .*

Then, for  $\nu$  going to  $\infty$ , the whole family  $(\rho^\nu, v^\nu)$  converges to the unique global solution of System (1.10) with initial data  $(\rho_0, v_0)$  given by Theorem 2.1 of [12], and we have

$$(1.11) \quad \operatorname{div} v^\nu = \mathcal{O}(\nu^{-1/2}) \quad \text{in } L_2(\mathbb{R}_+ \times \mathbb{T}^2) \cap L_\infty(\mathbb{R}_+; L_2(\mathbb{T}^2)),$$

and even  $\operatorname{div} v^\nu = \mathcal{O}(\nu^{\varepsilon-1})$  in  $L_{2-\varepsilon, \text{loc}}(0, T; L_\infty(\mathbb{T}^2))$  for all  $\varepsilon \in (0, 1)$  and  $T > 0$ .

**Remark 1.2.** To the best of our knowledge, Theorem 1.3 is the first example of a global-in-time result of convergence from (1.1) to (1.10) in the truly inhomogeneous framework (see our recent work in [13] for an example of almost global convergence).

**Remark 1.3.** The above results of existence, uniqueness and convergence are valid in  $\mathbb{T}^3$  either locally in time for large data, or globally under a suitable scaling invariant smallness condition on the velocity (no smallness is required for the density). The reader is referred to Appendix C for more details.

Let us report on the main ideas leading to our results in dimension two. Assuming that we are given a solution  $(\rho, v)$  to (1.1), the first step is to establish global-in-time a priori estimates for the  $H^1$  norm of  $v$  in terms of the data, of the parameters of the system and of a (given) upper bound of the density. The overall strategy has some similarities with our recent work [12] dedicated to System (1.10). However, the compressible situation is more complex (as the reader will judge by himself in the next section) since one cannot expect  $\nabla P$  to be in any Lebesgue space. For that reason, we shall consider the *viscous effective flux*  $G$  defined by

$$(1.12) \quad G := \nu \operatorname{div} v - P \quad \text{with } \nu := \lambda + 2\mu$$

since it has better regularity than  $\operatorname{div} v$  or  $P$  taken separately, as observed before by D. Hoff [22] and P.-L. Lions [27] when constructing intermediate or weak solutions. Rewriting the momentum equation in terms of  $G$  and  $\mathcal{P}v$  (the divergence-free part of  $v$ ) will spare us making integrability assumptions on  $\nabla \rho$ , in contrast with our recent work in [13].

The second key ingredient of that step is the following logarithmic interpolation inequality

$$(1.13) \quad \left( \int_{\mathbb{T}^2} \rho |v|^4 dx \right)^{\frac{1}{2}} \leq C \|\sqrt{\rho} v\|_2 \|\nabla v\|_2 \log^{\frac{1}{2}} \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right) \quad \text{with } \tilde{\rho} := \rho - 1,$$

that has been discovered by B. Desjardins [14] and is an appropriate substitute of the well-known Ladyzhenskaya inequality

$$\|v\|_4^2 \leq C \|v\|_2 \|\nabla v\|_2$$

since bounds are available on  $\|\sqrt{\rho} v\|_2$  (through (1.6)), but not on  $\|v\|_2$ .

Then, the main idea is to introduce a suitable modified energy functional that contains informations on the  $H^1$  norm of  $v$ , and may be bounded uniformly on  $\mathbb{R}_+$ . Our definition enables us, after tracking carefully the dependency of the estimates with respect to the viscosity coefficients, to exhibit global-in-time bounds depending only on the data and on  $\rho^* := \|\rho\|_\infty$ , if  $\nu$  is large enough (results in [14] were local).

The goal of the second step is to bound  $\rho^*$  in terms of the data. As in [14], we shall rather consider the following quantity

$$F := \log \rho - \nu^{-1} (-\Delta)^{-1} \operatorname{div} (\rho v)$$

that may be seen as an approximate damped mode associated to (1.1). The new achievement here is that, by combining with the first step and a bootstrap argument, one gets a *global-in-time* control on  $\rho^*$  in terms of the data only, provided that  $\nu$  is large enough.

Step 3 aims at proving that  $\operatorname{div} v$  and  $\operatorname{curl} v$  are in  $L_{r,loc}(\mathbb{R}_+; L_\infty)$  for some  $r > 1$ . To achieve it, the general idea is to use time weighted estimates to glean some regularity on  $v_t$ , then to transfer time regularity to space regularity thanks to elliptic estimates and functional embeddings. However, in contrast with the incompressible case studied in [12], it is no longer possible to discard the pressure term by means of the divergence free property, and it is actually more appropriate to work with the *convective derivative*  $\dot{v} := v_t + v \cdot \nabla v$ . In the end, we shall get bounds on  $\sqrt{\rho t} \dot{v}$  in  $L_{\infty,loc}(\mathbb{R}_+; L_2)$  and  $\sqrt{t} \nabla \dot{v}$  in  $L_{2,loc}(\mathbb{R}_+; L_2)$ , from which we will eventually bound  $\operatorname{div} v$  and  $\operatorname{curl} v$  in  $L_{r,loc}(\mathbb{R}_+; L_\infty)$ .

Steps 1 to 3 were formal a priori estimates for smooth solutions. To complete the proof of existence, we mollify the initial density so as to make it strictly positive and regular. Then, one can resort to classical results to construct a local-in-time smooth solution corresponding to those data. The difficulty is to establish that, indeed, the control of norms that has been obtained so far allows to extend the solution for all time. Once it has been done, the uniform bounds given by steps 1 to 3 allow to pass to the limit and to complete the proof of the global existence. In fact, since compared to weak solutions theory, more regularity is available on the velocity, passing to the limit is much more direct than in [19] or [27]. Furthermore, as the bounds from steps 1 to 3 have some uniformity with respect to  $\nu$ , similar arguments allow to justify the convergence of (1.1) to (1.10), whence Theorem 1.3.

Since steps 1 to 3 just give that  $\operatorname{div} v$  and  $\operatorname{curl} v$  are in  $L_{r,loc}(\mathbb{R}_+; L_\infty)$  for some  $r > 1$ , we miss by a little the property that  $\nabla v$  is in  $L_{1,loc}(\mathbb{R}_+; L_\infty)$  and  $v$  need not have a Lipschitz flow. Therefore, in contrast with what has been done for (1.10) in [12] or for (1.1) in [10], it is not clear whether recasting the compressible Navier-Stokes equations in Lagrangian coordinates may help to prove uniqueness. However, we know from the previous steps that  $\nabla \mathcal{P}v$  and  $G$  are in  $L_{2,loc}(\mathbb{R}_+; H^1)$ , whence  $\nabla v$  is in  $L_{2,loc}(\mathbb{R}_+; BMO)$ . In the particular case of a *linear* pressure law, this turns out to be enough to control the difference of two solutions in  $L_\infty(0, T; \dot{H}^{-1})$  for the density and  $L_2(0, T; L_2)$  for the velocity. The proof has some similarities with that of D. Hoff in [23] but does not require Lagrangian coordinates. In fact, we overcome that  $\nabla v \notin L_{1,loc}(\mathbb{R}_+; L_\infty)$  by combining the information that  $\nabla v \in L_{2,loc}(\mathbb{R}_+; BMO)$  with a suitable logarithmic interpolation inequality from [31].

Let us finally point out an interesting application of Theorem 1.1 pertaining to the case where the initial density has nontrivial vacuum regions.

**Corollary 1.1.** *Let the assumptions of Theorem 1.1 be in force, and denote by  $(\rho, v)$  a global solution given by Theorem 1.1. Let  $X$  be the (generalized) flow of  $v$ , defined by*

$$(1.14) \quad X(t, y) = y + \int_0^t X(\tau, v(\tau, y)) d\tau, \quad t \geq 0, \quad y \in \mathbb{T}^2.$$

*Then, the following results hold:*

- (1) *Let  $V_0 := \rho_0^{-1}(\{0\})$ . Then  $\rho_t^{-1}(\{0\}) = V_t$  with  $V_t := X(t, V_0)$ . Furthermore, if  $V_0$  is an open set with Lipschitz boundary, then  $V_t$  is an open set with  $C^{0, \alpha_t}$  regularity.*
- (2) *If  $\rho_0 = 1_{A_0}$  and  $A_t := X(t, A_0)$ , then  $\inf_{x \in A_t} \rho(t, x) > 0$  for all  $t > 0$ . Furthermore, if  $A_0$  is a Lipschitz open set, then  $A_t$  has  $C^{0, \alpha_t}$  regularity.*

*Above,  $\alpha_t > 0$  is a continuously decreasing function of  $t$  and is such that  $\alpha_0 = 1$ .*

*Proof.* By using the continuity of Riesz operator, we get

$$\|v\|_{LL} \leq \|v\|_2 + \|\operatorname{div} v\|_\infty + \|\operatorname{curl} v\|_\infty,$$

where  $LL$  stands for the space of bounded log-Lipschitz functions. Hence Theorem 1.1 ensures that  $v \in L_{loc}^1(\mathbb{R}_+; LL)$  and, applying [1, Th. 3.7] guarantees the existence and uniqueness of

a generalized flow fulfilling (1.14). Let us assume that  $\partial V_0$  coincides with  $\phi_0^{-1}(\{0\})$  for some Lipschitz function  $\phi_0 : \mathbb{T}^2 \rightarrow \mathbb{R}$ . Then,  $\partial V_t = \phi_t^{-1}(\{0\})$  where  $\phi$  solves the transport equation

$$(\partial_t + v \cdot \nabla)\phi = 0.$$

Now, [1, Th. 3.12] guarantees that  $\phi_t$  has regularity  $C^{0,\alpha_t}$  with

$$\alpha_t := \exp\left(-\int_0^t \|v\|_{LL} d\tau\right),$$

and one can conclude the proof of the first item.

The second item follows from similar arguments and from the fact that

$$\rho(t, X(t, y)) = \rho_0(y) \exp\left(\int_0^t \operatorname{div} v(\tau, X(\tau, y)) d\tau\right)$$

for all  $t \geq 0$  and  $y \in \mathbb{T}^2$ . □

We end this part proposing some conjecture, that probably requires further developments of the transport theory, the basic problem here being that the velocity field is not Lipschitz, thus preventing us to reformulate the equations in the Lagrangian coordinates without any loss of regularity:

**Conjecture.** *The solutions constructed in Theorem 1.1 (or in Theorem C.1 for the three-dimensional case) are unique for arbitrary strictly increasing convex pressure functions.*

The rest of the paper unfolds as follows. The next section is dedicated to the proof of regularity estimates for (1.1) assuming that the solution under consideration is smooth with density bounded away from zero, and that  $\nu$  is large enough (this corresponds to steps 1 to 3 above). In Section 3, we prove the existence part of our main theorem and also justify the convergence of (1.1) to (1.10) for  $\nu$  going to  $\infty$ , while Section 4 is dedicated to uniqueness. Some technical results like, in particular, Inequality (1.13) and time weighted estimates, and the case  $d = 3$  are presented in the appendix.

## 2. REGULARITY ESTIMATES

The present section is devoted to proving regularity estimates for the velocity field of a solution  $(\rho, v)$  to (1.1) in  $\mathbb{R}_+ \times \mathbb{T}^d$ . We focus on  $d = 2$ , the three dimensional case being postponed in appendix. We show three results: a control of the  $H^1$  norm of the velocity, a pointwise global-in-time bound for the density and, finally, a new estimate for the effective viscous flux and the divergence-free part of the velocity. This latter estimate is based on the *shift of integrability method* introduced in [12].

As a start, we normalize the potential energy  $e$  in such a way that  $e(1) = e'(1) = 0$ , setting

$$(2.1) \quad e(\rho) := \rho \int_1^\rho \frac{P(\varrho)}{\varrho^2} d\varrho - P(1)(\rho - 1).$$

Hence,  $\|e\|_1$  is essentially equivalent to  $\|\rho - 1\|_2^2$  and, in the case  $P(\rho) = \rho^\gamma$ , we have

$$e(\rho) = \rho \log \rho + 1 - \rho \quad \text{if } \gamma = 1, \quad \text{and} \quad e(\rho) = \frac{\rho^\gamma}{\gamma - 1} - \frac{\gamma \rho}{\gamma - 1} + 1 \quad \text{if } \gamma > 1.$$

We shall often use the notations  $e$  and  $P$  instead of  $e(\rho)$  and  $P(\rho)$ .



**2.1. Sobolev estimates for the velocity.** Here we derive a global-in-time  $H^1$  energy estimate that requires only a control on  $\sup \rho$ .

Throughout the proof, we denote  $\tilde{P} := P - \bar{P}$  and  $\tilde{G} := G - \bar{G}$  where  $\bar{P}$  and  $\bar{G}$  stand for the average of  $P$  and  $G$ . Note that we have

$$(2.2) \quad \tilde{G} = \nu \operatorname{div} v - \tilde{P}.$$

**Proposition 2.1.** *Consider a smooth solution  $(\rho, v)$  to (1.1) on  $[0, T] \times \mathbb{T}^2$  satisfying (1.5). Assume that the pressure law fulfills (1.9) and that, for some positive constant  $\rho^*$ ,*

$$(2.3) \quad 0 \leq \rho(t, x) \leq \rho^* \quad \text{for all } (t, x) \in [0, T] \times \mathbb{T}^2.$$

Let  $\dot{v} := v_t + v \cdot \nabla v$  be the material derivative of  $v$ , and  $h := \rho P' - P$ . There exist:

– a functional  $\mathcal{E}$  such that

$$(2.4) \quad \mathcal{E} \geq \frac{1}{2} \int_{\mathbb{T}^2} \left( \rho |v|^2 + \mu |\nabla \mathcal{P} v|^2 + \frac{1}{\nu} (\tilde{G}^2 + \tilde{P}^2) + 2e \right) dx,$$

– an absolute positive constant  $C$ ,

– a positive constant  $\nu_0$  depending<sup>3</sup> only on the pressure function  $P$ , on  $\mu$  and on  $\rho^*$ , such that if  $\nu \geq \nu_0$  then for all  $t \in [0, T]$ , we have

$$(2.5) \quad 1 + \frac{1}{\mu E_0} \left( \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \right) \leq \left( 1 + \frac{\mathcal{E}_0}{\mu E_0} \exp \left\{ C \left( 1 + \frac{(\rho^*)^2}{\mu^4} E_0^2 \log(e + \rho^*) \right) \right\} \right)^{\exp \left\{ C \frac{(\rho^*)^2}{\mu^4} E_0^2 \right\}},$$

with  $E_0$  defined in (1.6) and

$$\mathcal{D} := \int_{\mathbb{T}^2} \left( \frac{1}{4} \rho |\dot{v}|^2 + \frac{\mu^2}{4\rho^*} |\nabla^2 \mathcal{P} v|^2 + \frac{1}{8\rho^*} |\nabla G|^2 + \frac{1}{4\nu} \tilde{P}^2 + \left( \frac{\nu+h}{2} \right) (\operatorname{div} v)^2 + \frac{\mu}{2} |\nabla v|^2 \right) dx.$$

*Proof.* As in the work of B. Desjardins in [14], the proof consists in introducing a suitable ‘energy’ functional that contains  $H^1$  information on the velocity, then to combine with the logarithmic interpolation inequality (1.13). The novelty here is that we succeed in getting a *time-independent* control on the solution in terms of the data and of  $\rho^*$ .

*Step 1.* The goal of this step (which is independent of the dimension  $d$ ) is to bound:

$$(2.6) \quad \tilde{\mathcal{E}} = \frac{1}{2} \int_{\mathbb{T}^d} \left( \mu |\nabla \mathcal{P} v|^2 + \frac{1}{\nu} (\tilde{G}^2 + \tilde{P}^2 + \rho \int_\rho^1 \frac{P^2(\tau)}{\tau^2} d\tau) \right) dx.$$

To achieve it, we take the  $L_2$  scalar product of the momentum equation of (1.1) with  $\dot{v}$ , and get

$$(2.7) \quad \int_{\mathbb{T}^d} \rho |\dot{v}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} (\mu |\nabla v|^2 + (\lambda + \mu) (\operatorname{div} v)^2) dx + \int_{\mathbb{T}^d} \nabla P \cdot v_t dx = \int_{\mathbb{T}^d} (\rho \dot{v}) \cdot (v \cdot \nabla v) dx.$$

To handle the pressure term in the left-hand side, we start from

$$(2.8) \quad P_t + \operatorname{div} (Pv) + h \operatorname{div} v = 0.$$

<sup>3</sup>Here we find  $\nu_0 = \max \left( \mu, C \sqrt{\frac{\rho^* \log(e + \rho^*)}{\mu}} P(\rho^*), \frac{P(\rho^*)}{2}, 4\sqrt{\rho^* (1 + h(\rho^*))} \right).$

Therefore, integrating by parts yields

$$\int_{\mathbb{T}^d} \nabla P \cdot v_t dx = -\frac{d}{dt} \int_{\mathbb{T}^d} P \operatorname{div} v dx - \int_{\mathbb{T}^d} h (\operatorname{div} v)^2 dx + \int_{\mathbb{T}^d} P v \cdot \nabla \operatorname{div} v dx.$$

Since

$$-(\nu \operatorname{div} v)^2 = P^2 - G^2 - 2\nu P \operatorname{div} v \quad \text{and} \quad \nu \nabla \operatorname{div} v = \nabla(P + G),$$

we get after integrating by parts once to avoid the appearance of some  $\nabla P$  term,

$$(2.9) \quad \int_{\mathbb{T}^d} \nabla P \cdot v_t dx = -\frac{d}{dt} \int_{\mathbb{T}^d} P \operatorname{div} v dx + \frac{1}{\nu^2} \int_{\mathbb{T}^d} (P^2 - G^2) h dx + \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G dx \\ - \frac{1}{\nu} \int_{\mathbb{T}^d} \left( \frac{P^2}{2} + 2Ph \right) \operatorname{div} v dx.$$

Observing that, owing to the definition of  $G$  and to (2.8), we have

$$\bar{G} = -\bar{P} \quad \text{and} \quad \bar{P}' = - \int_{\mathbb{T}^d} h \operatorname{div} v dx,$$

we find that

$$(2.10) \quad \int_{\mathbb{T}^d} (P^2 - G^2) h dx = \nu \int_{\mathbb{T}^d} (\tilde{P} - \tilde{G}) \operatorname{div} v h dx + 2\nu \bar{P} \int_{\mathbb{T}^d} h \operatorname{div} v dx \\ = \nu^2 \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx - 2\nu \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx - \nu \frac{d}{dt} (\bar{P})^2.$$

Let the function  $k$  be the unique solution of

$$k - \rho k' = -\frac{P^2}{2} - 2Ph \quad \text{and} \quad k(1) = P^2(1).$$

Then, we have

$$(2.11) \quad - \int_{\mathbb{T}^d} \left( \frac{P^2}{2} + 2Ph \right) \operatorname{div} v dx = \int_{\mathbb{T}^d} (\partial_t k + \operatorname{div}(kv)) dx = \frac{d}{dt} \int_{\mathbb{T}^d} k dx.$$

Hence, plugging (2.9) and (2.10) in (2.11), we obtain

$$\int_{\mathbb{T}^d} \nabla P \cdot v_t dx = \frac{d}{dt} \int_{\mathbb{T}^d} \left( \frac{k - (\bar{P})^2}{\nu} - P \operatorname{div} v \right) dx - \frac{2}{\nu} \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx \\ + \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx + \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G dx.$$

Now, denoting

$$(2.12) \quad \check{\mathcal{E}} := \int_{\mathbb{T}^d} \left( \frac{\mu}{2} |\nabla v|^2 + \frac{\lambda + \mu}{2} (\operatorname{div} v)^2 + \frac{1}{\nu} (k - (\bar{P})^2) - \tilde{P} \operatorname{div} v \right) dx$$

and reverting to (2.7), we conclude that

$$(2.13) \quad \frac{d}{dt} \check{\mathcal{E}} + \int_{\mathbb{T}^d} \rho |\dot{v}|^2 dx + \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx \\ = \int_{\mathbb{T}^d} (\rho \dot{v}) \cdot (v \cdot \nabla v) dx + \frac{2}{\nu} \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx - \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G dx.$$

We claim that  $\check{\mathcal{E}} = \tilde{\mathcal{E}}$ . Indeed, we have

$$\check{\mathcal{E}} = \int_{\mathbb{T}^d} \left( \frac{\mu}{2} (|\nabla v|^2 - (\operatorname{div} v)^2) + \frac{1}{2\nu} \left( (\nu \operatorname{div} v)^2 - 2\nu \operatorname{div} v \tilde{P} + 2(k - (\bar{P})^2) \right) \right) dx$$

and

$$(2.14) \quad k(\rho) = P^2(\rho) - \frac{\rho}{2} \int_1^\rho \frac{P^2(\varrho)}{\varrho^2} d\varrho.$$

In order to get a control on the right-hand side of (2.13), let us rewrite the momentum equation in terms of the viscous effective flux  $G = \nu \operatorname{div} v - P$  as follows:

$$(2.15) \quad \mu(\Delta v - \nabla \operatorname{div} v) + \nabla G = \rho \dot{v}.$$

From it, we discover that

$$(2.16) \quad \mu^2 \|\Delta \mathcal{P}v\|_2^2 + \|\nabla G\|_2^2 = \|\rho \dot{v}\|_2^2 \leq \rho^* \|\sqrt{\rho} \dot{v}\|_2^2.$$

