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Basic and bibasic identities related to divisor functions

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Abstract. Using basic hypergeometric functions and partial fraction decomposition we give a new kind of generalization of identities due to Uchimura, Dilcher, Van Hamme, Prodinger, and Chen-Fu related to divisor functions. An identity relating Lambert series to Eulerian polynomials is also proved.

Keywords: divisor functions; Lambert series; Prodinger's identity; Uchimura's identity; Dilcher's identity; Carlitz's q -Eulerian polynomials.

2000 Mathematics Subject Classifications: 11B65, 30E05

1. Introduction

The divisor function $\sigma_m(n)$ for a natural number n is defined as the sum of the m th powers of the (positive) divisors of n , i.e., $\sigma_m(n) = \sum_{d|n} d^m$. Throughout this paper, we assume that $|q| < 1$. The generating function of $\sigma_m(n)$ has an explicit Lambert series expansion (see [5])

$$\sum_{n=1}^{\infty} \sigma_m(n) q^n = \sum_{n=1}^{\infty} \frac{n^m q^n}{1 - q^n}. \quad (1.1)$$

In 1981, Uchimura [24] rediscovered an identity due to Kluyver [19] (see also Dilcher [9]):

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{\binom{k+1}{2}}}{(q; q)_k (1 - q^k)} = \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k}, \quad (1.2)$$

where $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ for $n \geq 0$. Since then, many authors have given different generalizations of (1.2) (see [2, 4, 9, 11, 12, 14, 16–18, 21, 22, 24–26, 28]). For example, Van Hamme [26] gave a finite form of (1.2) as follows:

$$\sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1 - q^k} = \sum_{k=1}^n \frac{q^k}{1 - q^k}, \quad (1.3)$$

where the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

Uchimura [25] proved that

$$\sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1 - q^{k+m}} = \sum_{k=1}^n \frac{q^k}{1 - q^k} \begin{bmatrix} k+m \\ m \end{bmatrix}_q^{-1}, \quad m \geq 0. \quad (1.4)$$

Dilcher [9] established the following multiple series generalization of (4.8):

$$\sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2} + km}}{(1 - q^k)^m} = \sum_{1 \leq k_1 \leq \dots \leq k_m \leq n} \frac{q^{k_1 + \dots + k_m}}{(1 - q^{k_1}) \dots (1 - q^{k_m})}. \quad (1.5)$$

Prodinger [21] proved that

$$\sum_{\substack{k=0 \\ k \neq m}}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}}}{1 - q^{k-m}} = (-1)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \sum_{\substack{k=0 \\ k \neq m}}^n \frac{q^{k-m}}{1 - q^{k-m}}, \quad 0 \leq m \leq n. \quad (1.6)$$

Using partial fraction decomposition the second author [28, (7)] obtained the following common generalization of Dilcher's identity (1.5) and of some identities due to Fu and Lascoux [11, 12]:

$$\sum_{k=i}^n (-1)^{k-i} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{q^{\binom{k-i}{2} + km}}{(1 - zq^k)^m} = \frac{q^i (q; q)_{i-1} (q; q)_n}{(q; q)_i (zq; q)_n} h_{m-1} \left(\frac{q^i}{1 - zq^i}, \dots, \frac{q^n}{1 - zq^n} \right), \quad (1.7)$$

where $1 \leq i \leq n$ and $h_k(x_1, \dots, x_n)$ is the k -th homogeneous symmetric polynomial in x_1, \dots, x_n defined by

$$h_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \dots x_{i_k} = \sum_{\alpha_1 + \dots + \alpha_n = k} x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

We note that Ismail and Stanton [18, Theorem 2.2] have rediscovered the $i = 1$ case of (1.7) as well as some other results in [28].

In this paper we shall give a different kind of generalizations of (1.4)–(1.6). Our starting point is an identity of Chen and Fu [8, (3.3)], which corresponds to the $(r, x) = (0, 1)$ case of the following result.

Theorem 1.1. *Let $m, n \geq 0$ and $0 \leq r \leq m$. Then*

$$\sum_{k=0}^m \frac{(-1)^k p^{\binom{k+1}{2} - rk}}{(p; p)_k (p; p)_{m-k} (xp^k; q)_{n+1}} = x^r \sum_{k=0}^n \frac{(-1)^k q^{\binom{k+1}{2} + rk}}{(q; q)_k (q; q)_{n-k} (xq^k; p)_{m+1}}. \quad (1.8)$$

We shall give a proof of Theorem 1.1 along with its several consequences and variations in Section 2.

