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# COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH RIPPED DENSITY

RAPHAËL DANCHIN AND PIOTR BOGUSŁAW MUCHA

ABSTRACT. Here we prove the all-time propagation of the  $H^1$  regularity for the velocity field solution of the two-dimensional compressible Navier-Stokes equations, provided the volume (bulk) viscosity coefficient  $\nu$  is large enough. The initial velocity can be arbitrarily large and the initial density is just required to be bounded. In particular, one can consider a characteristic function of a set as an initial density.

Uniqueness of the solutions to the equations is shown, in the case of a perfect gas.

As a by-product of our results, we give a rigorous justification of the convergence to the inhomogeneous incompressible Navier-Stokes equations when  $\nu$  tends to infinity.

Similar results are proved in the three-dimensional case, under some scaling invariant smallness condition on the velocity field.

## 1. INTRODUCTION

We are concerned with the regularity and uniqueness issues of viscous compressible flows in the unit torus  $\mathbb{T}^d$  with  $d = 2, 3$ . The corresponding equations of motion read

$$(1.1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) - \mu \Delta v - (\lambda + \mu) \nabla \operatorname{div} v + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d. \end{cases}$$

The pressure  $P$  is a given function of the density. The real numbers  $\lambda$  and  $\mu$  designate the bulk and shear viscosity coefficients, respectively, and are assumed to satisfy

$$(1.2) \quad \mu > 0 \quad \text{and} \quad \nu := \lambda + 2\mu > 0.$$

The system is supplemented with the initial data

$$(1.3) \quad v|_{t=0} = v_0, \quad \rho|_{t=0} = \rho_0.$$

For smooth enough solutions of (1.1), the total mass and momentum are conserved through the evolution that is, for all  $t \geq 0$ ,

$$(1.4) \quad \int_{\mathbb{T}^d} \rho(t, x) dx = \int_{\mathbb{T}^d} \rho_0(x) \quad \text{and} \quad \int_{\mathbb{T}^d} (\rho v)(t, x) dx = \int_{\mathbb{T}^d} (\rho_0 v_0)(x) dx.$$

Furthermore, if we denote by  $e$  the potential energy of the fluid defined by the relation  $\rho e'' = P'$ , and introduce the total energy

$$E(t) := \int_{\mathbb{T}^d} \left( \frac{1}{2} \rho(t, x) |v(t, x)|^2 + e(\rho(t, x)) \right) dx,$$

then the following energy balance holds true:

$$(1.5) \quad E(t) + \int_0^t (\mu \|\nabla \mathcal{P} v(\tau)\|_2^2 + \nu \|\operatorname{div} v(\tau)\|_2^2) d\tau = E_0 := E(0) \quad \text{for all } t \in [0, T],$$

where  $\mathcal{P}$  denotes the  $L_2$ -projector onto the set of solenoidal vector-fields and  $\|\cdot\|_p$ , the norm in  $L_p(\mathbb{T}^d)$ .

Since the pioneering works by P.-L. Lions in [18] and E. Feireisl in [14], it is well understood that, in the case of an isentropic pressure law  $P(\rho) = \rho^\gamma$  with  $\gamma > d/2$ , any finite energy initial data (that is such that  $E_0 < \infty$ ) generates a global-in-time weak solution to (1.1) satisfying

$$(1.6) \quad E(t) + \int_0^t (\mu \|\nabla \mathcal{P}v(\tau)\|_2^2 + \nu \|\operatorname{div} v(\tau)\|_2^2) d\tau \leq E_0 \quad \text{for all } t \geq 0.$$

At the same time, even in the two-dimensional case, the regularity and uniqueness issues for those general weak solutions are widely open, in sharp contrast with the theory for incompressible homogeneous flows (see [17]).

On the other side of the coin, the global existence and uniqueness issues are by now quite clear in the strong solution framework provided the data are small perturbations of a linearly stable positive constant density state [4, 19, 20]. For large smooth data with density bounded away from zero, well-posedness holds true only for small time [5, 21, 23]. Positivity of density may be somewhat relaxed but some compatibility condition involving the initial velocity must be satisfied (see [2]).

Our aim here is to provide the reader with a complete global-in-time existence theory with propagation of the  $H^1$  Sobolev regularity for the velocity, assuming only that the initial density is bounded. We shall indeed achieve our goal provided that  $\nu$  is large enough (and, of course, that the velocity is small enough in the case  $d = 3$ ). A remarkable feature of our result is that, even though the density need not be positive, one can exhibit some gain of regularity for the velocity so that both  $\operatorname{div} u$  and  $\operatorname{curl} u$  are in  $L_{r,loc}(\mathbb{R}_+; L_\infty)$  for some  $r > 1$ . Although this does not imply that the full gradient of  $u$  is in  $L_{1,loc}(\mathbb{R}_+; L_\infty)$ , we will get uniqueness in the case where  $P(\rho) = \rho$  despite the fact that the system under consideration is quasilinear and partially hyperbolic.

As a by-product of our result of propagation of regularity, we shall get almost for free the all-time convergence when  $\nu$  tends to  $+\infty$  to the following inhomogeneous incompressible Navier-Stokes equation:

$$(1.7) \quad \begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) - \mu \Delta v + \nabla \Pi = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{T}^d. \end{cases}$$

To the best of our knowledge, this is the first example of a global-in-time result of convergence from (1.1) to (1.7) *in the truly inhomogeneous framework* (see also our recent work in [10]).

For expository purposes, we shall assume from now on that

$$(1.8) \quad \int_{\mathbb{T}^d} \rho_0(x) dx = 1 \quad \text{and} \quad \int_{\mathbb{T}^d} (\rho_0 v_0)(x) dx = 0,$$

which is actually not restrictive, as one can rescale the density function and use the Galilean invariance of the system to have those two conditions fulfilled.

Let us state our main global existence result in the case where the fluid domain is the two-dimensional torus.

**Theorem 1.1.** *Consider any nonnegative bounded function  $\rho_0$  and vector field  $v_0$  in  $H^1(\mathbb{T}^2)$  satisfying (1.8), and assume that the pressure law is  $P(\rho) = a\rho^\gamma$  for some  $a > 0$  and  $\gamma \geq 1$ .*

There exists  $\nu_0$  depending only on  $\gamma$ ,  $\mu$ ,  $E_0$ ,  $\|\nabla v_0\|_2$  and  $\|\rho_0\|_\infty$  such that if  $\nu \geq \nu_0$  then System (1.1) admits a global-in-time solution  $(\rho, v)$  fulfilling the energy balance (1.5),

$$\rho \in L_\infty(\mathbb{R}_+ \times \mathbb{T}^2) \cap \mathcal{C}(\mathbb{R}_+; L_p(\mathbb{T}^2)) \text{ for all } p < \infty, \text{ and } \sqrt{\rho}v \in \mathcal{C}(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$

In addition, we have, denoting  $\dot{v} := v_t + v \cdot \nabla v$ ,

$$(1.9) \quad v \in L_\infty(\mathbb{R}_+; H^1(\mathbb{T}^2)), \quad \nabla^2 \mathcal{P}v \in L_2(\mathbb{R}_+ \times \mathbb{T}^2), \quad \sqrt{\rho} \dot{v} \in L_2(\mathbb{R}_+ \times \mathbb{T}^2), \\ \sqrt{\rho t} \dot{v} \in L_{\infty, \text{loc}}(\mathbb{R}_+; L_2(\mathbb{T}^2)), \quad \sqrt{t} \nabla \dot{v} \in L_2(\mathbb{R}_+ \times \mathbb{T}^2),$$

and both  $\text{div } v$  and  $\text{curl } v$  are in  $L_{1+\varepsilon, \text{loc}}(\mathbb{R}_+; L_\infty(\mathbb{T}^2))$  for some  $\varepsilon > 0$ .

**Remark 1.1.** *The above existence result as well as its corollary, Theorem 1.3 (see just below), are valid in  $\mathbb{T}^3$  either locally in time for large data, or globally under a suitable scaling invariant smallness condition. The reader is referred to Appendix C for more details.*

In dimensions 2 and 3, in the case of a linear pressure law, our existence result is supplemented with uniqueness.

**Theorem 1.2.** *Under the above assumptions with  $\gamma = 1$ , then the solution constructed in Theorem 1.1 (resp. Theorem C.1 in the appendix) if  $d = 2$  (resp.  $d = 3$ ) is unique.*

By taking advantage of the fact that the estimates that have been proved in Theorem 1.1 have some uniformity with respect to  $\nu$ , we get the following result of convergence of the compressible Navier-Stokes equations to the incompressible and inhomogeneous Navier-Stokes equations.

**Theorem 1.3.** *Fix some initial data  $(\rho_0, v_0)$  in  $L_\infty(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$  satisfying in addition  $\text{div } v_0 = 0$  and  $\rho_0 \geq 0$ , and denote by  $(\rho^\nu, v^\nu)$  the corresponding global solution of (1.1) provided by Theorem 1.1 for  $\nu \geq \nu_0$ .*

*Then, for  $\nu$  going to  $\infty$ , the whole family  $(\rho^\nu, v^\nu)$  converges to the unique global solution of system (1.7) supplemented with initial data  $(\rho_0, v_0)$  given by Theorem 2.1 of [9], and we have*

$$(1.10) \quad \text{div } v^\nu = \mathcal{O}(\nu^{-1/2}) \text{ in } L_2(\mathbb{R}_+ \times \mathbb{T}^2) \cap L_\infty(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$

**Remark 1.2.** *For simplicity, we focus on the physically relevant case where the pressure function  $P$  is given by  $P(\rho) = a\rho^\gamma$  for some  $\gamma \geq 1$  and  $a > 0$ . However, most of our results remain true whenever:*

$$(1.11) \quad P \text{ is a } C^1 \text{ nonnegative function on } \mathbb{R}_+ \text{ such that } \rho \mapsto \rho^{-1}P(\rho) \text{ is nondecreasing.}$$

Let us review the main ideas leading to our results. Assuming that we are given a solution  $(\rho, v)$  to (1.1), the first step is to establish global-in-time a priori estimates for the  $H^1$  norm of  $v$  in terms of the data, of the parameters of the system and of an upper bound for the density  $\rho$ . That step is partly based on the work by B. Desjardins in [11] where the so-called *viscous effective flux*  $G$  defined by

$$(1.12) \quad G := \nu \text{div } v - P \quad \text{with } \nu := \lambda + 2\mu$$

plays a key role. Recall that, as observed before in e.g. the works by D. Hoff [15] and P.-L. Lions [18],  $G$  has better regularity than  $\text{div } v$  or  $P$  taken separately. This property is the cornerstone of the construction of weak solutions to (1.1). Furthermore, rewriting the momentum equation in terms of  $G$  rather than  $P$  will spare us making integrability assumptions on  $\nabla \rho$ , in contrast with our recent work in [10]. This will be the key to considering initial density with no regularity whatsoever. In fact, thanks to it, we shall define a modified energy functional that controls the  $H^1$  regularity of the velocity and also contains some information on the

density. Compared to Desjardins' work, the main breakthrough is that, by tracking carefully the dependency of the estimates with respect to  $\nu$  and by optimizing the use of the following logarithmic interpolation inequality

$$(1.13) \quad \left( \int_{\mathbb{T}^2} \rho |v|^4 dx \right)^{\frac{1}{2}} \leq C \|\sqrt{\rho}v\|_2 \|\nabla v\|_2 \log^{\frac{1}{2}} \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right) \quad \text{with } \tilde{\rho} := \rho - 1,$$

we achieve global-in-time bounds depending only on the data and on  $\rho^* := \|\rho\|_\infty$ , while the result in [11] was local. Note that, here, (1.13) is an appropriate substitute of the well-known Ladyzhenskaya inequality

$$\|v\|_4^2 \leq C \|v\|_2 \|\nabla v\|_2$$

since bounds are available on  $\|\sqrt{\rho}v\|_2$  (through (1.5)), but not on  $\|v\|_2$ .

Step 1 required an a priori upper bound for the density. The second step will enable us to discard that assumption. Again, it is partly based on the work by B. Desjardins in [11] where the quantity  $F := \log \rho + \nu^{-1} \Delta^{-1}(\rho v)$  that may be seen as an approximate damped mode associated to (1.1) is introduced. The new achievement here is that, by combining with the first step and an obvious bootstrap argument, one ends up with a control on  $\rho^*$  in terms of the data only, provided that  $\nu$  is large enough.

In order to have a chance to prove uniqueness of the solutions, we need to exhibit more regularity for  $\nabla v$ . Ideally, since the system under consideration is partially quasilinear hyperbolic of order one, it would be good to have  $\nabla v$  in  $L_{1,loc}(\mathbb{R}_+; L_\infty)$ . The goal of step 3 is to prove that  $\operatorname{div} v$  and  $\operatorname{curl} v$  are in  $L_{1,loc}(\mathbb{R}_+; L_\infty)$ , which is "almost" what we want. To achieve it, we shall adapt our recent work [9] dedicated to the inhomogeneous Navier-Stokes equations (1.7) to the compressible situation. The main idea is to use time weighted estimates to glean some regularity on  $v_t$ , then to transfer time regularity to space regularity by using elliptic estimates. The situation here is more complicated, though, owing to the pressure term that cannot be discarded by means of the divergence free property. Nonetheless, by using the *convective derivative*  $\dot{v}$  rather than  $v_t$ , we shall get bounds on  $\sqrt{\rho t} \dot{v}$  in  $L_{\infty,loc}(\mathbb{R}_+; L_2)$  and  $\sqrt{t} \nabla \dot{v}$  in  $L_{2,loc}(\mathbb{R}_+; L_2)$ . Then, putting together with elliptic estimates and functional embedding, one gets that  $\operatorname{div} v$  and  $\operatorname{curl} v$  are in  $L_{1+\varepsilon,loc}(\mathbb{R}_+; L_\infty)$  for some  $\varepsilon > 0$ .

Steps 1 to 3 were just formal a priori estimates for smooth solutions. To complete the proof of existence, we mollify the initial density so as to make it strictly positive and regular. Then, one can resort to classical results to construct a local-in-time smooth solution corresponding to those data. The difficulty is to establish that, indeed, the control of norms that has been obtained so far allows to extend the solution for all time. Once it has been done, the uniform bounds turn out to be enough to pass to the limit and to complete the proof of existence. Since, compared to weak solutions, more regularity is available on the velocity, passing to the limit is much more direct than in [14] or [18].

Remember that steps 1 to 3 give that  $\operatorname{div} v$  and  $\operatorname{curl} v$  are in  $L_{1+\varepsilon,loc}(\mathbb{R}_+; L_\infty)$  for some  $\varepsilon > 0$ . Hence, we miss by a little the property that  $\nabla v$  is in  $L_{1,loc}(\mathbb{R}_+; L_\infty)$  and  $v$  need not have a Lipschitz flow. Therefore, in contrast with what has been done for (1.7) or for (1.1) in [7], it is not clear whether recasting the compressible Navier-Stokes equations in Lagrangian coordinates may help to prove uniqueness. Nonetheless, in the particular case of a *linear* pressure law, we succeed in proving stability estimates directly for (1.1) in  $L_\infty(0, T; \dot{H}^{-1})$  for the density and  $L_2(0, T; L_2)$  for the velocity. The proof has some similarities with that of D. Hoff in [16] but does not require Lagrangian coordinates. Indeed we overcome the fact that  $\nabla v \notin L_{1,loc}(\mathbb{R}_+; L_\infty)$  by combining the information that  $\nabla \in L_{1+\varepsilon,loc}(\mathbb{R}_+; BMO)$  with a suitable logarithmic interpolation inequality from [22].

The rest of the paper unfolds as follows. The next section is dedicated to the proof of regularity estimates for (1.1) assuming that the solution under consideration is smooth and that  $\nu$  is large enough (this corresponds to steps 1 to 3 above). For better readability, we focus there on  $d = 2$ , and the most technical parts of that section are postponed in appendix. In Section 3, we prove the existence part of our main theorem (and also justify the convergence of (1.1) to (1.7) for  $\nu$  going to  $\infty$ ), while Section 4 is dedicated to uniqueness. Some technical results like, in particular, Inequality (1.13) and time weighted estimates, and the case  $d = 3$  are postponed in appendix.

## 2. REGULARITY ESTIMATES

The present section is devoted to proving regularity estimates for the velocity field of a solution  $(\rho, v)$  to (1.1) in  $\mathbb{R}_+ \times \mathbb{T}^d$ . We focus on  $d = 2$ , the three dimensional case being postponed in appendix.

As a start, we normalize the potential energy  $e$  in such a way that  $e(1) = e'(1) = 0$ , setting

$$(2.1) \quad e(\rho) := \rho \int_1^\rho \frac{P(\varrho)}{\varrho^2} d\varrho - P(1)(\rho - 1).$$

Hence,  $\|e\|_1$  is essentially equivalent to  $\|\rho - 1\|_2^2$  and, in the case  $P(\rho) = \rho^\gamma$ , we have

$$e(\rho) = \rho \log \rho + 1 - \rho \quad \text{if } \gamma = 1, \quad \text{and} \quad e(\rho) = \frac{\rho^\gamma}{\gamma - 1} - \frac{\gamma \rho}{\gamma - 1} + 1 \quad \text{if } \gamma > 1.$$

We shall often use the notations  $e$  and  $P$  instead of  $e(\rho)$  and  $P(\rho)$ .

**2.1. Sobolev estimates for the velocity.** Here we derive a global-in-time  $H^1$  energy estimate that requires only a control on  $\sup \rho$ . The overall strategy is inspired by [11].

Throughout the proof, we denote  $\tilde{P} := P - \bar{P}$  and  $\tilde{G} := G - \bar{G}$  where  $\bar{P}$  and  $\bar{G}$  stand for the average of  $P$  and  $G$ . Note that we have

$$(2.2) \quad \tilde{G} = \nu \operatorname{div} v - \tilde{P}.$$

**Proposition 2.1.** *Consider a smooth solution  $(\rho, v)$  to (1.1) on  $[0, T] \times \mathbb{T}^2$  satisfying (1.8). Assume that the pressure law fulfills (1.11) and that, for some positive constant  $\rho^*$ ,*

$$(2.3) \quad 0 \leq \rho(t, x) \leq \rho^* \quad \text{for all } (t, x) \in [0, T] \times \mathbb{T}^2.$$

*Let  $\dot{v} := v_t + v \cdot \nabla v$  be the material derivative of  $v$ , and  $h := \rho P' - P$ . There exist:*

*– a functional  $\mathcal{E}$  such that*

$$\mathcal{E} \geq \frac{1}{2} \int_{\mathbb{T}^2} \left( \rho |v|^2 + \mu |\nabla \mathcal{P} v|^2 + \frac{1}{\nu} (\tilde{G}^2 + \tilde{P}^2) + 2e \right) dx,$$

*– an absolute positive constant  $C$ ,*

*– a positive constant  $\nu_0$  depending<sup>1</sup> only on the pressure function  $P$ , on  $\mu$  and on  $\rho^*$ , such that if  $\nu \geq \nu_0$  then for all  $t \in [0, T]$ , we have*

$$(2.4) \quad 1 + \frac{1}{\mu E_0} \left( \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \right) \leq \left( 1 + \frac{\mathcal{E}_0}{\mu E_0} \exp \left\{ C \left( 1 + \frac{(\rho^*)^2}{\mu^4} E_0^2 \log(e + \rho^*) \right) \right\} \right)^{\exp \left\{ C \frac{(\rho^*)^2}{\mu^4} E_0^2 \right\}},$$

---

<sup>1</sup>Here we find  $\nu_0 = \max \left( \mu, C \sqrt{\frac{\rho^* \log(e + \rho^*)}{\mu}} P(\rho^*), \frac{P(\rho^*)}{2}, 4\sqrt{\rho^*(1 + h(\rho^*))} \right)$ .

with  $E_0$  defined in (1.5) and

$$\mathcal{D} := \int_{\mathbb{T}^2} \left( \frac{1}{4} \rho |\dot{v}|^2 + \frac{\mu^2}{4\rho^*} |\nabla^2 \mathcal{P} v|^2 + \frac{1}{8\rho^*} |\nabla G|^2 + \frac{1}{4\nu} \tilde{P}^2 + \left( \frac{\nu+h}{2} \right) (\operatorname{div} v)^2 + \frac{\mu}{2} |\nabla v|^2 \right) dx.$$

*Proof.* The beginning of the proof is independent of the dimension : we take the  $L_2$  inner product of the momentum equation of (1.1) with  $\dot{v}$ , and get

$$(2.5) \quad \int_{\mathbb{T}^d} \rho |\dot{v}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} (\mu |\nabla v|^2 + (\lambda + \mu) (\operatorname{div} v)^2) dx \\ + \int_{\mathbb{T}^d} \nabla P \cdot v_t dx = \int_{\mathbb{T}^d} (\rho \dot{v}) \cdot (v \cdot \nabla v) dx.$$

To handle the pressure term in the left-hand side, we start from

$$P_t + \operatorname{div}(Pv) + h \operatorname{div} v = 0.$$

Therefore, integrating by parts yields

$$\int_{\mathbb{T}^d} \nabla P \cdot v_t dx = -\frac{d}{dt} \int_{\mathbb{T}^d} P \operatorname{div} v dx - \int_{\mathbb{T}^d} h (\operatorname{div} v)^2 dx + \int_{\mathbb{T}^d} P v \cdot \nabla \operatorname{div} v dx.$$