Since we obviously have

$$\frac{2}{\nu} \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h \, dx \leq \frac{1}{2} \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h \, dx + \frac{2}{\nu^2} \int_{\mathbb{T}^d} \tilde{G}^2 h \, dx,$$

equality (2.13) and the fact that  $\check{\mathcal{E}} = \tilde{\mathcal{E}}$  imply that

$$\begin{aligned} \frac{d}{dt} \tilde{\mathcal{E}} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{2\rho^*} \|\Delta \mathcal{P}v\|_2^2 + \frac{1}{2\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h \, dx \\ \leq \frac{2}{\nu^2} \int_{\mathbb{T}^d} \tilde{G}^2 h \, dx - \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G \, dx + \|\sqrt{\rho} v \cdot \nabla v\|_2^2. \end{aligned}$$

To bound the last term in the right-hand side, we decompose  $\nabla v$  into

$$(2.17) \quad \nabla v = \nabla \mathcal{P}v - \frac{1}{\nu} \nabla^2 (-\Delta)^{-1} \tilde{G} - \frac{1}{\nu} \nabla^2 (-\Delta)^{-1} \tilde{P}.$$

Hence

$$\begin{aligned} (2.18) \quad \frac{d}{dt} \tilde{\mathcal{E}} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{2\rho^*} \|\Delta \mathcal{P}v\|_2^2 + \frac{1}{2\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h \, dx \leq \frac{2}{\nu^2} \int_{\mathbb{T}^d} \tilde{G}^2 h \, dx \\ - \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G \, dx + 3 \left( \|\sqrt{\rho} v \cdot \nabla \mathcal{P}v\|_2^2 + \frac{1}{\nu^2} \|\sqrt{\rho} v \cdot \nabla^2 (-\Delta)^{-1} \tilde{G}\|_2^2 + \frac{1}{\nu^2} \|\sqrt{\rho} v \cdot \nabla^2 (-\Delta)^{-1} \tilde{P}\|_2^2 \right). \end{aligned}$$

*Step 2: Bounding the right-hand side of (2.18) in dimension  $d = 2$ .* Hölder and Gagliardo-Nirenberg inequalities yield

$$\int_{\mathbb{T}^2} \rho |v \cdot \nabla \mathcal{P}v|^2 \, dx \leq C \sqrt{\rho^*} \left( \int_{\mathbb{T}^2} \rho |v|^4 \, dx \right)^{\frac{1}{2}} \|\nabla \mathcal{P}v\|_2 \|\nabla^2 \mathcal{P}v\|_2.$$

Since the density is not bounded from below, in order to bound the right-hand side, one has to take advantage of Inequality (1.13). We get

$$\begin{aligned} (2.19) \quad 3 \int_{\mathbb{T}^2} \rho |v \cdot \nabla \mathcal{P}v|^2 \, dx \leq C \sqrt{\rho^*} \|\sqrt{\rho} v\|_2 \|\nabla v\|_2 \|\nabla \mathcal{P}v\|_2 \|\nabla^2 \mathcal{P}v\|_2 \log^{\frac{1}{2}} \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right) \\ \leq \frac{\mu^2}{4\rho^*} \|\Delta \mathcal{P}v\|_2^2 + \frac{C(\rho^*)^2}{\mu^2} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \|\nabla \mathcal{P}v\|_2^2 \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right). \end{aligned}$$

Arguing similarly and using the fact that  $\nabla^2 (-\Delta)^{-1}$  maps  $L_4(\mathbb{T}^2)$  to itself, we get

$$\begin{aligned} (2.20) \quad \frac{3}{\nu^2} \int_{\mathbb{T}^2} \rho \left| v \cdot \left[ \nabla^2 (-\Delta)^{-1} \tilde{G} \right] \right|^2 \, dx \leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 \\ + \frac{C(\rho^*)^2}{\nu^4} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \|\tilde{G}\|_2^2 \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right), \end{aligned}$$

and also,

$$(2.21) \quad \begin{aligned} & \frac{3}{\nu^2} \int_{\mathbb{T}^2} \rho |v \cdot \nabla^2 \Delta^{-1} \tilde{P}|^2 dx \\ & \leq \frac{1}{4\nu} \|\tilde{P}\|_2^2 + \frac{C\rho^*}{\nu^3} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right) \|\tilde{P}\|_\infty^2. \end{aligned}$$

Finally, we have, thanks to Inequality (A.3),

$$\begin{aligned} -\frac{1}{\nu} \int_{\mathbb{T}^2} P v \cdot \nabla G dx & \leq \frac{1}{\nu} P^* \|v\|_2 \|\nabla G\|_2 \\ & \leq \frac{C}{\nu} P^* \log^{\frac{1}{2}}(e + \|\tilde{\rho}\|_2) \|\nabla v\|_2 \|\nabla G\|_2. \end{aligned}$$

Hence

$$(2.22) \quad -\frac{1}{\nu} \int_{\mathbb{T}^2} P v \cdot \nabla G dx \leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 + C \frac{\rho^*}{\nu^2} (P^*)^2 \|\nabla v\|_2^2 \log(e + \|\tilde{\rho}\|_2).$$

Therefore, plugging (2.19), (2.20), (2.21) and (2.22) in (2.18), we conclude that

$$(2.23) \quad \begin{aligned} & \frac{d}{dt} \tilde{\mathcal{E}} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\Delta P v\|_2^2 + \frac{1}{4\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} v)^2 h dx \\ & \leq \frac{C\rho^*}{\nu^2} \log(e + \|\tilde{\rho}\|_2) (P^*)^2 \|\nabla v\|_2^2 + \frac{1}{4\nu} \|\tilde{P}\|_2^2 + \frac{2}{\nu^2} \int_{\mathbb{T}^2} \tilde{G}^2 h dx \\ & + C \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \left( \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} + \frac{(\rho^*)^2}{\mu^2} \|\nabla P v\|_2^2 + \frac{(\rho^*)^2}{\nu^4} \|\tilde{G}\|_2^2 \right) \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right). \end{aligned}$$

*Step 3: Upgrading the energy functional  $\tilde{\mathcal{E}}$ .* In order to handle all the terms of the right-hand side of (2.23), one has to add up to  $\tilde{\mathcal{E}}$  a suitable multiple of the basic energy  $E$  and of the potential energy  $e$  so as to glean some time-decay for  $\|\tilde{P}\|_2$ . Indeed, we have

$$\partial_t e + \operatorname{div}(e v) + P \operatorname{div} v = 0.$$

Hence, integrating on  $\mathbb{T}^2$  and remembering that  $\nu \operatorname{div} v = \tilde{P} + \tilde{G}$  yields

$$(2.24) \quad \frac{d}{dt} \int_{\mathbb{T}^2} e dx + \frac{1}{\nu} \int_{\mathbb{T}^2} |\tilde{P}|^2 dx = -\frac{1}{\nu} \int_{\mathbb{T}^2} \tilde{P} \tilde{G} dx.$$

Now, denoting  $P^* := \|P(\rho)\|_\infty$ , we observe that for all  $\rho \geq 0$ , we have

$$\begin{aligned} \rho \int_1^\rho \frac{P^2(\tau)}{\tau^2} d\tau & \leq \rho P(\rho) \int_1^\rho \frac{P(\tau)}{\tau^2} d\tau = P(\rho) (e(\rho) + P(1)(\rho - 1)) \\ & \leq P^* (e(\rho) + P(1)\rho) - P(\rho)P(1). \end{aligned}$$

Hence, owing to (1.5),

$$(2.25) \quad \int_{\mathbb{T}^2} \rho(x) \left( \int_1^{\rho(x)} \frac{P^2(\tau)}{\tau^2} d\tau \right) dx \leq P^* (\|e\|_1 + P(1)) - \bar{P} P(1),$$

and thus

$$(2.26) \quad \tilde{\mathcal{E}} \geq \frac{1}{2} \int_{\mathbb{T}^2} \left( \mu |\nabla P v|^2 + \frac{1}{\nu} (\tilde{G}^2 + \tilde{P}^2) \right) dx - \frac{1}{2\nu} \left( P^* \|e\|_1 + P(1)(P^* - \bar{P}) \right).$$

Consequently, if we set

$$\begin{aligned}\mathcal{E} &:= \tilde{\mathcal{E}} + E + \|e\|_1 + \frac{1}{2\nu}(P^* - P(1))P(1) \\ &= \frac{1}{2} \int_{\mathbb{T}^2} \left( \rho|v|^2 + \mu|\nabla \mathcal{P}v|^2 + \frac{1}{\nu} \left( \tilde{G}^2 + \tilde{P}^2 + \left( \rho \int_{\rho}^1 \frac{P^2(\tau)}{\tau^2} d\tau \right) + (P^* - P(1))P(1) \right) + 4e \right) dx,\end{aligned}$$

then we have thanks to (2.26),

$$(2.27) \quad \mathcal{E} \geq \frac{1}{2} \left( \|\sqrt{\rho}v\|_2^2 + \mu\|\nabla \mathcal{P}v\|_2^2 + \frac{1}{\nu} (\|\tilde{G}\|_2^2 + \|\tilde{P}\|_2^2) \right) + \left( 2 - \frac{P^*}{2\nu} \right) \|e\|_1.$$

*Step 4. A global-in-time estimate.* In order to control the integral in the right-hand side of (2.24), one may use that

$$\frac{1}{\nu} \int_{\mathbb{T}^2} |\tilde{P}| |\tilde{G}| dx \leq \frac{1}{2\nu} \int_{\mathbb{T}^2} \tilde{P}^2 dx + \frac{1}{2\nu} \int_{\mathbb{T}^2} \tilde{G}^2 dx.$$

Then, Poincaré inequality implies that

$$\frac{1}{\nu} \int_{\mathbb{T}^2} \left( \frac{2}{\nu} h + 1 \right) \tilde{G}^2 dx \leq \frac{4\rho^*}{\nu} \left( \frac{2}{\nu} \|h\|_{\infty} + 1 \right) \frac{\|\nabla G\|_2^2}{4\rho^*}.$$

Using also the fact that  $\|\tilde{\rho}\|_2^2 = \|\rho\|_2^2 - 1 \leq (\rho^*)^2 - 1$ , we get from (2.23) that

$$\begin{aligned}\frac{d}{dt} \mathcal{E} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\nabla^2 \mathcal{P}v\|_2^2 + \frac{1}{4\rho^*} \left( 1 - \frac{4\rho^*}{\nu} \left( \frac{2}{\nu} \|h\|_{\infty} + 1 \right) \right) \|\nabla G\|_2^2 + \frac{1}{4\nu} \|\tilde{P}\|_2^2 \\ + \int_{\mathbb{T}^2} \left( \nu + \frac{h}{2} \right) (\operatorname{div} v)^2 dx + \mu \|\nabla \mathcal{P}v\|_2^2 - C \frac{\rho^* \log(e + \rho^*)}{\nu^2} (P^*)^2 \|\nabla v\|_2^2 \\ \leq C \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^2 \left( \frac{\rho^* \|\tilde{P}\|_{\infty}^2}{\nu^3} + \frac{(\rho^*)^2}{\mu^2} \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^2}{\nu^4} \|\tilde{G}\|_2^2 \right) \log \left( e + \rho^* + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right).\end{aligned}$$

Now, since

$$(2.28) \quad \mu \|\nabla v\|_2^2 + (\lambda + \mu) \|\operatorname{div} v\|_2^2 = \mu \|\nabla \mathcal{P}v\|_2^2 + \nu \|\operatorname{div} v\|_2^2,$$

we have if  $\nu \geq \mu$ ,

$$\nu \|\operatorname{div} v\|_2^2 + \mu \|\nabla \mathcal{P}v\|_2^2 \geq \mu \|\nabla v\|_2^2.$$

Therefore, because for all  $A \geq 0$ ,

$$\log(e + \rho^* + \rho^* A) \leq \log(e + \rho^*) + \log(1 + A) \leq \log(e + \rho^*) + A,$$

if one assumes that

$$(2.29) \quad 1 \geq 2C \frac{\rho^* \log(e + \rho^*) (P^*)^2}{\mu \nu^2} \quad \text{and} \quad \frac{8\rho^*}{\nu} \left( \frac{2}{\nu} \|h\|_{\infty} + 1 \right) \leq 1,$$

then the above inequalities imply that

$$\begin{aligned}(2.30) \quad \frac{d}{dt} \mathcal{E} + \mathcal{D} \leq C \rho^* \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^2 \frac{\|\tilde{P}\|_{\infty}^2}{\nu^3} \left( \log(e + \rho^*) + \frac{\|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right) \\ + C (\rho^*)^2 \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^2 \left( \frac{1}{\mu^2} \|\nabla \mathcal{P}v\|_2^2 + \frac{1}{\nu^4} \|\tilde{G}\|_2^2 \right) \left( \log(e + \rho^*) + \log \left( 1 + \frac{\|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right) \right)\end{aligned}$$

with

$$\mathcal{D} := \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\nabla^2 \mathcal{P}v\|_2^2 + \frac{1}{8\rho^*} \|\nabla G\|_2^2 + \frac{1}{4\nu} \|\tilde{P}\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} v)^2 (\nu + h) dx + \frac{\mu}{2} \|\nabla v\|_2^2.$$

So, finally, if one assumes that

$$(2.31) \quad \nu \geq \mu, \quad \nu^2 \geq 2C\mu^{-1}\rho^* \log(e+\rho^*)(P^*)^2, \quad \nu \geq 8\rho^*(2\nu^{-1}\|h\|_\infty + 1) \quad \text{and} \quad \nu \geq P^*/2,$$

the last condition ensuring that the coefficient of the last term in (2.27) is greater than 1, then (2.4) holds true, and thus

$$(2.32) \quad \|\sqrt{\rho}v\|_2^2 \leq 2\mathcal{E}, \quad \|\nabla \mathcal{P}v\|_2^2 \leq 2\mathcal{E}/\mu \quad \text{and} \quad \|\tilde{P}\|_2^2 + \|\tilde{G}\|_2^2 \leq 2\nu\mathcal{E}.$$

Thanks to that, Inequality (2.30) combined with the energy balance (1.6) and the fact that the map  $r \mapsto r \log^{1/2}(a+b/r)$  is nondecreasing on  $\mathbb{R}_+$  if  $a \geq 1$  and  $b \geq 0$ , imply that

$$\begin{aligned} \frac{d}{dt}\mathcal{E} + \mathcal{D} &\leq C \frac{(\rho^*)^2}{\mu^3} E_0 \|\nabla v\|_2^2 \mathcal{E} \log\left(1 + \frac{\mathcal{E}}{\mu E_0}\right) \\ &\quad + C \left( \frac{(\rho^*)^2}{\mu^3} E_0 \log(e + \rho^*) + \frac{\rho^* \|\tilde{P}\|_\infty^2}{\mu\nu^3} + \log(e + \rho^*) \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} \right) \|\nabla v\|_2^2 \mathcal{E}. \end{aligned}$$

Note that Condition (2.31) entails that

$$\frac{\rho^* \|\tilde{P}\|_\infty^2}{\mu\nu^3} + \log(e + \rho^*) \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} \leq 1.$$

Therefore applying Lemma A.1 with

$$A := 1, \quad B := \frac{1}{\mu E_0}, \quad f := C \frac{(\rho^*)^2}{\mu^3} E_0 \|\nabla v\|_2^2 \quad \text{and} \quad g := C \left(1 + \frac{(\rho^*)^2}{\mu^3} E_0 \log(e + \rho^*)\right) \|\nabla v\|_2^2,$$

we get

$$\begin{aligned} 1 + \frac{1}{\mu E_0} \left( \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \right) \\ \leq \left( 1 + \frac{\mathcal{E}_0}{\mu E_0} \exp \left\{ C \left( 1 + \frac{(\rho^*)^2}{\mu^3} E_0 \log(e + \rho^*) \right) \int_0^t \|\nabla v\|_2^2 d\tau \right\} \right)^{\exp \left\{ C \frac{(\rho^*)^2}{\mu^3} E_0 \int_0^t \|\nabla v\|_2^2 d\tau \right\}}, \end{aligned}$$

which, in light of the basic energy balance (1.6), yields (2.5).  $\square$

**Remark 2.1.** One has some freedom in the definition of  $\mathcal{E}$ , and lots of possibilities for bounding the right-hand side of (2.23). As a consequence, for small  $\nu$ , one can get a global, but time dependent control on  $\mathcal{E}$ . We chose not to treat that case here since the condition that  $\nu$  is large will be needed in the next step, so as to remove the a priori assumption that  $\rho$  is bounded.

**Remark 2.2.** Relation (2.2) and Inequality (2.4) imply that

$$\nu \|\operatorname{div} v\|_2^2 \leq 4\mathcal{E}.$$

**2.2. An upper bound for the density.** Here, we prove that, for large enough  $\nu$ , if the initial data fulfill the assumptions of the previous section, then we have a global-in-time control on the supremum of  $\rho$ . As in the previous subsection, we assume that we are given a smooth solution with strictly positive density, keeping in mind that the result below will be only applied to the family constructed in Section 3. For simplicity, we assume that  $P(\rho) = \rho^\gamma$  for some  $\gamma \geq 1$ .

**Proposition 2.2.** Consider a smooth solution  $(\rho, v)$  of (1.1) on  $[0, T] \times \mathbb{T}^2$  for some  $T < \infty$  pertaining to smooth initial data  $(\rho_0, v_0)$  such that  $\rho_0 > 0$  and  $\nu^{1/2} \|\operatorname{div} v_0\|_2 \leq K$ .

There exists  $\nu_0$  depending on  $K, \gamma, \mu, \|\rho_0\|_\infty, E_0$  and  $\|\nabla v_0\|_2$ , but independent of  $T$  such that if  $\nu \geq \nu_0$ , then

$$(2.33) \quad \sup_{t \in [0, T]} \|\rho(t)\|_\infty \leq 2e^{\frac{\gamma-1}{\gamma} E_0} \|\rho_0\|_\infty.$$

*Proof.* Throughout the proof, we denote slightly abusively the right-hand side of (2.33) by  $\rho^*$ . We start from the observation that as  $\rho$  is smooth and positive (by assumption), we may write, owing to (2.2),

$$\partial_t \log \rho + v \cdot \nabla \log \rho = -\frac{1}{\nu}(\tilde{P} + \tilde{G}).$$

Remember that the definition of  $\tilde{G}$  ensures that

$$\Delta \tilde{G} = \partial_t(\operatorname{div}(\rho v)) + \operatorname{div}(\operatorname{div}(\rho v \otimes v)).$$

Therefore, following [14] and introducing

$$F := \log \rho - \nu^{-1}(-\Delta)^{-1} \operatorname{div}(\rho v),$$

we discover that (with the summation convention over repeated indices),

$$(2.34) \quad \partial_t F + v \cdot \nabla F + \frac{1}{\nu} \tilde{P} = -\frac{1}{\nu} [v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i.$$

Since we have

$$P(\rho) \geq \gamma \log \rho + 1 \quad \text{for all } \rho > 0,$$

setting  $F^+ := \max(0, F)$  yields

$$(2.35) \quad \partial_t F^+ + v \cdot \nabla F^+ + \frac{\gamma}{\nu} F^+ \leq \frac{1}{\nu} [v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i + \frac{\gamma}{\nu^2} |(-\Delta)^{-1} \operatorname{div}(\rho v)| + \frac{1}{\nu} (\bar{P} - 1).$$

As  $P(\rho) = \rho^\gamma$ , we have  $\bar{P} - 1 = (\gamma - 1) \|e\|_1$ , so that the last term may be bounded by  $(\gamma - 1) E_0$ . Since the field  $v$  is assumed to be smooth, applying the maximal principle for the transport equation yields:

$$(2.36) \quad \|F^+(t)\|_\infty \leq e^{-\frac{\gamma}{\nu} t} \|F^+(0)\|_\infty + \frac{1}{\nu} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i(\tau)\|_\infty d\tau \\ + \frac{\gamma}{\nu^2} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|(-\Delta)^{-1} \operatorname{div}(\rho v)(\tau)\|_\infty d\tau + \frac{\gamma - 1}{\gamma} (1 - e^{-\frac{\gamma}{\nu} t}) E_0.$$

Using that the average of  $\rho v$  is zero, Sobolev embedding and the properties of continuity of Riesz operator imply that

$$(2.37) \quad \|(-\Delta)^{-1} \operatorname{div}(\rho v)\|_\infty \lesssim \|(-\Delta)^{-1} \nabla \operatorname{div}(\rho v)\|_4 \lesssim \|\rho v\|_4.$$

Then, we use again (1.13) and get

$$\|(-\Delta)^{-1} \operatorname{div}(\rho v)\|_\infty \lesssim (\rho^*)^{\frac{3}{4}} \|\sqrt{\rho} v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right),$$

whence, thanks to the energy balance (1.6) and the definition of  $\mathcal{E}$  (assuming that  $\nu \geq \mu$ ),

$$(2.38) \quad \|(-\Delta)^{-1} \operatorname{div}(\rho v)\|_\infty \leq C(\rho^*)^{\frac{3}{4}} E_0^{\frac{1}{4}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \mathcal{E}}{\mu E_0} \right).$$