Recently Liu [20, Proposition 4.1] has obtained the following formula, the left-hand side of which specializes to the Lambert series in (1.2) when $a = 1$:

$$\sum_{n=1}^{\infty} \frac{aq^n}{1 - aq^n} = \sum_{n=1}^{\infty} \frac{(1 - aq^{2n})a^n q^{n^2}}{(1 - q^n)(1 - aq^n)}. \quad (1.9)$$

Motivated by the identity (1.9), we shall generalize the expansion of the Lambert series (1.1) by using Carlitz's q -Eulerian polynomials $A_n(t; q)$. Recall (see [10]) that these q -Eulerian polynomials $A_n(t; q)$ are defined by

$$\sum_{j=0}^{\infty} t^j [j + 1]_q^n = \frac{A_n(t; q)}{(t; q)_{n+1}}, \quad (1.10)$$

where $[n]_q = (1 - q^n)/(1 - q)$ for any positive integer n . A well-known combinatorial interpretation for $A_n(t; q)$ is given by

$$A_n(t; q) = \sum_{\sigma \in S_n} t^{\text{des } \sigma} q^{\text{maj } \sigma},$$

where S_n is the set of permutations of $\{1, \dots, n\}$ and

$$\text{des } \sigma = \sum_{\sigma(i) > \sigma(i+1)} 1, \quad \text{maj } \sigma = \sum_{\sigma(i) > \sigma(i+1)} i.$$

The first values of the polynomials $A_n(t; q)$ are the following:

$$A_0(t; q) = A_1(t; q) = 1; \quad A_2(t; q) = 1 + tq; \quad A_3(t; q) = 1 + 2tq(q + 1) + t^2q^3.$$

We generalize Liu's result (1.9) as follows.

Theorem 1.2. *For $m \geq 0$ and $|a| \leq 1$, we have*

$$\sum_{n=1}^{\infty} \frac{a[n]_p^m q^n}{1 - aq^n} = \sum_{n=1}^{\infty} \frac{[n]_p^m (1 - aq^{2n}) a^n q^{n^2}}{(1 - q^n)(1 - aq^n)} + \sum_{k=1}^m \binom{m}{k} \sum_{n=1}^{\infty} \frac{[n]_p^{m-k} a^n p^{kn} q^{n^2+n} A_k(q^n; p)}{(q^n; p)_{k+1}}. \quad (1.11)$$

When $p = 1$ the polynomials $A_n(t) := A_n(t; 1)$ are the Eulerian polynomials (see [10, 23]). For example, we have

$$A_1(t) = 1, \quad A_2(t) = 1 + t, \quad A_3(t) = 1 + 4t + t^2, \quad A_4(t) = 1 + 11t + 11t^2 + t^3.$$

Clearly, the above theorem reduces to the following result for $p = 1$.

Corollary 1.3. For $m \geq 0$ and $|a| \leq 1$, we have

$$\sum_{n=1}^{\infty} \frac{an^m q^n}{1-aq^n} = \sum_{n=1}^{\infty} \frac{n^m(1-aq^{2n})a^n q^{n^2}}{(1-q^n)(1-aq^n)} + \sum_{k=1}^m \binom{m}{k} \sum_{n=1}^{\infty} \frac{n^{m-k} a^n q^{n^2+n} A_k(q^n)}{(1-q^n)^{k+1}}. \quad (1.12)$$

When $m = 0$, the above theorem reduces to Liu's formula (1.9). However, our proof for (1.11) is quite different from Liu's proof for (1.9).

Dividing both sides of (1.12) by a and setting $a = 0$ we obtain

$$\sum_{n=1}^{\infty} n^m q^n = \frac{q}{1-q} + \sum_{k=1}^m \binom{m}{k} \frac{q^2 A_k(q)}{(1-q)^{k+1}}.$$

This is equivalent to the $q = 1$ case of (1.10) by using the symmetry of the Eulerian polynomials $t^{n-1} A_n(t^{-1}) = A_n(t)$ ($n \geq 1$) and the recurrence relation

$$A_0(t) = 1, \quad A_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} (t-1)^{n-k-1} A_k(t).$$

In the next section we shall first give a simple proof of Theorem 1.1 using the q -series theory and then derive some consequences and variations. We will prove Theorem 1.2 in Section 3 and give a finite analogue of Uchimura's variation when $a = 1$ along with a connection between partitions and the number of divisors. Finally we give new generalizations of (1.4)–(1.6) in Section 4.