Since

$$-(\nu \operatorname{div} v)^2 = P^2 - G^2 - 2\nu P \operatorname{div} v \quad \text{and} \quad \nu \nabla \operatorname{div} v = \nabla(P + G),$$

we get after integrating by parts once to avoid the appearance of some  $\nabla P$  term,

$$(2.6) \quad \int_{\mathbb{T}^d} \nabla P \cdot v_t dx = -\frac{d}{dt} \int_{\mathbb{T}^d} P \operatorname{div} v dx + \frac{1}{\nu^2} \int_{\mathbb{T}^d} (P^2 - G^2) h dx + \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G dx \\ - \frac{1}{\nu} \int_{\mathbb{T}^d} \left( \frac{P^2}{2} + 2Ph \right) \operatorname{div} v dx.$$

Observing that

$$\tilde{G} = -\bar{P} \quad \text{and} \quad \bar{P}' = -\int_{\mathbb{T}^d} h \operatorname{div} v dx,$$

we find that

$$(2.7) \quad \int_{\mathbb{T}^d} (P^2 - G^2) h dx = \nu \int_{\mathbb{T}^d} (\tilde{P} - \tilde{G}) \operatorname{div} v h dx + 2\nu \bar{P} \int_{\mathbb{T}^d} h \operatorname{div} v dx \\ = \nu^2 \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx - 2\nu \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx - \nu \frac{d}{dt} (\bar{P})^2.$$

Let the function  $k$  be the unique solution of

$$k - \rho k' = -\frac{P^2}{2} - 2Ph \quad \text{and} \quad k(1) = P^2(1).$$

Then, we have

$$(2.8) \quad -\int_{\mathbb{T}^d} \left( \frac{P^2}{2} + 2Ph \right) \operatorname{div} v dx = \int_{\mathbb{T}^d} (\partial_t k + \operatorname{div}(kv)) dx = \frac{d}{dt} \int_{\mathbb{T}^d} k dx.$$

Hence, plugging (2.6) and (2.7) in (2.8), we obtain

$$\int_{\mathbb{T}^d} \nabla P \cdot v_t dx = \frac{d}{dt} \int_{\mathbb{T}^d} \left( \frac{k - (\bar{P})^2}{\nu} - P \operatorname{div} v \right) dx - \frac{2}{\nu} \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx \\ + \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx + \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G dx.$$

Now, denoting

$$\tilde{\mathcal{E}} := \int_{\mathbb{T}^d} \left( \frac{\mu}{2} |\nabla v|^2 + \frac{\lambda + \mu}{2} (\operatorname{div} v)^2 + \frac{1}{\nu} (k - (\bar{P})^2) - \tilde{P} \operatorname{div} v \right) dx$$

and reverting to (2.5), we conclude that

$$(2.9) \quad \begin{aligned} \frac{d}{dt} \tilde{\mathcal{E}} + \int_{\mathbb{T}^d} \rho |\dot{v}|^2 dx + \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx \\ = \int_{\mathbb{T}^d} (\rho \dot{v}) \cdot (v \cdot \nabla v) dx + \frac{2}{\nu} \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx - \frac{1}{\nu} \int_{\mathbb{T}^d} P v \cdot \nabla G dx. \end{aligned}$$

Observe that

$$\tilde{\mathcal{E}} = \int_{\mathbb{T}^d} \left( \frac{\mu}{2} (|\nabla v|^2 - (\operatorname{div} v)^2) + \frac{1}{2\nu} \left( (\nu \operatorname{div} v)^2 - 2\nu \operatorname{div} v \tilde{P} + 2(k - (\bar{P})^2) \right) \right) dx$$

and that

$$(2.10) \quad k(\rho) = P^2(\rho) - \frac{\rho}{2} \int_1^\rho \frac{P^2(\varrho)}{\varrho^2} d\varrho.$$

Hence we have

$$\tilde{\mathcal{E}} = \frac{1}{2} \int_{\mathbb{T}^d} \left( \mu |\nabla \mathcal{P} v|^2 + \frac{1}{\nu} \left( \tilde{G}^2 + \tilde{P}^2 + \rho \int_\rho^1 \frac{P^2(\tau)}{\tau^2} d\tau \right) \right) dx.$$

Let  $P^* := \|P(\rho)\|_\infty$ . Since we have for all  $\rho \geq 0$ ,

$$\begin{aligned} \rho \int_1^\rho \frac{P^2(\tau)}{\tau^2} d\tau \leq \rho P(\rho) \int_1^\rho \frac{P(\tau)}{\tau^2} d\tau = P(\rho) (e(\rho) + P(1)(\rho - 1)) \\ \leq P^* (e(\rho) + P(1)\rho) - P(\rho)P(1), \end{aligned}$$

we get

$$(2.11) \quad \int_{\mathbb{T}^d} \rho(x) \left( \int_1^{\rho(x)} \frac{P^2(\tau)}{\tau^2} d\tau \right) dx \leq P^* (\|e\|_1 + P(1)) - \bar{P}P(1),$$

and thus

$$(2.12) \quad \tilde{\mathcal{E}} \geq \frac{1}{2} \int_{\mathbb{T}^d} \left( \mu |\nabla \mathcal{P} v|^2 + \frac{1}{\nu} (\tilde{G}^2 + \tilde{P}^2) \right) dx - \frac{1}{2\nu} \left( P^* \|e\|_1 + P(1)(P^* - \bar{P}) \right).$$

In order to get a control on the right-hand side of (2.9), let us rewrite the momentum equation in terms of the viscous effective flux  $G = \nu \operatorname{div} v - P$  as follows:

$$(2.13) \quad \mu(\Delta v - \nabla \operatorname{div} v) + \nabla G = \rho \dot{v}.$$

From it, we discover that

$$(2.14) \quad \mu^2 \|\Delta \mathcal{P} v\|_2^2 + \|\nabla G\|_2^2 = \|\rho \dot{v}\|_2^2 \leq \rho^* \|\sqrt{\rho} \dot{v}\|_2^2.$$

Since we obviously have

$$\frac{2}{\nu} \int_{\mathbb{T}^d} \tilde{G} \operatorname{div} v h dx \leq \frac{1}{2} \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h dx + \frac{2}{\nu^2} \int_{\mathbb{T}^d} \tilde{G}^2 h dx,$$



equality (2.9) implies that

$$(2.15) \quad \begin{aligned} \frac{d}{dt} \tilde{\mathcal{E}} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{2\rho^*} \|\Delta \mathcal{P}v\|_2^2 + \frac{1}{2\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^d} (\operatorname{div} v)^2 h \, dx \\ \leq \|\sqrt{\rho} v \cdot \nabla v\|_2^2 + \frac{2}{\nu^2} \int_{\mathbb{T}^d} \tilde{G}^2 h \, dx - \frac{1}{\nu} \int_{\mathbb{T}^d} Pv \cdot \nabla G \, dx. \end{aligned}$$

To bound the first term in the right-hand side, we decompose  $\nabla v$  into

$$(2.16) \quad \nabla v = \nabla \mathcal{P}v - \frac{1}{\nu} \nabla^2 (-\Delta)^{-1} \tilde{G} - \frac{1}{\nu} \nabla^2 (-\Delta)^{-1} \tilde{P}.$$

Hence

$$\|\sqrt{\rho} v \cdot \nabla v\|_2^2 \leq 3 \left( \|\sqrt{\rho} v \cdot \nabla \mathcal{P}v\|_2^2 + \frac{1}{\nu^2} \|\sqrt{\rho} v \cdot \nabla^2 (-\Delta)^{-1} \tilde{G}\|_2^2 + \frac{1}{\nu^2} \|\sqrt{\rho} v \cdot \nabla^2 (-\Delta)^{-1} \tilde{P}\|_2^2 \right).$$

From this point, we assume that  $d = 2$ . Then, Hölder and Gagliardo-Nirenberg inequalities yield

$$\int_{\mathbb{T}^2} \rho |v \cdot \nabla \mathcal{P}v|^2 \, dx \leq C \sqrt{\rho^*} \left( \int_{\mathbb{T}^2} \rho |v|^4 \, dx \right)^{\frac{1}{2}} \|\nabla \mathcal{P}v\|_2 \|\nabla^2 \mathcal{P}v\|_2.$$

Since the density is not bounded from below, in order to bound the right-hand side, one has to take advantage of Inequality (1.13). We get

$$(2.17) \quad \begin{aligned} 3 \int_{\mathbb{T}^2} \rho |v \cdot \nabla \mathcal{P}v|^2 \, dx &\leq C \sqrt{\rho^*} \|\sqrt{\rho} v\|_2 \|\nabla v\|_2 \|\nabla \mathcal{P}v\|_2 \|\nabla^2 \mathcal{P}v\|_2 \log^{\frac{1}{2}} \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right) \\ &\leq \frac{\mu^2}{4\rho^*} \|\Delta \mathcal{P}v\|_2^2 + \frac{C(\rho^*)^2}{\mu^2} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \|\nabla \mathcal{P}v\|_2^2 \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right). \end{aligned}$$

Arguing similarly and using the fact that  $\nabla^2 (-\Delta)^{-1}$  maps  $L_4(\mathbb{T}^2)$  to itself, we get

$$(2.18) \quad \begin{aligned} \frac{3}{\nu^2} \int_{\mathbb{T}^2} \rho |v \cdot [\nabla^2 (-\Delta)^{-1} \tilde{G}]|^2 \, dx &\leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 \\ &\quad + \frac{C(\rho^*)^2}{\nu^4} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \|\tilde{G}\|_2^2 \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right), \end{aligned}$$

and also,

$$(2.19) \quad \begin{aligned} \frac{3}{\nu^2} \int_{\mathbb{T}^2} \rho |v \cdot \nabla^2 \Delta^{-1} \tilde{P}|^2 \, dx \\ \leq \frac{1}{4\nu} \|\tilde{P}\|_2^2 + \frac{C\rho^*}{\nu^3} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2} \right) \|\tilde{P}\|_\infty^2. \end{aligned}$$

Finally, we have, thanks to Inequality (A.3),

$$\begin{aligned} -\frac{1}{\nu} \int_{\mathbb{T}^2} Pv \cdot \nabla G \, dx &\leq \frac{1}{\nu} P^* \|v\|_2 \|\nabla G\|_2 \\ &\leq \frac{C}{\nu} P^* \log^{\frac{1}{2}}(e + \|\tilde{\rho}\|_2) \|\nabla v\|_2 \|\nabla G\|_2. \end{aligned}$$

Hence

$$(2.20) \quad -\frac{1}{\nu} \int_{\mathbb{T}^2} Pv \cdot \nabla G \, dx \leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 + C \frac{\rho^*}{\nu^2} (P^*)^2 \|\nabla v\|_2^2 \log(e + \|\tilde{\rho}\|_2).$$

Therefore, plugging (2.17), (2.18), (2.19) and (2.20) in (2.15), we conclude that

$$\begin{aligned}
(2.21) \quad & \frac{d}{dt} \tilde{\mathcal{E}} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\Delta \mathcal{P}v\|_2^2 + \frac{1}{4\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} v)^2 h \, dx \\
& \leq \frac{C\rho^*}{\nu^2} \log(e + \|\tilde{\rho}\|_2) (P^*)^2 \|\nabla v\|_2^2 + \frac{1}{4\nu} \|\tilde{P}\|_2^2 + \frac{2}{\nu^2} \int_{\mathbb{T}^2} \tilde{G}^2 h \, dx \\
& + C \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^2 \left( \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} + \frac{(\rho^*)^2}{\mu^2} \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^2}{\nu^4} \|\tilde{G}\|_2^2 \right) \log \left( e + \|\tilde{\rho}\|_2 + \frac{\|\rho\|_2 \|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right).
\end{aligned}$$

At this stage, in order to handle all the terms of the right-hand side, one may add up to  $\tilde{\mathcal{E}}$  some suitable multiple of the basic energy  $E$  and of a complementary relation involving  $e$  from which one can glean some time-decay for  $\|\tilde{P}\|_2$ . Indeed, let us start with

$$\partial_t e + \operatorname{div}(ev) + P \operatorname{div} v = 0.$$

Integrating on  $\mathbb{T}^2$  and remembering that  $\nu \operatorname{div} v = \tilde{P} + \tilde{G}$  yields

$$(2.22) \quad \frac{d}{dt} \int_{\mathbb{T}^2} e \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} |\tilde{P}|^2 \, dx = -\frac{1}{\nu} \int_{\mathbb{T}^2} \tilde{P} \tilde{G} \, dx.$$

Let us set

$$\begin{aligned}
\mathcal{E} & := \tilde{\mathcal{E}} + E + \|e\|_1 + \frac{1}{2\nu} (P^* - P(1))P(1) \\
& = \frac{1}{2} \int_{\mathbb{T}^2} \left( \rho |v|^2 + \mu |\nabla \mathcal{P}v|^2 + \frac{1}{\nu} \left( \tilde{G}^2 + \tilde{P}^2 + \left( \rho \int_\rho^1 \frac{P^2(\tau)}{\tau^2} d\tau \right) + (P^* - P(1))P(1) \right) + 4e \right) dx.
\end{aligned}$$

In order to control the integral in the right-hand side of (2.22), one may use that

$$\frac{1}{\nu} \int_{\mathbb{T}^2} |\tilde{P}| |\tilde{G}| \, dx \leq \frac{1}{2\nu} \int_{\mathbb{T}^2} \tilde{P}^2 \, dx + \frac{1}{2\nu} \int_{\mathbb{T}^2} \tilde{G}^2 \, dx.$$

Then, Poincaré inequality implies that

$$\frac{1}{\nu} \int_{\mathbb{T}^2} \left( \frac{2}{\nu} h + 1 \right) \tilde{G}^2 \, dx \leq \frac{4\rho^*}{\nu} \left( \frac{2}{\nu} \|h\|_\infty + 1 \right) \frac{\|\nabla G\|_2^2}{4\rho^*}.$$

Using also the fact that  $\|\tilde{\rho}\|_2^2 = \|\rho\|_2^2 - 1 \leq (\rho^*)^2 - 1$ , we get

$$\begin{aligned}
\frac{d}{dt} \mathcal{E} + \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\nabla^2 \mathcal{P}v\|_2^2 + \frac{1}{4\rho^*} \left( 1 - \frac{4\rho^*}{\nu} \left( \frac{2}{\nu} \|h\|_\infty + 1 \right) \right) \|\nabla G\|_2^2 + \frac{1}{4\nu} \|\tilde{P}\|_2^2 \\
+ \int_{\mathbb{T}^2} \left( \nu + \frac{h}{2} \right) (\operatorname{div} v)^2 \, dx + \mu \|\nabla \mathcal{P}v\|_2^2 - C \frac{\rho^* \log(e + \rho^*)}{\nu^2} (P^*)^2 \|\nabla v\|_2^2 \\
\leq C \|\sqrt{\rho}v\|_2^2 \|\nabla v\|_2^2 \left( \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} + \frac{(\rho^*)^2}{\mu^2} \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^2}{\nu^4} \|\tilde{G}\|_2^2 \right) \log \left( e + \rho^* + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho}v\|_2^2} \right).
\end{aligned}$$

Obviously, thanks to (2.11), we have

$$(2.23) \quad \mathcal{E} \geq \frac{1}{2} \left( \|\sqrt{\rho}v\|_2^2 + \mu \|\nabla \mathcal{P}v\|_2^2 + \frac{1}{\nu} (\|\tilde{G}\|_2^2 + \|\tilde{P}\|_2^2) \right) + \left( 2 - \frac{P^*}{2\nu} \right) \|e\|_1.$$

Now, since

$$(2.24) \quad \mu \|\nabla v\|_2^2 + (\lambda + \mu) \|\operatorname{div} v\|_2^2 = \mu \|\nabla \mathcal{P}v\|_2^2 + \nu \|\operatorname{div} v\|_2^2,$$

we have if  $\nu \geq \mu$ ,

$$\nu \|\operatorname{div} v\|_2^2 + \mu \|\nabla \mathcal{P}v\|_2^2 \geq \mu \|\nabla v\|_2^2.$$

Therefore, because for all  $A \geq 0$ ,

$$\log(e + \rho^* + \rho^* A) \leq \log(e + \rho^*) + \log(1 + A) \leq \log(e + \rho^*) + A,$$

if one assumes that

$$(2.25) \quad 1 \geq 2C \frac{\rho^* \log(e + \rho^*) (P^*)^2}{\mu \nu^2} \quad \text{and} \quad \frac{8\rho^*}{\nu} \left( \frac{2}{\nu} \|h\|_\infty + 1 \right) \leq 1,$$

then the above inequalities imply that

$$(2.26) \quad \frac{d}{dt} \mathcal{E} + \mathcal{D} \leq C \rho^* \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \frac{\|\tilde{P}\|_\infty^2}{\nu^3} \left( \log(e + \rho^*) + \frac{\|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right) \\ + C (\rho^*)^2 \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^2 \left( \frac{1}{\mu^2} \|\nabla \mathcal{P} v\|_2^2 + \frac{1}{\nu^4} \|\tilde{G}\|_2^2 \right) \left( \log(e + \rho^*) + \log \left( 1 + \frac{\|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right) \right)$$

with

$$\mathcal{D} := \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\nabla^2 \mathcal{P} v\|_2^2 + \frac{1}{8\rho^*} \|\nabla G\|_2^2 + \frac{1}{4\nu} \|\tilde{P}\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} v)^2 (\nu + h) dx + \frac{\mu}{2} \|\nabla v\|_2^2.$$

So, finally, if one assumes that

$$(2.27) \quad \nu \geq \mu, \quad \nu^2 \geq 2C \mu^{-1} \rho^* \log(e + \rho^*) (P^*)^2, \quad \nu \geq 8\rho^* (2\nu^{-1} \|h\|_\infty + 1) \quad \text{and} \quad \nu \geq P^*/2,$$

the last condition ensuring that the coefficient of the last term in (2.23) is greater than 1, then we have

$$(2.28) \quad \mathcal{E} \geq \frac{1}{2} \left( \|\sqrt{\rho} v\|_2^2 + \mu \|\nabla \mathcal{P} v\|_2^2 + \frac{1}{\nu} \|\tilde{G}\|_2^2 + \frac{1}{\nu} \|\tilde{P}\|_2^2 \right) + \|e\|_1,$$

and thus

$$\|\sqrt{\rho} v\|_2^2 \leq 2\mathcal{E}, \quad \|\nabla \mathcal{P} v\|_2^2 \leq 2\mathcal{E}/\mu \quad \text{and} \quad \|\tilde{G}\|_2^2 \leq 2\nu\mathcal{E}.$$

Thanks to that, inequality (2.26) combined with the energy balance (1.5) and the fact that the map  $r \mapsto r \log^{1/2}(a + b/r)$  is nondecreasing on  $\mathbb{R}_+$  if  $a \geq 1$  and  $b \geq 0$ , implies that

$$\frac{d}{dt} \mathcal{E} + \mathcal{D} \leq C \frac{(\rho^*)^2}{\mu^3} E_0 \|\nabla v\|_2^2 \mathcal{E} \log \left( 1 + \frac{\mathcal{E}}{\mu E_0} \right) \\ + C \left( \frac{(\rho^*)^2}{\mu^3} E_0 \log(e + \rho^*) + \frac{\rho^* \|\tilde{P}\|_\infty^2}{\mu \nu^3} + \log(e + \rho^*) \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} \right) \|\nabla v\|_2^2 \mathcal{E}.$$

Note that Condition (2.27) entails that

$$\frac{\rho^* \|\tilde{P}\|_\infty^2}{\mu \nu^3} + \log(e + \rho^*) \frac{\rho^* \|\tilde{P}\|_\infty^2}{\nu^3} \leq 1.$$

Therefore applying Lemma A.1 with

$$A := 1, \quad B := \frac{1}{\mu E_0}, \quad f := C \frac{(\rho^*)^2}{\mu^3} E_0 \|\nabla v\|_2^2 \quad \text{and} \quad g := C \left( 1 + \frac{(\rho^*)^2}{\mu^3} E_0 \log(e + \rho^*) \right) \|\nabla v\|_2^2,$$

we get

$$1 + \frac{1}{\mu E_0} \left( \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \right) \\ \leq \left( 1 + \frac{\mathcal{E}_0}{\mu E_0} \exp \left\{ C \left( 1 + \frac{(\rho^*)^2}{\mu^3} E_0 \log(e + \rho^*) \right) \int_0^t \|\nabla v\|_2^2 d\tau \right\} \right)^{\exp \left\{ C \frac{(\rho^*)^2}{\mu^3} E_0 \int_0^t \|\nabla v\|_2^2 d\tau \right\}},$$

which, in light of the basic energy conservation (1.5), yields (2.4).  $\square$

**Remark 2.1.** We do not really need  $\nu$  to be large, as one has some freedom on the definition of  $\mathcal{E}$ , and lots of possibilities for bounding the right-hand side of (2.21). For small  $\nu$ , one can get a global, but time dependent control on  $\mathcal{E}$ . We chose not to treat that case here since the condition that  $\nu$  is large will be needed in the next step, in order to remove the a priori assumption that  $\rho$  is bounded.

**2.2. An upper bound for the density.** Here, we prove that, for large enough  $\nu$ , if the initial data fulfill the assumptions of the previous section, then we have a global-in-time control on the supremum of  $\rho$ . For simplicity, we assume that  $P(\rho) = \rho^\gamma$  for some  $\gamma \geq 1$ .