Since  $[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i = [\tilde{v}^j, \Delta^{-1} \partial_i \partial_j] \rho v^i$  with  $\tilde{v} = v - \bar{v}$ , the second term in the r.h.s. of (2.36) may be bounded by means of Sobolev embedding and of the following Coifman, Lions, Meyer and Semmes inequality (from [6]) as follows:

$$(2.39) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim \|[\tilde{v}^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_{W^{1,3}} \lesssim \|\nabla v\|_{12} \|\rho v\|_4.$$

To handle  $\nabla v$ , we use that

$$\|\nabla v\|_{12} \lesssim \|\nabla \mathcal{P} v\|_{12} + \nu^{-1} (\|\tilde{G}\|_{12} + \|\tilde{P}\|_{12}) \\ \lesssim \|\nabla^2 \mathcal{P} v\|_2 + \nu^{-1} (\|\nabla G\|_2 + \|\tilde{P}\|_\infty).$$

Hence, using once more (1.13),

$$\begin{aligned} \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty &\lesssim (\rho^*)^{\frac{3}{4}} (\|\nabla^2 \mathcal{P}v\|_2 + \nu^{-1} \|\nabla G\|_2 + \nu^{-1} \|\tilde{P}\|_\infty) \\ &\quad \times \|\sqrt{\rho}v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right), \end{aligned}$$

whence, using the energy conservation (1.6) and the definition of  $\mathcal{E}$  and  $\mathcal{D}$ ,

$$(2.40) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim ((\rho^*)^{\frac{5}{4}} \mu^{-1} \mathcal{D}^{\frac{1}{2}} + (\rho^*)^{\frac{3}{4}} \nu^{-1} \|\tilde{P}\|_\infty) \times E_0^{\frac{1}{4}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \mathcal{E}}{\mu E_0} \right).$$

Plugging (2.38) and (2.40) in (2.36) and performing obvious simplifications, we end up with

$$(2.41) \quad \|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + \frac{\gamma-1}{\gamma} E_0 + C(\rho^*)^{\frac{3}{4}} \frac{E_0^{\frac{1}{4}}}{\nu} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \left( (\sqrt{\rho^*} \mu^{-1} \mathcal{D}^{\frac{1}{2}} + \nu^{-1} (\|\tilde{P}\|_\infty + \gamma)) \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \mathcal{E}}{\mu E_0} \right) \right) d\tau.$$

Let us consider the largest sub-interval  $[0, T_0]$  of  $[0, T]$  on which (2.33) is fulfilled. Then, Inequality (2.5) tells us that there exist  $\nu_0$  depending only on  $\|\rho_0\|_\infty$ ,  $\mu$  and  $\gamma$ , and  $C_0 > 0$  (depending also on  $K$ ,  $E_0$ ,  $\|\rho_0\|_\infty$ ,  $\|\nabla v_0\|_2$ ) so that we have for all  $t \in [0, T_0]$ , if  $\nu \geq \nu_0$ ,

$$(2.42) \quad \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \leq C_0.$$

Inequality (2.41) thus becomes for a possibly larger  $C_0$ ,

$$(2.43) \quad \|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + \frac{\gamma-1}{\gamma} E_0 + \frac{C_0}{\nu} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \left( \frac{\mathcal{D}^{\frac{1}{2}}(\tau)}{\mu} + \frac{1}{\nu} \right) \|\nabla v(\tau)\|_2^{\frac{1}{2}} d\tau.$$

From Hölder inequality, we have for all  $t \in [0, T]$ ,

$$\begin{aligned} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \mathcal{D}^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} d\tau &\leq C \left( \frac{\nu}{\gamma} \right)^{\frac{1}{4}} \left( \int_0^T \mathcal{D}(\tau) d\tau \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla v(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\ \text{and } \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|\nabla v(\tau)\|_2^{\frac{1}{2}} d\tau &\leq C \left( \frac{\nu}{\gamma} \right)^{\frac{3}{4}} \left( \int_0^T \|\nabla v(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}}. \end{aligned}$$

As the integrals in the right-hand side may be bounded in terms of the data according to the basic energy balance (1.6) and to (2.42), we eventually get (changing once again  $C_0$  if needed) if  $\nu \geq \nu_0$ :

$$\|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + C_0 \nu^{-\frac{1}{4}} + \frac{\gamma-1}{\gamma} E_0.$$

Of course, owing to the definition of  $F^+$  and to (2.5) and (2.38), we have

$$\log \rho \leq F^+ + \nu^{-1} \|\Delta^{-1} \operatorname{div}(\rho v)\|_\infty \leq F^+ + \nu^{-1} C_0.$$

Hence one can eventually conclude that

$$(2.44) \quad \log \rho^* \leq \log \rho_0^* + C_0 \nu^{-\frac{1}{4}} + \frac{\gamma-1}{\gamma} E_0.$$

Now, if  $\nu$  is so large as to satisfy also

$$C_0 \nu^{-\frac{1}{4}} < \log 2,$$



then (2.44) implies (2.33) with a strict inequality. So  $T_0 = T$ .  $\square$

**2.3. Weighted estimates.** That section is devoted to the proof of the following result, that is based on the estimates that have been established so far. For better readability, we postpone the most technical parts to the appendix.

**Proposition 2.3.** *Define  $\nu_0$  as in Proposition 2.2. Let  $T \geq 1$ . Then, smooth solutions to (1.1) on  $[0, T] \times \mathbb{T}^2$  fulfill, if  $\nu \geq \nu_0$ :*

$$(2.45) \quad \sup_{t \in [0, T]} \int_{\mathbb{T}^2} \rho |\dot{v}|^2 t \, dx + \int_0^T \int_{\mathbb{T}^2} (\mu |\nabla \mathcal{P} \dot{v}|^2 + \nu |\operatorname{div} \dot{v}|^2) t \, dx \, dt \leq C_0 T \exp\left(\frac{C_0 T}{\nu}\right),$$

where  $C_0$  depends on  $\rho^*$ ,  $\mu$ ,  $\mathcal{E}_0$  and on the pressure function, but is independent of  $\nu$  and  $T$ .

*Proof.* Here it will be convenient to use the two notations  $\dot{f}$  and  $\frac{D}{Dt}f$  to designate the convective derivative of  $f$ , and we shall denote  $A : B = \sum_{i,j} A_{ij} B_{ij}$  if  $A$  and  $B$  are two  $d \times d$  matrices. Finally, if  $v$  is a vector field on  $\mathbb{T}^d$  then  $(Dv)_{ij} := \partial_j v^i$  and  $(\nabla v)_{ij} := \partial_i v^j$  for  $1 \leq i, j \leq d$ .

The general principle is to rewrite the momentum equation as:

$$(2.46) \quad \rho \dot{v} - \mu \Delta v - (\nu - \mu) \nabla \operatorname{div} v + \nabla P = 0,$$

then to take the material derivative and test it by  $t \dot{v}$ . We get

$$(2.47) \quad \int_{\mathbb{T}^d} \left( \frac{D}{Dt}(\rho \dot{v}) - \mu \frac{D}{Dt} \Delta v - (\nu - \mu) \frac{D}{Dt} \nabla \operatorname{div} v + \frac{D}{Dt} \nabla P \right) \cdot (t \dot{v}) \, dx = 0.$$

The rest of the proof consists in describing each term of (2.47). To this end, we shall repeatedly use the fact that for all  $\nu \geq \nu_0$  (where  $\nu_0$  is given by Proposition 2.2), we have

$$(2.48) \quad \|\nabla v\|_{L_4(0, T \times \mathbb{T}^2)} \leq C_0.$$

Indeed, recall the decomposition

$$(2.49) \quad v = \mathcal{P}v - \frac{1}{\nu} \nabla (-\Delta)^{-1} (\tilde{G} + \tilde{P}).$$

Proposition 2.1 and Sobolev embeddings imply that

$$(2.50) \quad \|\nabla \mathcal{P}v\|_{L_4(0, T \times \mathbb{T}^2)} \lesssim \|\nabla \mathcal{P}v\|_{L_\infty(0, T; L_2)}^{1/2} \|\nabla^2 \mathcal{P}v\|_{L_\infty(0, T; L_2)}^{1/2} \leq C_0.$$

Furthermore, we have

$$(2.51) \quad \|\tilde{G}\|_{L_4(0, T \times \mathbb{T}^2)} \lesssim \|\tilde{G}\|_{L_\infty(0, T; L_2)}^{1/2} \|\nabla G\|_{L_2(0, T; L_2)}^{1/2} \leq \nu^{1/4} C_0,$$

and

$$(2.52) \quad \|\tilde{P}\|_{L_4(0, T \times \mathbb{T}^2)} \leq \|\tilde{P}\|_{L_2(0, T; L_2)}^{1/2} \|\tilde{P}\|_{L_\infty(0, T; L_\infty)}^{1/2} \leq \nu^{1/4} C_0.$$

*Step 1.* Obvious computations give (in any dimension):

$$(2.53) \quad \int_{\mathbb{T}^d} \frac{D}{Dt}(\rho \dot{v}) \cdot \dot{v} t \, dx = \frac{1}{2} \int_{\mathbb{T}^d} \left( \frac{D}{Dt}(\rho |\dot{v}|^2 t) + \dot{\rho} |\dot{v}|^2 t - \rho |\dot{v}|^2 \right) dx.$$

Integrating by parts, we see that

$$(2.54) \quad \int_{\mathbb{T}^d} \frac{D}{Dt}(\rho |\dot{v}|^2 t) \, dx = \frac{d}{dt} \int_{\mathbb{T}^d} \rho |\dot{v}|^2 t \, dx - \int_{\mathbb{T}^d} \operatorname{div} v (\rho |\dot{v}|^2 t) \, dx.$$

Thanks to the mass conservation equation, we have

$$(2.55) \quad \int_{\mathbb{T}^d} \dot{\rho} |\dot{v}|^2 t dx = - \int_{\mathbb{T}^d} \rho \operatorname{div} v |\dot{v}|^2 t dx,$$

whence

$$\int_{\mathbb{T}^d} \frac{D}{Dt} (\rho \dot{v}) \cdot \dot{v} t dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \rho |\dot{v}|^2 t dx - \frac{1}{2} \int_{\mathbb{T}^d} \rho |\dot{v}|^2 dx - \int_{\mathbb{T}^d} \rho \operatorname{div} v |\dot{v}|^2 t dx.$$

If  $d = 2$  then one can bound the last term using that

$$\begin{aligned} \int_{\mathbb{T}^2} \rho \operatorname{div} v |\dot{v}|^2 t dx &= \nu^{-1} \int_{\mathbb{T}^2} (\tilde{P} + \tilde{G}) \rho |\dot{v}|^2 t dx \\ &\leq \nu^{-1} \|\tilde{P}\|_\infty \|\sqrt{\rho t} \dot{v}\|_2^2 + \rho^* \nu^{-1} \|\tilde{G}\|_2 \|\sqrt{t} \dot{v}\|_4^2 \\ &\leq C_0 \nu^{-1} \|\sqrt{\rho t} \dot{v}\|_2^2 + C_0 \nu^{-1/2} \|\sqrt{t} \dot{v}\|_4^2. \end{aligned}$$

Since  $\int_{\mathbb{T}^2} \rho \dot{v} dx = 0$ , one can take advantage of the Poincaré inequality (A.2) with  $p = 2$  and get:

$$(2.56) \quad \|\sqrt{t} \dot{v}\|_4^2 \leq C \|\sqrt{t} \dot{v}\|_2 \|\sqrt{t} \nabla \dot{v}\|_2 \leq C(1 + \|\tilde{\rho}\|_2) \|\sqrt{t} \nabla \dot{v}\|_2^2 \leq C \rho^* \|\sqrt{t} \nabla \dot{v}\|_2^2.$$

Hence,

$$\int_{\mathbb{T}^2} \rho \operatorname{div} v |\dot{v}|^2 t dx \leq C_0 \left( \nu^{-1} \|\sqrt{\rho t} \dot{v}\|_2^2 + \nu^{-1/2} \|\sqrt{t} \nabla \dot{v}\|_2^2 \right),$$

and thus

$$(2.57) \quad \begin{aligned} \int_{\mathbb{T}^d} \frac{D}{Dt} (\rho \dot{v}) \cdot \dot{v} t dx &\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \rho |\dot{v}|^2 t dx - \frac{1}{2} \int_{\mathbb{T}^2} \rho |\dot{v}|^2 dx \\ &\quad - C_0 \left( \nu^{-1} \int_{\mathbb{T}^2} \rho t |\dot{v}|^2 dx + \nu^{-1/2} \int_{\mathbb{T}^2} t |\nabla \dot{v}|^2 dx \right). \end{aligned}$$

*Step 2.* To handle the second term of (2.47), we use that

$$(2.58) \quad -\frac{D}{Dt} \Delta v = -\operatorname{div} \frac{D}{Dt} \nabla v - \nabla v \cdot \nabla^2 v \quad \text{with} \quad (\nabla v \cdot \nabla^2 v)^i := \sum_{j,k} \partial_k v^j \partial_j \partial_k v^i.$$

Hence, testing (2.58) by  $t\dot{v}$  and integrating by parts yields for  $d = 2, 3$ ,

$$-\int_{\mathbb{T}^d} \frac{D}{Dt} \Delta v \cdot t\dot{v} dx = \int_{\mathbb{T}^d} \frac{D}{Dt} \nabla v : \nabla \dot{v} t dx - \int_{\mathbb{T}^d} (\nabla v \cdot \nabla^2 v) \cdot \dot{v} t dx.$$

Since

$$\frac{D}{Dt} \nabla v = \nabla \dot{v} - \nabla v \cdot \nabla v,$$

we get

$$(2.59) \quad -\int_{\mathbb{T}^d} \frac{D}{Dt} \Delta v \cdot t\dot{v} dx = \int_{\mathbb{T}^d} |\nabla \dot{v}|^2 t dx - \int_{\mathbb{T}^d} (\nabla v \cdot \nabla v) : \nabla \dot{v} t dx - \int_{\mathbb{T}^d} (\nabla v \cdot \nabla^2 v) \cdot \dot{v} t dx.$$

The first term is the main one. The other two terms are denoted by  $I_1$  and  $I_2$ , respectively. Bounding  $I_1$  is easy : using Hölder inequality yields

$$|I_1| = \left| \int_{\mathbb{T}^d} (\nabla v \cdot \nabla v) : \nabla \dot{v} t dx \right| \leq \|t^{1/4} \nabla v\|_4^2 \|\sqrt{t} \nabla \dot{v}\|_2.$$

Therefore, we have according to (2.48),

$$(2.60) \quad \left| \int_0^T I_1(t) dt \right| \leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)}.$$

Bounding  $I_2$  is much more involved. We eventually get (see the details in appendix):

$$(2.61) \quad \left| \int_0^T I_2 dt \right| \leq \left( T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \right. \\ \left. + \sqrt{T} (\nu^{-3/4} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + \nu^{-2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}) \right).$$

Plugging (2.60) and (2.61) in (2.59) and using (2.56) yields

$$(2.62) \quad -\mu \int_0^T \int_{\mathbb{T}^2} \left( \frac{D}{Dt} \Delta v \right) \cdot \dot{v} t dx dt \geq \mu \int_0^T \int_{\mathbb{T}^2} |\nabla \dot{v}|^2 t dx dt \\ - C_0 T^{1/4} (T^{1/4} + \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)}.$$

*Step 3.* In order to bound the third term of equation (2.47), we use the relation

$$(2.63) \quad -\frac{D}{Dt} \nabla \operatorname{div} v = -\nabla \frac{D}{Dt} \operatorname{div} v + \nabla v \cdot \nabla \operatorname{div} v.$$

We have to keep in mind that the right-hand side involves only the potential part  $\mathcal{Q}v$  of the velocity, since  $\operatorname{div} v = \operatorname{div} \mathcal{Q}v$ . This enables us to write that

$$\nabla \frac{D}{Dt} \operatorname{div} v = \nabla \operatorname{div} \dot{v} - \nabla (\operatorname{tr}(\nabla v \cdot \nabla \mathcal{Q}v)).$$

Hence, testing (2.63) with  $\dot{v} t$  and integrating by parts, we find that

$$(2.64) \quad -\int_{\mathbb{T}^d} \frac{D}{Dt} \nabla \operatorname{div} v \cdot \dot{v} t dx = \int_{\mathbb{T}^d} (\operatorname{div} \dot{v})^2 t dx - K_1 + K_2 \\ \text{with } K_1 := \int_{\mathbb{T}^d} \operatorname{Tr}(\nabla v \cdot \nabla \mathcal{Q}v) \operatorname{div} \dot{v} t dx \quad \text{and} \quad K_2 := \int_{\mathbb{T}^d} (\nabla v \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx.$$

Since  $\nu \mathcal{Q}v = -\nabla(-\Delta)^{-1}(\tilde{G} + \tilde{P})$ , using (2.48), (2.51) and (2.52), we find that, if  $d = 2$ ,

$$(2.65) \quad \nu \left| \int_0^T K_1 dt \right| \leq C \sqrt{T} \|\nabla v\|_{L_4(0,T;L_4)} \|\tilde{P} + \tilde{G}\|_{L_4(0,T;L_4)} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} \\ \leq C_0 \nu^{1/4} \sqrt{T} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}.$$

Bounding  $K_2$  will be performed in the appendix. In the end, we get

$$(2.66) \quad \nu \left| \int_0^T K_2 dt \right| \leq C_0 T^{1/4} \left( \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} + T^{1/4} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \right).$$

Thanks to (2.56), the conclusion of this step is that if  $\nu$  is large enough then

$$(2.67) \quad -(\nu - \mu) \int_0^T \int_{\mathbb{T}^2} \frac{D}{Dt} \nabla \operatorname{div} v \cdot \dot{v} t dx dt \geq (\nu - \mu) \int_0^T \int_{\mathbb{T}^2} (\operatorname{div} \dot{v})^2 t dx dt \\ - C_0 T^{1/4} \left( (\nu T)^{1/4} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} + (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \right).$$

*Step 4.* The last term under consideration in (2.46) is

$$(2.68) \quad \frac{D}{Dt} \nabla P = \nabla \frac{D}{Dt} P - \nabla v \cdot \nabla P.$$

Here the analysis is simple: since  $\dot{P} = -h \operatorname{div} v$ , we have

$$\begin{aligned} \int_{\mathbb{T}^d} \frac{D}{Dt} \nabla P \cdot \dot{v} t \, dx &= L_1 + L_2 \quad \text{with} \quad L_1 := \int_{\mathbb{T}^d} h \operatorname{div} v \operatorname{div} \dot{v} t \, dx \\ &\quad \text{and} \quad L_2 := - \int_{\mathbb{T}^d} \partial_i v^j \partial_j P \dot{v}^i t \, dx. \end{aligned}$$

On the one hand, we obviously have

$$(2.69) \quad |L_1| \leq \frac{\nu}{4} \int_{\mathbb{T}^d} (\operatorname{div} \dot{v})^2 t \, dx + T \nu^{-1} \|h\|_\infty^2 \int_{\mathbb{T}^d} (\operatorname{div} v)^2 \, dx \quad \text{for all } t \in [0, T].$$

On the other hand, integrating by parts a couple of times and using  $\operatorname{div} v = \nu^{-1}(\tilde{P} + \tilde{G})$  yields

$$\begin{aligned} L_2 &= \int_{\mathbb{T}^d} \tilde{P} \nabla \operatorname{div} v \cdot \dot{v} t \, dx + \int_{\mathbb{T}^d} \tilde{P} \nabla v : D \dot{v} t \, dx \\ &= \frac{1}{\nu} \int_{\mathbb{T}^d} \tilde{P} \nabla P \cdot \dot{v} t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^d} \tilde{P} \nabla G \cdot \dot{v} t \, dx + \int_{\mathbb{T}^d} \tilde{P} \nabla v : D \dot{v} t \, dx \\ &= -\frac{1}{2\nu} \int_{\mathbb{T}^d} \tilde{P}^2 \operatorname{div} \dot{v} t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^d} \tilde{P} \nabla G \cdot \dot{v} t \, dx + \int_{\mathbb{T}^d} \tilde{P} \nabla v : D \dot{v} t \, dx. \end{aligned}$$

Hence we have, if  $d = 2$ ,

$$\begin{aligned} \left| \int_0^T L_2(t) \, dt \right| &\leq \frac{1}{2\nu} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|t \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} \\ &\quad + \frac{1}{\nu} \|\tilde{P}\|_{L_\infty(0,T;L_4)} \|\nabla G\|_{L_2(0,T;L_2)} \|t \dot{v}\|_{L_2(0,T;L_4)} \\ &\quad + \|\tilde{P}\|_{L_\infty(0,T;L_\infty)} \|\nabla v\|_{L_2(0,T;L_2)} \|t \nabla \dot{v}\|_{L_2(0,T;L_2)}, \end{aligned}$$

whence, thanks to (2.52) and (2.56),

$$\left| \int_0^T L_2(t) \, dt \right| \leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.$$

So this step gives

$$\begin{aligned} (2.70) \quad \int_{\mathbb{T}^2} \frac{D}{Dt} \nabla P \cdot \dot{v} t \, dx &\geq -\frac{\nu}{4} \int_0^T \int_{\mathbb{T}^2} (\operatorname{div} \dot{v})^2 t \, dx \, dt \\ &\quad - \|h\|_\infty \nu^{-1} T \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 - C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}. \end{aligned}$$

*Step 5.* Plugging inequalities (2.57), (2.62), (2.67) and (2.70) in (2.47) (after integrating on  $[0, T]$ ) and using the fact that for a smooth solution, we have  $\sqrt{\rho t} \dot{v}|_{t=0} = 0$ , we discover that for large enough  $\nu$ ,

$$\begin{aligned} &\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^2 + 2\mu \int_0^T \|\sqrt{t} \nabla \mathcal{P} \dot{v}\|_2^2 \, dt + \frac{3\nu}{2} \int_0^T \|\sqrt{t} \operatorname{div} \dot{v}\|_2^2 \, dt \\ &\leq C_0 \left( \nu^{-1} \int_0^T \|\sqrt{\rho t} \dot{v}\|_2^2 \, dt + \nu^{-1/2} \int_0^T \|\sqrt{t} \nabla \dot{v}\|_2^2 \, dt \right) + \|\sqrt{\rho} \dot{v}\|_{L_2(0,T;L_2)}^2 + \|h\|_\infty T \nu^{-1} \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 \\ &\quad + C_0 T^{1/4} \left( (\nu T)^{1/4} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} + (T^{1/4} + \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \right) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)}. \end{aligned}$$

Taking advantage of inequality (2.5), we have

$$\|\sqrt{\rho}\dot{v}\|_{L_2(0,T;L_2)}^2 + \nu\|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 \leq C_0.$$

Furthermore, Young inequality implies that

$$\begin{aligned} C_0\sqrt{T}\|\sqrt{t}\nabla\dot{v}\|_{L_2(0,T;L_2)} &\leq \frac{\mu}{2}\|\sqrt{t}\nabla\dot{v}\|_{L_2(0,T;L_2)}^2 + C_0T \\ C_0\nu^{1/4}\sqrt{T}\|\sqrt{t}\operatorname{div}\dot{v}\|_{L_2(0,T;L_2)} &\leq \frac{\nu}{2}\|\sqrt{t}\operatorname{div}\dot{v}\|_{L_2(0,T;L_2)}^2 + C_0T\nu^{-1/2} \quad \text{and} \\ C_0T^{1/4}\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}\|\sqrt{t}\nabla\dot{v}\|_{L_2(0,T;L_2)} &\leq C_0T + \frac{1}{2}\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^2 + \frac{\mu}{2}\|\sqrt{t}\nabla\dot{v}\|_{L_2(0,T;L_2)}^2. \end{aligned}$$

In the end, we thus have if  $\nu$  is large enough and  $T \geq 1$ ,

$$\begin{aligned} X^2(t) + \frac{1}{2} \int_0^t Y^2(\tau) d\tau &\leq C_0T + C_0\nu^{-1} \int_0^T X^2 d\tau \quad \text{with} \\ X(t) &:= \|\sqrt{\rho s}\dot{v}\|_{L_\infty(0,t;L_2)} \quad \text{and} \quad Y(t) := (\mu\|\sqrt{t}\nabla\mathcal{P}\dot{v}\|_{L_2(0,T;L_2)}^2 + \nu\|\sqrt{t}\operatorname{div}\dot{v}\|_{L_2(0,T;L_2)}^2)^{\frac{1}{2}}. \end{aligned}$$

Then, applying Gronwall inequality completes the proof of the proposition.  $\square$

The following consequence of those weighted estimate will be fundamental in the proof of uniqueness.