2. Proof of Theorem 1.1 and some consequences

2.1. Proof of Theorem 1.1

Applying the q -binomial theorem [3, (3.3.6)–(3.3.7)]:

$$(x; q)_N = \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}} x^j, \quad (2.1)$$

$$\frac{1}{(x; q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix}_q x^j, \quad (2.2)$$

we have

$$\begin{aligned}
\sum_{k=0}^m \frac{(-1)^k p^{\binom{k+1}{2} - rk}}{(p; p)_k (p; p)_{m-k} (p^k x; q)_{n+1}} &= \sum_{k=0}^m \frac{(-1)^k p^{\binom{k+1}{2} - rk}}{(p; p)_k (p; p)_{m-k}} \sum_{j=0}^{\infty} \begin{bmatrix} n+j \\ j \end{bmatrix}_q (p^k x)^j \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} n+j \\ j \end{bmatrix}_q \frac{x^j}{(p; p)_m} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_p (-1)^k p^{\binom{k}{2} + (j-r+1)k} \\
&= \frac{1}{(p; p)_m} \sum_{j=0}^{\infty} \begin{bmatrix} n+j \\ j \end{bmatrix}_q x^j (p^{j-r+1}; p)_m \\
&= \frac{1}{(p; p)_m} \sum_{j=0}^{\infty} \begin{bmatrix} n+r+j \\ r+j \end{bmatrix}_q x^{r+j} (p^{j+1}; p)_m \\
&= \sum_{j=0}^{\infty} x^{r+j} \begin{bmatrix} m+j \\ m \end{bmatrix}_p \begin{bmatrix} n+r+j \\ n \end{bmatrix}_q. \tag{2.3}
\end{aligned}$$

Similarly, we have

$$\sum_{k=0}^n \frac{(-1)^k q^{\binom{k+1}{2} + rk} x^r}{(q; q)_k (q; q)_{n-k} (q^k x; p)_{m+1}} = \frac{x^r}{(q; q)_n} \sum_{j=0}^{\infty} \begin{bmatrix} m+j \\ j \end{bmatrix}_p x^j (q^{j+r+1}; q)_n,$$

which is equal to the right-hand side of (2.3). This completes the proof. \square

2.2. An infinite version

Letting $r = 0$ in (1.8), we obtain the following symmetric bibasic transformation formula.

Corollary 2.1. *For $m, n \geq 0$, there holds*

$$\sum_{k=0}^m \frac{(-1)^k p^{\binom{k+1}{2}}}{(p; p)_k (p; p)_{m-k} (xp^k; q)_{n+1}} = \sum_{k=0}^n \frac{(-1)^k q^{\binom{k+1}{2}}}{(q; q)_k (q; q)_{n-k} (xq^k; p)_{m+1}}. \tag{2.4}$$

In fact, we have the following infinite form of (2.4).

Theorem 2.2. *Let $|a|, |b|, |p|, |q| < 1$. Then*

$$\frac{(a; p)_{\infty}}{(p; p)_{\infty}} \sum_{k=0}^{\infty} \frac{(p/a; p)_k (x p^k; q)_{\infty}}{(p; p)_k (x p^k; q)_{\infty}} a^k = \frac{(b; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q/b; q)_k (x a q^k; p)_{\infty}}{(q; q)_k (x q^k; p)_{\infty}} b^k, \tag{2.5}$$

where $(a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n$.

Note that (2.5) may be deemed a bibasic extension of the second iteration of the Heine transformation. (The $p = q$ case can be seen to reduce to a special case of [13, Appendix (III.2)].)

Proof. Using the q -binomial theorem [3, Theorem 2.1]:

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}},$$

we can write the left-hand side of (2.5) as

$$\begin{aligned} \frac{(a; p)_{\infty}}{(p; p)_{\infty}} \sum_{k=0}^{\infty} \frac{(p/a; p)_k}{(p; p)_k} a^k \sum_{j=0}^{\infty} \frac{(b; q)_j}{(q; q)_j} (p^k x)^j &= \frac{(a; p)_{\infty}}{(p; p)_{\infty}} \sum_{j=0}^{\infty} \frac{(b; q)_j}{(q; q)_j} x^j \sum_{k=0}^{\infty} \frac{(p/a; p)_k}{(p; p)_k} (ap^j)^k \\ &= \frac{(a; p)_{\infty}}{(p; p)_{\infty}} \sum_{j=0}^{\infty} \frac{(b; q)_j (p^{j+1}; p)_{\infty}}{(q; q)_j (ap^j; p)_{\infty}} x^j \\ &= \sum_{j=0}^{\infty} \frac{(b; q)_j (a; p)_j}{(q; q)_j (p; p)_j} x^j. \end{aligned} \quad (2.6)$$

Therefore, by symmetry, both sides of (2.5) are equal to (2.6). \square

When $a = p^{m+1}$ and $b = q^{n+1}$, the identity (2.5) reduces to (2.4). Letting $p = q^2$, $r = 0$, $x = q^2$ and $m, n \rightarrow \infty$ in (1.8), we obtain

Corollary 2.3. *There holds*

$$\sum_{k=0}^{\infty} (-1)^k (q; q^2)_{k+1} q^{k(k+1)} = \sum_{k=0}^{\infty} \frac{q^{(2k+1)k}}{(q; q^2)_k} - \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{(2k+1)(k+1)}}{(q^2; q^2)_k}.$$