**Proposition 2.2.** Consider a smooth solution  $(\rho, v)$  of (1.1) on  $[0, T] \times \mathbb{T}^2$ . There exists  $\nu_0$  depending only on  $\gamma, \mu, \|\rho_0\|_\infty$  and  $\mathcal{E}_0$  but independent of  $T$  such that if  $\nu \geq \nu_0$ , then

$$(2.29) \quad \sup_{t \in [0, T]} \|\rho(t)\|_\infty \leq 2e^{\frac{\gamma-1}{\gamma} E_0} \|\rho_0\|_\infty.$$

*Proof.* Throughout the proof, we denote slightly abusively the right-hand side of (2.29) by  $\rho^*$ . We start from the observation that if  $\rho > 0$  then

$$\partial_t \log \rho + v \cdot \nabla \log \rho = -\operatorname{div} v = -\frac{1}{\nu} (\tilde{P} + \tilde{G}).$$

Remember that the definition of  $\tilde{G}$  ensures that

$$\Delta \tilde{G} = \partial_t (\operatorname{div}(\rho v)) + \operatorname{div}(\operatorname{div}(\rho v \otimes v)).$$

Therefore, following [11] and introducing

$$F := \log \rho + \nu^{-1} \Delta^{-1} \operatorname{div}(\rho v),$$

we discover that (with the summation convention over repeated indices),

$$(2.30) \quad \partial_t F + v \cdot \nabla F + \frac{1}{\nu} \tilde{P} = -\frac{1}{\nu} [v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i.$$

Since we have

$$P(\rho) \geq \gamma \log \rho + 1 \quad \text{for all } \rho > 0,$$

setting  $F^+ := \max(0, F)$  yields

$$(2.31) \quad \partial_t F^+ + v \cdot \nabla F^+ + \frac{\gamma}{\nu} F^+ \leq \frac{1}{\nu} [v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i + \frac{\gamma}{\nu^2} |(-\Delta)^{-1} \operatorname{div}(\rho v)| + \frac{1}{\nu} (\bar{P} - C).$$

As  $P(\rho) = \rho^\gamma$ , we have  $\bar{P} - 1 = (\gamma - 1)\|e\|_1$ , so that the last term may be bounded by  $(\gamma - 1)E_0$ . Then, performing a time integration in (2.31) yields

$$(2.32) \quad \begin{aligned} \|F^+(t)\|_\infty &\leq e^{-\frac{\gamma}{\nu} t} \|F^+(0)\|_\infty + \frac{1}{\nu} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i(\tau)\|_\infty d\tau \\ &\quad + \frac{\gamma}{\nu^2} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|(-\Delta)^{-1} \operatorname{div}(\rho v)(\tau)\|_\infty d\tau + \frac{\gamma-1}{\gamma} (1 - e^{-\frac{\gamma}{\nu} t}) E_0. \end{aligned}$$

Using that the average of  $\rho v$  is zero, Sobolev embedding and the properties of continuity of Riesz operator imply that

$$(2.33) \quad \|(-\Delta)^{-1} \operatorname{div}(\rho v)\|_\infty \lesssim \|(-\Delta)^{-1} \nabla \operatorname{div}(\rho v)\|_4 \lesssim \|\rho v\|_4.$$

Then we use again (1.13) and get

$$\|(-\Delta)^{-1} \operatorname{div}(\rho v)\|_\infty \lesssim (\rho^*)^{\frac{3}{4}} \|\sqrt{\rho} v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right),$$

whence, thanks to the energy balance (1.5) and the definition of  $\mathcal{E}$  (assuming that  $\nu \geq \mu$ ),

$$(2.34) \quad \|(-\Delta)^{-1} \operatorname{div}(\rho v)\|_\infty \leq C(\rho^*)^{\frac{3}{4}} E_0^{\frac{1}{4}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \mathcal{E}}{\mu E_0} \right).$$

Since  $[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i = [\tilde{v}^j, \Delta^{-1} \partial_i \partial_j] \rho v^i$  with  $\tilde{v} = v - \bar{v}$ , the second term in the r.h.s. of (2.32) may be bounded by means of Sobolev embedding and of the following Coifman, Lions, Meyer and Semmes inequality (from [3]) as follows:

$$(2.35) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim \|[\tilde{v}^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_{W^{1,3}} \lesssim \|\nabla v\|_{12} \|\rho v\|_4.$$

To handle  $\nabla v$ , we use that

$$\begin{aligned} \|\nabla v\|_{12} &\lesssim \|\nabla \mathcal{P}v\|_{12} + \nu^{-1} (\|\tilde{G}\|_{12} + \|\tilde{P}\|_{12}) \\ &\lesssim \|\nabla^2 \mathcal{P}v\|_2 + \nu^{-1} (\|\nabla G\|_2 + \|\tilde{P}\|_\infty). \end{aligned}$$

Hence, using once more (1.13),

$$\begin{aligned} \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty &\lesssim (\rho^*)^{\frac{3}{4}} (\|\nabla^2 \mathcal{P}v\|_2 + \nu^{-1} \|\nabla G\|_2 + \nu^{-1} \|\tilde{P}\|_\infty) \\ &\quad \times \|\sqrt{\rho} v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \|\nabla v\|_2^2}{\|\sqrt{\rho} v\|_2^2} \right), \end{aligned}$$

whence, using the energy conservation (1.5) and the definition of  $\mathcal{E}$  and  $\mathcal{D}$ ,

$$(2.36) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim ((\rho^*)^{\frac{5}{4}} \mu^{-1} \mathcal{D}^{\frac{1}{2}} + (\rho^*)^{\frac{3}{4}} \nu^{-1} \|\tilde{P}\|_\infty) \times E_0^{\frac{1}{4}} \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \mathcal{E}}{\mu E_0} \right).$$

Plugging (2.34) and (2.36) in (2.32) and performing obvious simplifications, we end up with

$$(2.37) \quad \|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + \frac{\gamma-1}{\gamma} E_0 + C(\rho^*)^{\frac{3}{4}} \frac{E_0^{\frac{1}{4}}}{\nu} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \left( (\sqrt{\rho^*} \mu^{-1} \mathcal{D}^{\frac{1}{2}} + \nu^{-1} (\|\tilde{P}\|_\infty + \gamma)) \|\nabla v\|_2^{\frac{1}{2}} \log^{\frac{1}{4}} \left( e + \rho^* + \frac{\rho^* \mathcal{E}}{\mu E_0} \right) \right) d\tau.$$

Now, let us consider the largest sub-interval  $[0, T_0]$  of  $[0, T]$  on which (2.29) is fulfilled. Then, Inequality (2.4) tells us that there exist  $\nu_0$  depending only on  $\|\rho_0\|_\infty$ ,  $\mu$  and  $\gamma$ , and  $C_0 > 0$  (depending also on  $E_0$ ,  $\mathcal{E}_0$ ) so that we have for all  $t \in [0, T]$ , if  $\nu \geq \nu_0$ ,

$$(2.38) \quad \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \leq C_0.$$

Inequality (2.37) thus becomes (taking a larger  $C_0$  as the case may be):

$$(2.39) \quad \|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + \frac{\gamma-1}{\gamma} E_0 + \frac{C_0}{\nu} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \left( \frac{\mathcal{D}^{\frac{1}{2}}(\tau)}{\mu} + \frac{\gamma}{\nu} \right) \|\nabla v(\tau)\|_2^{\frac{1}{2}} d\tau.$$

From Hölder inequality, we have for all  $t \in [0, T]$ ,

$$\begin{aligned} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \mathcal{D}^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} d\tau &\leq C \left( \frac{\nu}{\gamma} \right)^{\frac{1}{4}} \left( \int_0^T \mathcal{D}(\tau) d\tau \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla v(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\ \text{and } \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|\nabla v(\tau)\|_2^{\frac{1}{2}} d\tau &\leq C \left( \frac{\nu}{\gamma} \right)^{\frac{3}{4}} \left( \int_0^T \|\nabla v(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{4}}. \end{aligned}$$

As the integrals in the right-hand side may be bounded in terms of the data according to the basic energy balance (1.5) and to (2.38), we eventually get (changing once again  $C_0$  if needed) if  $\nu \geq \nu_0$ :

$$\|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + C_0\nu^{-\frac{3}{4}} + \frac{\gamma-1}{\gamma}E_0.$$

Of course, owing to the definition of  $F^+$  and to (2.4) and (2.34), we have

$$\log \rho \leq F^+ + \nu^{-1}\|\Delta^{-1}\operatorname{div}(\rho v)\|_\infty \leq F^+ + \nu^{-1}C_0.$$

Hence one can eventually conclude that

$$(2.40) \quad \log \rho^* \leq \log \rho_0^* + C_0\nu^{-\frac{3}{4}} + \frac{\gamma-1}{\gamma}E_0.$$

Now, if  $\nu$  is so large as to satisfy also

$$C_0\nu^{-\frac{3}{4}} < \log 2,$$

then (2.40) combined with a bootstrap argument implies that we have (2.29) on  $[0, T]$ .  $\square$

**2.3. Weighted estimates.** That section is devoted to the proof of the following result, that is based on the estimates that have been established so far. For better readability, we postpone the most technical parts of the proof to the appendix.

**Proposition 2.3.** *Define  $\nu_0$  as in Proposition 2.2. Then, smooth solutions to (1.1) on  $[0, T] \times \mathbb{T}^2$  fulfill, if  $\nu \geq \nu_0$ :*

$$(2.41) \quad \sup_{t \in [0, T]} \int_{\mathbb{T}^2} \rho |\dot{v}|^2 t \, dx + \int_0^T \int_{\mathbb{T}^2} (\mu |\nabla \mathcal{P} \dot{v}|^2 + \nu |\operatorname{div} \dot{v}|^2) t \, dx \, dt \leq C_0 T e^{\frac{C_0 T}{\nu}},$$

where  $C_0$  depends on  $\rho^*$ ,  $\mu$ ,  $\mathcal{E}_0$  and on the pressure function, but is independent of  $\nu$  and  $T$ .

*Proof.* Here it will be convenient to use the two notations  $\dot{f}$  and  $\frac{D}{Dt}f$  to designate the convective derivative of  $f$ , and we shall denote  $A : B = \sum_{i,j} A_{ij}B_{ij}$  if  $A$  and  $B$  are two  $d \times d$  matrices. Finally, if  $v$  is a vector field on  $\mathbb{T}^d$  then  $(Dv)_{ij} := \partial_j v^i$  and  $(\nabla v)_{ij} := \partial_i v^j$  for  $1 \leq i, j \leq d$ .

The general principle is to rewrite the momentum equation as:

$$(2.42) \quad \rho \dot{v} - \mu \Delta v - (\nu - \mu) \nabla \operatorname{div} v + \nabla P = 0,$$

then to take the material derivative and test it by  $t \dot{v}$ . We get

$$(2.43) \quad \int_{\mathbb{T}^d} \left( \frac{D}{Dt}(\rho \dot{v}) - \mu \frac{D}{Dt} \Delta v - (\nu - \mu) \frac{D}{Dt} \nabla \operatorname{div} v + \frac{D}{Dt} \nabla P \right) \cdot (t \dot{v}) \, dx = 0.$$

The rest of the proof consists in describing each term of (2.43). To this end, we shall repeatedly use the fact that for all  $\nu \geq \nu_0$  (where  $\nu_0$  is given by Proposition 2.2), we have

$$(2.44) \quad \|\nabla v\|_{L_4(0, T \times \mathbb{T}^2)} \leq C_0.$$

Indeed, recall the decomposition

$$(2.45) \quad v = \mathcal{P}v - \frac{1}{\nu} \nabla (-\Delta)^{-1} (\tilde{G} + \tilde{P}).$$

Proposition 2.1 and Sobolev embeddings imply that

$$(2.46) \quad \|\nabla \mathcal{P}v\|_{L_4(0, T \times \mathbb{T}^2)} \lesssim \|\nabla \mathcal{P}v\|_{L_\infty(0, T; L_2)}^{1/2} \|\nabla^2 \mathcal{P}v\|_{L_\infty(0, T; L_2)}^{1/2} \leq C_0.$$

Furthermore, we have

$$(2.47) \quad \|\tilde{G}\|_{L_4(0,T;\mathbb{T}^2)} \lesssim \|\tilde{G}\|_{L_\infty(0,T;L_2)}^{1/2} \|\nabla G\|_{L_2(0,T;L_2)}^{1/2} \leq \nu^{1/4} C_0,$$

and

$$(2.48) \quad \|\tilde{P}\|_{L_4(0,T;\mathbb{T}^2)} \leq \|\tilde{P}\|_{L_2(0,T;L_2)}^{1/2} \|\tilde{P}\|_{L_\infty(0,T;L_\infty)}^{1/2} \leq \nu^{1/4} C_0.$$

*Step 1.* Obvious computations give (in any dimension):

$$(2.49) \quad \int_{\mathbb{T}^d} \frac{D}{Dt}(\rho\dot{v}) \cdot \dot{v} t \, dx = \frac{1}{2} \int_{\mathbb{T}^d} \left( \frac{D}{Dt}(\rho|\dot{v}|^2 t) + \dot{\rho}|\dot{v}|^2 t - \rho|\dot{v}|^2 \right) dx.$$

Integrating by parts, we see that

$$(2.50) \quad \int_{\mathbb{T}^d} \frac{D}{Dt}(\rho|\dot{v}|^2 t) \, dx = \frac{d}{dt} \int_{\mathbb{T}^d} \rho|\dot{v}|^2 t \, dx - \int_{\mathbb{T}^d} \operatorname{div} v(\rho|\dot{v}|^2 t) \, dx.$$

Thanks to the mass conservation equation, we have

$$(2.51) \quad \int_{\mathbb{T}^d} \dot{\rho}|\dot{v}|^2 t \, dx = - \int_{\mathbb{T}^d} \rho \operatorname{div} v|\dot{v}|^2 t \, dx,$$

whence

$$\int_{\mathbb{T}^d} \frac{D}{Dt}(\rho|\dot{v}|^2 t) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \rho|\dot{v}|^2 t \, dx - \frac{1}{2} \int_{\mathbb{T}^d} \rho|\dot{v}|^2 t \, dx - \int_{\mathbb{T}^d} \rho \operatorname{div} v|\dot{v}|^2 t \, dx.$$

If  $d = 2$  then one can bound the last term using that

$$\begin{aligned} \int_{\mathbb{T}^2} \rho \operatorname{div} v|\dot{v}|^2 t \, dx &= \nu^{-1} \int_{\mathbb{T}^2} (\tilde{P} + \tilde{G})\rho|\dot{v}|^2 t \, dx \\ &\leq \nu^{-1} \|\tilde{P}\|_\infty \|\sqrt{\rho t} \dot{v}\|_2^2 + \rho^* \nu^{-1} \|\tilde{G}\|_2 \|\sqrt{t} \dot{v}\|_4^2 \\ &\leq C_0 \nu^{-1} \|\sqrt{\rho t} \dot{v}\|_2^2 + C_0 \nu^{-1/2} \|\sqrt{t} \dot{v}\|_4^2. \end{aligned}$$

Since  $\int_{\mathbb{T}^2} \rho \dot{v} \, dx = 0$ , one can take advantage of the Poincaré inequality (A.2) with  $p = 2$  and get:

$$(2.52) \quad \|\sqrt{t} \dot{v}\|_4^2 \leq C \|\sqrt{t} \dot{v}\|_2 \|\sqrt{t} \nabla \dot{v}\|_2 \leq C(1 + \|\tilde{\rho}\|_2) \|\sqrt{t} \nabla \dot{v}\|_2^2 \leq C \rho^* \|\sqrt{t} \nabla \dot{v}\|_2^2.$$

Hence,

$$\int_{\mathbb{T}^2} \rho \operatorname{div} v|\dot{v}|^2 t \, dx \leq C_0 \left( \nu^{-1} \|\sqrt{\rho t} \dot{v}\|_2^2 + \nu^{-1/2} \|\sqrt{t} \nabla \dot{v}\|_2^2 \right),$$

and thus

$$(2.53) \quad \begin{aligned} \int_{\mathbb{T}^2} \frac{D}{Dt}(\rho|\dot{v}|^2 t) \, dx &\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \rho|\dot{v}|^2 t \, dx - \frac{1}{2} \int_{\mathbb{T}^2} \rho|\dot{v}|^2 t \, dx \\ &\quad - C_0 \left( \nu^{-1} \int_{\mathbb{T}^2} \rho t |\dot{v}|^2 \, dx + \nu^{-1/2} \int_{\mathbb{T}^2} t |\nabla \dot{v}|^2 \, dx \right). \end{aligned}$$

*Step 2.* The second term of (2.43) rewrites

$$(2.54) \quad -\frac{D}{Dt}\Delta v = -\operatorname{div} \frac{D}{Dt}\nabla v - \nabla v \cdot \nabla^2 v \quad \text{with} \quad (\nabla v \cdot \nabla^2 v)^i := \sum_{j,k} \partial_k v^j \partial_j \partial_k v^i.$$

Testing (2.54) by  $t\dot{v}$  and integrating by parts yields for  $d = 2, 3$ ,

$$-\int_{\mathbb{T}^d} \frac{D}{Dt}\Delta v \cdot t\dot{v} dx = \int_{\mathbb{T}^d} \frac{D}{Dt}\nabla v : \nabla \dot{v} t dx - \int_{\mathbb{T}^d} (\nabla v \cdot \nabla^2 v) \cdot \dot{v} t dx.$$

Since

$$\frac{D}{Dt}\nabla v = \nabla \dot{v} - \nabla v \cdot \nabla v,$$

we get

$$(2.55) \quad -\int_{\mathbb{T}^d} \frac{D}{Dt}\Delta v \cdot t\dot{v} dx = \int_{\mathbb{T}^d} |\nabla \dot{v}|^2 t dx - \int_{\mathbb{T}^d} (\nabla v \cdot \nabla v) : \nabla \dot{v} t dx - \int_{\mathbb{T}^d} (\nabla v \cdot \nabla^2 v) \cdot \dot{v} t dx.$$

The first term is the main one. The other two terms are denoted by  $I_1$  and  $I_2$ , respectively.

Bounding  $I_1$  is easy : using Hölder inequality yields

$$|I_1| = \left| \int_{\mathbb{T}^d} (\nabla v \cdot \nabla v) : \nabla \dot{v} t dx \right| \leq \|t^{1/4} \nabla v\|_4^2 \|\sqrt{t} \nabla \dot{v}\|_2.$$

Therefore, we have according to (2.44),

$$(2.56) \quad \left| \int_0^T I_1(t) dt \right| \leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)}.$$

Bounding  $I_2$  is much more involved. We eventually get (see the details in appendix):

$$(2.57) \quad \left| \int_0^T I_2 dt \right| \leq \left( T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \right. \\ \left. + \sqrt{T} (\nu^{-3/4} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + \nu^{-2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}) \right).$$

Plugging (2.56) and (2.57) in (2.55) and using (2.52) yields

$$(2.58) \quad -\mu \int_0^T \int_{\mathbb{T}^2} \left( \frac{D}{Dt}\Delta v \right) \cdot \dot{v} t dx dt \geq \mu \int_0^T \int_{\mathbb{T}^2} |\nabla \dot{v}|^2 t dx dt \\ - C_0 T^{1/4} (T^{1/4} + \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^2)}.$$

*Step 3.* Now, we consider the third term from equation (2.43), namely

$$(2.59) \quad -\frac{D}{Dt}\nabla \operatorname{div} v = -\nabla \frac{D}{Dt}\operatorname{div} v + \nabla v \cdot \nabla \operatorname{div} v.$$

To control the right-hand side, we have to keep in mind that it involves only the potential part  $\mathcal{Q}v$  of the velocity, since  $\operatorname{div} v = \operatorname{div} \mathcal{Q}v$ . This enables us to write that

$$\nabla \frac{D}{Dt}\operatorname{div} v = \nabla \operatorname{div} \dot{v} - \nabla(\operatorname{tr}(\nabla v \cdot \nabla \mathcal{Q}v)).$$

Hence, testing (2.59) with  $\dot{v} t$  and integrating by parts, we find that

$$(2.60) \quad -\int_{\mathbb{T}^d} \frac{D}{Dt}\nabla \operatorname{div} v \cdot \dot{v} t dx = \int_{\mathbb{T}^d} (\operatorname{div} \dot{v})^2 t dx - K_1 + K_2 \\ \text{with} \quad K_1 := \int_{\mathbb{T}^d} \operatorname{Tr}(\nabla v \cdot \nabla \mathcal{Q}v) \operatorname{div} \dot{v} t dx \quad \text{and} \quad K_2 := \int_{\mathbb{T}^d} (\nabla v \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx.$$



Since  $\nu \mathcal{Q}v = -\nabla(-\Delta)^{-1}(\tilde{G} + \tilde{P})$ , using (2.44), (2.47) and (2.48), we find that, if  $d = 2$ ,

$$(2.61) \quad \begin{aligned} \nu \left| \int_0^T K_1 dt \right| &\leq C\sqrt{T} \|\nabla v\|_{L_4(0,T;L_4)} \|\tilde{P} + \tilde{G}\|_{L_4(0,T;L_4)} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} \\ &\leq C_0 \nu^{1/4} \sqrt{T} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}. \end{aligned}$$

Bounding  $K_2$  will be performed in the appendix. In the end, we get

$$(2.62) \quad \nu \left| \int_0^T K_2 dt \right| \leq C_0 T^{1/4} \left( \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} + T^{1/4} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \right).$$

Thanks to (2.52), the conclusion of this step is that if  $\nu$  is large enough then

$$(2.63) \quad \begin{aligned} -(\nu - \mu) \int_0^T \int_{\mathbb{T}^2} \frac{D}{Dt} \nabla \operatorname{div} v \cdot \dot{v} t dx dt &\geq (\nu - \mu) \int_0^T \int_{\mathbb{T}^2} (\operatorname{div} \dot{v})^2 t dx dt \\ &- C_0 T^{1/4} \left( (\nu T)^{1/4} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} + (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \right). \end{aligned}$$

*Step 4.* The last term under consideration in (2.42) is

$$(2.64) \quad \frac{D}{Dt} \nabla P = \nabla \frac{D}{Dt} P - \nabla v \cdot \nabla P.$$

Here the analysis is simple: since  $\dot{P} = -h \operatorname{div} v$ , we have

$$\begin{aligned} \int_{\mathbb{T}^d} \frac{D}{Dt} \nabla P \cdot \dot{v} t dx &= L_1 + L_2 \quad \text{with} \quad L_1 := \int_{\mathbb{T}^d} h \operatorname{div} v \operatorname{div} \dot{v} t dx \\ &\quad \text{and} \quad L_2 := - \int_{\mathbb{T}^d} \partial_i v^j \partial_j P \dot{v}^i t dx. \end{aligned}$$