**Corollary 2.1.** *Let  $(\rho, v)$  be a smooth solution of (1.1) on  $[0, T] \times \mathbb{T}^2$  and assume that  $\nu \geq \nu_0$ . Then, we have for all  $\varepsilon \in (0, 1)$ ,*

$$(2.71) \quad \|\operatorname{div} v\|_{L_{2-\varepsilon}(0,T;L_\infty(\mathbb{T}^2))} \leq C_{0,T} \nu^{-1+\varepsilon} \quad \text{and} \quad \|\nabla \mathcal{P}v\|_{L_{2-\varepsilon}(0,T;L_\infty(\mathbb{T}^2))} \leq C_{0,T},$$

for some  $C_{0,T} > 0$  depending on  $\varepsilon$ ,  $T$ ,  $\mu$  and on  $\mathcal{E}_0$ , but not on  $\nu$ .

*Proof.* Remember that

$$(2.72) \quad \mu(\nabla \operatorname{div} v - \Delta v) + \nabla G = \rho \dot{v}.$$

Hence, we have

$$\nabla(\sqrt{t}G) = \mathcal{Q}(\sqrt{t}\rho\dot{v}) \quad \text{and} \quad -\mu\Delta(\sqrt{t}\mathcal{P}v) = \mathcal{P}(\sqrt{t}\rho\dot{v}).$$

Then, combining the fact that  $\mathcal{P}$  and  $\mathcal{Q}$  map  $L_p(\mathbb{T}^2)$  to itself for all  $1 < p < \infty$  with routine interpolation inequalities yields for all  $t \in [0, T]$  and  $p \in [2, \infty)$ ,

$$\|\nabla(\sqrt{t}G)\|_p + \|\nabla^2(\sqrt{t}\mathcal{P}v)\|_p \leq C_{\rho,T} \|\sqrt{\rho t}\dot{v}\|_2^{\frac{2}{p}} \|\nabla(\sqrt{t}\dot{v})\|_2^{1-\frac{2}{p}},$$

which already gives, after time integration and use of Inequality (2.45), that

$$\int_0^T \left( \|\nabla(\sqrt{t}G)\|_p^{\frac{2p}{p-2}} + \|\nabla^2(\sqrt{t}\mathcal{P}v)\|_p^{\frac{2p}{p-2}} \right) dt \leq C_{0,T} \quad \text{if } \nu \geq \nu_0.$$

Next, in order to bound  $\tilde{G}$ , one can argue again by interpolation writing that

$$\|\tilde{G}\|_\infty \lesssim (\nu^{-\frac{1}{2}}\|\tilde{G}\|_2)^\theta (\sqrt{t}\|\nabla G\|_p)^{1-\theta} t^{\frac{\theta-1}{2}} \nu^{\frac{\theta}{2}} \quad \text{with} \quad \theta := \frac{p-2}{2p-2}.$$

Since the previous computations ensure that

$$t \mapsto (\sqrt{t}\|\nabla G\|_p)^{1-\theta} \quad \text{is in} \quad L_{\frac{4p-4}{p-2},loc}(\mathbb{R}_+)$$

and, because  $t \mapsto t^{\frac{\theta-1}{2}}$  is in  $L_{q,loc}(\mathbb{R}_+)$  for all  $q < \frac{4p-4}{p}$  and, finally,  $t \mapsto \|\tilde{G}(t)\|_2$  is bounded according to Proposition 2.1, one can conclude that for all  $\varepsilon \in (0, 1)$ ,

$$\int_0^t \|\tilde{G}\|_\infty^{2-\varepsilon} dt \leq C_{0,T} \nu^\varepsilon.$$

Bounding  $\nabla \mathcal{P}v$  in  $L_\infty(\mathbb{T}^2)$  follows from similar arguments. Finally, as  $\operatorname{div} v = \nu^{-1}(\tilde{G} + \tilde{P})$  and  $P$  is bounded, one gets the desired inequality for  $\operatorname{div} v$ .  $\square$

### 3. THE PROOF OF EXISTENCE IN DIMENSIONS 2 AND 3

This section is mainly devoted to the construction of solutions fulfilling Theorem 1.1 (or the corresponding statement in dimension 3, see the appendix). The main two difficulties we have to face is that the initial density has no regularity whatsoever and is not positive. To fit in the classical literature devoted to the compressible Navier-Stokes equations, one has to mollify the initial data and to make the density strictly positive. Although this procedure does not disturb the a priori estimates we proved hitherto, the state-of-the-art on the topics just ensures the existence of a smooth solution corresponding to the regularized data on some *finite* time interval. As a first, we thus have to justify that, indeed, the estimates we proved so far ensure that the approximate smooth solution is global, if  $\nu$  is large enough. Then, resorting to rather classical compactness arguments will enable us to conclude the proof of Theorem 1.1.

At the end of the section, we justify the convergence from (1.1) to (1.10), namely we prove Theorem 1.3. The passing to the limit therein is very similar to Step 4 of the proof of existence.

*Step 1.* The original initial data are:

$$(3.1) \quad \rho_0 \in L_\infty(\mathbb{T}^d) \text{ and } v_0 \in H^1(\mathbb{T}^d).$$

First, we want to change the initial density in such a way that it is bounded away from zero and still has total mass equal to one. To this end, we introduce for any  $\delta \in (0, 1)$ ,

$$(3.2) \quad \tilde{\rho}_0^\delta = \max\{\rho_0, \delta\} \quad \text{and then} \quad \check{\rho}_0^\delta = \min\{\xi_\delta, \tilde{\rho}_0^\delta\},$$

where  $\xi_\delta \geq 1$  is fixed so that

$$(3.3) \quad \int_{\mathbb{T}^d} \check{\rho}_0^\delta dx = 1.$$

Clearly, we have  $\xi_\delta \rightarrow \rho_0^* := \|\rho_0\|_\infty$  when  $\delta \rightarrow 0$ , and thus

$$(3.4) \quad \delta \leq \check{\rho}_0^\delta \leq \rho_0^* \quad \text{and} \quad \check{\rho}_0^\delta \rightarrow \rho_0 \quad \text{pointwise.}$$

Then, we smooth out both  $\check{\rho}_0^\delta$  and  $v_0^\delta$  as follows:

$$(3.5) \quad \rho_0^\delta = \pi_\delta * \check{\rho}_0^\delta \quad \text{and} \quad v_0^\delta = \pi_\delta * v_0,$$

where  $(\pi_\delta)_{\delta>0}$  is a family of positive mollifiers.

Let us emphasize that the total mass of  $\rho_0^\delta$  is still equal to one, and that  $\rho_0^\delta \geq \delta$ .

*Step 2.* We solve (1.1) with data  $(\rho_0^\delta, v_0^\delta)$  according to the classical literature. For example, one may use the following result (see [9, 30, 32]):

**Theorem 3.1.** *Let  $\rho_0 \in W_p^1(\mathbb{T}^d)$  and  $v_0 \in W_p^{2-2/p}(\mathbb{T}^d)$  for some  $p > d$ , with  $d = 2, 3$ . Assume that  $\rho_0 > 0$ . Then there exists  $T_* > 0$  depending only on the norms of the data, and on  $\inf_{\mathbb{T}^d} \rho_0$  such that (1.1) supplemented with data  $\rho_0$  and  $v_0$  has a unique solution  $(\rho, v)$  on the time interval  $[0, T_*]$ , satisfying<sup>4</sup>*

$$(3.6) \quad v \in W_p^{1,2}(0, T_* \times \mathbb{T}^d) \quad \text{and} \quad \rho \in \mathcal{C}([0, T_*]; W_p^1(\mathbb{T}^d)).$$

Let  $(\rho^\delta, v^\delta)$  be the maximal solution pertaining to data  $(\rho_0^\delta, v_0^\delta)$  provided by the above statement, and let  $T^\delta$  be the largest time so that  $(\rho^\delta, v^\delta)$  fulfills (3.6) for all  $T < T^\delta$ . Since  $(\rho^\delta, v^\delta)$  is smooth and with no vacuum, it satisfies all the formal estimates we proved so far, with the same constants *independent of  $\delta$* . In particular,  $\operatorname{div} v^\delta$  is in  $L_1(0, T; L_\infty)$  for all  $T < T^\delta$ , which implies that  $\rho^\delta$  is bounded from below and above, according to:

$$(3.7) \quad \delta \exp \left\{ - \int_0^T \|\operatorname{div} v^\delta\|_\infty dt \right\} \leq \rho^\delta(t, x) \leq \rho_0^* \exp \left\{ \int_0^T \|\operatorname{div} v^\delta\|_\infty dt \right\}.$$

*Step 3.* Our goal here is to prove that the solution  $(\rho^\delta, v^\delta)$  is actually global. To achieve it, we argue by contradiction, assuming that  $T^\delta$  is finite.

The classical estimates for the continuity equations imply that for all  $T < T^\delta$  (dropping exponents  $\delta$  on  $(\rho^\delta, v^\delta)$ , for better readability):

$$(3.8) \quad \|\nabla \rho(T)\|_p \leq \|\nabla \rho_0^\delta\|_p + C \int_0^T (\|\nabla v\|_\infty \|\nabla \rho\|_p + \|\nabla \operatorname{div} v\|_p) dt.$$

Observe that the previous sections ensure that we have  $\sqrt{\rho t} \dot{v} \in L_\infty(0, T^\delta; L_2)$  and  $\sqrt{t} \nabla \dot{v} \in L_2(0, T^\delta; L_2)$ . Combining with straightforward interpolation arguments and Hölder inequality, we deduce that

$$(3.9) \quad \rho \dot{v} \in L_a(0, T^\delta; L_p(\mathbb{T}^2)) \quad \text{for all } 2 < p < \infty \quad \text{and} \quad a < p' \quad \text{if } d = 2,$$

$$(3.10) \quad \rho \dot{v} \in L_a(0, T^\delta; L_p(\mathbb{T}^3)) \quad \text{for all } p \in ]2, 6[ \quad \text{and} \quad \frac{1}{a} = \frac{5}{4} - \frac{3}{2p} \quad \text{if } d = 3.$$

Remembering that  $\Delta v + \nu \nabla \operatorname{div} v - \nabla P = -\rho \dot{v}$ , we thus get

$$(3.11) \quad \Delta v + \nu \nabla \operatorname{div} v - \nabla P \in L_a(0, T; L_p),$$

whence, using  $L_p$  estimates for the Riesz operator and the fact that  $P = \rho^\gamma$  with  $\rho$  bounded, one may conclude that, uniformly with respect to  $\delta$ , we have for all  $t < T^\delta$ ,

$$(3.12) \quad \|\nabla^2 v(t)\|_p \leq C \|\nabla \rho(t)\|_p + h(t) \quad \text{with} \quad h \in L_a(0, T^\delta).$$

Hence we have for all  $T < T^\delta$ ,

$$(3.13) \quad \|\nabla \rho(T)\|_p \leq \left( \|\nabla \rho_0^\delta\|_p + \int_0^T h(t) dt \right) \exp \left\{ \int_0^T C(1 + \|\nabla v\|_\infty) dt \right\}.$$

---

<sup>4</sup>Recall that  $W_p^{1,2}(0, T_* \times \mathbb{T}^d)$  designates the set of functions  $v : [0, T_*] \times \mathbb{T}^d \rightarrow \mathbb{R}^d$  such that  $v \in W_p^1(0, T_*; L_p(\mathbb{T}^d)) \cap L_p(0, T_*; W_p^2(\mathbb{T}^d))$ , and  $W_p^{2-2/p}(\mathbb{T}^d)$ , the corresponding trace space on  $t = 0$  (that may be identified to the Besov space  $B_{p,p}^{2-2/p}(\mathbb{T}^d)$ ).

In order to close the estimates, we have to bound  $\|\nabla v\|_\infty$ . Since  $p > d$  and  $\nabla v$  is bounded in  $L_1(0, T; BMO)$  (recall Corollary 2.1), one may start from the following well known logarithmic inequality:

$$\|\nabla v\|_\infty \leq C \|\nabla v\|_{BMO} \log \left( e + \frac{\|\nabla v\|_{W_p^1}}{\|\nabla v\|_{BMO}} \right),$$

which, in light of (3.12), implies that

$$\|\nabla v\|_\infty \leq C \|\nabla v\|_{BMO} \log \left( e + \frac{h + \|\nabla \rho\|_p}{\|\nabla v\|_{BMO}} \right).$$

Hence, plugging that inequality in (3.8), we discover that for all  $T < T^\delta$ ,

$$\|\nabla \rho(T)\|_p \leq \|\nabla \rho_0^\delta\|_p + \int_0^T h \, dt + C \int_0^T \left( 1 + \|\nabla v\|_{BMO} \log \left( e + \frac{h + \|\nabla \rho\|_p}{\|\nabla v\|_{BMO}} \right) \right) \|\nabla \rho\|_p \, dt.$$

Since

$$\|\nabla v\|_{BMO} \log \left( e + \frac{h}{\|\nabla v\|_{BMO}} \right) \leq C \max(h, \|\nabla v\|_{BMO})$$

and

$$\|\nabla v\|_{BMO} \log \left( e + \frac{\|\nabla \rho\|_p}{\|\nabla v\|_{BMO}} \right) \leq C(1 + \|\nabla v\|_{BMO}) \log(e + \|\nabla \rho\|_p),$$

we get

$$\|\nabla \rho(T)\|_p \leq \|\nabla \rho_0^\delta\|_p + \int_0^T h(1 + C\|\nabla \rho\|_p) \, dt + C \int_0^T (1 + \|\nabla v\|_{BMO}) \log(e + \|\nabla \rho\|_p) \|\nabla \rho\|_p \, dt.$$

From this and Osgood lemma, one can conclude (as  $T^\delta$  is finite) that  $\nabla \rho$  and  $\nabla u$  belong to  $L_\infty(0, T^\delta; L_p)$  and  $L_1(0, T^\delta; L_\infty)$ , respectively.

Putting together with (3.12), this leads to

$$(3.14) \quad \rho_t = -\operatorname{div}(v\rho) \in L^a(0, T^\delta; L_p).$$

Hence, by Sobolev embedding, one can conclude that there exists  $\alpha > 0$  such that

$$(3.15) \quad \rho \in C^\alpha([0, T^\delta] \times \mathbb{T}^d).$$

Now, one can go back to the momentum equation of (1.1), written in the form

$$(3.16) \quad \rho v_t - \mu \Delta v - \nu \nabla \operatorname{div} v = -\nabla P - \rho v \cdot \nabla v.$$

Thanks to (3.7) and (3.15), one may apply Theorem 2.2. of [9] and get

$$(3.17) \quad \|v\|_{W_p^{1,2}(0, T^\delta \times \mathbb{T}^d)} \leq C_\delta \left( \|\nabla P\|_{L_p(0, T^\delta \times \mathbb{T}^d)} + \|v \cdot \nabla v\|_{L_p(0, T^\delta \times \mathbb{T}^d)} \right).$$

For general  $p > 2$  if  $d = 2$ , or  $2 < p < 6$  if  $d = 3$ , we do not know how to prove directly that  $v \cdot \nabla v$  is in  $L_p(0, T^\delta \times \mathbb{T}^d)$ , and we shall need several steps.

More precisely, if  $d = 2$ , then one may use the fact that for all  $p < q < \infty$ ,

$$\|\nabla v\|_{p^*} \leq C \|\nabla^2 v\|_p^{1/2} \|v\|_q^{1/2} \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{q} \right)$$

which, combined with the fact that  $v \in L_\infty(0, T^\delta; H^1(\mathbb{T}^2))$  (from Proposition 2.1) and thus  $v \in L_\infty(0, T^\delta; L_r(\mathbb{T}^2))$  for all  $r < \infty$ , and (3.12) implies that  $v \cdot \nabla v \in L_{2a}(0, T^\delta; L_p(\mathbb{T}^2))$ . Hence the right-hand side of (3.16) belongs to  $L_{2a}(0, T^\delta; L_p(\mathbb{T}^2))$ , and Theorem 2.2. of [9] implies that

$$\partial_t v \in L_{2a}(0, T^\delta; L_p(\mathbb{T}^2)) \quad \text{and} \quad \nabla^2 v \in L_{2a}(0, T^\delta; L_p(\mathbb{T}^2)).$$



Starting from that new information and arguing as above entails that the right-hand side of (3.16) belongs to  $L_{4a}(0, T^\delta; L_p(\mathbb{T}^2))$ , and so on. After a finite number of steps, we eventually reach  $v \in W_p^{1,2}(0, T^\delta \times \mathbb{T}^2)$ .

For the 3D case we note that the information that  $v \in L_\infty(0, T^\delta; H^1(\mathbb{T}^2))$  implies that

$$(3.18) \quad v \in L_\infty(0, T; L_6(\mathbb{T}^3)).$$

Hence, to bound  $v \cdot \nabla v$  in  $L_p(\mathbb{T}^3)$ , we need to have  $\nabla v$  in  $L_k$  with  $k$  such that  $\frac{1}{6} + \frac{1}{k} = \frac{1}{p}$  (remember that  $2 < p < 6$  in the 3D case). By interpolation and the definition of  $a$  in (3.10), we have

$$(3.19) \quad \|\nabla v\|_k \leq C \|\nabla^2 v\|_p^{1-a/4} \|v\|_6^{a/4}.$$

Hence

$$\|\nabla v\|_k^{\frac{4a}{4-a}} \leq C \|\nabla^2 v\|_p^a \|v\|_6^{\frac{4a^2}{16-4a}},$$

and  $v \cdot \nabla v$  is thus in  $L^{4a/(4-a)}(0, T^\delta; L_p(\mathbb{T}^3))$  which, in view of Theorem 2.2. of [9] yields

$$\partial_t v \in L_{4a/(4-a)}(0, T^\delta; L_p(\mathbb{T}^3)) \quad \text{and} \quad \nabla^2 v \in L_{4a/(4-a)}(0, T^\delta; L_p(\mathbb{T}^3)).$$

Again, after a finite number of steps, we achieve

$$(3.20) \quad \|v\|_{W_p^{1,2}(0, T^\delta \times \mathbb{T}^d)} < \infty.$$

Now, thanks to the trace theorem and the estimates that we proved for  $\rho$ , one may conclude that, if  $T^\delta$  is finite, then

$$\sup_{T < T^\delta} (\|v(T)\|_{W_p^{2-2/p}} + \|\rho(T)\|_{W_p^1}) < \infty \quad \text{and} \quad \inf_{T < T^\delta} \rho(T) > 0.$$

Thanks to that information, one may solve System (1.1) supplemented with initial data  $(\rho(T), v(T))$  whenever  $T < T^\delta$ , and the existence time  $T_*$  provided by Theorem 3.1 is independent of  $T$ . In that way, taking  $T = T^\delta - T_*/2$ , we get a continuation of the solution beyond  $T^\delta$ , thus contradicting the definition of  $T^\delta$ .