Writing (1.8) as

$$\begin{aligned} \sum_{k=1}^m \frac{(-1)^k p^{\binom{k+1}{2} - rk}}{(p; p)_k (p; p)_{m-k} (p^k x; q)_{n+1}} - \sum_{k=1}^n \frac{x^r (-1)^k q^{\binom{k+1}{2} + rk}}{(q; q)_k (q; q)_{n-k} (q^k x; p)_{m+1}} \\ = \frac{x^r (p; p)_m (xq; q)_n - (q; q)_n (xp; p)_m}{(p; p)_m (q; q)_n (xp; p)_m (xq; q)_n (1-x)}, \end{aligned} \quad (2.7)$$

and letting $x \rightarrow 1$ in (2.7) and applying l'Hôpital's rule, we are led to the following result:

Corollary 2.4. *For $m, n \geq 0$ and $0 \leq r \leq m$, there holds*

$$\begin{aligned} \sum_{k=1}^m \frac{(-1)^k p^{\binom{k+1}{2} - rk}}{(p; p)_k (p; p)_{m-k} (p^k; q)_{n+1}} - \sum_{k=1}^n \frac{(-1)^k q^{\binom{k+1}{2} + rk}}{(q; q)_k (q; q)_{n-k} (q^k; p)_{m+1}} \\ = \frac{1}{(p; p)_m (q; q)_n} \left(-r + \sum_{k=1}^n \frac{q^k}{1-q^k} - \sum_{k=1}^m \frac{p^k}{1-p^k} \right). \end{aligned} \quad (2.8)$$

Note that, by the symmetry of m and n , the identity (2.8) also holds for $-n \leq r \leq 0$. In particular, when $m = n$ and $p = q$, we have

Corollary 2.5. For $n \geq 0$ and $|r| \leq n$, there holds

$$\sum_{k=1}^n \frac{(-1)^{k-1} q^{\binom{k+1}{2} - rk} (1 - q^{2rk})}{(q; q)_k (q; q)_{n-k} (q^k; q)_{n+1}} = \frac{r}{(q; q)_n^2}.$$

Taking $r = 0$ in (2.8), we obtain the following bibasic generalization of [16, Corollary 6.4].

Corollary 2.6. For $m, n \geq 0$, there holds

$$\begin{aligned} & \sum_{k=1}^m \frac{(-1)^k p^{\binom{k+1}{2}}}{(p; p)_k (p; p)_{m-k} (p^k; q)_{n+1}} - \sum_{k=1}^n \frac{(-1)^k q^{\binom{k+1}{2}}}{(q; q)_k (q; q)_{n-k} (q^k; p)_{m+1}} \\ &= \frac{1}{(p; p)_m (q; q)_n} \left(\sum_{k=1}^n \frac{q^k}{1 - q^k} - \sum_{k=1}^m \frac{p^k}{1 - p^k} \right). \end{aligned} \quad (2.9)$$

Letting $p = q^2$ and $m = n$ in (2.9), we have

Corollary 2.7. For $n \geq 0$, there holds

$$\begin{aligned} & \sum_{k=1}^n \frac{(-1)^k q^{k(k+1)}}{(q^2; q^2)_k (q^2; q^2)_{n-k} (q^{2k}; q)_{n+1}} - \sum_{k=1}^n \frac{(-1)^k q^{\binom{k+1}{2}}}{(q; q)_k (q; q)_{n-k} (q^k; q^2)_{n+1}} \\ &= \frac{1}{(q^2; q^2)_n (q; q)_n} \sum_{k=1}^n \frac{q^k}{1 - q^{2k}}. \end{aligned} \quad (2.10)$$

Letting n tend to ∞ in (2.10), we obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{(-1)^k (q; q^2)_k q^{k(k+1)}}{1 - q^{2k}} - \sum_{k=1}^{\infty} \frac{q^{(2k+1)k}}{(q; q^2)_k (1 - q^{2k})} + \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{(2k+1)(k+1)}}{(q^2; q^2)_k (1 - q^{2k+1})} \\ &= \sum_{k=1}^{\infty} \frac{q^k}{1 - q^{2k}}. \end{aligned}$$

Note that (see [12, (3.1)]) the coefficient of q^n in

$$\sum_{k=1}^{\infty} \frac{q^k}{1 - q^{2k}} = \sum_{k=1}^{\infty} \frac{q^{2k-1}}{1 - q^{2k-1}} = q + q^2 + 2q^3 + q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + 3q^9 + \dots$$

counts the number of (positive) odd divisors of n .