On the one hand, we obviously have

$$(2.65) \quad |L_1| \leq \frac{\nu}{4} \int_{\mathbb{T}^d} (\operatorname{div} \dot{v})^2 t dx + T \nu^{-1} \|h\|_\infty^2 \int_{\mathbb{T}^d} (\operatorname{div} v)^2 dx.$$

On the other hand, integrating by parts a couple of times and using  $\operatorname{div} v = \nu^{-1}(\tilde{P} + \tilde{G})$  yields

$$\begin{aligned} L_2 &= \int_{\mathbb{T}^d} \tilde{P} \nabla \operatorname{div} v \cdot \dot{v} t dx + \int_{\mathbb{T}^d} \tilde{P} \nabla v : D \dot{v} t dx \\ &= \frac{1}{\nu} \int_{\mathbb{T}^d} \tilde{P} \nabla P \cdot \dot{v} t dx + \frac{1}{\nu} \int_{\mathbb{T}^d} \tilde{P} \nabla G \cdot \dot{v} t dx + \int_{\mathbb{T}^d} \tilde{P} \nabla v : D \dot{v} t dx \\ &= -\frac{1}{2\nu} \int_{\mathbb{T}^d} \tilde{P}^2 \operatorname{div} \dot{v} t dx + \frac{1}{\nu} \int_{\mathbb{T}^d} \tilde{P} \nabla G \cdot \dot{v} t dx + \int_{\mathbb{T}^d} \tilde{P} \nabla v : D \dot{v} t dx. \end{aligned}$$

Hence we have, if  $d = 2$ ,

$$\begin{aligned} \left| \int_0^T L_2(t) dt \right| &\leq \frac{1}{2\nu} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|t \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} \\ &\quad + \frac{1}{\nu} \|\tilde{P}\|_{L_\infty(0,T;L_4)} \|\nabla G\|_{L_2(0,T;L_2)} \|t \dot{v}\|_{L_2(0,T;L_4)} \\ &\quad + \|\tilde{P}\|_{L_\infty(0,T;L_\infty)} \|\nabla v\|_{L_2(0,T;L_2)} \|t \nabla \dot{v}\|_{L_2(0,T;L_2)}, \end{aligned}$$

whence, thanks to (2.48) and (2.52),

$$\left| \int_0^T L_2(t) dt \right| \leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.$$

So this step gives

$$(2.66) \quad \int_{\mathbb{T}^2} \frac{D}{Dt} \nabla P \cdot \dot{v} t \, dx \geq -\frac{\nu}{4} \int_0^T \int_{\mathbb{T}^2} (\operatorname{div} \dot{v})^2 t \, dx \, dt - h(\rho^*) \nu^{-1} T \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2.$$

*Step 5.* Plugging inequalities (2.53), (2.58), (2.63) and (2.66) in (2.43) (after integrating on  $[0, T]$ ) and using the fact that for a smooth solution, we have  $\sqrt{\rho t} \dot{v}|_{t=0} = 0$ , we discover that for large enough  $\nu$ ,

$$\begin{aligned} & \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^2 + 2\mu \int_0^T \|\sqrt{t} \nabla \mathcal{P} \dot{v}\|_2^2 \, dt + \frac{3\nu}{2} \int_0^T \|\sqrt{t} \operatorname{div} \dot{v}\|_2^2 \, dt \\ & \leq C_0 \left( \nu^{-1} \int_0^T \|\sqrt{\rho t} \dot{v}\|_2^2 + \nu^{-1/2} \int_0^T \|\sqrt{t} \nabla \dot{v}\|_2^2 \, dt \right) + \|\sqrt{\rho} \dot{v}\|_{L_2(0,T;L_2)}^2 + \|h\|_\infty T \nu^{-1} \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 \\ & \quad + C_0 T^{1/4} ((\nu T)^{1/4} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} + (T^{1/4} + \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^2)}). \end{aligned}$$

Taking advantage of inequality (2.4), we have

$$\|\sqrt{\rho} \dot{v}\|_{L_2(0,T;L_2)}^2 + \nu \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 \leq C_0.$$

Furthermore, Young inequality implies that

$$\begin{aligned} C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} & \leq \frac{\mu}{2} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}^2 + C_0 T \\ C_0 \nu^{1/4} \sqrt{T} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} & \leq \frac{\nu}{2} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}^2 + C_0 T \nu^{-1/2} \quad \text{and} \end{aligned}$$

$$C_0 T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \leq C_0 T + \frac{1}{2} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^2 + \frac{\mu}{2} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}^2.$$

In the end, we thus have if  $\nu$  is large enough and  $T \geq 1$ ,

$$X^2(t) + \frac{1}{2} \int_0^t Y^2(\tau) \, d\tau \leq C_0 T + C_0 \nu^{-1} \int_0^t X^2 \, d\tau \quad \text{with}$$

$$X(t) := \|\sqrt{\rho s} \dot{v}\|_{L_\infty(0,t;L_2)} \quad \text{and} \quad Y(t) := (\mu \|\sqrt{t} \nabla \mathcal{P} \dot{v}\|_{L_2(0,T;L_2)}^2 + \nu \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}^2)^{\frac{1}{2}}.$$

Then, applying Gronwall inequality completes the proof of the proposition.  $\square$

As a consequence of those weighted estimates, one can bound  $\operatorname{div} v$  and  $\nabla \mathcal{P} v$  in  $L_1(0, T; L_\infty)$  as follows:

**Corollary 2.1.** *Let  $(\rho, v)$  be a smooth solution of (1.1) on  $[0, T] \times \mathbb{T}^2$  and assume that  $\nu \geq \nu_0$ . Then we have for all  $\varepsilon \in [0, 1/2[$ ,*

$$(2.67) \quad \int_0^T \|\operatorname{div} v\|_\infty^{1+\varepsilon} \, dt \leq C_{0,T,\varepsilon} \nu^{-5/6} \quad \text{and} \quad \int_0^T \|\nabla \mathcal{P} v\|_\infty^{1+\varepsilon} \, dt \leq C_{0,T,\varepsilon}$$

for some  $C_{0,T,\varepsilon} > 0$  depending on  $\varepsilon$ ,  $T$ ,  $\mu$  and on the data, but not on  $\nu$ .

*Proof.* From (1.12) and the previous section, one can write that

$$\nu \|\operatorname{div} v\|_{L_{1+\varepsilon}(0,T;L_\infty)} \leq \|\tilde{P}\|_{L_{1+\varepsilon}(0,T;L_\infty)} + \|\tilde{G}\|_{L_{1+\varepsilon}(0,T;L_\infty)} \leq C_0 T^{\frac{1}{1+\varepsilon}} + \|\tilde{G}\|_{L_{1+\varepsilon}(0,T;L_\infty)}.$$

Gagliardo-Nirenberg inequality implies that

$$\|\tilde{G}\|_\infty \lesssim \|\tilde{G}\|_2^{1/3} \|\nabla \tilde{G}\|_4^{2/3}.$$

Hence, remembering that

$$(2.68) \quad \mu(\nabla \operatorname{div} v - \Delta v) + \nabla G = \rho \dot{v},$$

we get

$$\|\tilde{G}\|_\infty \lesssim \|\tilde{G}\|_2^{1/3} \|\rho \dot{v}\|_4^{2/3} \lesssim (\rho^*)^{2/3} \|\tilde{G}\|_2^{1/3} t^{-1/3} \|\sqrt{t} \dot{v}\|_4^{2/3}.$$

Then, integrating in time and using Hölder inequality, we discover that

$$\int_0^T \|\tilde{G}\|_\infty^{1+\varepsilon} dt \lesssim (\rho^*)^{\frac{2(1+\varepsilon)}{3}} \nu^{\frac{1+\varepsilon}{6}} (\nu^{-1/2} \|\tilde{G}\|_{L_\infty(0,T;L_2)})^{\frac{1+\varepsilon}{3}} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}^{2(1+\varepsilon)/3} \left( \int_0^T t^{-\frac{1+\varepsilon}{2-\varepsilon}} dt \right)^{2/3},$$

whence, using (2.4) to bound  $\tilde{G}$  in  $L_\infty(0,T;L_2)$ ,

$$\|\tilde{G}\|_{L_{1+\varepsilon}(0,T;L_\infty)} \leq C_{0,T,\varepsilon} \nu^{1/6} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}^{2/3}.$$

The last term may be bounded thanks to (2.41) and to (2.52).

Exactly in the same way, we have,

$$\|\nabla \mathcal{P}v\|_\infty \lesssim \|\nabla \mathcal{P}v\|_2^{1/3} \|\nabla^2 \mathcal{P}v\|_4^{2/3}.$$

The last term may be bounded thanks to (2.68) and (2.52), while the first one is bounded from (2.4).  $\square$

### 3. THE PROOF OF EXISTENCE IN DIMENSIONS 2 AND 3

This section is mainly devoted to the construction of solutions fulfilling Theorem 1.1 (or the corresponding statement in dimension 3, see the appendix). The main two difficulties we have to face is that the initial density has no regularity whatsoever and is not positive. To fit in the classical literature devoted to the compressible Navier-Stokes equations, one has to mollify the initial data and to make the density strictly positive. Although this procedure does not disturb the a priori estimates we proved hitherto, the state-of-the-art on the topics just ensures the existence of a smooth solution corresponding to the regularized data on some *finite* time interval. As a first, we thus have to justify that, indeed, the estimates we proved so far ensure that smooth solution to be global, if  $\nu$  is large enough. Then, resorting to rather classical compactness arguments will enable us to conclude the proof of Theorem 1.1.

At the end of the section, we justify the convergence from (1.1) to (1.7) since passing to the limit therein is very similar to Step 4 of the proof of existence.

*Step 1.* The original initial data are:

$$(3.1) \quad \rho_0 \in L_\infty(\mathbb{T}^d) \text{ and } v_0 \in H^1(\mathbb{T}^d).$$

First, we want to change the initial density in such a way that it is bounded away from zero and still has total mass equal to one. To this end, we introduce for any  $\delta \in (0, 1)$ ,

$$(3.2) \quad \tilde{\rho}_0^\delta = \max\{\rho_0, \delta\} \quad \text{and then} \quad \check{\rho}_0^\delta = \min\{\xi_\delta, \tilde{\rho}_0^\delta\},$$

where  $\xi_\delta \geq 1$  is fixed so that

$$(3.3) \quad \int_{\mathbb{T}^d} \check{\rho}_0^\delta dx = 1.$$

Clearly, we have  $\xi_\delta \rightarrow \rho_0^* := \|\rho_0\|_\infty$  when  $\delta \rightarrow 0$ , and thus

$$(3.4) \quad \delta \leq \check{\rho}_0^\delta \leq \rho_0^* \quad \text{and} \quad \check{\rho}_0^\delta \rightarrow \rho_0 \text{ pointwise.}$$

Then we smooth out both  $\check{\rho}_0^\delta$  and  $v_0^\delta$  as follows:

$$(3.5) \quad \rho_0^\delta = \pi_\delta * \check{\rho}_0^\delta \quad \text{and} \quad v_0^\delta = \pi_\delta * v_0,$$

where  $(\pi_\delta)_{\delta>0}$  is a family of positive mollifiers.

Let us emphasize that the total mass of  $\rho_0^\delta$  is still equal to one, and that  $\rho_0^\delta \geq \delta$ .

*Step 2.* We solve (1.1) with data  $(\rho_0^\delta, v_0^\delta)$  according to the classical literature. For example, one may use the following result (see [6, 21, 23]):

**Theorem 3.1.** *Let  $\rho_0 \in W_p^1(\mathbb{T}^d)$  and  $v_0 \in W_p^{2-2/p}(\mathbb{T}^d)$  for some  $p > d$ , with  $d = 2, 3$ . Assume that  $\rho_0 > 0$ . Then there exists  $T_0 > 0$  depending only on the norms of the data, and on  $\inf_{\mathbb{T}^d} \rho_0$  such that (1.1) supplemented with data  $\rho_0$  and  $v_0$  has a unique solution  $(\rho, v)$  on the time interval  $[0, T_0]$ , satisfying<sup>2</sup>*

$$(3.6) \quad v \in W_p^{1,2}(0, T_0 \times \mathbb{T}^d) \quad \text{and} \quad \rho \in \mathcal{C}([0, T_0]; W_p^1(\mathbb{T}^d)).$$

Let us denote by  $(\rho^\delta, v^\delta)$  the maximal solution pertaining to data  $(\rho_0^\delta, v_0^\delta)$  provided by the above statement, and by  $T^\delta$  its existence time (that is  $(\rho^\delta, v^\delta)$  fulfills (3.6) for all  $T < T^\delta$ ). Since the solution is rather smooth, it satisfies all the formal estimates we proved so far, with the same constants *independent of  $\delta$*  and for all  $T < T^\delta$ . In particular,  $\operatorname{div} v^\delta$  is in  $L_1(0, T; L_\infty)$  for all  $T < T^\delta$ , which implies that  $\rho^\delta$  is bounded from below and above, according to the following inequalities:

$$(3.7) \quad \delta \exp\left\{-\int_0^T \|\operatorname{div} v^\delta\|_\infty dt\right\} \leq \rho^\delta(t, x) \leq \rho_0^* \exp\left\{\int_0^T \|\operatorname{div} v^\delta\|_\infty dt\right\}.$$

*Step 3.* Our goal here is to prove that the solution  $(\rho^\delta, v^\delta)$  is actually global. To achieve it, we shall argue by contradiction, assuming that  $T^\delta$  is finite.

Now, the classical estimates for the continuity equations implies that for all  $T < T^\delta$  (dropping exponents  $\delta$  on  $(\rho^\delta, v^\delta)$ , for better readability):

$$(3.8) \quad \|\nabla \rho(T)\|_p \leq \|\nabla \rho_0^\delta\|_p + C \int_0^T (\|\nabla v\|_\infty \|\nabla \rho\|_p + \|\nabla \operatorname{div} v\|_p) dt.$$

Observe that the previous sections ensure that, uniformly with respect to  $T$  and  $\delta$ , we have  $\sqrt{\rho t} \dot{v} \in L_\infty(0, T^\delta; L_2)$  and  $\sqrt{t} \nabla \dot{v} \in L_2(0, T^\delta; L_2)$ . Combining with straightforward interpolation arguments and Hölder inequality, we deduce that

$$(3.9) \quad \rho \dot{v} \in L_a(0, T^\delta; L_p(\mathbb{T}^2)) \quad \text{for all } 2 < p < \infty \quad \text{and} \quad a < p' \quad \text{if } d = 2,$$

$$(3.10) \quad \rho \dot{v} \in L_a(0, T^\delta; L_p(\mathbb{T}^3)) \quad \text{for all } p \in ]2, 6[ \quad \text{and} \quad \frac{1}{a} = \frac{5}{4} - \frac{3}{2p} \quad \text{if } d = 3.$$

Remembering that  $\Delta v + \nu \nabla \operatorname{div} v - \nabla P = -\rho \dot{v}$ , we thus get

$$(3.11) \quad \Delta v + \nu \nabla \operatorname{div} v - \nabla P \in L_a(0, T; L_p),$$

whence, using  $L_p$  estimates for the Riesz operator and the fact that  $P = \rho^\gamma$  with  $\rho$  bounded, one may conclude that, uniformly with respect to  $\delta$ , we have for all  $t < T^\delta$ ,

$$(3.12) \quad \|\nabla^2 v(t)\|_p \leq C \|\nabla \rho(t)\|_p + h(t) \quad \text{with} \quad h \in L_a(0, T^\delta).$$

Hence we have for all  $T < T^\delta$ ,

$$(3.13) \quad \|\nabla \rho(T)\|_p \leq \left( \|\nabla \rho_0^\delta\|_p + \int_0^T h(t) dt \right) \exp\left\{ \int_0^T C(1 + \|\nabla v\|_\infty) dt \right\}.$$

<sup>2</sup>Recall that  $W_p^{1,2}(0, T_0 \times \mathbb{T}^d)$  designates the set of functions  $v : [0, T_0] \times \mathbb{T}^d \rightarrow \mathbb{R}^d$  such that  $v \in W_p^1(0, T_0; L_p(\mathbb{T}^d)) \cap L_p(0, T_0; W_p^2(\mathbb{T}^d))$ , and  $W_p^{2-2/p}(\mathbb{T}^d)$ , the corresponding trace space on  $t = 0$  (that may be identified to the Besov space  $B_{p,p}^{2-2/p}(\mathbb{T}^d)$ ).

In order to close the estimates, we have to bound  $\|\nabla v\|_\infty$ . Since  $p > d$  and  $\nabla v$  is bounded in  $L_1(0, T; BMO)$  independently of  $\delta$  (recall Corollary 2.1), one may start from the following well known logarithmic inequality:

$$\|\nabla v\|_\infty \leq C \|\nabla v\|_{BMO} \log \left( e + \frac{\|\nabla v\|_{W_p^1}}{\|\nabla v\|_{BMO}} \right),$$

which, in light of (3.12), implies that

$$\|\nabla v\|_\infty \leq C \|\nabla v\|_{BMO} \log \left( e + \frac{h + \|\nabla \rho\|_p}{\|\nabla v\|_{BMO}} \right).$$

Hence, plugging that inequality in (3.8), we discover that for all  $T < T^\delta$ ,

$$\|\nabla \rho(T)\|_p \leq \|\nabla \rho_0^\delta\|_p + \int_0^T h \, dt + C \int_0^T \left( 1 + \|\nabla v\|_{BMO} \log \left( e + \frac{h + \|\nabla \rho\|_p}{\|\nabla v\|_{BMO}} \right) \right) \|\nabla \rho\|_p \, dt.$$

Since

$$\|\nabla v\|_{BMO} \log \left( e + \frac{h}{\|\nabla v\|_{BMO}} \right) \leq C \max(h, \|\nabla v\|_{BMO})$$

and

$$\|\nabla v\|_{BMO} \log \left( e + \frac{\|\nabla \rho\|_p}{\|\nabla v\|_{BMO}} \right) \leq C(1 + \|\nabla v\|_{BMO}) \log(e + \|\nabla \rho\|_p),$$

we get

$$\|\nabla \rho(T)\|_p \leq \|\nabla \rho_0^\delta\|_p + \int_0^T h(1 + C\|\nabla \rho\|_p) \, dt + C \int_0^T (1 + \|\nabla v\|_{BMO}) \log(e + \|\nabla \rho\|_p) \|\nabla \rho\|_p \, dt.$$

From this and Osgood lemma, one can conclude (as  $T^\delta$  is finite) that  $\nabla \rho$  and  $\nabla u$  belong to  $L_\infty(0, T^\delta; L_p)$  and  $L_1(0, T^\delta; L_\infty)$ , respectively.

Putting together with (3.12), this leads to

$$(3.14) \quad \rho_t = -\operatorname{div}(v\rho) \in L^a(0, T^\delta; L_p).$$

Hence, by Sobolev embedding, one can conclude that there exists  $\alpha > 0$  such that

$$(3.15) \quad \rho \in C^\alpha([0, T^\delta] \times \mathbb{T}^d).$$

Now, one can go back to the momentum equation of (1.1), written in the form

$$(3.16) \quad \rho v_t - \mu \Delta v - \nu \nabla \operatorname{div} v = -\nabla P - \rho v \cdot \nabla v.$$

Thanks to (3.7) and (3.15), one may apply Theorem 2.2. of [6] and get

$$(3.17) \quad \|v\|_{W_p^{1,2}(0, T^\delta \times \mathbb{T}^d)} \leq C_\delta \left( \|\nabla P\|_{L_p(0, T^\delta \times \mathbb{T}^d)} + \|v \cdot \nabla v\|_{L_p(0, T^\delta \times \mathbb{T}^d)} \right).$$

For general  $p > 2$  if  $d = 2$ , or  $2 < p < 6$  if  $d = 3$ , we do not know how to prove directly that  $v \cdot \nabla v$  is in  $L_p(0, T^\delta \times \mathbb{T}^d)$ , and we shall need several steps.

More precisely, if  $d = 2$ , then one may use the fact that for all  $p < q < \infty$ ,

$$\|\nabla v\|_{p^*} \leq C \|\nabla^2 v\|_p^{1/2} \|v\|_q^{1/2} \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{q} \right)$$

which, combined with the fact that  $v \in L_\infty(0, T^\delta; H^1(\mathbb{T}^2))$  (from Proposition 2.1) and thus  $v \in L_\infty(0, T^\delta; L_r(\mathbb{T}^2))$  for all  $r < \infty$ , and (3.12) implies that  $v \cdot \nabla v \in L_{2a}(0, T^\delta; L_p(\mathbb{T}^2))$ . Since now, we know that the right-hand side of (3.16) belongs to  $L_{2a}(0, T^\delta; L_p(\mathbb{T}^2))$ , Theorem 2.2. of [6] implies that

$$\partial_t v \in L_{2a}(0, T^\delta; L_p(\mathbb{T}^2)) \quad \text{and} \quad \nabla^2 v \in L_{2a}(0, T^\delta; L_p(\mathbb{T}^2)).$$

Starting from that new information and arguing as above entails that the right-hand side of (3.16) belongs to  $L_{4a}(0, T^\delta; L_p(\mathbb{T}^2))$ , and so on. After a finite number of steps, we eventually achieve  $v \in W_p^{1,2}(0, T^\delta \times \mathbb{T}^2)$ .