Hence  $T^\delta = +\infty$ . In other words, the solution  $(\rho^\delta, v^\delta)$  is global and all the estimates of the previous sections are true on  $\mathbb{R}_+$ . Furthermore, it is clear that they are uniform with respect to  $\delta$ .

*Step 4.* The previous step ensures uniform boundedness of  $(\rho^\delta, v^\delta)$  in the desired existence space. The last step is to prove the convergence of a subsequence. Since we have more regularity than in the classical weak solutions theory, one can pass to the limit by following the steps therein. However, this would give some restriction on the pressure laws that one can consider (typically, if  $P = a\rho^\gamma$ , then one has to assume that  $\gamma > d/2$ ). In our case, the higher regularity of the velocity will enable us to pass to the limit for rather general pressure laws, and by means of a much more elementary method.

To start with, let us observe that, up to extraction, we have

$$(3.21) \quad v^\delta \rightarrow v \quad \text{in} \quad L_2(0, T \times \mathbb{T}^d) \quad \text{for all} \quad T > 0.$$

Indeed, since  $(v^\delta)$  and  $(\sqrt{t} v_t^\delta)$  are bounded in  $L_2(0, T; L_2)$ , Lemma 3.2 of [12] implies that  $(v^\delta)$  is bounded in  $H^{\frac{1}{2}-\alpha}(0, T; L_2(\mathbb{T}^d))$  for all  $\alpha > 0$ , which, combined with the fact that  $(v^\delta)$  is also bounded in  $L_2(0, T; H^1(\mathbb{T}^d))$  implies that

$$(v^\delta) \quad \text{is bounded in} \quad H^{\frac{1}{4}}(0, T \times \mathbb{T}^d).$$

This entails (3.21) by standard compact Sobolev embedding.

This is still not enough to pass to the limit in the pressure term of the momentum equation. To achieve it, we shall exhibit some strong convergence property for the effective viscous flux  $G^\delta$ .

From (3.11) and uniform estimates given by the previous sections, one knows that

$$(3.22) \quad (G^\delta) \text{ is bounded in } L_\infty(0, T; L_2) \cap L_2(0, T; H^1) \text{ for all finite } T > 0,$$

which already yields weak convergence.

To get strong convergence, one can take advantage of uniform estimates for  $(G_t^\delta)$ : from the previous step, Sobolev embeddings and the relation

$$P_t^\delta = -\operatorname{div}(P^\delta v^\delta) - h^\delta \operatorname{div} v^\delta,$$

we gather that  $(P_t^\delta)$  is bounded in  $L_2(0, T; W_p^{-1})$  for all finite  $T > 0$  and  $p < \infty$  (or just  $p \leq 6$  if  $d = 3$ ). Furthermore, we also know that  $\sqrt{t} \operatorname{div} v^\delta$  is bounded in  $L_2(0, T; L_2)$ . Since  $\operatorname{div}(v^\delta \cdot \nabla v^\delta)$  is bounded in  $L_2(0, T; W_p^{-1})$  (again, use the previous step), one may conclude that

$$\sqrt{t} G_t^\delta \text{ is bounded in } L_2(0, T; W_p^{-1}).$$

By suitable modification of Lemma 3.2 of [7], we deduce that

$$(G^\delta) \text{ is bounded in } H^{\frac{1}{2}-\alpha}(0, T; W_p^{-1}) \text{ for all } \alpha > 0,$$

and interpolating with (3.22) allows to get that  $(G^\delta)$  is bounded in  $H^\beta(0, T \times \mathbb{T}^d)$  for some small enough  $\beta > 0$ . So, finally, up to extraction, we have

$$(3.23) \quad G^\delta \rightarrow G \text{ in } L_2(0, T \times \mathbb{T}^d) \text{ for all } T > 0.$$

We are now in a good position to prove the strong convergence of the density. After suitable relabelling, the previous considerations ensure that there exists a sub-sequence  $(\rho^n, v^n)_{n \in \mathbb{N}}$  of  $(\rho^\delta, v^\delta)$  such that, for all  $T > 0$ ,

$$(3.24) \quad \rho^n \rightharpoonup^* \rho \text{ in } L_\infty(0, T \times \mathbb{T}^d) \text{ and } v^n \rightarrow v \text{ in } L_2(0, T \times \mathbb{T}^d).$$

Since for all  $n \in \mathbb{N}$ , we have

$$(3.25) \quad \rho_t^n + \operatorname{div}(\rho^n v^n) = 0,$$

the limit  $(\rho, v)$  fulfills

$$(3.26) \quad \rho_t + \operatorname{div}(\rho v) = 0.$$

At this point, let us emphasize that, since  $\operatorname{div} v \in L_1(0, T; L_\infty)$  (another consequence of the uniform estimates provided by the previous step) and  $\rho \in L_\infty(0, T \times \mathbb{T}^d)$ , one can assert that  $\rho$  is actually a *renormalized* solution of (3.26) (apply Theorem II.2 of [18]), and thus fulfills

$$(3.27) \quad (\rho \log \rho)_t + \operatorname{div}(\rho \log \rho v) + \rho \operatorname{div} v = 0.$$

Of course, since  $(\rho^n, v^n)$  is smooth, we also have

$$(3.28) \quad (\rho^n \log \rho^n)_t + \operatorname{div}(\rho^n \log \rho^n v^n) + \rho^n \operatorname{div} v^n = 0.$$

Then, remembering the definition of  $G^n$ , we get

$$(3.29) \quad (\rho^n \log \rho^n)_t + \operatorname{div}(\rho^n \log \rho^n v^n) + \nu^{-1} \rho^n P(\rho^n) + \nu^{-1} \rho^n G^n = 0$$

and the limit version

$$(3.30) \quad (\rho \log \rho)_t + \operatorname{div}(\rho \log \rho v) + \nu^{-1} \rho P(\rho) + \nu^{-1} \rho G = 0.$$

Denote by  $\overline{\rho \log \rho}$  and  $\overline{\rho P(\rho)}$  the weak limits of  $\rho^n \log \rho^n$  and  $\rho^n P(\rho^n)$ , respectively. Since functions  $z \mapsto z \log z$  and  $z \mapsto zP(z)$  are convex, we know that

$$(3.31) \quad \overline{\rho \log \rho} \geq \rho \log \rho \quad \text{and} \quad \overline{\rho P(\rho)} \geq \rho P(\rho).$$

Furthermore, integrating (3.29) and (3.30) on  $[0, T] \times \mathbb{T}^d$ , we find that

$$\begin{aligned} \nu \left( \int_{\mathbb{T}^d} (\rho^n \log \rho^n - \rho \log \rho)(T) dx - \int_{\mathbb{T}^d} (\rho_0^n \log \rho_0^n - \rho_0 \log \rho_0) dx \right) \\ + \int_0^T \int_{\mathbb{T}^d} (\rho^n P^n - \rho P) dx dt + \int_0^T \int_{\mathbb{T}^d} (\rho^n G^n - \rho G) dx dt = 0. \end{aligned}$$

By construction, the term pertaining to the initial data tends to zero. Furthermore, since  $(G^n)$  converges strongly to  $G$ , the last term also tends to zero. This leads us to

$$\nu \int_{\mathbb{T}^d} (\overline{\rho \log \rho} - \rho \log \rho)(T) dx + \int_0^T \int_{\mathbb{T}^d} (\overline{\rho P} - \rho P) dx dt = 0.$$

Combining with (3.31), one may now conclude that

$$(3.32) \quad \overline{\rho \log \rho} = \rho \log \rho.$$

Since the function  $z \mapsto z \log z$  is strictly convex we find by standard arguments that  $(\rho^n)$  converges strongly and pointwise to  $\rho$ . Hence one can pass to the limit in all the nonlinear terms (in particular in the pressure one) of the momentum equation, and conclude that  $(\rho, v)$  is indeed a solution to (1.1).

Besides, classical arguments that may be found in [18] ensure that  $\rho \in \mathcal{C}(\mathbb{R}_+; L_p)$  for all  $p < \infty$ , and that strong convergence holds true in the corresponding space. Thanks to that information, since (1.6) is fulfilled with data  $(\rho_0^n, v_0^n)$  by the sequence  $(\rho^n, v^n)_{n \in \mathbb{N}}$ , one may pass to the limit and see that  $(\rho, v)$  satisfies (1.6) as well. Finally, since the internal energy  $e$  is continuous with respect to time (a consequence of the strong convergence of  $\rho$ ), one may reproduce the argument that has been used in [12] so as to prove that  $\sqrt{\rho} v \in \mathcal{C}(\mathbb{R}_+; L_2)$ . This completes the proof of our existence theorems in dimensions 2 and 3.  $\square$

**Proof of Theorem 1.3.** We end this section with a fast justification of the convergence of solutions to (1.1) to those of (1.10) when  $\nu$  goes to  $\infty$ , leading to Theorem 1.3. As the proof goes along the lines of that of Theorem 1.1, we just indicate the main steps. The starting point is the estimate provided by Proposition 2.1 which ensures in particular (1.11), that  $(\nabla G^\nu)$  is bounded in  $L_2(\mathbb{R}_+ \times \mathbb{T}^2)$  and that  $(v^\nu)$  is bounded in  $L_\infty(\mathbb{R}_+; H^1)$ , while Proposition 2.2 guarantees that  $(\rho^\nu)$  is bounded in  $L_\infty(\mathbb{R}_+ \times \mathbb{T}^2)$ . Hence, there exists  $(\rho, v) \in L_\infty(\mathbb{R}_+ \times \mathbb{T}^2) \times L_\infty(\mathbb{R}_+; H^1)$  and a subsequence  $(\rho^n, v^n)$  of  $(\rho^\nu, v^\nu)$  such that

$$\rho^n \rightharpoonup^* \rho \quad \text{in } L_\infty(\mathbb{R}_+ \times \mathbb{T}^2) \quad \text{and} \quad v^n \rightharpoonup v \quad \text{in } L_\infty(\mathbb{R}_+; H^1).$$

As in the proof of existence, in order to get some compactness, one may look at time weighted estimates. More specifically, we know from Proposition 2.3 that if  $\nu \geq \nu_0$  then

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^2} \rho |\dot{v}^\nu|^2 t dx + \int_0^T \int_{\mathbb{T}^2} (\mu |\nabla \mathcal{P} \dot{v}^\nu|^2 + \nu |\operatorname{div} \dot{v}^\nu|^2) t dx dt \leq C_0 T e^{\frac{C_0 T}{\nu}},$$

and this ensures that  $(v^\nu)$  is bounded in, say,  $H^{1/4}(0, T \times \mathbb{T}^2)$  for all  $T > 0$ . Hence, we actually have (extracting one more subsequence as the case may be),

$$v^n \rightarrow v \quad \text{in } L_{2,loc}(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$

Next, arguing exactly as in the proof of existence, we get that, for all finite  $T > 0$  and  $p < \infty$ ,  $(G^\nu)$  is bounded in  $L_\infty(0, T; L_2) \cap L_2(0, T; H^1)$  and  $(\sqrt{t} G_t^\nu)$  is bounded in  $L_2(0, T; W_p^{-1})$  from which we deduce that  $(G^\nu)$  is bounded in  $H^{\frac{1}{2}-\alpha}(0, T; W_p^{-1})$  for all  $\alpha > 0$  and, eventually

$$G^n \rightarrow G \quad \text{in} \quad L_{loc}^2(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$

Putting together all those results of convergence, one gets

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \quad \text{and} \quad \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \mu \Delta \mathcal{P}v + \nabla G = 0.$$

Since we know in addition (from (1.11)) that  $\operatorname{div} v = 0$ , one can conclude that  $(\rho, v, \nabla G)$  satisfies (1.10). Finally, from the uniform bounds that are available for  $(\rho^\nu, v^\nu)$ , one may check that  $(\rho, v, \nabla G)$  has the regularity of the solution constructed in Theorem 2.1 of [12], which is unique. Hence the whole family  $(\rho^\nu, v^\nu)$  converges to  $(\rho, v)$ .  $\square$

#### 4. THE PROOF OF UNIQUENESS

Here we show the uniqueness of the solutions we constructed in the paper, both in dimensions 2 and 3. The main difficulty we have to face is that having  $\operatorname{div} v$  and  $\nabla \mathcal{P}v$  in  $L_1(0, T; L_\infty)$  (see Corollary 2.1) *does not* ensure that  $\nabla v$  is in  $L_1(0, T; L_\infty)$  so that, in contrast with our recent work [10], one cannot reformulate System (1.1) in Lagrangian coordinates to prove uniqueness. However, we do have  $\nabla v$  is in  $L_r(0, T; BMO)$  for some  $r > 1$ , which will turn out to be enough to prove uniqueness *provided that the pressure law is linear*. Actually, we encounter the same difficulty as in D. Hoff's paper [23]: we need, at some point, to bound the difference of the pressures in  $H^{-1}$  by the norm of the difference of the densities in  $H^{-1}$ , which is impossible if  $P$  is nonlinear.

**Proposition 4.1.** *Assume that  $P(\rho) = a\rho$  for some  $a > 0$ , and consider two finite energy solutions  $(\rho, v)$  and  $(\bar{\rho}, \bar{v})$  of (1.1) on  $[0, T_0] \times \mathbb{T}^d$  ( $d = 2, 3$ ) with bounded density, satisfying (1.4) and emanating from the same initial data. If, in addition,  $v$  and  $\bar{v}$  are in  $L_\infty(0, T_0; H^1)$ ,  $\sqrt{t} \nabla \dot{v}$  and  $\sqrt{t} \nabla \dot{\bar{v}}$  are in  $L_2(0, T_0 \times \mathbb{T}^d)$ ,  $\sqrt{\rho t} \dot{v}$ ,  $\sqrt{\bar{\rho} t} \dot{\bar{v}}$  belong to  $L_\infty(0, T_0; L_2)$ ,*

$$(4.1) \quad \nabla \bar{v} \in L_2(0, T_0; L_3(\mathbb{T}^d)) \quad \text{and} \quad \int_0^{T_0} (1 + |\log t|) \|\nabla \bar{v}(t)\|_{BMO} dt < \infty,$$

*then  $(\bar{\rho}, \bar{v}) \equiv (\rho, v)$  on  $[0, T_0] \times \mathbb{T}^d$ .*

*Proof.* The general scheme of the proof is the same in dimensions 2 or 3. Assume that  $a = 1$  for notational simplicity and consider two solutions  $(\rho, v)$  and  $(\bar{\rho}, \bar{v})$  to (1.1) corresponding to the same initial data  $(\rho_0, v_0)$ . The system for the difference

$$\delta \rho := \rho - \bar{\rho} \quad \text{and} \quad \delta v := v - \bar{v}$$

reads

$$(4.2) \quad \begin{cases} \delta \rho_t + \operatorname{div}(\delta \rho \bar{v} + \rho \delta v) = 0, \\ \rho \delta v_t + \rho v \cdot \nabla \delta v - \mu \Delta \delta v - (\lambda + \mu) \nabla \operatorname{div} \delta v + \nabla \delta \rho = \delta \rho \dot{\bar{v}} + \rho \delta v \cdot \nabla \bar{v}. \end{cases}$$

In order to show that  $\delta \rho \equiv 0$  and  $\delta v \equiv 0$ , we shall perform estimates in  $L_\infty(0, T; \dot{H}^{-1})$  for  $\delta \rho$ , and in  $L_2(0, T \times \mathbb{T}^d)$  for  $\sqrt{\rho} \delta v$ . To this end, we set  $\phi := -(-\Delta)^{-1} \delta \rho$  (which makes sense, since  $\int_{\mathbb{T}^d} \delta \rho dx = 0$ ) so that

$$(4.3) \quad \|\nabla \phi\|_2 = \|\delta \rho\|_{\dot{H}^{-1}} = \|\delta \rho\|_{H^{-1}}.$$

Now, testing the first equation of (4.2) by  $\phi$  yields

$$\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_2^2 \leq \left| \int_{\mathbb{T}^d} (\bar{v} \cdot \nabla \phi \delta \rho + \rho \delta v \cdot \nabla \phi) dx \right|.$$

The last term is bounded as follows:

$$\left| \int_{\mathbb{T}^d} \rho \delta v \cdot \nabla \phi dx \right| \leq \sqrt{\rho^*} \|\sqrt{\rho} \delta v\|_2 \|\nabla \phi\|_2.$$

Regarding the first one, observe that (with the usual summation convention)

$$\int_{\mathbb{T}^d} \bar{v} \cdot \nabla \phi \delta \rho dx = \int_{\mathbb{T}^d} \bar{v}^j \partial_j \phi \Delta \phi dx = - \int_{\mathbb{T}^d} \partial_k \bar{v}^j \partial_j \phi \partial_k \phi dx + \frac{1}{2} \int_{\mathbb{T}^d} \operatorname{div} \bar{v} |\nabla \phi|^2 dx.$$

Hence, we have

$$\left| \int_{\mathbb{T}^d} \bar{v} \cdot \nabla \phi \delta \rho dx \right| \leq C \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi \otimes \nabla \phi\|_{\mathcal{H}^1}.$$

Now, in light of the following inequality (see e.g. [31, Thm. D]),

$$(4.4) \quad \|f\|_{\mathcal{H}^1} \leq C \|f\|_1 (|\log \|f\|_1| + \log(e + \|f\|_\infty)),$$

we discover that

$$(4.5) \quad \left| \int_{\mathbb{T}^d} \bar{v} \cdot \nabla \phi \delta \rho dx \right| \leq C \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi\|_2^2 (|\log \|\nabla \phi\|_2^2| + \log(e + \|\nabla \phi\|_\infty^2)).$$

Since the densities are bounded by  $\rho^*$ , we have

$$\|\nabla \phi(t)\|_\infty \leq C \rho^* \quad \text{for all } t \in [0, T_0].$$

Hence Inequality (4.5) implies that for some constant  $C$  depending only on  $\rho^*$ ,

$$(4.6) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_2^2 \leq C \left( \|\sqrt{\rho} \delta v\|_2 + \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi\|_2 (1 + |\log \|\nabla \phi\|_2|) \right) \|\nabla \phi\|_2.$$

Since  $\nabla \phi(0) = 0$ , this gives after integration that for all  $t \in [0, T_0]$ ,

$$(4.7) \quad \|\nabla \phi(t)\|_2 \leq C \left( \int_0^t \|\sqrt{\rho} \delta v\|_2 d\tau + \int_0^t \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi\|_2 (1 + |\log \|\nabla \phi\|_2|) d\tau \right).$$

Hence, using (4.3) and denoting  $Z(t) := \sup_{\tau \leq t} \tau^{-1/2} \|\delta \rho(\tau)\|_{\dot{H}^{-1}}$ , we get after using Cauchy-Schwarz inequality, for all  $T \in [0, T_0]$ ,

$$(4.8) \quad Z(T) \leq C \left( \sup_{t \in [0, T]} \int_0^t \|\nabla \bar{v}\|_{\text{BMO}} Z (1 + |\log \tau| + |\log Z|) d\tau + \|\sqrt{\rho} \delta v\|_{L_2(0, T \times \mathbb{T}^d)} \right).$$

In order to control the difference of the velocities, we introduce the solution  $w$  to the following backward parabolic system:

$$(4.9) \quad \begin{cases} \rho w_t + \rho v \cdot \nabla w + \mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = -\rho \delta v, \\ w|_{t=T} = 0. \end{cases}$$

Solving the above system is not part of the classical theory for linear parabolic systems, as the coefficients are rough and may vanish. However, if  $\rho$  and  $v$  are regular with  $\rho$  bounded away from zero, this is well known, and the case we are interested may be achieved by a regularizing process of  $\rho$  and  $v$ , after using Inequality (4.12) below for the corresponding regular solutions.