3. Proof of Theorem 1.2 and some variations

3.1. Proof of Theorem 1.2

Assume that $|aq| < 1$. As $[n+j]_p = [n]_p + p^n[j]_p$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a[n]_p^m q^n}{1-aq^n} &= \sum_{n \geq 1} [n]_p^m \sum_{k \geq 1} a^k q^{nk} \\
&= \sum_{k \geq n \geq 1} [n]_p^m a^k q^{nk} + \sum_{n > k \geq 1} [n]_p^m a^k q^{nk} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} [n]_p^m a^{n+k} q^{n(n+k)} + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} [n+j]_p^m a^n q^{n(n+j)} \\
&= \sum_{n=1}^{\infty} [n]_p^m \sum_{k=0}^{\infty} a^{n+k} q^{n(n+k)} + \sum_{n=1}^{\infty} [n]_p^m \sum_{j=1}^{\infty} a^n q^{n(n+j)} \\
&\quad + \sum_{k=1}^m \binom{m}{k} [n]_p^{m-k} \sum_{n=1}^{\infty} a^n p^{nk} \sum_{j=1}^{\infty} [j]_p^k q^{n(n+j)}. \tag{3.1}
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k=0}^{\infty} a^{n+k} q^{n(n+k)} + \sum_{j=1}^{\infty} a^n q^{n(n+j)} &= \frac{a^n q^{n^2}}{1-aq^n} + \frac{a^n q^{n^2+n}}{1-q^n} \\
&= \frac{(1-aq^{2n})a^n q^{n^2}}{(1-q^n)(1-aq^n)}, \tag{3.2}
\end{aligned}$$

and, by (1.10),

$$\sum_{j=1}^{\infty} [j]_p^k q^{n(n+j)} = q^{n^2+n} \sum_{j=1}^{\infty} [j]_p^k q^{n(j-1)} = \frac{q^{n^2+n} A_k(q^n; p)}{(q^n; p)_{k+1}}. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we complete the proof of (1.11).

Proposition 3.1. *For $m \geq 0$ and $|a| \leq 1$, we have*

$$\sum_{n=1}^{\infty} \binom{n}{m} \frac{aq^n}{1-aq^n} = \sum_{n=1}^{\infty} \binom{n}{m} \frac{(1-aq^{2n})a^n q^{n^2}}{(1-q^n)(1-aq^n)} + \sum_{k=1}^m \sum_{n=1}^{\infty} \binom{n}{m-k} \frac{a^n q^{n(n+k)}}{(1-q^n)^{k+1}}. \tag{3.4}$$

Proof. Similarly to the proof of Theorem 1.2, applying the Chu-Vandermonde convolution formula

$$\binom{n+j}{m} = \sum_{k=0}^m \binom{n}{m-k} \binom{j}{k},$$

and the binomial theorem

$$\sum_{j=k}^{\infty} \binom{j}{k} q^j = \frac{q^k}{(1-q)^{k+1}},$$

we can prove (3.4). □

3.2. A refinement of Uchimura's variation

In fact, Liu's formula (1.9) is a special case of Agarwal's identity, see [1, 6]:

$$\sum_{n=0}^{\infty} \frac{t^n}{1-xq^n} = \sum_{n=0}^{\infty} \frac{(1-xtq^{2n})x^n t^n q^{n^2}}{(1-xq^n)(1-tq^n)}.$$

Another variation of (1.9) when $a = 1$ is Uchimura's identity [25]:

$$\sum_{k \geq 1} \frac{q^k}{1-q^k} = \sum_{m=1}^{+\infty} m q^m \prod_{j=m+1}^{+\infty} (1-q^j). \quad (3.5)$$

For $n \in \mathbb{N}$ let $d(n)$ be the number of positive divisors of n . A partition of n is a non decreasing sequence of positive integers $\pi = (n_1, n_2, \dots, n_l)$ such that $n_1 \geq n_2 \geq \dots \geq n_l > 0$ and $n = n_1 + \dots + n_l$. The number of parts l is called the length of π . The largest part n_1 is denoted by $g(\pi)$ and the smallest part n_l is denoted by $s(\pi)$. A partition into distinct parts is called odd (resp. even) if its length is odd (resp. even). Let $t(n)$ be the sum of smallest parts of odd partitions of n minus the sum of smallest parts of even partitions of n . Bressoud and Subbarao [7] noticed that Uchimura's identity is equivalent to

$$d(n) = t(n), \quad (3.6)$$

and gave a combinatorial proof of (3.5). It is interesting to note that (3.6) was rediscovered by Wang et al. [27]. We can give a further refinement of (3.6).

Let $d(n, N)$ be the number of divisors $\leq N$ of n . Let $\mathcal{P}(n, N)$ be the set of partitions π of n into distinct parts such that $g(\pi) - s(\pi) \leq N - 1$. Let $t(n, N)$ be the sum of smallest parts of odd partitions of $\mathcal{P}(n, N)$ minus the sum of smallest parts of even partitions of $\mathcal{P}(n, N)$. We have the following refinement of (3.6).