For the 3D case we note that the information that  $v \in L_\infty(0, T^\delta; H^1(\mathbb{T}^2))$  implies that

$$(3.18) \quad v \in L_\infty(0, T; L_6(\mathbb{T}^3)).$$

Hence, to bound  $v \cdot \nabla v$  in  $L_p(\mathbb{T}^3)$ , we need to have  $\nabla v$  in  $L_k$  with  $k$  such that  $\frac{1}{6} + \frac{1}{k} = \frac{1}{p}$  (remember that  $2 < p < 6$  in the 3D case). By interpolation and the definition of  $a$  in (3.10), we have

$$(3.19) \quad \|\nabla v\|_k \leq C \|\nabla^2 v\|_p^{1-a/4} \|v\|_6^{a/4}.$$

Hence

$$\|\nabla v\|_k^{\frac{4a}{4-a}} \leq C \|\nabla^2 v\|_p^a \|v\|_6^{\frac{4a^2}{16-4a}},$$

and  $v \cdot \nabla v$  is thus in  $L^{4a/(4-a)}(0, T^\delta; L_p(\mathbb{T}^3))$  which, in view of Theorem 2.2. of [6] yields

$$\partial_t v \in L_{4a/(4-a)}(0, T^\delta; L_p(\mathbb{T}^3)) \quad \text{and} \quad \nabla^2 v \in L_{4a/(4-a)}(0, T^\delta; L_p(\mathbb{T}^3)).$$

Again, after a finite number of steps, we achieve

$$(3.20) \quad \|v\|_{W_p^{1,2}(0, T^\delta \times \mathbb{T}^d)} < \infty.$$

Now, thanks to the trace theorem and the estimates that we proved for  $\rho$ , one may conclude that, if  $T^\delta$  is finite, then

$$\sup_{T < T^\delta} (\|v(T)\|_{W_p^{2-2/p}} + \|\rho(T)\|_{W_p^1}) < \infty \quad \text{and} \quad \inf_{T < T^\delta} \rho(T) > 0.$$

Thanks to that information, one may solve System (1.1) supplemented with initial data  $(\rho(T), v(T))$  whenever  $T < T^\delta$ , and the existence time  $T_0$  provided by Theorem 3.1 is independent of  $T$ . In that way, taking  $T = T^\delta - T_0/2$ , we get a continuation of the solution beyond  $T^\delta$ , thus contradicting the definition of  $T^\delta$ .

Hence  $T^\delta = +\infty$ . In other words, the solution  $(\rho^\delta, v^\delta)$  is global and all the estimates of the previous sections are true on all interval  $[0, T]$ , and are uniform with respect to  $\delta$ .

*Step 4.* The previous step ensures uniform boundedness of  $(\rho^\delta, v^\delta)$  in the desired existence space. The last step is to prove the convergence of a subsequence. Since we have more regularity than in the classical weak solutions theory, one can pass to the limit by following the steps therein. However, this would give some restriction on  $\gamma$ , if  $P = \rho^\gamma$  (namely  $\gamma > d/2$ ). In our case, the higher regularity of the velocity will enable us to pass to the limit for any  $\gamma \geq 1$  (even for more general pressure laws) and by means of a much more elementary method.

To start with, let us observe that, up to extraction, we have

$$(3.21) \quad v^\delta \rightarrow v \quad \text{in} \quad L_2(0, T \times \mathbb{T}^d) \quad \text{for all} \quad T > 0.$$

Indeed, since  $(v^\delta)$  and  $(\sqrt{t} v_t^\delta)$  are bounded in  $L_2(0, T; L_2)$ , Lemma 3.2 of [9] implies that  $(v^\delta)$  is bounded in  $H^{\frac{1}{2}-\alpha}(0, T; L_2(\mathbb{T}^d))$  for all  $\alpha > 0$ , which, combined with the fact that  $(v^\delta)$  is also bounded in  $L_2(0, T; H^1(\mathbb{T}^d))$  implies that

$$(v^\delta) \quad \text{is bounded in} \quad H^{\frac{1}{4}}(0, T \times \mathbb{T}^d).$$

This entails (3.21) by standard compact Sobolev embedding.

However, this is not enough to pass to the limit in the pressure term of the momentum equation. To achieve it, we shall exhibit some strong convergence property for the effective viscous flux  $G^\delta$ .

From (3.11) and uniform estimates given by the previous sections, one knows that

$$(3.22) \quad (G^\delta) \text{ is bounded in } L_\infty(0, T; L_2) \cap L_2(0, T; H^1) \text{ for all finite } T > 0,$$

which already yields weak convergence.

To get strong convergence, we need to glean some compactness, and this will be achieved by looking at uniform estimates for  $(G_t^\delta)$ .

Now, from the previous step, Sobolev embeddings and the relation

$$P_t^\delta = -\operatorname{div}(P^\delta v^\delta) - h^\delta \operatorname{div} v^\delta,$$

we gather that  $(P_t^\delta)$  is bounded in  $L_\infty(0, T; W^{-1,p})$  for all finite  $T > 0$ . Furthermore, we also have the information that  $\sqrt{t} \operatorname{div} \dot{v}^\delta$  is bounded in  $L_2(0, T; L_2)$ . Since  $\operatorname{div}(v^\delta \cdot \nabla v^\delta)$  is bounded in  $L_2(0, T; W_p^{-1})$  (again, use the previous step), one may conclude that

$$\sqrt{t} G_t^\delta \text{ is bounded in } L_2(0, T; W_p^{-1}).$$

By suitable modification of Lemma 3.2 of [7], we deduce that

$$(G^\delta) \text{ is bounded in } H^{\frac{1}{2}-\alpha}(0, T; W_p^{-1}) \text{ for all } \alpha > 0,$$

and interpolating with (3.22) allows to get that  $(G^\delta)$  is bounded in  $H^\beta(0, T \times \mathbb{T}^d)$  for some small enough  $\beta > 0$ . So, finally, up to extraction, we have

$$(3.23) \quad G^\delta \rightarrow G \text{ in } L_2(0, T \times \mathbb{T}^d) \text{ for all } T > 0.$$

We are now in a good position to prove the strong convergence of the density. After suitable relabelling, the previous considerations ensure that there exists a sub-sequence  $(\rho^n, v^n)_{n \in \mathbb{N}}$  of  $(\rho^\delta, v^\delta)$  such that, for all  $T > 0$ ,

$$(3.24) \quad \rho^n \rightharpoonup^* \rho \text{ in } L_\infty(0, T \times \mathbb{T}^d) \text{ and } v^n \rightarrow v \text{ in } L_2(0, T \times \mathbb{T}^d).$$

Since for all  $n \in \mathbb{N}$ , we have

$$(3.25) \quad \rho_t^n + \operatorname{div}(\rho^n v^n) = 0,$$

the limit  $(\rho, v)$  fulfills

$$(3.26) \quad \rho_t + \operatorname{div}(\rho v) = 0.$$

At this point, let us emphasize that, since  $\operatorname{div} v \in L_1(0, T; L_\infty)$  (another consequence of the uniform estimates provided by the previous step) and  $\rho \in L_\infty(0, T \times \mathbb{T}^d)$ , one can assert that  $\rho$  is actually a *renormalized* solution of (3.26) (apply Theorem II.2 of [13]), and thus fulfills

$$(3.27) \quad (\rho \log \rho)_t + \operatorname{div}(\rho \log \rho v) + \rho \operatorname{div} v = 0.$$

Of course, since  $(\rho^n, v^n)$  is smooth, we also have

$$(3.28) \quad (\rho^n \log \rho^n)_t + \operatorname{div}(\rho^n \log \rho^n v^n) + \rho^n \operatorname{div} v^n = 0.$$

Then, remembering the definition of  $G^n$ , we get

$$(3.29) \quad (\rho^n \log \rho^n)_t + \operatorname{div}(\rho^n \log \rho^n v^n) + \nu^{-1} \rho^n P(\rho^n) + \nu^{-1} \rho^n G^n = 0$$

and the limit version

$$(3.30) \quad (\rho \log \rho)_t + \operatorname{div}(\rho \log \rho v) + \nu^{-1} \rho P(\rho) + \nu^{-1} \rho G = 0.$$

Denote by  $\overline{\rho \log \rho}$  and  $\overline{\rho P(\rho)}$  the weak limits of  $\rho^n \log \rho^n$  and  $\rho^n P(\rho^n)$ , respectively. Since functions  $z \mapsto z \log z$  and  $z \mapsto zP(z)$  are convex, we know that

$$(3.31) \quad \overline{\rho \log \rho} \geq \rho \log \rho \quad \text{and} \quad \overline{\rho P(\rho)} \geq \rho P(\rho).$$

Furthermore, integrating (3.29) and (3.30) on  $[0, T] \times \mathbb{T}^d$ , we find that

$$\begin{aligned} \nu \left( \int_{\mathbb{T}^d} (\rho^n \log \rho^n - \rho \log \rho)(T) dx - \int_{\mathbb{T}^d} (\rho_0^n \log \rho_0^n - \rho_0 \log \rho_0) dx \right) \\ + \int_0^T \int_{\mathbb{T}^d} (\rho^n P^n - \rho P) dx dt + \int_0^T \int_{\mathbb{T}^d} (\rho^n G^n - \rho G) dx dt = 0. \end{aligned}$$

By construction, the term pertaining to the initial data tends to zero. Furthermore, since  $(G^n)$  converges strongly to  $G$ , the last term also tends to zero. This leads us to

$$\int_{\mathbb{T}^d} (\overline{\rho \log \rho} - \rho \log \rho)(T) dx + \int_0^T \int_{\mathbb{T}^d} (\overline{\rho P} - \rho P) dx dt = 0.$$

Combining with (3.31), one may now conclude that

$$(3.32) \quad \overline{\rho \log \rho} = \rho \log \rho.$$

Since the function  $z \mapsto z \log z$  is strictly convex we find by standard arguments that  $(\rho^n)$  converges strongly and pointwise to  $\rho$ . Hence one can pass to the limit in all the nonlinear terms (in particular in the pressure one) of the momentum equation, and to conclude that  $(\rho, v)$  is indeed a solution to (1.1).

Besides, classical arguments that may be found in [13] ensure that  $\rho \in \mathcal{C}(\mathbb{R}_+; L_p)$  for all  $p < \infty$ , and that strong convergence holds true in the corresponding space. Thanks to that information, since (1.5) is fulfilled with data  $(\rho_0^n, v_0^n)$  by the sequence  $(\rho^n, v^n)_{n \in \mathbb{N}}$ , one may pass to the limit and see that  $(\rho, v)$  satisfies (1.5) as well. Finally, since the internal energy  $e$  is continuous with respect to time (a consequence of the strong convergence of  $\rho$ ), one may reproduce the argument that has been used in [9] so as to prove that  $\sqrt{\rho} v \in \mathcal{C}(\mathbb{R}_+; L_2)$ . This completes the proof of our existence theorems in dimensions 2 and 3.  $\square$

We end this section with a fast justification of the convergence of solutions to (1.1) to those of (1.7) when  $\nu$  goes to  $\infty$ , leading to Theorem 1.3. As the proof goes along the lines of that of Theorem 1.1, we just indicate the main steps. The starting point is the estimate provided by Proposition 2.1 which ensures in particular (1.10), that  $(\nabla G^\nu)$  is bounded in  $L_2(\mathbb{R}_+ \times \mathbb{T}^2)$  and that  $(v^\nu)$  is bounded in  $L_\infty(\mathbb{R}_+; H^1)$ , while Proposition 2.2 guarantees that  $(\rho^\nu)$  is bounded in  $L_\infty(\mathbb{R}_+ \times \mathbb{T}^2)$ . Hence, there exists  $(\rho, v) \in L_\infty(\mathbb{R}_+ \times \mathbb{T}^2) \times L_\infty(\mathbb{R}_+; H^1)$  and a subsequence  $(\rho^n, v^n)$  of  $(\rho^\nu, v^\nu)$  such that

$$\rho^n \rightharpoonup^* \rho \quad \text{in} \quad L_\infty(\mathbb{R}_+ \times \mathbb{T}^2) \quad \text{and} \quad v^n \rightharpoonup v \quad \text{in} \quad L_\infty(\mathbb{R}_+; H^1).$$

As in the proof of existence, in order to get some compactness, one may look at time weighted estimates. More specifically, we know from Proposition 2.3 that if  $\nu \geq \nu_0$  then

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^2} \rho |\dot{v}^\nu|^2 t dx + \int_0^T \int_{\mathbb{T}^2} (\mu |\nabla \mathcal{P} \dot{v}^\nu|^2 + \nu |\operatorname{div} \dot{v}^\nu|^2) t dx dt \leq C_0 T e^{\frac{C_0 T}{\nu}},$$

and this ensures that  $(v^\nu)$  is bounded in, say,  $H^{1/4}(0, T \times \mathbb{T}^2)$  for all  $T > 0$ . Hence, we actually have (extracting one more subsequence as the case may be),

$$v^n \rightarrow v \quad \text{in} \quad L_{loc}^2(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$



Next, arguing exactly as in the proof of existence, we get that, for all finite  $T > 0$ ,

$(G^\nu)$  is bounded in  $L_\infty(0, T; L_2) \cap L_2(0, T; H^1)$  and  $(\sqrt{t}G_t^\nu)$  is bounded in  $L_2(0, T; W_p^{-1})$

from which we deduce that  $(G^\nu)$  is bounded in  $H^{\frac{1}{2}-\alpha}(0, T; W_p^{-1})$  for all  $\alpha > 0$  and, eventually

$$G^n \rightarrow G \quad \text{in} \quad L_{loc}^2(\mathbb{R}_+; L_2(\mathbb{T}^2)).$$

Putting together all those results of convergence, one gets

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \quad \text{and} \quad \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \mu \Delta \mathcal{P}v + \nabla G = 0.$$

Since we know in addition (from (1.10)) that  $\operatorname{div} v = 0$ , one can conclude that  $(\rho, v, \nabla G)$  satisfies (1.7). Finally, from the uniform bounds that are available for  $(\rho^\nu, v^\nu)$ , one may check that  $(\rho, v, \nabla G)$  has the regularity of the solution constructed in Theorem 2.1 of [9], which is unique. Hence the whole family  $(\rho^\nu, v^\nu)$  converges to  $(\rho, v)$ .  $\square$

#### 4. THE PROOF OF UNIQUENESS

Here we show the uniqueness of the solutions we constructed in the paper, both in dimensions 2 and 3. The main difficulty we have to face is that having  $\operatorname{div} v$  and  $\nabla \mathcal{P}v$  in  $L^1(0, T; L_\infty)$  (see Corollary 2.1) *does not* ensure that  $\nabla v$  is in  $L^1(0, T; L_\infty)$  so that, in contrast with our recent work [7], it is not clear that one can reformulate System (1.1) in Lagrangian coordinates so as to prove uniqueness. However, we do have  $\nabla v$  is in  $L_r(0, T; BMO)$  for some  $r > 1$ , which will turn out to be enough to prove uniqueness *provided that the pressure law is linear*. Actually, we encounter the same difficulty as in D. Hoff's paper [16]: since we have to estimate the difference of the densities in  $L_\infty(0, T; H^{-1})$ , we need, at some point, to bound the difference of the pressures in  $H^{-1}$  from the norm of the difference of the densities in  $H^{-1}$ .

Here is the main statement of this section.

**Proposition 4.1.** *Assume that  $P(\rho) = a\rho$  for some  $a > 0$ , and consider two finite energy solutions  $(\rho, v)$  and  $(\bar{\rho}, \bar{v})$  of (1.1) on  $[0, T_0] \times \mathbb{T}^d$  ( $d = 2, 3$ ) with bounded density and emanating from the same initial data. If, in addition,  $v$  and  $\bar{v}$  are in  $L_\infty(0, T_0; H^1)$ ,  $\sqrt{t} \nabla \dot{v}$  and  $\sqrt{t} \nabla \dot{\bar{v}}$  are in  $L_2(0, T_0 \times \mathbb{T}^d)$ ,  $\sqrt{\rho t} \dot{v}$ ,  $\sqrt{\bar{\rho} t} \dot{\bar{v}}$  belong to  $L_\infty(0, T_0; L_2)$ ,*

$$(4.1) \quad \nabla \bar{v} \in L_2(0, T_0; L_3) \quad \text{and} \quad \int_0^{T_0} (1 + |\log t|) \|\nabla \bar{v}(t)\|_{BMO} dt < \infty,$$

then  $(\bar{\rho}, \bar{v}) \equiv (\rho, v)$  on  $[0, T_0] \times \mathbb{T}^d$ .

*Proof.* The general scheme of the proof is the same in dimensions 2 or 3. Assume that  $a = 1$  for notational simplicity and consider two solutions  $(\rho, v)$  and  $(\bar{\rho}, \bar{v})$  to (1.1) corresponding to the same initial data  $(\rho_0, v_0)$ . The system for the difference

$$\delta\rho := \rho - \bar{\rho} \quad \text{and} \quad \delta v := v - \bar{v}$$

reads

$$(4.2) \quad \begin{cases} \delta\rho_t + \operatorname{div}(\delta\rho \bar{v} + \rho \delta v) = 0, \\ \rho \delta v_t + \rho v \cdot \nabla \delta v - \mu \Delta \delta v - (\lambda + \mu) \nabla \operatorname{div} \delta v + \nabla \delta\rho = \delta\rho \dot{\bar{v}} + \rho \delta v \cdot \nabla \bar{v}. \end{cases}$$

In order to show that  $\delta\rho \equiv 0$  and  $\delta v \equiv 0$ , we shall perform suitable estimates in  $\dot{H}^{-1}$  for  $\delta\rho(t)$ , and for  $\sqrt{\bar{\rho}} \delta v$  in  $L_2(0, T \times \mathbb{T}^d)$ . To this end, we set  $\phi := -(-\Delta)^{-1} \delta\rho$  (which makes sense, since  $\int_{\mathbb{T}^d} \delta\rho dx = 0$ ) so that

$$(4.3) \quad \|\nabla \phi\|_2 = \|\delta\rho\|_{\dot{H}^{-1}} = \|\delta\rho\|_{H^{-1}}.$$

Now, testing the first equation of (4.2) by  $\phi$  yields

$$\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_2^2 \leq \left| \int_{\mathbb{T}^d} (\bar{v} \cdot \nabla \phi \delta \rho + \rho \delta v \cdot \nabla \phi) dx \right|.$$

The last term is bounded as follows:

$$\left| \int_{\mathbb{T}^d} \rho \delta v \cdot \nabla \phi dx \right| \leq \sqrt{\rho^*} \|\sqrt{\rho} \delta v\|_2 \|\nabla \phi\|_2.$$

Regarding the first one, observe that (with the usual summation convention)

$$\int_{\mathbb{T}^d} \bar{v} \cdot \nabla \phi \delta \rho dx = \int_{\mathbb{T}^d} \bar{v}^j \partial_j \phi \Delta \phi dx = - \int_{\mathbb{T}^d} \partial_k \bar{v}^j \partial_j \phi \partial_k \phi dx + \frac{1}{2} \int_{\mathbb{T}^d} \operatorname{div} \bar{v} |\nabla \phi|^2 dx.$$

Hence, we have

$$\left| \int_{\mathbb{T}^d} \bar{v} \cdot \nabla \phi \delta \rho dx \right| \leq C \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi \otimes \nabla \phi\|_{\mathcal{H}^1}.$$

Now, in light of the following inequality (see e.g. [22, Thm. D]),

$$(4.4) \quad \|f\|_{\mathcal{H}^1} \leq C \|f\|_1 (|\log \|f\|_1| + \log(e + \|f\|_\infty)),$$

we discover that

$$(4.5) \quad \left| \int_{\mathbb{T}^d} \bar{v} \cdot \nabla \phi \delta \rho dx \right| \leq C \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi\|_2^2 (|\log \|\nabla \phi\|_2^2| + \log(e + \|\nabla \phi\|_\infty^2)).$$

Since the densities are bounded by  $\rho^*$ , we have

$$\|\nabla \phi(t)\|_\infty \leq C \rho^* \quad \text{for all } t \in [0, T_0].$$

Hence Inequality (4.5) implies that for some constant  $C$  depending only on  $\rho^*$ ,

$$(4.6) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_2^2 \leq C \left( \|\sqrt{\rho} \delta v\|_2 + \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi\|_2 (1 + |\log \|\nabla \phi\|_2|) \right) \|\nabla \phi\|_2.$$

Since the initial quantity is zero, after integration, this gives for all  $t \in [0, T_0]$ ,

$$(4.7) \quad \|\nabla \phi(t)\|_2 \leq C \left( \int_0^t \|\sqrt{\rho} \delta v\|_2 d\tau + \int_0^t \|\nabla \bar{v}\|_{\text{BMO}} \|\nabla \phi\|_2 (1 + |\log \|\nabla \phi\|_2|) d\tau \right).$$

Hence, using (4.3) and denoting  $Z(t) := \sup_{\tau \leq t} \tau^{-1/2} \|\delta \rho(\tau)\|_{\dot{H}^{-1}}$ , we get after using Cauchy-Schwarz inequality, for all  $T \in [0, T_0]$ ,

$$(4.8) \quad Z(T) \leq C \left( \sup_{t \in [0, T]} \int_0^t \|\nabla \bar{v}\|_{\text{BMO}} Z (1 + |\log \tau| + |\log Z|) d\tau + \|\sqrt{\rho} \delta v\|_{L_2(0, T \times \mathbb{T}^d)} \right).$$

In order to control the difference of the velocities, we introduce the solution  $w$  to the following backward parabolic system:

$$(4.9) \quad \begin{cases} \rho w_t + \rho v \cdot \nabla w + \mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = -\rho \delta v, \\ w|_{t=T} = 0. \end{cases}$$

Solving the above system is not part of the classical theory for linear parabolic systems, as the coefficients are rough and may vanish. However, if  $\rho$  and  $v$  are regular with  $\rho$  bounded away from zero, this is well known, and the case we are interested may be achieved by a regularizing process of  $\rho$  and  $v$ , after using Inequality (4.12) below for the corresponding regular solutions.