Now, testing the equation by  $w$ , we find that

$$(4.10) \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^d} \rho |w|^2 dx + \int_0^T \int_{\mathbb{T}^d} (\mu |\nabla \mathcal{P} w|^2 + \nu (\operatorname{div} w)^2) dx dt \leq \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 dx dt.$$

Next, we test (4.9) by  $w_t$  and take advantage of the usual elliptic estimates given by

$$\mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = -\rho \dot{w} - \rho \delta v$$

that ensure that

$$\mu^2 \|\nabla^2 \mathcal{P} w\|_2^2 + \nu^2 \|\nabla \operatorname{div} w\|_2^2 = \|\rho \dot{w} + \rho \delta v\|_2^2 \leq \rho^* \|\sqrt{\rho} (\dot{w} + \delta v)\|_2^2$$

in order to get

$$(4.11) \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^d} (\mu |\nabla \mathcal{P} w(t)|^2 + \nu (\operatorname{div} w(t))^2) dx \\ + \int_0^T \int_{\mathbb{T}^d} \left( \rho |w_t|^2 + \frac{\mu^2}{6\rho^*} |\nabla^2 \mathcal{P} w|^2 + \frac{\nu^2}{6\rho^*} |\nabla \operatorname{div} w|^2 \right) dx dt \leq \frac{3}{2} \int_0^T \int_{\mathbb{T}^d} (\rho |\delta v|^2 + \rho |v \cdot \nabla w|^2) dx dt.$$

If  $d = 2$  then we bound the last term as follows:

$$\begin{aligned} \int_{\mathbb{T}^2} \rho |v \cdot \nabla w|^2 dx &\leq \sqrt{\rho^*} \|\rho^{1/4} v\|_4^2 \|\nabla w\|_4^2 \\ &\leq C \sqrt{\rho^*} \|\rho^{1/4} v\|_4^2 \|\nabla w\|_2 \|\nabla^2 w\|_2 \\ &\leq C \frac{(\rho^*)^2}{\mu^2} \|\rho^{1/4} v\|_4^4 \|\nabla w\|_2^2 + \frac{\mu^2}{12\rho^*} \|\nabla^2 w\|_2^2. \end{aligned}$$

If  $d = 3$ , then we rather write that

$$\begin{aligned} \int_{\mathbb{T}^3} \rho |v \cdot \nabla w|^2 dx &\leq (\rho^*)^{3/4} \|\sqrt{\rho} v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\nabla w\|_2^{1/2} \|\nabla^2 w\|_2^{3/2} \\ &\leq C (\rho^*)^3 \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^6 \|\nabla w\|_2^2 + \frac{\mu^2}{12\rho^*} \|\nabla^2 w\|_2^2. \end{aligned}$$

Hence, using the properties of regularity of  $v$ , plugging the above inequality in (4.11), then resorting to Gronwall inequality, we get

$$(4.12) \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^d} (\rho |w|^2 + \mu |\nabla \mathcal{P} w|^2 + \nu (\operatorname{div} w)^2) dx \\ + \int_0^T \int_{\mathbb{T}^d} (\mu |\nabla \mathcal{P} w|^2 + \nu (\operatorname{div} w)^2 + \mu^2 |\nabla^2 \mathcal{P} w|^2 + \nu^2 |\nabla \operatorname{div} w|^2 + \rho |w_t|^2) dx dt \\ \leq C_T \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 dx dt,$$

with  $C_T$  depending only on the norms of the two solutions on  $[0, T]$ .

Let us next test (4.2) by  $w$ . We get

$$(4.13) \quad \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} \delta \rho \operatorname{div} w dx dt \leq \int_0^T \int_{\mathbb{T}^d} (\delta \rho \dot{v} \cdot w + \rho (\delta v \cdot \nabla \bar{v}) \cdot w) dx dt.$$

One can bound the first term of the right-hand side as follows:

$$\begin{aligned}
\left| \int_0^T \int_{\mathbb{T}^d} \delta \rho \dot{v} \cdot w \, dx \, dt \right| &= \left| \int_0^T \int_{\mathbb{T}^d} t^{-1/2} \delta \rho \, t^{1/2} \dot{v} \cdot w \, dx \, dt \right| \\
&\leq \|t^{-1/2} \delta \rho\|_{L_\infty(0,T;\dot{H}^{-1})} \|t^{1/2} \nabla(\dot{v} \cdot w)\|_{L_1(0,T;L_2)} \\
&\leq \|t^{-1/2} \delta \rho\|_{L_\infty(0,T;\dot{H}^{-1})} \left( \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \|w\|_{L_2(0,T;L_\infty)} \right. \\
&\quad \left. + \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \|\nabla w\|_{L_2(0,T;L_3)} \right).
\end{aligned}$$

For bounding the last term of (4.13), one can just use the fact that

$$\int_0^T \int_{\mathbb{T}^d} \rho(\delta v \cdot \nabla \bar{v}) \cdot w \, dx \, dt \leq \sqrt{\rho^*} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \|\nabla \bar{v}\|_{L_2(0,T;L_3)} \|w\|_{L_\infty(0,T;L_6)}.$$

Finally, we note that

$$\int_0^T \int_{\mathbb{T}^d} \delta \rho \operatorname{div} w \, dx \, dt \leq T \|t^{-1/2} \delta \rho\|_{L_\infty(0,T;\dot{H}^{-1})} \|\nabla \operatorname{div} w\|_{L_2(0,T;L_2)}.$$

Plugging the above three inequalities in (4.13), we get

$$\begin{aligned}
(4.14) \quad \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 \, dx \, dt &\leq \|t^{-1/2} \delta \rho\|_{L_\infty(0,T;\dot{H}^{-1})} \left( T \|\nabla \operatorname{div} w\|_{L_2(0,T;L_2)} \right. \\
&\quad \left. + C \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \|w\|_{L_2(0,T;L_\infty)} + C \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \|\nabla w\|_{L_2(0,T;L_3)} \right) \\
&\quad + C \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \|\nabla \bar{v}\|_{L_2(0,T;L_3)} \|w\|_{L_\infty(0,T;L_6)}.
\end{aligned}$$

Observe that our assumptions on  $\bar{v}$  guarantee that we have

$$(4.15) \quad \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} + \|\nabla \bar{v}\|_{L_2(0,T;L_3)} \leq C_T.$$

Next, we have to bound the terms containing  $w$  in (4.14) by means of the data. Since  $\int_{\mathbb{T}^d} \rho w \, dx$  need not be zero, Poincaré inequality (A.2) becomes

$$\|w\|_2 \leq \left| \int_{\mathbb{T}^d} \rho w \, dx \right| + \rho^* \|\nabla w\|_2.$$

To bound the mean value of  $\rho w$ , we note that integrating (4.9) on  $[t, T] \times \mathbb{T}^d$  readily gives

$$\int_{\mathbb{T}^d} (\rho w)(t, x) \, dx = \int_t^T \int_{\mathbb{T}^d} (\rho \delta v)(\tau, x) \, dx \, d\tau.$$

Therefore we have

$$\left| \int_{\mathbb{T}^d} (\rho w)(t) \, dx \right| \leq \sqrt{\rho^*} T^{1/2} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \quad \text{for all } t \in [0, T],$$

whence

$$(4.16) \quad \|w(t)\|_2 \leq C_{\rho^*} (\|\nabla w(t)\|_2 + T^{1/2} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}) \quad \text{for all } t \in [0, T].$$

Then, combining with (4.12), we end up with

$$\|w\|_{L_2(0,T;H^1)} \leq C_{0,T} T^{1/2} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \quad \text{and} \quad \|\nabla w\|_{L_2(0,T;\dot{H}^1)} \leq C_{0,T} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}.$$

By interpolation and Sobolev embedding, it follows that for small enough  $\varepsilon$  if  $d = 2$  (and  $\varepsilon = 1/4$  if  $d = 3$ ), we have

$$(4.17) \quad \|w\|_{L_2(0,T;L_\infty)} \leq C_\varepsilon T^{1/2-\varepsilon} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}.$$

Likewise, we have

$$\begin{aligned} \|\nabla w\|_{L_2(0,T;L_3(\mathbb{T}^2))} &\lesssim \|\nabla w\|_{L_2(0,T;L_2(\mathbb{T}^2))}^{2/3} \|\nabla^2 w\|_{L_2(0,T;L_2(\mathbb{T}^2))}^{1/3} \\ \text{and } \|\nabla w\|_{L_2(0,T;L_3(\mathbb{T}^3))} &\lesssim \|\nabla w\|_{L_2(0,T;L_2(\mathbb{T}^3))}^{1/2} \|\nabla^2 w\|_{L_2(0,T;L_2(\mathbb{T}^3))}^{1/2}, \end{aligned}$$

whence

$$(4.18) \quad \|\nabla w\|_{L_2(0,T;L_3)} \leq CT^\alpha \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \quad \text{with } \alpha = 1/3 \text{ if } d = 2, \quad \alpha = 1/4 \text{ if } d = 3.$$

Finally, using once more that  $H^1(\mathbb{T}^d) \hookrightarrow L_6(\mathbb{T}^d)$  for  $d = 2, 3$ , we get after plugging all the above inequalities in (4.14), for all  $T \in [0, T_0]$ ,

$$\|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}^2 \leq CT^{1/3} \left( \|t^{-1/2} \delta \rho\|_{L^\infty(0,T;\dot{H}^{-1})} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} + \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}^2 \right).$$

Clearly, the above inequality implies that, if  $T$  is small enough then

$$(4.19) \quad \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \leq CT^{1/3} Z(T).$$

Plugging that inequality in (4.8) and assuming that  $T$  is small enough, we obtain

$$Z(t) \leq C_T \int_0^t (1 + \|\nabla \bar{v}(\tau)\|_{BMO} (1 + |\log \tau| + |\log Z(\tau)|)) Z(\tau) d\tau \quad \text{for all } t \in [0, T].$$

Then, Osgood lemma (see e.g. [1, Lem. 3.4]) implies that  $Z \equiv 0$  on  $[0, T]$ , and thus, owing to (4.19), that  $\sqrt{\rho} \delta v \equiv 0$  on  $[0, T]$ .

Now, since  $\sqrt{\rho} \delta v$  and  $\delta \rho$  are zero, the second equation of (4.2) becomes

$$\rho \delta v_t + \rho v \cdot \nabla \delta v - \mu \Delta \delta v - (\lambda + \mu) \nabla \operatorname{div} \delta v = 0,$$

which implies that

$$\frac{1}{2} \|\sqrt{\rho} \delta v\|_{L^\infty(0,T;L_2)}^2 + \int_0^T (\mu \|\nabla \mathcal{P} \delta v\|_2^2 + \nu \|\operatorname{div} \delta v\|_2^2) dx = 0.$$

Since  $\int_{\mathbb{T}^d} \rho \delta v dx = 0$ , one gets (in light of Inequality (A.2)) that  $\delta v = 0$  on  $[0, T]$ , which completes the proof of uniqueness.

To complete the proof of Theorem 1.2, it suffices to observe that Inequality (2.71) implies Assumption (4.1) in Proposition 4.1.  $\square$

## APPENDIX A. SOME INEQUALITIES

The following Osgood type lemma has been used in Section 2.

**Lemma A.1.** *Let  $f$  and  $g$  be two locally integrable nonnegative functions on  $\mathbb{R}_+$ , and assume that the a.e. differentiable function  $X : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfies*

$$X' \leq f X \log(A + BX) + g X \quad \text{for some } A \geq 1 \text{ and } B \geq 0.$$

*Then we have for all  $t \geq 0$ ,*

$$A + BX(t) \leq \left( A + B e^{\int_0^t g d\tau} X(0) \right)^{\exp \int_0^t f d\tau}.$$

*Proof.* It suffices to prove the inequality on  $[0, T]$  for all  $T \geq 0$ . Setting  $Y(t) := e^{-\int_0^t g d\tau} X(t)$ , then  $Z(t) := C_T Y(t)$  with  $C_T := \exp \int_0^T g d\tau$ , we have for all  $t \in [0, T]$ ,

$$(A + BZ)' = BZ' \leq BZ \log(A + BZ) f \leq (A + BZ) \log(A + BZ) f.$$



Therefore, integrating once,

$$\log(\log(A + BZ(t))) \leq \log(\log(A + BZ(0))) + \int_0^t f \, d\tau \quad \text{for all } t \in [0, T].$$

Then considering  $t = T$  and taking exp twice gives

$$A + BZ(T) \leq (A + BZ(0))^{\exp \int_0^T f \, d\tau}.$$

Reverting to the original function  $X$  gives exactly what we want at  $t = T$ .  $\square$

We also used the following Poincaré inequality.

**Lemma A.2.** *Let  $\rho$  be in  $L_{p'}(\mathbb{T}^d)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $2 \leq p \leq \frac{2d}{d-2}$  if  $d \geq 3$  ( $2 \leq p < +\infty$  if  $d = 2$ ). Assume that*

$$(A.1) \quad \int_{\mathbb{T}^d} \rho b \, dx = 0 \quad \text{and} \quad M := \int_{\mathbb{T}^d} \rho \, dx > 0.$$

*There exists a constant  $C_p$  depending on  $p$  and on  $d$  (and with  $C_2 = 1$ ), such that*

$$(A.2) \quad \|b\|_2 \leq \left(1 + \frac{C_p}{M} \|\rho - c\|_{p'}\right) \|\nabla b\|_2 \quad \text{for any real number } c.$$

*Furthermore, in dimension  $d = 2$ , we have*

$$(A.3) \quad \|b\|_2 \leq C \log^{\frac{1}{2}} \left(e + \frac{\|\rho - c\|_2}{M}\right) \|\nabla b\|_2.$$

*Proof.* Let  $\bar{b}$  be the average of  $b$  and  $\tilde{b} := b - \bar{b}$ . Then we have by Poincaré inequality,

$$(A.4) \quad \|b\|_2 \leq |\bar{b}| + \|\tilde{b}\|_2 \leq |\bar{b}| + \|\nabla b\|_2.$$

Now, hypothesis (A.1) implies that for all real number  $c$ , we have

$$(A.5) \quad -M\bar{b} = \int_{\mathbb{T}^d} (\rho - c)\tilde{b} \, dx.$$

Therefore, by Sobolev inequality,

$$(A.6) \quad M|\bar{b}| \leq \|\rho - c\|_{p'} \|\tilde{b}\|_p \leq C_p \|\rho - c\|_{p'} \|\nabla b\|_2$$

and, clearly,  $C_2 = 1$ . This gives (A.2).

To handle the endpoint case  $d = 2$  and  $p = +\infty$ , decompose  $\tilde{b}$  into Fourier series:

$$\tilde{b}(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \hat{b}_k e^{2i\pi k \cdot x},$$

and set for any integer  $n$ ,

$$\tilde{b}_n(x) := \sum_{1 \leq |k| \leq n} \hat{b}_k e^{2i\pi k \cdot x}.$$

By Cauchy-Schwarz inequality, it is easy to prove that

$$(A.7) \quad \|\tilde{b}_n\|_\infty \leq C \sqrt{\log n} \|\nabla b\|_2.$$

Because the average of  $\tilde{b}_n$  is 0, one may write, thanks to (A.5) that for all  $c \in \mathbb{R}$ ,

$$-M\bar{b} = \int_{\mathbb{T}^2} (\rho - c)\tilde{b} \, dx = \int_{\mathbb{T}^2} \rho \tilde{b}_n \, dx + \int_{\mathbb{T}^2} (\rho - c)(\tilde{b} - \tilde{b}_n) \, dx.$$

Therefore, using Hölder and Poincaré inequality, and also (A.7),

$$\begin{aligned} M|\bar{b}| &\leq \|\rho\|_1 \|\tilde{b}_n\|_\infty + \|\rho - c\|_2 \|\tilde{b} - \tilde{b}_n\|_2 \\ &\leq C \left( \sqrt{\log n} M + n^{-1} \|\rho - c\|_2 \right) \|\nabla b\|_2. \end{aligned}$$

Then taking  $n \approx \|\rho - c\|_2/M$  gives

$$(A.8) \quad |\bar{b}| \leq C \log^{\frac{1}{2}} \left( e + \frac{\|\rho - c\|_2}{M} \right) \|\nabla b\|_2,$$

which, combined with (A.4) yields (A.3).  $\square$

We used the following version of Desjardins' estimate in [14].

**Lemma A.3.** *Let  $\rho \in L_\infty(\mathbb{T}^2)$  with  $\rho \geq 0$ , and  $u \in H^1(\mathbb{T}^2)$ . Then, there exists a universal constant  $C$  such that for all real number  $c$ , we have*

$$(A.9) \quad \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} \leq C \|\sqrt{\rho}u\|_2 \|\nabla u\|_2 \log^{\frac{1}{2}} \left( e + \frac{\|\rho - c\|_2}{M} + \frac{\|\rho\|_2 \|\nabla u\|_2^2}{\|\sqrt{\rho}u\|_2^2} \right).$$

*Proof.* Let  $\tilde{u} := u - \bar{u}$  and fix some  $n \in \mathbb{N}$ . Then, keeping the same notation as in the above lemma and using Hölder inequality,

$$\begin{aligned} \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} &= \left( \int_{\mathbb{T}^2} (\bar{u} + \tilde{u}_n + (\tilde{u} - \tilde{u}_n))^2 \rho u^2 dx \right)^{\frac{1}{2}} \\ &\leq |\bar{u}| \|\sqrt{\rho}u\|_2 + \|\sqrt{\rho}u\|_2 \|\tilde{u}_n\|_\infty + \|\rho\|_2^{\frac{1}{4}} \|\tilde{u} - \tilde{u}_n\|_8 \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{4}}. \end{aligned}$$

We thus have, using Young inequality and embedding  $\dot{H}^{\frac{3}{4}}(\mathbb{T}^2) \hookrightarrow L_8(\mathbb{T}^2)$ ,

$$(A.10) \quad \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} \leq 2 \|\sqrt{\rho}u\|_2 (|\bar{u}| + \|\tilde{u}_n\|_\infty) + C \|\rho\|_2^{\frac{1}{2}} \|\tilde{u} - \tilde{u}_n\|_{\dot{H}^{\frac{3}{4}}}^2.$$

Hence, taking advantage of (A.7) and of

$$(A.11) \quad \|\tilde{u} - \tilde{u}_n\|_{\dot{H}^{\frac{3}{4}}} \leq n^{-1/4} \|\nabla u\|_2,$$

then plugging (A.11) in (A.10), we get

$$\left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} \lesssim \|\sqrt{\rho}u\|_2 |\bar{u}| + (\sqrt{\log n} \|\sqrt{\rho}u\|_2 + n^{-\frac{1}{2}} \|\rho\|_2^{\frac{1}{2}} \|\nabla u\|_2) \|\nabla u\|_2.$$

Taking  $n \approx \frac{\|\rho\|_2 \|\nabla u\|_2^2}{\|\sqrt{\rho}u\|_2^2}$  and using (A.8) to bound  $|\bar{u}|$  yields the desired inequality.  $\square$

## APPENDIX B. END OF THE PROOF OF TIME WEIGHTED ESTIMATES IN THE 2D CASE

We here provide the reader with the proofs of Inequalities (2.61) and (2.66).

**Proof of (2.61).** We use (2.49) to bound  $I_2$  as follows:

$$(B.1) \quad I_2 = \int_{\mathbb{T}^2} \nabla v \cdot \nabla^2 \left( \mathcal{P}v - \frac{1}{\nu} \nabla(-\Delta)^{-1}(\tilde{G} + \tilde{P}) \right) \cdot \dot{v} t \, dx =: I_{21} + I_{22} + I_{23}.$$

From (2.72), we know that

$$(B.2) \quad \mu^2 \|\sqrt{t} \nabla^2 \mathcal{P}v(t)\|_2^2 + \|\sqrt{t} \nabla G(t)\|_2^2 \leq \rho^* \|\sqrt{\rho t} \dot{v}\|_2^2.$$

Since we have

$$\frac{\mu}{(\rho^*)^{1/4}} \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_4(0,T;L_2)} \leq \left( \frac{\mu T^{1/2}}{\sqrt{\rho^*}} \|\nabla^2 \mathcal{P}v\|_{L_2(0,T;L_2)} \right)^{1/2} \left( \mu \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_\infty(0,T;L_2)} \right)^{1/2}$$

and a similar inequality for  $\nabla G$ , combining (B.2) and Proposition 2.1 yields

$$(B.3) \quad \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_4(0,T;L_2)} + \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \leq C_0 T^{1/4} \|\sqrt{t} \rho \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}.$$

Therefore, putting together with (2.48), we gather that

$$(B.4) \quad \begin{aligned} \left| \int_0^T I_{21} \, dt \right| &\leq \|\nabla v\|_{L_4(0,T \times \mathbb{T}^2)} \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ &\leq C_0 T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}. \end{aligned}$$

Term  $I_{22}$  is almost the same: taking into account (B.3), we obtain

$$(B.5) \quad \begin{aligned} \left| \int_0^T I_{22} \, dt \right| &\leq \nu^{-1} \|\nabla v\|_{L_4(0,T \times \mathbb{T}^2)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ &\leq C_0 \nu^{-1} T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}. \end{aligned}$$

To handle  $I_{23}$ , we integrate by parts several times and get (with the summation convention for repeated indices and the notation  $\psi := (-\Delta)^{-1} \tilde{P}$ ):

$$\begin{aligned} I_{23} &= -\frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ijk}^3 \psi \dot{v}^i t \, dx \\ &= \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k \operatorname{div} v \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx \\ &= \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{P} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx \\ &= -\frac{1}{\nu^2} \int_{\mathbb{T}^2} \tilde{P} \partial_{ikk}^2 \psi \dot{v}^i t \, dx - \frac{1}{\nu^2} \int_{\mathbb{T}^2} \tilde{P} \partial_{ik}^2 \psi \partial_k \dot{v}^i t \, dx \\ &\quad + \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx. \end{aligned}$$

As  $\psi := (-\Delta)^{-1} \tilde{P}$ , integrating by parts one more time in the first term of the right-hand side just above, we conclude that

$$(B.6) \quad \begin{aligned} I_{23} &= -\frac{1}{2\nu^2} \int_{\mathbb{T}^2} \tilde{P}^2 \operatorname{div} \dot{v} t \, dx - \frac{1}{\nu^2} \int_{\mathbb{T}^2} \tilde{P} \partial_{ik}^2 \psi \partial_k \dot{v}^i t \, dx \\ &\quad + \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx. \end{aligned}$$