Theorem 3.2. *For all natural numbers n and N , there holds*

$$d(n, N) = t(n, N) - t(n - N, N). \quad (3.7)$$

For example, when $n = 9$ and $N = 3$, we have

$$\mathcal{P}(9, 3) = \{(9), (5, 4), (4, 3, 2)\}; \quad \mathcal{P}(6, 3) = \{(6), (4, 2), (3, 2, 1)\},$$

and so $t(9, 3) - t(6, 3) = (9 + 2 - 4) - (6 + 1 - 2) = 2$, which is the number of divisors of 9 less than or equal to 3.

Proof. Rewrite (2.2) as follows:

$$\frac{1}{(zq; q)_N} = \sum_{m=0}^{\infty} \frac{(q^{m+1}; q)_{N-1}}{(q; q)_{N-1}} z^m q^m.$$

Differentiating the above identity with respect to z , we obtain

$$\frac{1}{(zq; q)_N} \sum_{k=1}^N \frac{q^k}{1 - zq^k} = \sum_{m=0}^{\infty} \frac{m(q^{m+1}; q)_{N-1}}{(q; q)_{N-1}} z^{m-1} q^m.$$

Multiplying the two sides by $(zq; q)_N$ and setting $z = 1$, we get

$$\sum_{k=1}^N \frac{q^k}{1 - q^k} = \sum_{m=1}^{\infty} m q^m (q^{m+1}; q)_{N-1} - \sum_{m=1}^{\infty} m q^{m+N} (q^{m+1}; q)_{N-1}. \quad (3.8)$$

The proof then follows from extracting the coefficients of q^n in both sides of (3.8). \square

Remark 3.3. *It is possible to give a combinatorial proof of (3.7) by generalizing Bressoud and Subarao's combinatorial proof of (3.6).*

4. Generalizations of (1.4), (1.5) and (1.6)

4.1. Two generalizations of Uchimura's identity

In this section, we give two generalizations of Uchimura's identity (1.4).

Theorem 4.1. *For $m, n \geq 0$ and $0 \leq r \leq m$, there holds*

$$\begin{aligned} & \sum_{k=1}^m \frac{(-1)^k q^{\binom{k+1}{2} - rk}}{(q; q)_k (q; q)_{m-k} (xq^k; q)_{n+1}} - x^r \sum_{k=1}^n \frac{(-1)^k q^{\binom{k+1}{2} + rk}}{(q; q)_k (q; q)_{n-k} (xq^k; q)_{m+1}} \\ &= \frac{1}{(q; q)_m (q; q)_n} \left(-\frac{1 - x^r}{1 - x} + \sum_{k=1}^n \frac{q^k (q; q)_{k-1}}{(xq; q)_k} - x^r \sum_{k=1}^m \frac{q^k (q; q)_{k-1}}{(xq; q)_k} \right). \end{aligned} \quad (4.1)$$

Proof. When $p = q$, the identity (1.8) may be rewritten as

$$\begin{aligned} & \sum_{k=1}^m \frac{(-1)^k q^{\binom{k+1}{2} - rk}}{(q; q)_k (q; q)_{m-k} (xq^k; q)_{n+1}} - x^r \sum_{k=1}^n \frac{(-1)^k q^{\binom{k+1}{2} + rk}}{(q; q)_k (q; q)_{n-k} (xq^k; q)_{m+1}} \\ &= \frac{x^r}{(q; q)_n (x; q)_{m+1}} - \frac{1}{(q; q)_m (x; q)_{n+1}}. \end{aligned} \quad (4.2)$$

On the other hand, since

$$\sum_{k=1}^n \frac{q^k (q; q)_{k-1}}{(xq; q)_k} = \sum_{k=1}^n \frac{1}{1 - x} \left(\frac{(q; q)_{k-1}}{(xq; q)_{k-1}} - \frac{(q; q)_k}{(xq; q)_k} \right) = \frac{1}{1 - x} \left(1 - \frac{(q; q)_n}{(xq; q)_n} \right),$$

we have

$$\begin{aligned} \sum_{k=1}^n \frac{q^k (q; q)_{k-1}}{(xq; q)_k} - x^r \sum_{k=1}^m \frac{q^k (q; q)_{k-1}}{(xq; q)_k} &= \frac{1-x^r}{1-x} + \frac{1}{1-x} \left(\frac{x^r (q; q)_m}{(xq; q)_m} - \frac{(q; q)_n}{(xq; q)_n} \right) \\ &= \frac{1-x^r}{1-x} + \frac{x^r (q; q)_m}{(x; q)_{m+1}} - \frac{(q; q)_n}{(x; q)_{n+1}}. \end{aligned} \quad (4.3)$$