Now, testing the equation by  $w$ , we find that

$$(4.10) \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^d} \rho |w|^2 dx + \int_0^T \int_{\mathbb{T}^d} (\mu |\nabla \mathcal{P}w|^2 + \nu (\operatorname{div} w)^2) dx dt \leq \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 dx dt.$$

Next, we test (4.9) by  $w_t$  and take advantage of the usual elliptic estimates given by

$$\mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = -\rho \dot{w} - \rho \delta v$$

that ensure that

$$\mu^2 \|\nabla^2 \mathcal{P}w\|_2^2 + \nu^2 \|\nabla \operatorname{div} w\|_2^2 = \|\rho \dot{w} + \rho \delta v\|_2^2 \leq \rho^* \|\sqrt{\rho} (\dot{w} + \delta v)\|_2^2$$

in order to get

$$(4.11) \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^d} (\mu |\nabla \mathcal{P}w(t)|^2 + \nu (\operatorname{div} w(t))^2) dx \\ + \int_0^T \int_{\mathbb{T}^d} \left( \rho |w_t|^2 + \frac{\mu^2}{6\rho^*} |\nabla^2 \mathcal{P}w|^2 + \frac{\nu^2}{6\rho^*} |\nabla \operatorname{div} w|^2 \right) dx dt \leq \frac{3}{2} \int_0^T \int_{\mathbb{T}^d} (\rho |\delta v|^2 + \rho |v \cdot \nabla w|^2) dx dt.$$

If  $d = 2$  then we bound the last term as follows:

$$\int_{\mathbb{T}^2} \rho |v \cdot \nabla w|^2 dx \leq \sqrt{\rho^*} \|\rho^{1/4} v\|_4^2 \|\nabla w\|_4^2 \\ \leq C \sqrt{\rho^*} \|\rho^{1/4} v\|_4^2 \|\nabla w\|_2 \|\nabla^2 w\|_2 \\ \leq C \frac{(\rho^*)^2}{\mu^2} \|\rho^{1/4} v\|_4^4 \|\nabla w\|_2^2 + \frac{\mu^2}{12\rho^*} \|\nabla^2 w\|_2^2.$$

If  $d = 3$ , then we rather write that

$$\int_{\mathbb{T}^3} \rho |v \cdot \nabla w|^2 dx \leq (\rho^*)^{3/4} \|\sqrt{\rho} v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\nabla w\|_2^{1/2} \|\nabla^2 w\|_2^{3/2} \\ \leq C (\rho^*)^3 \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^6 \|\nabla w\|_2^2 + \frac{\mu^2}{12\rho^*} \|\nabla^2 w\|_2^2.$$

Hence, using the properties of regularity of  $v$ , plugging the above inequality in (4.11), then using Gronwall inequality, we get

$$(4.12) \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^d} (\rho |w|^2 + \mu |\nabla \mathcal{P}w|^2 + \nu (\operatorname{div} w)^2) dx \\ + \int_0^T \int_{\mathbb{T}^d} (\mu |\nabla \mathcal{P}w|^2 + \nu (\operatorname{div} w)^2 + \mu^2 |\nabla^2 \mathcal{P}w|^2 + \nu^2 |\nabla \operatorname{div} w|^2 + \rho |w_t|^2) dx dt \\ \leq C_T \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 dx dt,$$

with  $C_T$  depending only on the norms of the two solutions on  $[0, T]$ .

Let us next test (4.2) by  $w$ . We get

$$(4.13) \quad \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} \delta \rho \operatorname{div} w dx dt \leq \int_0^T \int_{\mathbb{T}^d} (\delta \rho \dot{v} \cdot w + \rho (\delta v \cdot \nabla \bar{v}) \cdot w) dx dt.$$

One can bound the first term of the right-hand side as follows:

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}^d} \delta\rho \dot{v} \cdot w \, dx \, dt \right| &= \left| \int_0^T \int_{\mathbb{T}^d} t^{-1/2} \delta\rho t^{1/2} \dot{v} \cdot w \, dx \, dt \right| \\ &\leq \|t^{-1/2} \delta\rho\|_{L_\infty(0,T;\dot{H}^{-1})} \|t^{1/2} \nabla(\dot{v} \cdot w)\|_{L_1(0,T;L_2)} \\ &\leq \|t^{-1/2} \delta\rho\|_{L_\infty(0,T;\dot{H}^{-1})} \left( \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \|w\|_{L_2(0,T;L_\infty)} \right. \\ &\quad \left. + \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \|\nabla w\|_{L_2(0,T;L_3)} \right). \end{aligned}$$

For bounding the last term of (4.13), one can just use the fact that

$$\int_0^T \int_{\mathbb{T}^d} \rho(\delta v \cdot \nabla \bar{v}) \cdot w \, dx \, dt \leq \sqrt{\rho^*} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \|\nabla \bar{v}\|_{L_2(0,T;L_3)} \|w\|_{L_\infty(0,T;L_6)}.$$

Finally, we note that

$$\int_0^T \int_{\mathbb{T}^d} \delta\rho \operatorname{div} w \, dx \, dt \leq T \|t^{-1/2} \delta\rho\|_{L_\infty(0,T;\dot{H}^{-1})} \|\nabla \operatorname{div} w\|_{L_2(0,T;L_2)}.$$

Plugging the above three inequalities in (4.13), we get

$$(4.14) \quad \int_0^T \int_{\mathbb{T}^d} \rho |\delta v|^2 \, dx \, dt \leq \|t^{-1/2} \delta\rho\|_{L_\infty(0,T;\dot{H}^{-1})} \left( T \|\nabla \operatorname{div} w\|_{L_2(0,T;L_2)} \right. \\ \left. + C \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \|w\|_{L_2(0,T;L_\infty)} + C \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \|\nabla w\|_{L_2(0,T;L_3)} \right) \\ + C \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \|\nabla \bar{v}\|_{L_2(0,T;L_3)} \|w\|_{L_\infty(0,T;L_6)}.$$

Observe that our assumptions on  $\bar{v}$  guarantee that we have

$$(4.15) \quad \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} + \|\nabla \bar{v}\|_{L_2(0,T;L_3)} \leq C_T.$$

Next, we have to bound the terms containing  $w$  in (4.14) by means of the data. Since  $\int_{\mathbb{T}^d} \rho w \, dx$  need not be zero, Poincaré inequality (A.2) becomes

$$\|w\|_2 \leq \left| \int_{\mathbb{T}^d} \rho w \, dx \right| + \rho^* \|\nabla w\|_2.$$

To bound the mean value of  $\rho w$ , we note that integrating (4.9) on  $[t, T] \times \mathbb{T}^d$  readily gives

$$\int_{\mathbb{T}^d} (\rho w)(t, x) \, dx = \int_t^T \int_{\mathbb{T}^d} (\rho \delta v)(\tau, x) \, dx \, d\tau.$$

Therefore we have

$$\left| \int_{\mathbb{T}^d} (\rho w)(t) \, dx \right| \leq \sqrt{\rho^*} T^{1/2} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \quad \text{for all } t \in [0, T],$$

whence

$$(4.16) \quad \|w(t)\|_2 \leq C_{\rho^*} (\|\nabla w(t)\|_2 + T^{1/2} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}) \quad \text{for all } t \in [0, T].$$

Then, combining with (4.12), we end up with

$$\|w\|_{L_2(0,T;H^1)} \leq C_{0,T} T^{1/2} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \quad \text{and} \quad \|\nabla w\|_{L_2(0,T;\dot{H}^1)} \leq C_{0,T} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}.$$

By interpolation and Sobolev embedding, it follows that for small enough  $\varepsilon$  if  $d = 2$  (and  $\varepsilon = 1/4$  if  $d = 3$ ), we have

$$(4.17) \quad \|w\|_{L_2(0,T;L_\infty)} \leq C_\varepsilon T^{1/2-\varepsilon} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}.$$

Likewise, we have

$$\begin{aligned} \|\nabla w\|_{L_2(0,T;L_3(\mathbb{T}^2))} &\lesssim \|\nabla w\|_{L_2(0,T;L_2(\mathbb{T}^2))}^{2/3} \|\nabla^2 w\|_{L_2(0,T;L_2(\mathbb{T}^2))}^{1/3} \\ \text{and } \|\nabla w\|_{L_2(0,T;L_3(\mathbb{T}^3))} &\lesssim \|\nabla w\|_{L_2(0,T;L_2(\mathbb{T}^3))}^{1/2} \|\nabla^2 w\|_{L_2(0,T;L_2(\mathbb{T}^3))}^{1/2}, \end{aligned}$$

whence

$$(4.18) \quad \|\nabla w\|_{L_2(0,T;L_3)} \leq CT^\alpha \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \quad \text{with } \alpha = 1/3 \text{ if } d = 2, \quad \alpha = 1/4 \text{ if } d = 3.$$

Finally, using once more that  $H^1(\mathbb{T}^d) \hookrightarrow L_6(\mathbb{T}^d)$  for  $d = 2, 3$ , we get after plugging all the above inequalities in (4.14), for all  $T \in [0, T_0]$ ,

$$\|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}^2 \leq CT^{1/3} \left( \|t^{-1/2} \delta \rho\|_{L^\infty(0,T;\dot{H}^{-1})} \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} + \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)}^2 \right).$$

Clearly, the above inequality implies that, if  $T$  is small enough then

$$(4.19) \quad \|\sqrt{\rho} \delta v\|_{L_2(0,T;L_2)} \leq CT^{1/3} Z(T).$$

Plugging that inequality in (4.8) and assuming that  $T$  is small enough, we obtain

$$Z(t) \leq C_T \int_0^t (1 + |\log \tau|) \|\nabla \bar{v}(\tau)\|_{BMO} Z(\tau) (1 + |\log Z(\tau)|) d\tau \quad \text{for all } t \in [0, T].$$

Then, Osgood lemma (see e.g. [1, Lem. 3.4]) implies that  $Z \equiv 0$  on  $[0, T]$ , and thus, owing to (4.19), that  $\sqrt{\rho} \delta v \equiv 0$  on  $[0, T]$ .

Now, since  $\sqrt{\rho} \delta v$  and  $\delta \rho$  are zero, the second equation of (4.2) becomes

$$\rho \delta v_t + \rho v \cdot \nabla \delta v - \mu \Delta \delta v - (\lambda + \mu) \nabla \operatorname{div} \delta v = 0,$$

which implies that

$$\frac{1}{2} \|\sqrt{\rho} \delta v\|_{L^\infty(0,T;L_2)}^2 + \int_0^T (\mu \|\nabla \mathcal{P} \delta v\|_2^2 + \nu \|\operatorname{div} \delta v\|_2^2) dx = 0.$$

Since  $\int_{\mathbb{T}^d} \rho \delta v dx = 0$ , this implies (in light of Inequality (A.2)) that  $\delta v = 0$  on  $[0, T]$ , which completes the proof of uniqueness.

To complete the proof of Theorem 1.2, it suffices to observe that Condition (2.67) implies Assumption (4.1) in Proposition 4.1.  $\square$

## APPENDIX A. SOME INEQUALITIES

The following Osgood type lemma has been used a number of times.

**Lemma A.1.** *Let  $f$  and  $g$  be two locally integrable nonnegative functions on  $\mathbb{R}_+$ , and assume that the a.e. differentiable function  $X : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfies*

$$X' \leq f X \log(A + BX) + g X \quad \text{for some } A \geq 1 \text{ and } B \geq 0.$$

Then we have for all  $t \geq 0$ ,

$$A + BX(t) \leq \left( A + B e^{\int_0^t g d\tau} X(0) \right)^{\exp \int_0^t f d\tau}.$$

*Proof.* It suffices to prove the inequality on  $[0, T]$  for all  $T \geq 0$ . Setting  $Y(t) := e^{-\int_0^t g d\tau} X(t)$ , then  $Z(t) := C_T Y(t)$  with  $C_T := \exp \int_0^T g d\tau$ , we have for all  $t \in [0, T]$ ,

$$BZ' \leq BZ \log(A + BZ) f \leq (A + BZ) \log(A + BZ) f.$$

Therefore, integrating once,

$$\log(\log(A + BZ(t))) \leq \log(\log(A + BZ(0))) + \int_0^t f \, d\tau \quad \text{for all } t \in [0, T].$$

Then considering  $t = T$  and taking exp twice gives

$$A + BZ(T) \leq (A + BZ(0))^{\exp \int_0^T f \, d\tau}.$$

Reverting to the original function  $X$  gives exactly what we want at  $t = T$ .  $\square$

We also used the following Poincaré inequality.

**Lemma A.2.** *Let  $\rho$  be in  $L_{p'}(\mathbb{T}^d)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $2 \leq p \leq \frac{2d}{d-2}$  if  $d \geq 3$  ( $2 \leq p < +\infty$  if  $d = 2$ ). Assume that*

$$(A.1) \quad \int_{\mathbb{T}^d} \rho b \, dx = 0 \quad \text{and} \quad M := \int_{\mathbb{T}^d} \rho \, dx > 0.$$

There exists a constant  $C_p$  depending on  $p$  and on  $d$  (and with  $C_2 = 1$ ), such that

$$(A.2) \quad \|b\|_2 \leq \left(1 + \frac{C_p}{M} \|\rho - c\|_{p'}\right) \|\nabla b\|_2 \quad \text{for any real number } c.$$

Furthermore, in dimension  $d = 2$ , we have

$$(A.3) \quad \|b\|_2 \leq C \log^{\frac{1}{2}} \left( e + \frac{\|\rho - c\|_2}{M} \right) \|\nabla b\|_2.$$

*Proof.* Let  $\bar{b}$  be the average of  $b$  and  $\tilde{b} := b - \bar{b}$ . Then we have by Poincaré inequality,

$$(A.4) \quad \|b\|_2 \leq |\bar{b}| + \|\tilde{b}\|_2 \leq |\bar{b}| + \|\nabla b\|_2.$$

Now, hypothesis (A.1) implies that for all real number  $c$ , we have

$$(A.5) \quad -M\bar{b} = \int_{\mathbb{T}^d} (\rho - c)\tilde{b} \, dx.$$

Therefore, by Sobolev embedding,

$$(A.6) \quad M|\bar{b}| \leq \|\rho - c\|_{p'} \|\tilde{b}\|_p \leq C_p \|\rho - c\|_{p'} \|\nabla b\|_2$$

and, clearly,  $C_2 = 1$ . This gives (A.2).

To handle the endpoint case  $d = 2$  and  $p = +\infty$ , decompose  $\tilde{b}$  into Fourier series:

$$\tilde{b}(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \hat{b}_k e^{2i\pi k \cdot x},$$

and set for any integer  $n$ ,

$$\tilde{b}_n(x) := \sum_{1 \leq |k| \leq n} \hat{b}_k e^{2i\pi k \cdot x}.$$

By Cauchy-Schwarz inequality, it is easy to prove that

$$(A.7) \quad \|\tilde{b}_n\|_\infty \leq C \sqrt{\log n} \|\nabla b\|_2.$$

Because the average of  $\tilde{b}_n$  is 0, one may write, thanks to (A.5) that for all  $c \in \mathbb{R}$ ,

$$-M\bar{b} = \int_{\mathbb{T}^2} (\rho - c)\tilde{b} \, dx = \int_{\mathbb{T}^2} \rho \tilde{b}_n \, dx + \int_{\mathbb{T}^2} (\rho - c)(\tilde{b} - \tilde{b}_n) \, dx.$$

Therefore, using Hölder and Poincaré inequality, and also (A.7),

$$\begin{aligned} M|\bar{b}| &\leq \|\rho\|_1 \|\tilde{b}_n\|_\infty + \|\rho - c\|_2 \|\tilde{b} - \tilde{b}_n\|_2 \\ &\leq C \left( \sqrt{\log n} M + n^{-1} \|\rho - c\|_2 \right) \|\nabla b\|_2. \end{aligned}$$

Then taking  $n \approx \|\rho - c\|_2/M$  gives

$$(A.8) \quad |\bar{b}| \leq C \log^{\frac{1}{2}} \left( e + \frac{\|\rho - c\|_2}{M} \right) \|\nabla b\|_2,$$

which, combined with (A.4) yields (A.3).  $\square$

We used the following version of Desjardins' estimate in [11].

**Lemma A.3.** *Let  $\rho \in L_\infty(\mathbb{T}^2)$  with  $\rho \geq 0$ , and  $u \in H^1(\mathbb{T}^2)$ . Then, we have for some universal constant  $C$ ,*

$$(A.9) \quad \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} \leq C \|\sqrt{\rho}u\|_2 \|\nabla u\|_2 \log^{\frac{1}{2}} \left( e + \frac{\|\rho - c\|_2}{M} + \frac{\|\rho\|_2 \|\nabla u\|_2^2}{\|\sqrt{\rho}u\|_2^2} \right).$$

*Proof.* Let  $\tilde{u} := u - \bar{u}$  and fix some  $n \in \mathbb{N}$ . Then, keeping the same notation as in the above lemma and using Hölder inequality,

$$\begin{aligned} \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} &= \left( \int_{\mathbb{T}^2} (\bar{u} + \tilde{u}_n + (\tilde{u} - \tilde{u}_n))^2 \rho u^2 dx \right)^{\frac{1}{2}} \\ &\leq |\bar{u}| \|\sqrt{\rho}u\|_2 + \|\sqrt{\rho}u\|_2 \|\tilde{u}_n\|_\infty + \|\rho\|_2^{\frac{1}{2}} \|\tilde{u} - \tilde{u}_n\|_8 \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{4}}. \end{aligned}$$

We thus have, using Young inequality and embedding  $\dot{H}^{\frac{3}{4}}(\mathbb{T}^2) \hookrightarrow L_8(\mathbb{T}^2)$ ,

$$(A.10) \quad \left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} \leq 2 \|\sqrt{\rho}u\|_2 (|\bar{u}| + \|\tilde{u}_n\|_\infty) + C \|\rho\|_2^{\frac{1}{2}} \|\tilde{u} - \tilde{u}_n\|_{\dot{H}^{\frac{3}{4}}}^2.$$

Hence, taking advantage of (A.7) and of

$$(A.11) \quad \|\tilde{u} - \tilde{u}_n\|_{\dot{H}^{\frac{3}{4}}} \leq n^{-1/4} \|\nabla u\|_2.$$

Plugging (A.11) in (A.10), we get

$$\left( \int_{\mathbb{T}^2} \rho u^4 dx \right)^{\frac{1}{2}} \lesssim \|\sqrt{\rho}u\|_2 |\bar{u}| + (\sqrt{\log n} \|\sqrt{\rho}u\|_2 + n^{-\frac{1}{2}} \|\rho\|_2^{\frac{1}{2}} \|\nabla u\|_2) \|\nabla u\|_2.$$

Then, taking  $n \approx \frac{\|\rho\|_2 \|\nabla u\|_2^2}{\|\sqrt{\rho}u\|_2^2}$  and using (A.8) to bound  $|\bar{u}|$  yields the desired inequality.  $\square$

## APPENDIX B. END OF THE PROOF OF TIME WEIGHTED ESTIMATES IN THE 2D CASE

We here provide the reader with the proofs of Inequalities (2.57) and (2.62).