Hence, using Hölder inequality and the continuity of  $\nabla^2(-\Delta)^{-1}$  on  $L_4(\mathbb{T}^2)$ , we get

$$\begin{aligned} \left| \int_0^T I_{23}(t) dt \right| &\lesssim \frac{\sqrt{T}}{\nu^2} \left( \|\tilde{P}\|_{L_4(0,T;\mathbb{T}^2)}^2 \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)} \right. \\ &\quad + \|\nabla G\|_{L_2(0,T;\mathbb{T}^2)} \|\tilde{P}\|_{L_\infty(0,T;L_4)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ &\quad \left. + \nu \|\nabla v\|_{L_2(0,T;\mathbb{T}^2)} \|\tilde{P}\|_{L_\infty(0,T;\mathbb{T}^2)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)} \right). \end{aligned}$$

Hence, thanks to (1.6) and (2.52), one can conclude that

$$(B.7) \quad \left| \int_0^T I_{23}(t) dt \right| \leq C_0 \sqrt{T} (\nu^{-1} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + \nu^{-2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}).$$

**Proof of (2.66).** We use the decomposition  $K_2 = K_{2,1} + K_{2,2} + K_{2,3}$  with

$$\begin{aligned} K_{2,1} &:= \int_{\mathbb{T}^2} (\nabla \mathcal{P} v \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx, & K_{2,2} &:= -\frac{1}{\nu} \int_{\mathbb{T}^2} (\nabla^2(-\Delta)^{-1} \tilde{G} \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx \\ \text{and } K_{2,3} &:= -\frac{1}{\nu} \int_{\mathbb{T}^2} (\nabla^2(-\Delta)^{-1} \tilde{P} \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx. \end{aligned}$$

In order to handle  $K_{2,1}$ , we integrate by parts (note that  $\operatorname{div} \mathcal{P} v = 0$ ) and use the fact that  $\nu \operatorname{div} v = \tilde{P} + \tilde{G}$ . We get, with the usual summation convention

$$K_{2,1} = -\frac{\sqrt{t}}{\nu} \int_{\mathbb{T}^2} \partial_i (\mathcal{P} v)^j \tilde{P} \partial_j \dot{v}^i \sqrt{t} dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_i (\mathcal{P} v)^j \sqrt{t} \partial_j G \dot{v}^i \sqrt{t} dx.$$

Therefore,

$$\begin{aligned} \nu \left| \int_0^T K_{2,1} dt \right| &\leq \sqrt{T} \|\nabla \mathcal{P} v\|_{L_2(0,T;L_2)} \|\tilde{P}\|_{L_\infty(0,T;L_\infty)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\ &\quad + \|\nabla \mathcal{P} v\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ (B.8) \quad &\leq C_0 (\sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}) \cdot \end{aligned}$$

Next, integrating by parts in  $K_{2,2}$  and using  $\nu \operatorname{div} v = \tilde{P} + \tilde{G}$  gives

$$\begin{aligned} \nu K_{2,2} &= - \int_{\mathbb{T}^2} \sqrt{t} \nabla G \cdot \sqrt{t} \dot{v} \operatorname{div} v dx + \sqrt{t} \int_{\mathbb{T}^2} \nabla^2(-\Delta)^{-1} \tilde{G} \cdot \sqrt{t} \nabla \dot{v} \operatorname{div} v dx \\ &= - \int_{\mathbb{T}^2} \sqrt{t} \nabla G \cdot \sqrt{t} \dot{v} \operatorname{div} v dx + \frac{\sqrt{t}}{\nu} \int_{\mathbb{T}^2} \nabla^2(-\Delta)^{-1} \tilde{G} \cdot \sqrt{t} \nabla \dot{v} \tilde{G} dx \\ &\quad + \frac{\sqrt{t}}{\nu} \int_{\mathbb{T}^2} \nabla^2(-\Delta)^{-1} \tilde{G} \cdot \sqrt{t} \nabla \dot{v} \tilde{P} dx \end{aligned}$$

from which we get

$$\begin{aligned} \nu \left| \int_0^T K_{2,2} dt \right| &\leq \|\operatorname{div} v\|_{L_4(0,T;L_4)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \\ &\quad + \nu^{-1} \sqrt{T} \|\tilde{G}\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} (\|\tilde{G}\|_{L_4(0,T;L_4)} + \|\tilde{P}\|_{L_4(0,T;L_4)}) \\ (B.9) \quad &\leq C_0 T^{1/4} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-1/2} \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}. \end{aligned}$$

Finally, using again the notation  $\psi := (-\Delta)^{-1}\tilde{P}$ , we have

$$\begin{aligned}\nu^2 K_{2,3} &= - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j \tilde{P} t \dot{v}^i dx - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j G t \dot{v}^i dx \\ &= \int_{\mathbb{T}^2} \partial_i \partial_j^2 \psi \tilde{P} t \dot{v}^i dx + \int_{\mathbb{T}^2} \partial_i \partial_j \psi \tilde{P} t \partial_j \dot{v}^i dx - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j G t \dot{v}^i dx \\ &= \frac{1}{2} \int_{\mathbb{T}^2} \tilde{P}^2 t \operatorname{div} \dot{v} dx + \int_{\mathbb{T}^2} \partial_i \partial_j \psi \tilde{P} t \partial_j \dot{v}^i dx - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j G t \dot{v}^i dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\nu^2 \left| \int_0^T K_{2,3} dt \right| &\lesssim \sqrt{T} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\ &\quad + \|\tilde{P}\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ (B.10) \quad &\leq C_0 \sqrt{\nu T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + C_0 (\nu T)^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}.\end{aligned}$$

Plugging (2.5), (B.3), (2.48), (2.51) and (2.52) in (B.8), (B.9) and (C.29) yields (2.66).

### APPENDIX C. THE THREE-DIMENSIONAL CASE

This section is devoted to extending our existence result to the three-dimensional torus. For expository purpose, we focus on the global-in-time issue for small data, although a similar statement may be proved locally in time for large data.

**Theorem C.1.** *Let  $v_0$  be in  $H^1(\mathbb{T}^3)$  and  $\rho_0$  be a bounded nonnegative function on  $\mathbb{T}^3$ . Assume that  $P(\rho) = a\rho^\gamma$  for some  $\gamma \geq 1$  and  $a > 0$ . There exists  $\nu_0 > 0$  depending only on  $\mu$ ,  $\gamma$ ,  $a$  and on the norms of the data, and  $c_0 > 0$  such that if*

$$(C.1) \quad \mu \|\nabla \mathcal{P} v_0\|_2^2 + \frac{1}{\nu} \|\tilde{P}_0\|_2^2 + \nu \|\operatorname{div} v_0\|_2^2 \leq c_0 \frac{\mu^5}{(\rho^*)^3 E_0},$$

then System (1.1) has a unique global solution with the same properties as in Theorem 1.1.

The general strategy is basically the same as for the two-dimensional case, except that the smallness condition spares our using the logarithmic interpolation inequality (that does not hold for  $d = 3$ ). We just underline what has to be changed in the main steps of the proof.

**Step 1: Sobolev estimates for the velocity.** The counterpart of Proposition 2.1 reads:

**Proposition C.1.** *Let  $(\rho, v)$  be a smooth solution of (1.1) on  $[0, T] \times \mathbb{T}^3$ , fulfilling (1.5) and (2.3). Assume that  $P$  satisfies (1.9) and denote  $P^* := \|P\|_\infty$ . Under condition (C.1) and for large enough  $\nu$ , there exists a constant  $C$  such that for all  $t \in [0, T]$ , we have*

$$\begin{aligned}\mu \|\nabla \mathcal{P} v(t)\|_2^2 + \frac{1}{\nu} (\|\tilde{G}(t)\|_2^2 + \|\tilde{P}(t)\|_2^2) + \frac{P^*}{\nu} \|(\sqrt{\rho} v)(t)\|_2^2 + \frac{P^*}{\nu} \|e(t)\|_1 \\ + \int_0^t \left( \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{\rho^*} \|\nabla^2 \mathcal{P} v\|_2^2 + \frac{1}{\rho^*} \|\nabla G\|_2^2 + \nu \|\operatorname{div} v\|_2^2 + \frac{\mu P^*}{\nu} \|\nabla v\|_2^2 \right) d\tau \\ \leq C \left( \mu \|\nabla \mathcal{P} v_0\|_2^2 + \frac{1}{\nu} \|\tilde{P}_0\|_2^2 + \nu \|\operatorname{div} v_0\|_2^2 \right).\end{aligned}$$

*Proof.* In order to be able to consider general initial data *with large energy*, it is suitable to modify the definition of  $\mathcal{E}$  as follows:

$$(C.2) \quad \mathcal{E} := \tilde{\mathcal{E}} + \frac{P^*}{\nu} E + \frac{P(1)}{2\nu} (P^* - P(1)),$$

where  $\tilde{\mathcal{E}}$  is still defined by (2.12). Owing to (2.26) that is also valid for  $d = 3$ , we have

$$(C.3) \quad \mathcal{E} \geq \frac{\mu}{2} \|\mathcal{P}v\|_2^2 + \frac{1}{2\nu} (\|\tilde{G}\|_2^2 + \|\tilde{P}\|_2^2) + \frac{P^*}{2\nu} \|\sqrt{\rho}v\|_2^2 + \frac{P^*}{2\nu} \|e\|_1.$$

Then, we start from Inequality (2.18) that is valid in any dimension and, instead of (1.13), we use that

$$(C.4) \quad \left( \int_{\mathbb{T}^3} \rho |v|^4 dx \right)^{1/2} \leq \left( \int_{\mathbb{T}^3} \rho |v|^2 dx \right)^{1/4} \left( \int_{\mathbb{T}^3} \rho |v|^6 dx \right)^{1/4} \lesssim (\rho^*)^{1/4} \|\sqrt{\rho}v\|_2^{1/2} \|\nabla v\|_2^{3/2}.$$

One can thus bound the right-hand side of (2.17) as follows:

$$\begin{aligned} 3 \int_{\mathbb{T}^3} \rho |v \cdot \nabla \mathcal{P}v|^2 dx &\leq 3\sqrt{\rho^*} \|\rho^{1/4}v\|_4^2 \|\nabla \mathcal{P}v\|_4^2 \\ &\leq C\rho^{*3/4} \|\sqrt{\rho}v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\nabla \mathcal{P}v\|_2^{1/2} \|\nabla^2 \mathcal{P}v\|_2^{3/2} \\ &\leq \frac{\mu^2}{4\rho^*} \|\Delta \mathcal{P}v\|_2^2 + C \left( \frac{\rho^*}{\mu} \right)^6 \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^6 \|\nabla \mathcal{P}v\|_2^2, \\ \frac{3}{\nu^2} \int_{\mathbb{T}^3} \rho \left| v \cdot \left[ \nabla^2 (-\Delta)^{-1} \tilde{G} \right] \right|^2 dx &\leq C(\rho^*)^{1/2} \nu^{-2} \|\rho^{1/4}v\|_4^2 \|\tilde{G}\|_4^2 \\ &\leq C(\rho^*)^{3/4} \nu^{-2} \|\sqrt{\rho}v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\tilde{G}\|_2^{1/2} \|\nabla G\|_2^{3/2} \\ &\leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 + C \frac{(\rho^*)^6}{\nu^8} \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^6 \|\tilde{G}\|_2^2, \end{aligned}$$

and also, thanks to Inequality (A.2),

$$\begin{aligned} \frac{3}{\nu^2} \int_{\mathbb{T}^3} \rho |v \cdot \nabla^2 \Delta^{-1} \tilde{P}|^2 dx &\leq C \frac{\rho^*}{\nu^2} \|v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\tilde{P}\|_4^2 \\ &\leq C \frac{(\rho^*)^{3/2}}{\nu^2} \|\nabla v\|_2^2 \|\tilde{P}\|_2 \|\tilde{P}\|_\infty \\ &\leq \frac{\mu P^*}{4\nu} \|\nabla v\|_2^2 + C \frac{(\rho^*)^3}{\mu \nu^3} \frac{\|\tilde{P}\|_\infty^2}{P^*} \|\nabla v\|_2^2 \|\tilde{P}\|_2^2. \end{aligned}$$

Next, instead of (2.22), we write that, in light of Inequality (A.2) with  $p = 2$ , we have

$$\begin{aligned} -\frac{1}{\nu} \int_{\mathbb{T}^3} P v \cdot \nabla G &\leq \frac{1}{\nu} \|P\|_\infty (1 + \|\tilde{\rho}\|_2) \|\nabla v\|_2 \|\nabla G\|_2 \\ &\leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 + 2 \frac{\rho^*}{\nu^2} \|P\|_\infty^2 (1 + \|\tilde{\rho}\|_2)^2 \|\nabla v\|_2^2. \end{aligned}$$

Therefore, the right-hand side of Inequality (2.23) becomes

$$\begin{aligned} 2 \frac{(\rho^*)^3}{\nu^2} \|P\|_\infty^2 \|\nabla v\|_2^2 + \frac{2}{\nu^2} \int_{\mathbb{T}^3} \tilde{G}^2 h dx \\ + C \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^6 \left( \left( \frac{\rho^*}{\mu} \right)^6 \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^6}{\nu^8} \|\tilde{G}\|_2^2 \right) + C \frac{(\rho^*)^3}{\mu \nu^3} \frac{\|\tilde{P}\|_\infty^2}{P^*} \|\nabla v\|_2^2 \|\tilde{P}\|_2^2. \end{aligned}$$

Then, following the computations leading to (2.30) and assuming that  $\nu$  satisfies

$$(C.5) \quad \nu \geq 8(\rho^*)^3 \frac{P^*}{\mu} \quad \text{and} \quad \nu^2 \geq 8\|h\|_\infty \rho^*,$$

we get,

$$\frac{d}{dt}\mathcal{E} + \mathcal{D} \leq C \left( \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^6 \left( \left( \frac{\rho^*}{\mu} \right)^6 \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^6}{\nu^8} \|\tilde{G}\|_2^2 \right) + \frac{(\rho^*)^3}{\mu\nu^3} \frac{\|\tilde{P}\|_\infty^2}{P^*} \|\nabla v\|_2^2 \|\tilde{P}\|_2^2 \right)$$

with

$$\mathcal{D} := \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\nabla^2 \mathcal{P}v\|_2^2 + \frac{1}{8\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} v)^2 (\nu + h) dx + \frac{\mu P^*}{2\nu} \|\nabla v\|_2^2.$$

Hence, remembering Inequality (C.3), one can conclude that if  $\nu$  satisfies (C.5), then we have the differential inequality

$$(C.6) \quad \frac{d}{dt}\mathcal{E} + \mathcal{D} \leq C \|\nabla v\|_2^2 \mathcal{E} \left( \frac{(\rho^*)^{3/2} \|\tilde{P}\|_\infty^2}{\nu^3 \mu^{1/2}} + \frac{(\rho^*)^6}{\mu^9} E_0 \mathcal{E}^2 \right).$$

Setting

$$X(t) = \mathcal{E}(t) + \int_0^t \mathcal{D} d\tau, \quad f(t) := C \|\nabla v(t)\|_2^2, \quad A := \frac{(\rho^*)^3 P^*}{\mu\nu^2} \quad \text{and} \quad B := \frac{(\rho^*)^6}{\mu^9} E_0,$$

Inequality (C.6) rewrites

$$\frac{d}{dt}X \leq (AX + BX^3)f(t).$$

This may be integrated into

$$\frac{X(t)}{\sqrt{1 + cX^2(t)}} \leq \frac{X(0)}{\sqrt{1 + cX^2(0)}} e^{A \int_0^t f(\tau) d\tau} \quad \text{with} \quad c := \frac{B}{A}.$$

Bounding  $f$  according to (1.6), we see that under the smallness condition

$$(C.7) \quad \mathcal{E}_0^2 < \frac{A}{B} \frac{1}{e^{\frac{2CAE_0}{\mu}} - 1},$$

we have

$$(C.8) \quad X^2(t) \leq \frac{X^2(0)}{1 + cX^2(0)} \left( \frac{e^{\frac{2CAE_0}{\mu}}}{1 - \frac{cX^2(0)}{1 + cX^2(0)} e^{\frac{2CAE_0}{\mu}}} \right) \quad \text{for all } t \geq 0.$$

Note that the largeness condition (C.5) on  $\nu$  guarantees that the argument of the exponential function above is very small. Therefore, the smallness condition (C.7) may be simplified into

$$\mathcal{E}_0 \ll \frac{\mu^5}{(\rho^*)^3 E_0}.$$

If (C.1) holds true, then that latter condition is fulfilled for  $\nu$  large compared to  $E_0^2$ .  $\square$

**Remark C.1.** *Note that the smallness condition means that one can take the initial energy as large as we want provided that  $\nu$  is large enough, but that  $\operatorname{div} v_0$  must be  $\mathcal{O}(\nu^{-1/2})$ . At the same time, there is no smallness condition on  $\rho_0 - 1$  whatsoever.*

**Step 2: Upper bound for the density.** In order to adapt Proposition 2.2 to the case  $d = 3$ , the only changes are in (2.38) and (2.40). As regards (2.38), one may still start from (2.37) then combine with (C.4) in order to get

$$(C.9) \quad \|(-\Delta)^{-1}(\rho v)\|_\infty \leq C(\rho^*)^{\frac{7}{8}} \|\sqrt{\rho} v\|_2^{\frac{1}{4}} \|\nabla v\|_2^{\frac{3}{4}} \leq C(\rho^*)^{\frac{7}{8}} E_0^{\frac{1}{8}} \|\nabla v\|_2^{\frac{3}{4}}.$$

Next, instead of (2.39), in order to bound the commutator term, we write that

$$(C.10) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim \|[\tilde{v}^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_{W^{1, \frac{24}{7}}} \lesssim \|\nabla v\|_6 \|\rho v\|_8.$$

Now, combining Hölder inequalities, Sobolev embedding and interpolation inequalities yields

$$\begin{aligned} \|\rho v\|_8 &\leq (\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|v\|_{\frac{10}{12}}^{\frac{9}{10}} \\ &\leq C(\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|\nabla v\|_{\frac{10}{5}}^{\frac{9}{10}} \\ &\leq C(\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|\nabla v\|_2^{\frac{27}{40}} \|\nabla v\|_6^{\frac{9}{40}}. \end{aligned}$$

Therefore, in  $\mathbb{T}^3$ , Inequality (2.39) becomes

$$\|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \leq C(\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|\nabla v\|_2^{\frac{27}{40}} \|\nabla v\|_6^{\frac{9}{40}}.$$

In order to bound the last term, we use that

$$\begin{aligned} \|\nabla v\|_6 &\lesssim \|\nabla \mathcal{P} v\|_6 + \nu^{-1} (\|\tilde{G}\|_6 + \|\tilde{P}\|_6) \\ &\lesssim \|\nabla^2 \mathcal{P} v\|_2 + \nu^{-1} (\|\nabla G\|_2 + \|\tilde{P}\|_\infty). \end{aligned}$$

Hence, using the energy conservation (1.6) and the definition of  $\mathcal{E}$  and  $\mathcal{D}$ ,

$$(C.11) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim (\rho^*)^{\frac{19}{20}} E_0^{\frac{1}{20}} \|\nabla v\|_2^{\frac{27}{40}} \left( \left( \frac{\sqrt{\rho^*} \mathcal{D}}{\mu} \right)^{\frac{49}{80}} + \left( \frac{\|\tilde{P}\|_\infty}{\nu} \right)^{\frac{49}{40}} \right).$$

Plugging inequalities (C.9) and (C.11) in (2.36), we get

$$(C.12) \quad \begin{aligned} \|F^+(t)\|_\infty &\leq \|F^+(0)\|_\infty + \frac{\gamma-1}{\gamma} E_0 + C \frac{\gamma}{\nu^2} (\rho^*)^{\frac{7}{8}} E_0^{\frac{1}{8}} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|\nabla v(\tau)\|_2^{\frac{3}{4}} d\tau \\ &\quad + C \frac{(\rho^*)^{\frac{19}{20}}}{\nu} E_0^{\frac{1}{20}} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|\nabla v\|_2^{\frac{27}{40}} \left( \left( \frac{\sqrt{\rho^*} \mathcal{D}(\tau)}{\mu} \right)^{\frac{49}{80}} + \left( \frac{\|\tilde{P}\|_\infty}{\nu} \right)^{\frac{49}{40}} \right) d\tau. \end{aligned}$$

Since the integrals in the right-hand side may be bounded in terms of the data according to the basic energy inequality (1.6) and to (2.42), we eventually get if  $\nu$  is large enough:

$$\|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + C_0 \nu^{-\frac{27}{80}} + \frac{\gamma-1}{\gamma} E_0$$

with  $C_0$  depending only on  $E_0$ ,  $\mathcal{E}_0$ ,  $\|\rho_0\|_\infty$ ,  $\mu$  and  $\gamma$ . From this point, one can conclude as in the two-dimensional case that (2.33) is fulfilled if  $\nu$  is large enough.