Combining (4.2) and (4.3), we complete the proof of Theorem 4.1. \square

Theorem 4.2. *For $m, n \geq 0$, there holds*

$$\begin{aligned} \sum_{k=1}^m \frac{(-1)^k q^{nk + \binom{k+1}{2}}}{(q; q)_k (q; q)_{m-k} (xq^k; q)_{n+1}} - \sum_{k=1}^n \frac{(-1)^k q^{mk + \binom{k+1}{2}}}{(q; q)_k (q; q)_{n-k} (xq^k; q)_{m+1}} \\ = \frac{1}{(q; q)_m (q; q)_n} \left(\sum_{k=1}^n \frac{q^k (q; q)_{k-1}}{(xq; q)_k} - \sum_{k=1}^m \frac{q^k (q; q)_{k-1}}{(xq; q)_k} \right). \end{aligned} \quad (4.4)$$

Proof. Letting $v \rightarrow \infty$ and $z \rightarrow 0$ in the following identity (see [15, 16])

$$\sum_{k=0}^n \frac{(q/z; q)_k (vq^m; q)_k (z; q)_{n-k} (xz; q)_m}{(q; q)_k (v; q)_k (q; q)_{n-k} (xq^k; q)_{m+1}} z^k = \sum_{k=0}^m \frac{(q/z; q)_k (vq^n; q)_k (z; q)_{m-k} (xz; q)_n}{(q; q)_k (v; q)_k (q; q)_{m-k} (xq^k; q)_{n+1}} z^k,$$

we have

$$\begin{aligned} \sum_{k=1}^m \frac{(-1)^k q^{nk + \binom{k+1}{2}}}{(q; q)_k (q; q)_{m-k} (xq^k; q)_{n+1}} - \sum_{k=1}^n \frac{(-1)^k q^{mk + \binom{k+1}{2}}}{(q; q)_k (q; q)_{n-k} (xq^k; q)_{m+1}} \\ = \frac{1}{(q; q)_n (x; q)_{m+1}} - \frac{1}{(q; q)_m (x; q)_{n+1}}. \end{aligned}$$

The proof of Theorem 4.2 then follows from the $r = 0$ case of (4.3). \square

It is easy to see that, when $m = 0$ and $x = q^m$, both (4.1) (with $r = 0$) and (4.4) reduce to (1.4).

4.2. A generalization of Prodinger's identity

In this section, we give a generalization of Prodinger's identity (1.6).

Theorem 4.3. *For $n \geq 0$ and $0 \leq m, r \leq n$, there holds*

$$\sum_{\substack{k=0 \\ k \neq m}}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2} - rk}}{1 - q^{k-m}} = (-1)^m q^{\binom{m+1}{2} - rm} \begin{bmatrix} n \\ m \end{bmatrix}_q \left(r + \sum_{\substack{k=0 \\ k \neq m}}^n \frac{q^{k-m}}{1 - q^{k-m}} \right). \quad (4.5)$$

Proof. From the partial fraction decomposition

$$\sum_{k=0}^n \frac{(-1)^k q^{\binom{k+1}{2} - rk}}{(q; q)_k (q; q)_{n-k} (1 - xq^k)} = \frac{x^r}{(x; q)_{n+1}},$$

which can also be obtained by letting $n = 0$ and then replacing (m, p) by (n, q) in (1.8), we derive that

$$\begin{aligned} & \sum_{\substack{k=0 \\ k \neq m}}^n \frac{(-1)^k q^{\binom{k+1}{2} - rk}}{(q; q)_k (q; q)_{n-k} (1 - xq^k)} \\ &= \frac{x^r}{(x; q)_{n+1}} - \frac{(-1)^m q^{\binom{m+1}{2} - rm}}{(q; q)_m (q; q)_{n-m} (1 - xq^m)} \\ &= \frac{x^r (q; q)_m (q; q)_{n-m} - (-1)^m q^{\binom{m+1}{2} - rm} (x; q)_m (xq^{m+1}; q)_{n-m}}{(1 - xq^m) (x; q)_m (xq^{m+1}; q)_{n-m} (q; q)_m (q; q)_{n-m}}. \end{aligned} \quad (4.6)$$

Letting $x \rightarrow q^{-m}$ in (4.6) and applying l'Hôpital's rule, we obtain

$$\begin{aligned} & \sum_{\substack{k=0 \\ k \neq m}}^n \frac{(-1)^k q^{\binom{k+1}{2} - rk}}{(q; q)_k (q; q)_{n-k} (1 - q^{k-m})} \\ &= \frac{r q^{-rm} (q; q)_m (q; q)_{n-m} + (-1)^m q^{\binom{m+1}{2} - rm} (q^{-m}; q)_m (q; q)_{n-m} \sum_{k=0, k \neq m}^n \frac{q^{k-m}}{1 - q^{k-m}}}{-(q^{-m}; q)_m (q; q)_{n-m} (q; q)_m (q; q)_{n-m}}, \end{aligned} \quad (4.7)$$

which is equivalent to (4.5). \square

Letting $m = 0$ in (4.5), we get the following generalization of Van Hamme's identity (1.3).