**Proof of (2.57).** We use (2.45) to bound  $I_2$  as follows:

$$I_2 = \int_{\mathbb{T}^2} \nabla v \cdot \nabla^2 \left( \mathcal{P}v - \frac{1}{\nu} \nabla(-\Delta)^{-1}(\tilde{G} + \tilde{P}) \right) \cdot \dot{v} t \, dx =: I_{21} + I_{22} + I_{23}.$$

From (2.68), we know that

$$(B.1) \quad \mu^2 \|\sqrt{t} \nabla^2 \mathcal{P}v(t)\|_2^2 + \|\sqrt{t} \nabla G(t)\|_2^2 \leq \rho^* \|\sqrt{\rho t} \dot{v}\|_2^2.$$

Since we have

$$\frac{\mu}{(\rho^*)^{1/4}} \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_4(0,T;L_2)} \leq \left( \frac{\mu T^{1/2}}{\sqrt{\rho^*}} \|\nabla^2 \mathcal{P}v\|_{L_2(0,T;L_2)} \right)^{1/2} \left( \mu \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_\infty(0,T;L_2)} \right)^{1/2}$$

and a similar inequality for  $\nabla G$ , combining (B.1) and Proposition 2.1 yields

$$(B.2) \quad \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_4(0,T;L_2)} + \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \leq C_0 T^{1/4} \|\sqrt{t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}.$$

Therefore, putting together with (2.44), we gather that

$$(B.3) \quad \left| \int_0^T I_{21} \, dt \right| \leq \|\nabla v\|_{L_4(0,T \times \mathbb{T}^2)} \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ \leq C_0 T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}.$$

Term  $I_{22}$  is almost the same: taking into account (B.2), we obtain

$$(B.4) \quad \left| \int_0^T I_{22} \, dt \right| \leq \nu^{-1} \|\nabla v\|_{L_4(0,T \times \mathbb{T}^2)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ \leq C_0 \nu^{-1} T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}.$$

To handle  $I_{23}$ , we integrate by parts several times and get (with the summation convention for repeated indices and the notation  $\psi := (-\Delta)^{-1} \tilde{P}$ ):

$$I_{23} = -\frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ijk}^3 \psi \dot{v}^i t \, dx \\ = \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k \operatorname{div} v \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx \\ = \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{P} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx \\ = -\frac{1}{\nu^2} \int_{\mathbb{T}^2} \tilde{P} \partial_{ik}^2 \psi \dot{v}^i t \, dx - \frac{1}{\nu^2} \int_{\mathbb{T}^2} \tilde{P} \partial_{ik}^2 \psi \partial_k \dot{v}^i t \, dx \\ + \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx.$$

Remembering that  $\psi := (-\Delta)^{-1} \tilde{P}$  and integrating by parts one more time in the first term of the right-hand side just above, we conclude that

$$(B.5) \quad I_{23} = -\frac{1}{2\nu^2} \int_{\mathbb{T}^2} \tilde{P}^2 \operatorname{div} \dot{v}^i t \, dx - \frac{1}{\nu^2} \int_{\mathbb{T}^2} \tilde{P} \partial_{ik}^2 \psi \partial_k \dot{v}^i t \, dx \\ + \frac{1}{\nu^2} \int_{\mathbb{T}^2} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t \, dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t \, dx.$$



Hence, using Hölder inequality and the continuity of  $\nabla^2(-\Delta)^{-1}$  on  $L_4(\mathbb{T}^2)$ , we get

$$\left| \int_0^T I_{23}(t) dt \right| \lesssim \frac{\sqrt{T}}{\nu^2} \left( \|\tilde{P}\|_{L_4(0,T \times \mathbb{T}^2)}^2 \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^2)} \right. \\ \left. + \|\nabla G\|_{L_2(0,T \times \mathbb{T}^2)} \|\tilde{P}\|_{L_\infty(0,T;L_4)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \right. \\ \left. + \nu \|\nabla v\|_{L_2(0,T \times \mathbb{T}^2)} \|\tilde{P}\|_{L_\infty(0,T \times \mathbb{T}^2)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^2)} \right).$$

Hence, thanks to (1.5) and (2.48), one can conclude that

$$(B.6) \quad \left| \int_0^T I_{23}(t) dt \right| \leq C_0 \sqrt{T} (\nu^{-1} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + \nu^{-2} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}).$$

**Proof of (2.62).** We use the decomposition  $K_2 = K_{2,1} + K_{2,2} + K_{2,3}$  with

$$K_{2,1} := \int_{\mathbb{T}^2} (\nabla \mathcal{P}v \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx, \quad K_{2,2} := -\frac{1}{\nu} \int_{\mathbb{T}^2} (\nabla^2(-\Delta)^{-1} \tilde{G} \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx \\ \text{and } K_{2,3} := -\frac{1}{\nu} \int_{\mathbb{T}^2} (\nabla^2(-\Delta)^{-1} \tilde{P} \cdot \nabla \operatorname{div} v) \cdot \dot{v} t dx.$$

In order to handle  $K_{2,1}$ , we integrate by parts (note that  $\operatorname{div} \mathcal{P}v = 0$ ) and use the fact that  $\nu \operatorname{div} v = \tilde{P} + \tilde{G}$ . We get, with the usual summation convention

$$K_{2,1} = -\frac{\sqrt{t}}{\nu} \int_{\mathbb{T}^2} \partial_i (\mathcal{P}v)^j \tilde{P} \partial_j \dot{v}^i \sqrt{t} dx + \frac{1}{\nu} \int_{\mathbb{T}^2} \partial_i (\mathcal{P}v)^j \sqrt{t} \partial_j \dot{v}^i \sqrt{t} dx.$$

Therefore,

$$(B.7) \quad \nu \left| \int_0^T K_{2,1} dt \right| \leq \sqrt{T} \|\nabla \mathcal{P}v\|_{L_2(0,T;L_2)} \|\tilde{P}\|_{L_\infty(0,T;L_\infty)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\ + \|\nabla \mathcal{P}v\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \\ \leq C_0 (\sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)}.$$

Next, integrating by parts in  $K_{2,2}$  and using  $\nu \operatorname{div} v = \tilde{P} + \tilde{G}$  gives

$$\nu K_{2,2} = - \int_{\mathbb{T}^2} \sqrt{t} \nabla G \cdot \sqrt{t} \dot{v} \operatorname{div} v dx + \sqrt{t} \int_{\mathbb{T}^2} \nabla^2(-\Delta)^{-1} \tilde{G} \cdot \sqrt{t} \nabla \dot{v} \operatorname{div} v dx \\ = - \int_{\mathbb{T}^2} \sqrt{t} \nabla G \cdot \sqrt{t} \dot{v} \operatorname{div} v dx + \frac{\sqrt{t}}{\nu} \int_{\mathbb{T}^2} \nabla^2(-\Delta)^{-1} \tilde{G} \cdot \sqrt{t} \nabla \dot{v} \tilde{G} dx \\ + \frac{\sqrt{t}}{\nu} \int_{\mathbb{T}^2} \nabla^2(-\Delta)^{-1} \tilde{G} \cdot \sqrt{t} \nabla \dot{v} \tilde{P} dx$$

from which we get

$$(B.8) \quad \nu \left| \int_0^T K_{2,2} dt \right| \leq \|\operatorname{div} v\|_{L_4(0,T;L_4)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \\ + \nu^{-1} \sqrt{T} \|\tilde{G}\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} (\|\tilde{G}\|_{L_4(0,T;L_4)} + \|\tilde{P}\|_{L_4(0,T;L_4)}) \\ \leq C_0 T^{1/4} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_4)} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-1/2} \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.$$

Finally, using again the notation  $\psi := (-\Delta)^{-1}\tilde{P}$ , we have

$$\begin{aligned}\nu^2 K_{2,3} &= - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j \tilde{P} t v^i dx - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j G t v^i dx \\ &= \int_{\mathbb{T}^2} \partial_i \partial_j^2 \psi \tilde{P} t v^i dx + \int_{\mathbb{T}^2} \partial_i \partial_j \psi \tilde{P} t \partial_j v^i dx - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j G t v^i dx \\ &= \frac{1}{2} \int_{\mathbb{T}^2} \tilde{P}^2 t \operatorname{div} v dx + \int_{\mathbb{T}^2} \partial_i \partial_j \psi \tilde{P} t \partial_j v^i dx - \int_{\mathbb{T}^2} \partial_i \partial_j \psi \partial_j G t v^i dx.\end{aligned}$$

Therefore,

$$\begin{aligned}(B.9) \quad \nu^2 \left| \int_0^T K_{2,3} dt \right| &\lesssim \sqrt{T} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|\sqrt{t} \nabla v\|_{L_2(0,T;L_2)} \\ &\quad + \|\tilde{P}\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla G\|_{L_4(0,T;L_2)} \|\sqrt{t} v\|_{L_2(0,T;L_4)} \\ &\leq C_0 \sqrt{\nu T} \|\sqrt{t} \nabla v\|_{L_2(0,T;L_2)} + C_0 (\nu T)^{1/4} \|\sqrt{\rho t} v\|_{L_\infty(0,T;L_2)}^{1/2} \|\sqrt{t} v\|_{L_2(0,T;L_4)}.\end{aligned}$$

Plugging (2.4), (B.2), (2.44), (2.47) and (2.48) in (B.7), (B.8) and (C.29) yields (2.62).

### APPENDIX C. THE THREE-DIMENSIONAL CASE

This section is devoted to extending our existence result to the three-dimensional torus. For expository purpose, we focus on the global-in-time issue for small data, although a similar statement may be proved locally in time for large data.

**Theorem C.1.** *Let  $v_0 \in H^1(\mathbb{T}^3)$  and  $\rho_0$  be a bounded and nonnegative function on  $\mathbb{T}^3$ . There exists  $\nu_0 > 0$  depending only on  $\mu$ ,  $\gamma$  and on the norms of the data, and  $c_0 > 0$  such that if*

$$(C.1) \quad \mu \|\nabla \mathcal{P} v_0\|_2^2 + \frac{1}{\nu} \|\tilde{P}_0\|_2^2 + \nu \|\operatorname{div} v_0\|_2^2 \leq c_0 \frac{\mu^5}{(\rho^*)^3 E_0},$$

then System (1.1) has a global solution  $(\rho, v)$  having the same properties as in Theorem 1.1.

The general strategy is basically the same as for the two-dimensional case, except that the smallness condition spares our using the logarithmic interpolation inequality. We just point out the main steps, and what has to be changed.

**Step 1: Sobolev estimates for the velocity.** The counterpart of Proposition 2.1 reads:

**Proposition C.1.** *Let  $(\rho, v)$  be a smooth solution of (1.1) on  $[0, T] \times \mathbb{T}^3$ , fulfilling (1.8) and (2.3). Under condition (C.1) and for large enough  $\nu$ , there exists a constant  $C_0$  depending only on the data such that for all  $t \in [0, T]$ , we have*

$$\begin{aligned}\mu \|\mathcal{P} v(t)\|_2^2 + \frac{1}{\nu} (\|\tilde{G}(t)\|_2^2 + \|\tilde{P}(t)\|_2^2) + \frac{P^*}{\nu} \|(\sqrt{\rho} v)(t)\|_2^2 + \frac{P^*}{\nu} \|e(t)\|_1 \\ + \int_0^t \left( \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{\rho^*} \|\nabla^2 \mathcal{P} v\|_2^2 + \frac{1}{\rho^*} \|\nabla G\|_2^2 + \nu \|\operatorname{div} v\|_2^2 + \frac{\mu P^*}{\nu} \|\nabla v\|_2^2 \right) d\tau \leq C_0.\end{aligned}$$

*Proof.* In order to be able to consider general initial data *with large energy*, it is suitable to modify the definition of  $\mathcal{E}$  as follows: keeping the same definition for  $\tilde{\mathcal{E}}$ , we set

$$(C.2) \quad \mathcal{E} := \tilde{\mathcal{E}} + \frac{P^*}{\nu} E + \frac{P(1)}{2\nu} (P^* - P(1)),$$

which ensures that

$$(C.3) \quad \mathcal{E} \geq \frac{\mu}{2} \|\mathcal{P} v\|_2^2 + \frac{1}{2\nu} (\|\tilde{G}\|_2^2 + \|\tilde{P}\|_2^2) + \frac{P^*}{2\nu} \|\sqrt{\rho} v\|_2^2 + \frac{P^*}{2\nu} \|e\|_1.$$

Then, we start from Inequality (2.15) that is valid in any dimension and, instead of (1.13), we use that

$$(C.4) \quad \left( \int_{\mathbb{T}^3} \rho |v|^4 dx \right)^{1/2} \leq \left( \int_{\mathbb{T}^3} \rho |v|^2 dx \right)^{1/4} \left( \int_{\mathbb{T}^3} \rho |v|^6 dx \right)^{1/4} \lesssim (\rho^*)^{1/4} \|\sqrt{\rho} v\|_2^{1/2} \|\nabla v\|_2^{3/2}.$$

One can thus bound the right-hand side of (2.16) as follows:

$$\begin{aligned} 3 \int_{\mathbb{T}^3} \rho |v \cdot \nabla \mathcal{P}v|^2 dx &\leq 3\sqrt{\rho^*} \|\rho^{1/4} v\|_4^2 \|\nabla \mathcal{P}v\|_4^2 \\ &\leq C\rho^{*3/4} \|\sqrt{\rho} v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\nabla \mathcal{P}v\|_2^{1/2} \|\nabla^2 \mathcal{P}v\|_2^{3/2} \\ &\leq \frac{\mu^2}{4\rho^*} \|\Delta \mathcal{P}v\|_2^2 + C\left(\frac{\rho^*}{\mu}\right)^6 \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^6 \|\nabla \mathcal{P}v\|_2^2, \\ \frac{3}{\nu^2} \int_{\mathbb{T}^3} \rho |v \cdot [\nabla^2(-\Delta)^{-1}\tilde{G}]|^2 dx &\leq C(\rho^*)^{1/2} \nu^{-2} \|\rho^{1/4} v\|_4^2 \|\tilde{G}\|_4^2 \\ &\leq C(\rho^*)^{3/4} \nu^{-2} \|\sqrt{\rho} v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\tilde{G}\|_2^{1/2} \|\nabla G\|_2^{3/2} \\ &\leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 + C\frac{(\rho^*)^6}{\nu^8} \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^6 \|\tilde{G}\|_2^2, \end{aligned}$$

and also, thanks to Inequality (A.2),

$$\begin{aligned} \frac{3}{\nu^2} \int_{\mathbb{T}^3} \rho |v \cdot \nabla^2 \Delta^{-1} \tilde{P}|^2 dx &\leq C\frac{\rho^*}{\nu^2} \|v\|_2^{1/2} \|\nabla v\|_2^{3/2} \|\tilde{P}\|_4^2 \\ &\leq C\frac{(\rho^*)^{3/2}}{\nu^2} \|\nabla v\|_2^2 \|\tilde{P}\|_2 \|\tilde{P}\|_\infty \\ &\leq \frac{\mu P^*}{4\nu} \|\nabla v\|_2^2 + C\frac{(\rho^*)^3}{\mu\nu^3} \frac{\|\tilde{P}\|_\infty^2}{P^*} \|\nabla v\|_2^2 \|\tilde{P}\|_2^2. \end{aligned}$$

Next, instead of (2.20), we write that, in light of Inequality (A.2) with  $p = 2$ , we have

$$\begin{aligned} -\frac{1}{\nu} \int_{\mathbb{T}^3} P v \cdot \nabla G &\leq \frac{1}{\nu} \|P\|_\infty (1 + \|\tilde{\rho}\|_2) \|\nabla v\|_2 \|\nabla G\|_2 \\ &\leq \frac{1}{8\rho^*} \|\nabla G\|_2^2 + 2\frac{\rho^*}{\nu^2} \|P\|_\infty^2 (1 + \|\tilde{\rho}\|_2)^2 \|\nabla v\|_2^2. \end{aligned}$$

Therefore, the right-hand side of Inequality (2.21) becomes

$$\begin{aligned} 2\frac{(\rho^*)^3}{\nu^2} \|P\|_\infty^2 \|\nabla v\|_2^2 + \frac{2}{\nu^2} \int_{\mathbb{T}^3} \tilde{G}^2 h dx \\ + C\|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^6 \left( \left(\frac{\rho^*}{\mu}\right)^6 \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^6}{\nu^8} \|\tilde{G}\|_2^2 \right) + C\frac{(\rho^*)^3}{\mu\nu^3} \frac{\|\tilde{P}\|_\infty^2}{P^*} \|\nabla v\|_2^2 \|\tilde{P}\|_2^2. \end{aligned}$$

Then, following the computations leading to (2.26) and assuming that  $\nu$  satisfies

$$(C.5) \quad \nu \geq 8(\rho^*)^3 \frac{P^*}{\mu} \quad \text{and} \quad \nu^2 \geq 8\|h\|_\infty \rho^*,$$

we get,

$$\frac{d}{dt} \mathcal{E} + \mathcal{D} \leq C \left( \|\sqrt{\rho} v\|_2^2 \|\nabla v\|_2^6 \left( \left(\frac{\rho^*}{\mu}\right)^6 \|\nabla \mathcal{P}v\|_2^2 + \frac{(\rho^*)^6}{\nu^8} \|\tilde{G}\|_2^2 \right) + \frac{(\rho^*)^3}{\mu\nu^3} \frac{\|\tilde{P}\|_\infty^2}{P^*} \|\nabla v\|_2^2 \|\tilde{P}\|_2^2 \right)$$

with

$$\mathcal{D} := \frac{1}{4} \|\sqrt{\rho} \dot{v}\|_2^2 + \frac{\mu^2}{4\rho^*} \|\nabla^2 \mathcal{P}v\|_2^2 + \frac{1}{8\rho^*} \|\nabla G\|_2^2 + \frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} v)^2 (\nu + h) dx + \frac{\mu P^*}{2\nu} \|\nabla v\|_2^2.$$

Finally, using Inequality (C.3), we discover that if  $\nu$  satisfies (C.5), then we have the differential inequality

$$(C.6) \quad \frac{d}{dt} \mathcal{E} + \mathcal{D} \leq C \|\nabla v\|_2^2 \mathcal{E} \left( \frac{(\rho^*)^{3/2} \|\tilde{P}\|_\infty^2}{\nu^3 \mu^{1/2}} + \frac{(\rho^*)^6}{\mu^9} E_0 \mathcal{E}^2 \right).$$

Setting

$$X(t) = \mathcal{E}(t) + \int_0^t \mathcal{D} d\tau, \quad f(t) := C \|\nabla v(t)\|_2^2, \quad A := \frac{(\rho^*)^3 P^*}{\mu \nu^2} \quad \text{and} \quad B := \frac{(\rho^*)^6}{\mu^9} E_0,$$

Inequality (C.6) rewrites

$$\frac{d}{dt} X \leq (AX + BX^3) f(t).$$

This may be integrated into

$$\frac{X(t)}{\sqrt{1 + cX^2(t)}} \leq \frac{X(0)}{\sqrt{1 + cX^2(0)}} e^{A \int_0^t f(\tau) d\tau} \quad \text{with} \quad c := \frac{B}{A}.$$

Bounding  $f$  according to (1.5), we see that under the smallness condition

$$(C.7) \quad \mathcal{E}_0^2 < \frac{A}{B} \frac{1}{e^{\frac{2CAE_0}{\mu}} - 1},$$

we have

$$(C.8) \quad X^2(t) \leq \frac{X^2(0)}{1 + cX^2(0)} \left( \frac{e^{\frac{2CAE_0}{\mu}}}{1 - \frac{cX^2(0)}{1+cX^2(0)} e^{\frac{2CAE_0}{\mu}}} \right) \quad \text{for all } t \geq 0.$$

Note that the largeness condition (C.5) on  $\nu$  guarantees that the argument of the exponential function above is very small. Therefore, the smallness condition (C.7) may be simplified into

$$\mathcal{E}_0 \ll \frac{\mu^5}{(\rho^*)^3 E_0}.$$

For that latter condition to be fulfilled for  $\nu$  large enough compared to  $E_0^2$ , it suffices that (C.1) holds true.  $\square$

**Remark C.1.** *Note that the smallness condition means that one can take the initial energy as large as we want provided that  $\nu$  is large enough, but that  $\operatorname{div} v_0$  must be  $\mathcal{O}(\nu^{-1/2})$ . At the same time, there is no smallness condition on  $\rho_0 - 1$  whatsoever.*

**Step 2: Upper bound for the density.** In order to adapt Proposition 2.2 to the case  $d = 3$ , the only changes are in (2.34) and (2.36). As regards (2.34), one may still start from (2.33) then combine with (C.4) in order to get

$$(C.9) \quad \|(-\Delta)^{-1}(\rho v)\|_\infty \leq C(\rho^*)^{\frac{7}{8}} \|\sqrt{\rho} v\|_2^{\frac{1}{4}} \|\nabla v\|_2^{\frac{3}{4}} \leq C(\rho^*)^{\frac{7}{8}} E_0^{\frac{1}{8}} \|\nabla v\|_2^{\frac{3}{4}}.$$

Next, instead of (2.35), in order to bound the commutator term, we write that

$$(C.10) \quad \|[v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_\infty \lesssim \|[\tilde{v}^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i\|_{W^{1, \frac{24}{7}}} \lesssim \|\nabla v\|_6 \|\rho v\|_8.$$

Now, combining Hölder inequalities, Sobolev embedding and interpolation inequalities yields

$$\begin{aligned} \|\rho v\|_8 &\leq (\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|v\|_{\frac{10}{12}}^{\frac{9}{10}} \\ &\leq C(\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|\nabla v\|_{\frac{12}{5}}^{\frac{9}{10}} \\ &\leq C(\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|\nabla v\|_2^{\frac{27}{40}} \|\nabla v\|_6^{\frac{9}{40}}. \end{aligned}$$

Therefore, in  $\mathbb{T}^3$ , Inequality (2.35) becomes

$$\| [v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i \|_\infty \leq C(\rho^*)^{\frac{19}{20}} \|\sqrt{\rho} v\|_2^{\frac{1}{10}} \|\nabla v\|_2^{\frac{27}{40}} \|\nabla v\|_6^{\frac{9}{40}}.$$

In order to bound the last term, we use that

$$\begin{aligned} \|\nabla v\|_6 &\lesssim \|\nabla \mathcal{P}v\|_6 + \nu^{-1} (\|\tilde{G}\|_6 + \|\tilde{P}\|_6) \\ &\lesssim \|\nabla^2 \mathcal{P}v\|_2 + \nu^{-1} (\|\nabla G\|_2 + \|\tilde{P}\|_\infty). \end{aligned}$$

Hence, using the energy conservation (1.5) and the definition of  $\mathcal{E}$  and  $\mathcal{D}$ ,

$$(C.11) \quad \| [v^j, (-\Delta)^{-1} \partial_i \partial_j] \rho v^i \|_\infty \lesssim (\rho^*)^{\frac{19}{20}} E_0^{\frac{1}{20}} \|\nabla v\|_2^{\frac{27}{40}} \left( \left( \frac{\sqrt{\rho^*} \mathcal{D}}{\mu} \right)^{\frac{49}{80}} + \left( \frac{\|\tilde{P}\|_\infty}{\nu} \right)^{\frac{49}{40}} \right).$$

Plugging inequalities (C.9) and (C.11) in (2.32), we get

$$(C.12) \quad \begin{aligned} \|F^+(t)\|_\infty &\leq \|F^+(0)\|_\infty + \frac{\gamma-1}{\gamma} E_0 + C \frac{\gamma}{\nu^2} (\rho^*)^{\frac{7}{8}} E_0^{\frac{1}{8}} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|\nabla v(\tau)\|_2^{\frac{3}{4}} d\tau \\ &\quad + C \frac{(\rho^*)^{\frac{19}{20}}}{\nu} E_0^{\frac{1}{20}} \int_0^t e^{-\frac{\gamma}{\nu}(t-\tau)} \|\nabla v\|_2^{\frac{27}{40}} \left( \left( \frac{\sqrt{\rho^*} \mathcal{D}(\tau)}{\mu} \right)^{\frac{49}{80}} + \left( \frac{\|\tilde{P}\|_\infty}{\nu} \right)^{\frac{49}{40}} \right) d\tau. \end{aligned}$$

Since the integrals in the right-hand side may be bounded in terms of the data according to the basic energy inequality (1.5) and to (2.38), we eventually get if  $\nu$  is large enough:

$$\|F^+(t)\|_\infty \leq \|F^+(0)\|_\infty + C_0 \nu^{-\frac{27}{80}} + \frac{\gamma-1}{\gamma} E_0$$

with  $C_0$  depending only on  $E_0$ ,  $\mathcal{E}_0$ ,  $\|\rho_0\|_\infty$ ,  $\mu$  and  $\gamma$ . From this point, one can conclude as in the two-dimensional case that (2.29) is fulfilled if  $\nu$  is large enough.