**Step 3: Time weighted estimates.** As in the 2D case, the starting point is Identity (2.47). However, Inequality (2.48) that has been used all the time has to be replaced with an estimate for  $t^{1/8} \nabla v$  in  $L_4(0, T \times \mathbb{T}^3)$ : we write that the previous steps and (B.2) imply that

$$(C.13) \quad \begin{aligned} \|t^{1/8} \nabla \mathcal{P} v\|_{L_4(0, T \times \mathbb{T}^3)} &\leq \|t^{1/4} \nabla \mathcal{P} v\|_{L_\infty(0, T; L_3)}^{1/2} \|\nabla \mathcal{P} v\|_{L_2(0, T; L_6)}^{1/2} \\ &\lesssim \|\nabla \mathcal{P} v\|_{L_\infty(0, T; L_2)}^{1/4} \|\sqrt{t} \nabla^2 \mathcal{P} v\|_{L_\infty(0, T; L_2)}^{1/4} \|\nabla^2 \mathcal{P} v\|_{L_2(0, T \times \mathbb{T}^3)}^{1/2} \\ &\leq C_0 \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0, T; L_2)}^{1/4}, \end{aligned}$$



where the meaning of  $C_0$  is the same as in the two-dimensional case.

Similarly, we have

$$\begin{aligned}
\|t^{1/8}\nabla^2(-\Delta)^{-1}\tilde{G}\|_{L_4(0,T;\mathbb{T}^3)} &\lesssim \|\tilde{G}\|_{L_4(0,T;\mathbb{T}^3)} \\
&\lesssim \|\tilde{G}\|_{L_\infty(0,T;L_2)}^{1/4} \|\sqrt{t}\nabla G\|_{L_\infty(0,T;L_2)}^{1/4} \|\nabla G\|_{L_2(0,T;\mathbb{T}^3)}^{1/2} \\
(C.14) \quad &\leq C_0\nu^{1/8} \|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/4}.
\end{aligned}$$

Since (2.52) is valid in any dimension, one can conclude that

$$(C.15) \quad \|t^{1/8}\nabla v\|_{L_4(0,T;\mathbb{T}^3)} \leq C_0(\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{-3/4}T^{1/8}).$$

*Substep 1.* Compared to  $d = 2$ , the only change lies in the estimate for  $\int_{\mathbb{T}^3} \rho \operatorname{div} v |\dot{v}|^2 t dx$ . Now, still using that  $\operatorname{div} v = \nu^{-1}(\tilde{P} + \tilde{G})$ , we write that

$$\begin{aligned}
\int_{\mathbb{T}^3} \rho \operatorname{div} v |\dot{v}|^2 t dx &\leq \nu^{-1} \int_{\mathbb{T}^2} (\tilde{P} + \tilde{G}) \rho |\dot{v}|^2 t dx \\
&\leq \nu^{-1} (\|\tilde{P}\|_\infty \|\sqrt{\rho t}\dot{v}\|_2^2 + \sqrt{\rho^*} \|\tilde{G}\|_3 \|\sqrt{t}\dot{v}\|_6 \|\sqrt{\rho t}\dot{v}\|_2) \\
&\leq C_0\nu^{-1} (\|\sqrt{\rho t}\dot{v}\|_2^2 + \|\tilde{G}\|_2^{1/2} \|\nabla G\|_2^{1/2} \|\sqrt{t}\nabla \dot{v}\|_2 \|\sqrt{\rho t}\dot{v}\|_2).
\end{aligned}$$

The first term may be treated as in the 2D case. As for the second one, we use the fact that (2.72) ensures that

$$\|\nabla G\|_2 \leq \sqrt{\rho^*} \|\sqrt{\rho}\dot{v}\|_2.$$

Hence, using Proposition 2.1 to bound  $\|\tilde{G}\|_2$ , we get

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^2} \rho \operatorname{div} v |\dot{v}|^2 t dx dt &\leq C_0 \left( \nu^{-1} \int_0^T \|\sqrt{\rho t}\dot{v}\|_2^2 + \nu^{-3/4} \int_0^T \|\sqrt{\rho}\dot{v}\|_2^{1/2} \|\sqrt{\rho t}\dot{v}\|_2 \|\sqrt{t}\nabla \dot{v}\|_2 dt \right) \\
&\leq \frac{C_0}{\nu} \left( \int_0^T \|\sqrt{\rho t}\dot{v}\|_2^2 + \frac{1}{\sqrt{\nu}} \int_0^T \|\sqrt{\rho}\dot{v}\|_2 \|\sqrt{\rho t}\dot{v}\|_2^2 dt \right) + \frac{\mu}{2} \int_0^T \|\sqrt{t}\nabla \dot{v}\|_2^2 dt.
\end{aligned}$$

In the end, we thus obtain

$$\begin{aligned}
(C.16) \quad \int_{\mathbb{T}^3} \frac{D}{Dt}(\rho \dot{v}) \cdot (t \dot{v}) dx &\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \rho |\dot{v}|^2 t dx - \frac{1}{2} \int_{\mathbb{T}^3} \rho |\dot{v}|^2 dx \\
&\quad - \frac{C_0}{\nu} \left( \int_0^T \|\sqrt{\rho t}\dot{v}\|_2^2 + \frac{1}{\sqrt{\nu}} \int_0^T \|\sqrt{\rho}\dot{v}\|_2 \|\sqrt{\rho t}\dot{v}\|_2^2 dt \right) - \frac{\mu}{2} \int_0^T \|\sqrt{t}\nabla \dot{v}\|_2^2 dt.
\end{aligned}$$

*Substep 2.* Instead of (2.60), we use that, by virtue of (C.15),

$$(C.17) \quad \left| \int_0^T I_1(t) dt \right| \leq C_0 T^{1/4} (\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-3/2} T^{1/4}) \|\sqrt{t}\nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^3)}.$$

For bounding  $I_2$ , we decompose it into three parts, as in (B.1). For  $I_{2,1}$ , we write that

$$\left| \int_0^T I_{21}(t) dt \right| \leq \|\sqrt{t}\nabla v\|_{L_\infty(0,T;L_3)} \|\nabla^2 \mathcal{P}v\|_{L_2(0,T;\mathbb{T}^3)} \|\sqrt{t}\dot{v}\|_{L_2(0,T;L_6)}.$$

Let us notice that

$$(C.18) \quad \|t^{1/4}\nabla v\|_{L_\infty(0,T;L_3)} \leq C_0(\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}).$$

Indeed, as already used for proving (C.13) and (C.14), we have

$$(C.19) \quad \|t^{1/4} \nabla \mathcal{P} v\|_{L_\infty(0,T;L_3)} \leq C_0 \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2},$$

$$(C.20) \quad \|t^{1/4} \tilde{G}\|_{L_\infty(0,T;L_3)} \leq C_0 \nu^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2},$$

and we have the obvious inequality

$$(C.21) \quad \|\tilde{P}\|_{L_\infty(0,T;L_3)} \leq \|\tilde{P}\|_{L_\infty(0,T;L_2)}^{2/3} \|\tilde{P}\|_{L_\infty(0,T \times \mathbb{T}^3)}^{1/3} \leq \nu^{1/3} C_0.$$

Hence, combining Sobolev embedding and (C.18), we obtain

$$(C.22) \quad \left| \int_0^T I_{21} dt \right| \leq C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

In order to bound  $I_{22}$ , we now write that

$$\left| \int_0^T I_{22} dt \right| \lesssim \nu^{-1} \|\sqrt{t} \nabla v\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)}.$$

Therefore, using the previous section and (C.18), we get

$$(C.23) \quad \left| \int_0^T I_{22} dt \right| \leq C_0 \nu^{-1/2} T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

To bound  $I_{23}$ , we use (B.6) as in the two-dimensional case. The first two terms of the decomposition may be bounded as before. For the third one, we use the fact that

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}^3} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t dx \right| &\lesssim \sqrt{T} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\tilde{P}\|_{L_\infty(0,T;L_3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \\ &\leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)} \end{aligned}$$

and that

$$\begin{aligned} \left| \int_{\mathbb{T}^3} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t dx \right| &\leq T^{3/8} \|t^{1/8} \nabla v\|_{L_4(0,T \times \mathbb{T}^3)} \|\tilde{P}\|_{L_4(0,T \times \mathbb{T}^3)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)} \\ &\leq C_0 \nu^{1/4} T^{3/8} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{-3/4} T^{1/8}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}. \end{aligned}$$

Hence, one can conclude that

$$(C.24) \quad \left| \int_0^T I_{23} dt \right| \leq C_0 \left( \frac{\sqrt{T}}{\nu^{3/2}} + \frac{T^{3/8}}{\nu^{3/4}} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} \right) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

Putting together all the estimates of the second substep, we get

$$(C.25) \quad -\mu \int_0^T \int_{\mathbb{T}^3} \left( \frac{D}{Dt} \Delta v \right) \cdot \dot{v} t dx dt \geq \mu \int_0^T \int_{\mathbb{T}^3} |\nabla \dot{v}|^2 t dx dt \\ - C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

*Substep 3.* To bound  $K_1$  (defined in (2.64)), we write that

$$\nu \left| \int_0^T K_1 dt \right| \leq CT^{1/4} \|t^{1/8} \nabla v\|_{L_4(0,T;L_4)} \|t^{1/8} (\tilde{P} + \tilde{G})\|_{L_4(0,T;L_4)} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)},$$

whence

$$(C.26) \quad \nu \left| \int_0^T K_1 dt \right| \leq C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{-3/4} T^{1/8}) \\ \times (\nu^{1/8} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{1/4} T^{1/8}) \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}.$$

We decompose  $K_2$  as in the case  $d = 2$ . To bound  $K_{2,1}$ , we write that

$$(C.27) \quad \nu \left| \int_0^T K_{2,1} dt \right| \leq \sqrt{T} \|\nabla \mathcal{P} v\|_{L_2(0,T \times \mathbb{T}^3)} \|\tilde{P}\|_{L_\infty(0,T \times \mathbb{T}^3)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)} \\ + \|\sqrt{t} \nabla \mathcal{P} v\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \\ \leq C_0 (\sqrt{T} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

For  $K_{2,2}$ , we have

$$(C.28) \quad \nu \left| \int_0^T K_{2,2} dt \right| \leq \|\sqrt{t} \operatorname{div} v\|_{L_\infty(0,T;L_3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \\ + \nu^{-1} T^{1/4} \|t^{1/8} \tilde{G}\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} (\|t^{1/8} \tilde{G}\|_{L_4(0,T;L_4)} + \|t^{1/8} \tilde{P}\|_{L_4(0,T;L_4)}) \\ \leq C_0 T^{1/4} (T^{1/8} \nu^{-3/4} + \nu^{-3/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.$$

Finally, we have

$$(C.29) \quad \nu^2 \left| \int_0^T K_{2,3} dt \right| \lesssim \sqrt{T} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\ + \sqrt{T} \|\tilde{P}\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \\ \leq C_0 \sqrt{\nu T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.$$

Plugging (2.5), (B.3), (2.48), (2.51) and (2.52) in (C.27), (C.28) and (C.29) yields

$$\nu \left| \int_0^T K_2 dt \right| \leq C_0 (\sqrt{T} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

The conclusion of this step is that, if  $\nu$  is large enough then

$$(C.30) \quad -(\nu - \mu) \int_0^T \int_{\mathbb{T}^3} \frac{D}{Dt} \nabla \operatorname{div} v \cdot \dot{v} t dx dt \geq (\nu - \mu) \int_0^T \int_{\mathbb{T}^3} (\operatorname{div} \dot{v})^2 t dx dt \\ - C_0 (\nu^{1/4} \sqrt{T} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\ - C_0 (\nu^{1/8} T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{3/8} \sqrt{T}) \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}.$$

*Substep 4.* Term  $L_1$  may still be bounded according to Inequality (2.69). As for  $L_2$ , we have

$$\begin{aligned} \left| \int_0^T L_2(t) dt \right| &\leq \frac{1}{2\nu} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|t \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} \\ &\quad + \frac{1}{\nu} \|\tilde{P}\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T;L_2)} \|t \dot{v}\|_{L_2(0,T;L_6)} \\ &\quad + \|\tilde{P}\|_{L_\infty(0,T;L_\infty)} \|\nabla v\|_{L_2(0,T;L_2)} \|t \nabla \dot{v}\|_{L_2(0,T;L_2)} \\ &\leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}. \end{aligned}$$

So this step gives

$$\begin{aligned} \text{(C.31)} \quad \int_{\mathbb{T}^3} \frac{D}{Dt} \nabla P \cdot \dot{v} t dx &\geq -\frac{\nu}{4} \int_0^T \int_{\mathbb{T}^3} (\operatorname{div} \dot{v})^2 t dx dt \\ &\quad - \|h\|_\infty T \nu^{-1} \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 - C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}. \end{aligned}$$

*Substep 5.* Combining Inequalities (2.57), (C.25), (C.30) and (C.31) yields for large  $\nu$ ,

$$\begin{aligned} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^2 &+ 2\mu \int_0^T \|\sqrt{t} \nabla \mathcal{P} \dot{v}\|_2^2 dt + \frac{3\nu}{2} \int_0^T \|\sqrt{t} \operatorname{div} \dot{v}\|_2^2 dt \leq 2 \int_0^T \|\operatorname{div} v\|_\infty \|\sqrt{\rho t} \dot{v}\|_2^2 dt \\ &+ \|\sqrt{\rho} \dot{v}\|_{L_2(0,T;L_2)}^2 + \|h\|_\infty T \nu^{-1} \|\operatorname{div} v\|_{L_2(0,T;\mathbb{T}^2)}^2 + C_0 T^{1/4} (\nu^{1/8} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \\ &+ \nu^{3/8} T^{1/4}) \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} + C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}. \end{aligned}$$

Playing with Young inequality and Gronwall Lemma yields Prop. 2.3 for  $d = 3$ .

It is now easy to adapt Corollary 2.1 to the 3D case: we start from

$$\|\tilde{G}\|_\infty \lesssim \|\tilde{G}\|_2^{1/4} \|\nabla G\|_6^{3/4}.$$

Hence, remembering (2.72) and using the embedding  $\dot{H}^1(\mathbb{T}^3) \hookrightarrow L_6(\mathbb{T}^3)$ ,

$$\|\tilde{G}\|_\infty \lesssim (\rho^*)^{3/4} \|\tilde{G}\|_2^{1/4} t^{-3/8} \|\sqrt{t} \nabla \dot{v}\|_2^{3/4}.$$

Therefore, as in the 2D case,

$$\|\tilde{G}\|_{L_{1+\varepsilon}(0,T;L_\infty)}^{1+\varepsilon} \lesssim (\rho^*)^{\frac{3(1+\varepsilon)}{4}} \nu^{\frac{1+\varepsilon}{8}} (\nu^{-\frac{1}{2}} \|\tilde{G}\|_{L_\infty(0,T;L_2)})^{\frac{1+\varepsilon}{4}} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}^{\frac{3(1+\varepsilon)}{4}} \left( \int_0^T t^{-\frac{3+3\varepsilon}{5-3\varepsilon}} dt \right)^{\frac{5-3\varepsilon}{8}},$$

and one can thus conclude that  $\operatorname{div} v$  is in  $L_{1+\varepsilon}(0,T;L_\infty)$  provided that  $\varepsilon < 1/3$ . Bounding  $\mathcal{P}v$  is left to the reader.

**Acknowledgments.** This work was partially supported by ANR-15-CE40-0011. The second author (P.B.M.) has been partly supported by the Polish National Science Centre's grant No 2018/29/B/ST1/00339.

## REFERENCES

- [1] H. Bahouri, J.-Y. Chemin and R. Danchin: *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, **343**, Springer (2011).
- [2] D. Bresch and P.-E. Jabin: Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math.* **188** (2018), no. 2, 577–684.
- [3] T. Buckmaster and V. Vicol: Nonuniqueness of weak solutions to the Navier-Stokes equation, *Ann. of Math.* **189** (2019), no. 1, 101–144.

- [4] E. Chiodaroli and O. Kreml: On the energy dissipation rate of solutions to the compressible isentropic Euler system. *Arch. Ration. Mech. Anal.* **214** (2014), 1019–1049.
- [5] Y. Cho, H.J. Choe and H. Kim: Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.* **83** (2004), no. 2, 243–275.
- [6] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes: Compensated compactness and Hardy spaces, *J. Math. Pures Appl.*, **72** (1993), no. 3, 247–286.
- [7] R. Danchin: Global existence in critical spaces for compressible Navier-Stokes equations, *Inventiones Mathematicae*, **141** (2000), no. 3, 579–614.
- [8] R. Danchin: Local theory in critical spaces for compressible viscous and heat-conductive gases, *Comm. Partial Differential Equations*, **26** (2001), no. 7-8, 1183–1233.
- [9] R. Danchin: On the solvability of the compressible Navier-Stokes system in bounded domains, *Nonlinearity*, **23**(2), (2010), 383–408.
- [10] R. Danchin, F. Fanelli and M. Paicu: A well-posedness result for viscous compressible fluids with only bounded density, *Analysis and PDEs*, **13** (2020), no 1, 275–316.
- [11] R. Danchin and P.B. Mucha: Compressible Navier-Stokes system : large solutions and incompressible limit, *Advances in Mathematics*, **320** (2017), 904–925.
- [12] R. Danchin and P.B. Mucha: The incompressible Navier-Stokes equations in vacuum, *Communications on Pure and Applied Mathematics*, **52** (2019), 1351–1385.
- [13] R. Danchin and P.B. Mucha: From compressible to incompressible inhomogeneous flows in the case of large data, *Tunisian Journal of Mathematics*, **1** (2019), no. 1, 127–149.
- [14] B. Desjardins: Regularity of weak solutions of the compressible isentropic Navier-Stokes equations, *Comm. Partial Differential Equations*, **22** (1997), no. 5-6, 977–1008.
- [15] B. Desjardins: Global existence results for the incompressible density-dependent Navier-Stokes equations in the whole space, *Differential Integral Equations*, **10** (1997), no. 3, 587–598.
- [16] C. De Lellis and L. Székelyhidi: On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* **195** (2010), no. 1, 225–260.
- [17] C. De Lellis and L. Székelyhidi: The Euler equations as a differential inclusion. *Ann. of Math.* **170** (2009), no. 3, 1417–1436.
- [18] R. Di Perna and P.-L. Lions: Ordinary differential equations, transport theory and Sobolev spaces, *Inventiones Mathematicae*, **98** (1989), 511–547.
- [19] E. Feireisl: *Dynamics of viscous compressible fluids*. Oxford Lecture Series in Mathematics and its Applications, **26**. Oxford University Press, Oxford, 2004.
- [20] E. Feireisl: On weak-strong uniqueness for the compressible Navier-Stokes system with non-monotone pressure law. *Comm. Partial Differential Equations* **44** (2019), no. 3, 271–278.
- [21] P. Germain: Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. *J. Math. Fluid Mech.* **13** (2011), 137–146.
- [22] D. Hoff: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations*, **120** (1995), no. 1, 215–254.
- [23] D. Hoff: Uniqueness of weak solutions of the Navier-Stokes equations of multidimensional, compressible flow, *SIAM J. Math. Anal.*, **37** (2006), no. 6, 1742–1760.
- [24] N. Itaya: On the initial value problem of the motion of compressible viscous fluid, especially on the problem of uniqueness, *J. Math. Kyoto Univ.*, **16** (1976), no. 2, 413–427.
- [25] A.V. Kazhikhov: Correctness in the large of initial-boundary value problem for model system of equations of a viscous gas. *Din. Sploshnoi Sredy*, **21** (1975), 18–47.
- [26] O. Ladyzhenskaya: Solution ‘in the large’ of the non-stationary boundary value problem for the Navier-Stokes system with two space variables, *Comm. Pure Appl. Math.*, **12** (1959), 427–433.
- [27] P.-L. Lions: *Mathematical Topics in Fluid Mechanics*, Oxford Science Publications, Vol. 2, Compressible models, The Clarendon Press, Oxford University Press, New-York (1998).
- [28] A. Matsumura and T. Nishida: The initial value problem for equations of motion of compressible viscous and heat conductive gases, *J. Math. Kyoto Univ.*, **20**, (1980), 67–104.
- [29] P.B. Mucha: Stability of nontrivial solutions of the Navier-Stokes system on the three dimensional torus, *J. Differential Equations*, **172**, (2001), no. 2, 359–375.
- [30] P.B. Mucha: The Cauchy problem for the compressible Navier-Stokes equations in the  $L_p$ -framework, *Nonlinear Anal.*, **52**, (2003), no. 4 1379–1392.
- [31] P.B. Mucha: Transport equation: extension of classical results for  $\operatorname{div} b$  in  $BMO$ , *J. Differential Equations*, **249** (2010), no. 8, 1871–1883.

- [32] V. Solonnikov: Solvability of the initial boundary value problem for the equations of motion of a viscous compressible fluid, *Journal of Soviet Mathematics*, **14** (1980) 1120–1132.
- [33] V.A. Vaigant, A.V. Kazhikhov: On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid. (Russian) *Sibirsk. Mat. Zh.* **36** (1995), no. 6, 1283–1316, ii; translation in *Siberian Math. J.* **36** (1995), no. 6, 1108–1141.

(R. Danchin) UNIVERSITÉ PARIS-EST, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE.

*Email address:* `raphael.danchin@u-pec.fr`

(P.B. Mucha) INSTYTUT MATEMATYKI STOSOWANEJ I MECHANIKI, UNIWERSYTET WARSZAWSKI, UL. BANACHA 2, 02-097 WARSZAWA, POLAND.

*Email address:* `p.mucha@mimuw.edu.pl`