Corollary 4.4. *For $n \geq 0$ and $0 \leq r \leq n$, there holds*

$$\sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2} - rk}}{1 - q^k} = r + \sum_{k=1}^n \frac{q^k}{1 - q^k}. \quad (4.8)$$

Letting $n \rightarrow \infty$ in (4.8), we obtain

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{\binom{k+1}{2} - rk}}{(q; q)_k (1 - q^k)} = r + \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k}, \quad r \geq 0,$$

which is a generalization of Uchimura's identity (1.2).

4.3. A new generalization of Dilcher's identity

Here we shall give a new generalization of Dilcher's identity (1.5).

Theorem 4.5. *For $m, n \geq 1$, and $0 \leq r \leq m + n - 1$, there holds*

$$\begin{aligned}
& \sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2} + k(m-r)}}{(1-q^k)^m} \\
&= \binom{r}{m} + \binom{r}{m-1} \sum_{k_1=1}^n \frac{q^{k_1}}{1-q^{k_1}} + \binom{r}{m-2} \sum_{1 \leq k_1 \leq k_2 \leq n} \frac{q^{k_1+k_2}}{(1-q^{k_1})(1-q^{k_2})} \\
&+ \cdots + \binom{r}{0} \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \frac{q^{k_1+\cdots+k_m}}{(1-q^{k_1}) \cdots (1-q^{k_m})}. \tag{4.9}
\end{aligned}$$

Proof. We proceed by induction on m and on n . For $m = 1$, the identity (4.9) reduces to (4.8). Suppose that (4.9) is true for some $m \geq 1$. We need to show that it is also true for $m + 1$, namely,

$$\begin{aligned}
& \sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2} + k(m+1-r)}}{(1-q^k)^{m+1}} \\
&= \binom{r}{m+1} + \binom{r}{m} \sum_{k_1=1}^n \frac{q^{k_1}}{1-q^{k_1}} + \binom{r}{m-1} \sum_{1 \leq k_1 \leq k_2 \leq n} \frac{q^{k_1+k_2}}{(1-q^{k_1})(1-q^{k_2})} \\
&+ \cdots + \binom{r}{0} \sum_{1 \leq k_1 \leq \cdots \leq k_{m+1} \leq n} \frac{q^{k_1+\cdots+k_{m+1}}}{(1-q^{k_1}) \cdots (1-q^{k_{m+1}})}, \quad 0 \leq r \leq m + n. \tag{4.10}
\end{aligned}$$

We shall prove this induction step (4.10) by induction on n , following the proofs in [17] and [9, Theorem 4]. For $n = 1$, the left-hand side of (4.10) is equal to

$$\frac{q^{m+1-r}}{(1-q)^{m+1}},$$

while the right-hand side of (4.10) is given by

$$\sum_{k=0}^{m+1} \binom{r}{m+1-k} \left(\frac{q}{1-q} \right)^k = \left(\frac{q}{1-q} \right)^{m+1-r} \left(1 + \frac{q}{1-q} \right)^r = \frac{q^{m+1-r}}{(1-q)^{m+1}},$$

since $0 \leq r \leq m + 1$. This proves that (4.10) holds for $n = 1$. We now assume that (4.10) holds for $n - 1$. In order to show that it also holds for n , we need to check that the

difference between (4.10) for n and (4.10) for $n - 1$ is a true identity. This difference is

$$\begin{aligned}
& \sum_{k=1}^n (-1)^{k-1} \frac{q^{\binom{k}{2} + k(m+1-r)}}{(1-q^k)^{m+1}} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \right) \\
&= \binom{r}{m} \frac{q^n}{1-q^n} + \binom{r}{m-1} \frac{q^n}{1-q^n} \sum_{1 \leq k_1 \leq n} \frac{q^{k_1}}{1-q^{k_1}} \\
&+ \cdots + \binom{r}{0} \frac{q^n}{1-q^n} \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \frac{q^{k_1 + \cdots + k_m}}{(1-q^{k_1}) \cdots (1-q^{k_m})}. \tag{4.11}
\end{aligned}$$

Applying the relation

$$\frac{q^{\binom{k}{2} + k(m+1-r)}}{1-q^k} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \right) = \frac{q^{\binom{k}{2} + k(m-r) + n}}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

one sees that (4.11) is just the induction hypothesis (4.9). This proves that (4.10) holds for all $n \geq 1$, and consequently (4.9) holds for all $m \geq 1$ and all $n \geq 1$. \square

By the q -binomial theorem (2.1), we have

$$\sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} = 1 \quad (n \geq 1).$$

Hence, by Dilcher's identity (1.5), from (4.9) we deduce the following result.

Corollary 4.6. *For $m, n \geq 1$, and $0 \leq r \leq m + n - 1$, there holds*

$$\sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2} + k(m-r)}}{(1-q^k)^m} = \sum_{j=0}^m \binom{r}{m-j} \sum_{k=1}^n (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2} + kj}}{(1-q^k)^j}.$$

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