**Step 3: Time weighted estimates.** As in the 2D case, the starting point is Identity (2.43). However, Inequality (2.44) that has been used all the time has to be replaced with an estimate for  $t^{1/8} \nabla v$  in  $L_4(0, T \times \mathbb{T}^3)$ : we write that the previous steps and to (B.1) imply that

$$(C.13) \quad \begin{aligned} \|t^{1/8} \nabla \mathcal{P}v\|_{L_4(0, T \times \mathbb{T}^3)} &\leq \|t^{1/4} \nabla \mathcal{P}v\|_{L_\infty(0, T; L_3)}^{1/2} \|\nabla \mathcal{P}v\|_{L_2(0, T; L_6)}^{1/2} \\ &\lesssim \|\nabla \mathcal{P}v\|_{L_\infty(0, T; L_2)}^{1/4} \|\sqrt{t} \nabla^2 \mathcal{P}v\|_{L_\infty(0, T; L_2)}^{1/4} \|\nabla^2 \mathcal{P}v\|_{L_2(0, T \times \mathbb{T}^3)}^{1/2} \\ &\leq C_0 \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0, T; L_2)}^{1/4}. \end{aligned}$$

Similarly, we have

$$(C.14) \quad \begin{aligned} \|t^{1/8} \nabla^2 (-\Delta)^{-1} \tilde{G}\|_{L_4(0, T \times \mathbb{T}^3)} &\lesssim \|\tilde{G}\|_{L_4(0, T \times \mathbb{T}^3)} \\ &\lesssim \|\tilde{G}\|_{L_\infty(0, T; L_2)}^{1/4} \|\sqrt{t} \nabla G\|_{L_\infty(0, T; L_2)}^{1/4} \|\nabla G\|_{L_2(0, T \times \mathbb{T}^3)}^{1/2} \\ &\leq C_0 \nu^{1/8} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0, T; L_2)}^{1/4}. \end{aligned}$$

Since (2.48) is valid in any dimension, one can conclude that

$$(C.15) \quad \|t^{1/8}\nabla v\|_{L_4(0,T;\mathbb{T}^3)} \leq C_0(\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{-3/4}T^{1/8}).$$

*Substep 1.* Compared to  $d = 2$ , the only change lies in the estimate for  $\int_{\mathbb{T}^3} \rho \operatorname{div} v |\dot{v}|^2 t \, dx$ . Now, still using that  $\operatorname{div} v = \nu^{-1}(\tilde{P} + \tilde{G})$ , we write that

$$\begin{aligned} \int_{\mathbb{T}^3} \rho \operatorname{div} v |\dot{v}|^2 t \, dx &\leq \nu^{-1} \int_{\mathbb{T}^2} (\tilde{P} + \tilde{G}) \rho |\dot{v}|^2 t \, dx \\ &\leq \nu^{-1} (\|\tilde{P}\|_\infty \|\sqrt{\rho t}\dot{v}\|_2^2 + \sqrt{\rho^*} \|\tilde{G}\|_3 \|\sqrt{t}\dot{v}\|_6 \|\sqrt{\rho t}\dot{v}\|_2) \\ &\leq C_0 \nu^{-1} (\|\sqrt{\rho t}\dot{v}\|_2^2 + \|\tilde{G}\|_2^{1/2} \|\nabla G\|_2^{1/2} \|\sqrt{t}\nabla \dot{v}\|_2 \|\sqrt{\rho t}\dot{v}\|_2). \end{aligned}$$

The first term may be treated as in the 2D case. As for the second one, we use the fact that (2.68) ensures that

$$\|\nabla G\|_2 \leq \sqrt{\rho^*} \|\sqrt{\rho}\dot{v}\|_2.$$

Hence, using Proposition 2.1 to bound  $\|\tilde{G}\|_2$ , we get

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} \rho \operatorname{div} v |\dot{v}|^2 t \, dx \, dt &\leq C_0 \left( \nu^{-1} \int_0^T \|\sqrt{\rho t}\dot{v}\|_2^2 + \nu^{-3/4} \int_0^T \|\sqrt{\rho}\dot{v}\|_2^{1/2} \|\sqrt{\rho t}\dot{v}\|_2 \|\sqrt{t}\nabla \dot{v}\|_2 \, dt \right) \\ &\leq \frac{C_0}{\nu} \left( \int_0^T \|\sqrt{\rho t}\dot{v}\|_2^2 + \frac{1}{\sqrt{\nu}} \int_0^T \|\sqrt{\rho}\dot{v}\|_2 \|\sqrt{\rho t}\dot{v}\|_2^2 \, dt \right) + \frac{\mu}{2} \int_0^T \|\sqrt{t}\nabla \dot{v}\|_2^2 \, dt. \end{aligned}$$

In the end, we thus obtain

$$(C.16) \quad \begin{aligned} \int_{\mathbb{T}^3} \frac{D}{Dt} (\rho |\dot{v}|^2 t) \, dx &\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \rho |\dot{v}|^2 \, dx - \frac{1}{2} \int_{\mathbb{T}^3} \rho |\dot{v}|^2 \, dx \\ &\quad - \frac{C_0}{\nu} \left( \int_0^T \|\sqrt{\rho t}\dot{v}\|_2^2 + \frac{1}{\sqrt{\nu}} \int_0^T \|\sqrt{\rho}\dot{v}\|_2 \|\sqrt{\rho t}\dot{v}\|_2^2 \, dt \right) - \frac{\mu}{2} \int_0^T \|\sqrt{t}\nabla \dot{v}\|_2^2 \, dt. \end{aligned}$$

*Substep 2.* We have thanks to (C.15):

$$(C.17) \quad \left| \int_0^T I_1(t) \, dt \right| \leq C_0 T^{1/4} (\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-3/2} T^{1/4}) \|\sqrt{t}\nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^3)}.$$

For bounding  $I_2$ , we decompose it into three parts, as for  $d = 2$ . For  $I_{2,1}$ , we write that

$$\left| \int_0^T I_{21}(t) \, dt \right| \leq \|\sqrt{t}\nabla v\|_{L_\infty(0,T;L_3)} \|\nabla^2 \mathcal{P}v\|_{L_2(0,T;\mathbb{T}^3)} \|\sqrt{t}\dot{v}\|_{L_2(0,T;L_6)}.$$

Let us notice that

$$(C.18) \quad \|t^{1/4}\nabla v\|_{L_\infty(0,T;L_3)} \leq C_0 (\|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4})$$

that stems from the fact that, as already used for proving (C.13) and (C.14), we have

$$(C.19) \quad \|t^{1/4}\nabla \mathcal{P}v\|_{L_\infty(0,T;L_3)} \leq C_0 \|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2},$$

$$(C.20) \quad \|t^{1/4}\tilde{G}\|_{L_\infty(0,T;L_3)} \leq C_0 \nu^{1/4} \|\sqrt{\rho t}\dot{v}\|_{L_\infty(0,T;L_2)}^{1/2},$$

and from the obvious inequality

$$(C.21) \quad \|\tilde{P}\|_{L_\infty(0,T;L_3)} \leq \|\tilde{P}\|_{L_\infty(0,T;L_2)}^{2/3} \|\tilde{P}\|_{L_\infty(0,T;\mathbb{T}^3)}^{1/3} \leq \nu^{1/3} C_0.$$

Then, using Sobolev embedding and (C.18), we obtain

$$(C.22) \quad \left| \int_0^T I_{21} dt \right| \leq C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

In order to bound  $I_{22}$ , we now write that

$$\left| \int_0^T I_{22} dt \right| \lesssim \nu^{-1} \|\sqrt{t} \nabla v\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)}.$$

Therefore, using the previous section and (C.18), we get

$$(C.23) \quad \left| \int_0^T I_{22} dt \right| \leq C_0 \nu^{-1/2} T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

To bound  $I_{23}$ , we use (B.5) as in the two-dimensional case. The first two terms of the decomposition may be bounded as before. For the third one, we use the fact that

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}^3} \partial_k \tilde{G} \partial_{ik}^2 \psi \dot{v}^i t dx \right| &\lesssim \sqrt{T} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\tilde{P}\|_{L_\infty(0,T;L_3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \\ &\leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)} \end{aligned}$$

and that

$$\begin{aligned} \left| \int_{\mathbb{T}^3} \partial_k v^j \partial_{ik}^2 \psi \partial_j \dot{v}^i t dx \right| &\leq T^{3/8} \|t^{1/8} \nabla v\|_{L_4(0,T \times \mathbb{T}^3)} \|\tilde{P}\|_{L_4(0,T \times \mathbb{T}^3)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)} \\ &\leq C_0 \nu^{1/4} T^{3/8} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{-3/4} T^{1/8}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}. \end{aligned}$$

Hence, one can conclude that

$$(C.24) \quad \left| \int_0^T I_{23} dt \right| \leq C_0 \left( \frac{\sqrt{T}}{\nu^{3/2}} + \frac{T^{3/8}}{\nu^{3/4}} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} \right) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

Putting together all the estimates of the second step, we get

$$(C.25) \quad -\mu \int_0^T \int_{\mathbb{T}^3} \left( \frac{D}{Dt} \Delta v \right) \cdot \dot{v} t dx dt \geq \mu \int_0^T \int_{\mathbb{T}^3} |\nabla \dot{v}|^2 t dx dt - C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{-2/3} T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}.$$

*Substep 3.* To bound  $K_1$ , we write that

$$\nu \left| \int_0^T K_1 dt \right| \leq C T^{1/4} \|t^{1/8} \nabla v\|_{L_4(0,T;L_4)} \|t^{1/8} (\tilde{P} + \tilde{G})\|_{L_4(0,T;L_4)} \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)},$$

whence

$$(C.26) \quad \nu \left| \int_0^T K_1 dt \right| \leq C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{-3/4} T^{1/8}) \times (\nu^{1/8} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/4} + \nu^{1/4} T^{1/8}) \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}.$$

We decompose  $K_2$  as in the case  $d = 2$ . To bound  $K_{2,1}$ , we write that

$$(C.27) \quad \begin{aligned} \nu \left| \int_0^T K_{2,1} dt \right| &\leq \sqrt{T} \|\nabla \mathcal{P} v\|_{L_2(0,T \times \mathbb{T}^3)} \|\tilde{P}\|_{L_\infty(0,T \times \mathbb{T}^3)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)} \\ &\quad + \|\sqrt{t} \nabla \mathcal{P} v\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T \times \mathbb{T}^3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \\ &\leq C_0 (\sqrt{T} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T \times \mathbb{T}^3)}. \end{aligned}$$

For  $K_{2,2}$ , we have

$$\begin{aligned}
\nu \left| \int_0^T K_{2,2} dt \right| &\leq \|\sqrt{t} \operatorname{div} v\|_{L_\infty(0,T;L_3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \|\nabla G\|_{L_2(0,T;\mathbb{T}^3)} \\
&\quad + \nu^{-1} T^{1/4} \|t^{1/8} \tilde{G}\|_{L_4(0,T;L_4)} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} (\|t^{1/8} \tilde{G}\|_{L_4(0,T;L_4)} + \|t^{1/8} \tilde{P}\|_{L_4(0,T;L_4)}) \\
\text{(C.28)} \quad &\leq C_0 T^{1/4} (T^{1/8} \nu^{-3/4} + \nu^{-3/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\nu^2 \left| \int_0^T K_{2,3} dt \right| &\lesssim \sqrt{T} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\
&\quad + \sqrt{T} \|\tilde{P}\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T;\mathbb{T}^3)} \|\sqrt{t} \dot{v}\|_{L_2(0,T;L_6)} \\
\text{(C.29)} \quad &\leq C_0 \sqrt{\nu T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.
\end{aligned}$$

Plugging (2.4), (B.2), (2.44), (2.47) and (2.48) in (C.27), (C.28) and (C.29) yields

$$\nu \left| \int_0^T K_2 dt \right| \leq C_0 (\sqrt{T} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;\mathbb{T}^3)}.$$

The conclusion of this step is that, if  $\nu$  is large enough then

$$\begin{aligned}
\text{(C.30)} \quad &-(\nu - \mu) \int_0^T \int_{\mathbb{T}^3} \frac{D}{Dt} \nabla \operatorname{div} v \cdot \dot{v} t dx dt \geq (\nu - \mu) \int_0^T \int_{\mathbb{T}^3} (\operatorname{div} \dot{v})^2 t dx dt \\
&\quad - C_0 (\nu^{1/4} \sqrt{T} + T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)} \\
&\quad - C_0 (\nu^{1/8} T^{1/4} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + \nu^{3/8} \sqrt{T}) \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)}.
\end{aligned}$$

*Substep 4.* Term  $L_1$  may still be bounded according to Inequality (2.65). As for  $L_2$ , we have

$$\begin{aligned}
\left| \int_0^T L_2(t) dt \right| &\leq \frac{1}{2\nu} \|\tilde{P}\|_{L_4(0,T;L_4)}^2 \|t \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} \\
&\quad + \frac{1}{\nu} \|\tilde{P}\|_{L_\infty(0,T;L_3)} \|\nabla G\|_{L_2(0,T;L_2)} \|t \dot{v}\|_{L_2(0,T;L_6)} \\
&\quad + \|\tilde{P}\|_{L_\infty(0,T;L_\infty)} \|\nabla v\|_{L_2(0,T;L_2)} \|t \nabla \dot{v}\|_{L_2(0,T;L_2)} \\
&\leq C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.
\end{aligned}$$

So this step gives

$$\begin{aligned}
\text{(C.31)} \quad &\int_{\mathbb{T}^3} \frac{D}{Dt} \nabla P \cdot \dot{v} t dx \geq -\frac{\nu}{4} \int_0^T \int_{\mathbb{T}^3} (\operatorname{div} \dot{v})^2 t dx dt \\
&\quad - \|h\|_\infty T \nu^{-1} \|\operatorname{div} v\|_{L_2(0,T;L_2)}^2 - C_0 \sqrt{T} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.
\end{aligned}$$

*Substep 5.* Combining Inequalities (2.53), (C.25), (C.30) and (C.31) yields for large  $\nu$ ,

$$\begin{aligned}
&\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^2 + 2\mu \int_0^T \|\sqrt{t} \nabla \mathcal{P} \dot{v}\|_2^2 dt + \frac{3\nu}{2} \int_0^T \|\sqrt{t} \operatorname{div} \dot{v}\|_2^2 dt \leq 2 \int_0^T \|\operatorname{div} v\|_\infty \|\sqrt{\rho t} \dot{v}\|_2^2 dt \\
&\quad + \|\sqrt{\rho} \dot{v}\|_{L_2(0,T;L_2)}^2 + \|h\|_\infty T \nu^{-1} \|\operatorname{div} v\|_{L_2(0,T;\mathbb{T}^2)}^2 + C_0 T^{1/4} (\nu^{1/8} \|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} \\
&\quad + \nu^{3/8} T^{1/4}) \|\sqrt{t} \operatorname{div} \dot{v}\|_{L_2(0,T;L_2)} + C_0 T^{1/4} (\|\sqrt{\rho t} \dot{v}\|_{L_\infty(0,T;L_2)}^{1/2} + T^{1/4}) \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}.
\end{aligned}$$

Playing with Young inequality and Gronwall Lemma yields Prop. 2.3 for  $d = 3$ .



It is now easy to adapt Corollary 2.1 to the 3D case: we start from

$$\|\tilde{G}\|_\infty \lesssim \|\tilde{G}\|_2^{1/4} \|\nabla G\|_6^{3/4}.$$

Hence, remembering (2.68) and using the embedding  $\dot{H}^1(\mathbb{T}^3) \hookrightarrow L_6(\mathbb{T}^3)$ ,

$$\|\tilde{G}\|_\infty \lesssim (\rho^*)^{3/4} \|\tilde{G}\|_2^{1/4} t^{-3/8} \|\sqrt{t} \nabla \dot{v}\|_2^{3/4}.$$

Therefore, as in the 2D case,

$$\|\tilde{G}\|_{L_{1+\varepsilon}(0,T;L_\infty)}^{1+\varepsilon} \lesssim (\rho^*)^{\frac{3(1+\varepsilon)}{4}} \nu^{\frac{1+\varepsilon}{8}} (\nu^{-\frac{1}{2}} \|\tilde{G}\|_{L_\infty(0,T;L_2)})^{\frac{1+\varepsilon}{4}} \|\sqrt{t} \nabla \dot{v}\|_{L_2(0,T;L_2)}^{\frac{3(1+\varepsilon)}{4}} \left( \int_0^T t^{-\frac{3+3\varepsilon}{5-3\varepsilon}} dt \right)^{\frac{5-3\varepsilon}{8}},$$

and one can thus conclude that  $\operatorname{div} v$  is in  $L_{1+\varepsilon}(0,T;L_\infty)$  provided that  $\varepsilon < 1/3$ . Bounding  $\mathcal{P}v$  is left to the reader.

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#### REFERENCES

- [1] H. Bahouri, J.-Y. Chemin and R. Danchin: *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, **343**, Springer (2011).
- [2] Y. Cho, H.J. Choe and H. Kim: Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.* **83** (2004), no. 2, 243–275.
- [3] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes: Compensated compactness and Hardy spaces, *J. Math. Pures Appl.*, **72** (1993), no. 3, 247–286.
- [4] R. Danchin: Global existence in critical spaces for compressible Navier-Stokes equations, *Inventiones Mathematicae*, **141** (2000), n. 3, 579–614.
- [5] R. Danchin: Local theory in critical spaces for compressible viscous and heat-conductive gases, *Comm. Partial Differential Equations*, **26** (2001), no. 7-8, 1183–1233.
- [6] R. Danchin: On the solvability of the compressible Navier-Stokes system in bounded domains, *Nonlinearity*, **23**(2), (2010), 383–408.
- [7] R. Danchin, F. Fanelli and M. Paicu: A well-posedness result for viscous compressible fluids with only bounded density, to appear in *Analysis and PDEs*.
- [8] R. Danchin and P.B. Mucha: Compressible Navier-Stokes system : large solutions and incompressible limit, *Advances in Mathematics*, **320** (2017), 904–925.
- [9] R. Danchin and P.B. Mucha: The incompressible Navier-Stokes equations in vacuum, *Communications on Pure and Applied Mathematics*, to appear.
- [10] R. Danchin and P.B. Mucha: From compressible to incompressible inhomogeneous flows in the case of large data, *Tunisian Journal of Mathematics*, **1** (2019), no. 1, 127–149.
- [11] B. Desjardins: Regularity of weak solutions of the compressible isentropic Navier-Stokes equations, *Comm. Partial Differential Equations*, **22**(5-6), (1997), 977–1008.
- [12] B. Desjardins: Global existence results for the incompressible density-dependent Navier-Stokes equations in the whole space, *Differential Integral Equations*, **10** (1997), no. 3, 587–598.
- [13] R. Di Perna and P.-L. Lions: Ordinary differential equations, transport theory and Sobolev spaces, *Inventiones Mathematicae*, **98** (1989) 511–547.
- [14] E. Feireisl: *Dynamics of viscous compressible fluids*. Oxford Lecture Series in Mathematics and its Applications, **26**. Oxford University Press, Oxford, 2004.
- [15] D. Hoff: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations*, **120** (1995), no. 1, 215–254.
- [16] D. Hoff: Uniqueness of weak solutions of the Navier-Stokes equations of multidimensional, compressible flow, *SIAM J. Math. Anal.*, **37** (2006), no. 6, 1742–1760.
- [17] O. Ladyzhenskaya: Solution ‘in the large’ of the non-stationary boundary value problem for the Navier-Stokes system with two space variables, *Comm. Pure Appl. Math.*, **12**, (1959) 427–433.

- [18] P.-L. Lions: *Mathematical Topics in Fluid Mechanics*, Oxford Science Publications, Vol. 2, Compressible models, The Clarendon Press, Oxford University Press, New-York (1998).
- [19] A. Matsumura and T. Nishida: The initial value problem for equations of motion of compressible viscous and heat conductive gases, *J. Math. Kyoto Univ.*, **20**, (1980), 67–104.
- [20] P.B. Mucha: Stability of nontrivial solutions of the Navier-Stokes system on the three dimensional torus, *J. Differential Equations*, **172**, (2001), n. 2, 359–375.
- [21] P.B. Mucha: The Cauchy problem for the compressible Navier-Stokes equations in the  $L_p L_p$ -framework, *Nonlinear Anal.*, **52**(4), (2003), 1379–1392.
- [22] P.B. Mucha: Transport equation: extension of classical results for  $\operatorname{div} b$  in  $BMO$ , *J. Differential Equations*, **249** (2010), no. 8, 1871–1883.
- [23] V. Solonnikov: Solvability of the initial boundary value problem for the equations of motion of a viscous compressible fluid, *Journal of Soviet Mathematics*, **14** (1980) 1120–1132.

(R. Danchin) UNIVERSITÉ PARIS-EST, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE.

*E-mail address:* `raphael.danchin@u-pec.fr`

(P.B. Mucha) INSTYTUT MATEMATYKI STOSOWANEJ I MECHANIKI, UNIWERSYTET WARSZAWSKI, UL. BANACHA 2, 02-097 WARSZAWA, POLAND.

*E-mail address:* `p.mucha@mimuw.edu.pl`