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# SYMMETRIC UNIMODAL EXPANSIONS OF EXCEDANCES IN COLORED PERMUTATIONS

HEESUNG SHIN AND JIANG ZENG

ABSTRACT. We consider several generalizations of the classical  $\gamma$ -positivity of Eulerian polynomials (and their derangement analogues) using generating functions and combinatorial theory of continued fractions. For the symmetric group, we prove an expansion formula for inversions and excedances as well as a similar expansion for derangements. We also prove the  $\gamma$ -positivity for Eulerian polynomials for derangements of type  $B$ . More general expansion formulae are also given for Eulerian polynomials for  $r$ -colored derangements. Our results answer and generalize several recent open problems in the literature.

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1. INTRODUCTION AND MAIN RESULTS IN  $\mathfrak{S}_n$ 

Let  $\mathfrak{S}_n$  be the set of permutations of  $[n] = \{1, \dots, n\}$ . For any permutation  $\sigma \in \mathfrak{S}_n$ , written as the word  $\sigma = \sigma(1) \dots \sigma(n)$ , the entry  $i \in [n]$  is called a *descent* of  $\sigma$  if  $i < n$  and  $\sigma(i) > \sigma(i+1)$ ; the entry  $i \in [n]$  is called an *excedance* (resp. *drop*, *fixed-point*) of  $\sigma$  if  $i < \sigma(i)$  (resp.  $i > \sigma(i)$ ,  $i = \sigma(i)$ ). Denote the number of descents, excedances, drops, and fixed-points in  $\sigma$  by  $\text{des } \sigma$ ,  $\text{exc } \sigma$ ,  $\text{drop } \sigma$ , and  $\text{fix } \sigma$  respectively. It is well known [FS70] that the statistics  $\text{des}$ ,  $\text{exc}$  and  $\text{drop}$  have the same distribution on  $\mathfrak{S}_n$  and their common enumerative polynomial is the Eulerian polynomial:

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{drop } \sigma}. \quad (1.1)$$

Define the set  $\text{DD}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{des } \sigma = k \text{ and } \text{dd}(\sigma 0) = 0\}$ , where  $\text{dd}(\sigma 0)$  is the number of *double descents* in the word  $\sigma(1) \dots \sigma(n)0$ , i.e., indices  $i$ ,  $1 < i \leq n$ , such that  $\sigma(i-1) > \sigma(i) > \sigma(i+1)$  with  $\sigma(n+1) = 0$ . For example,

$$\text{DD}_{4,1} = \{1324, 1423, 2314, 2413, 3412, 2134, 3124, 4123\}.$$

Foata and Schützenberger [FS70] proved the following expansion formula

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} |\text{DD}_{n,k}| t^k (1+t)^{n-1-2k}. \quad (1.2)$$

A finite sequence of positive integers  $a_0, \dots, a_n$  is *unimodal* if there exists an index  $i'$  such that  $a_i \leq a_{i+1}$  if  $i < i'$  and  $a_i > a_{i+1}$  otherwise; it is *log-concave* if  $a_i^2 \geq a_{i-1}a_{i+1}$  for all  $i \in [1, n-1]$ . A polynomial  $p(x) = a_0 + a_1x + \dots + a_nx^n$  with nonnegative coefficients is unimodal (resp. log-concave) if and only if the sequence of its coefficients is unimodal (resp. log-concave). It is well known that a polynomial with nonnegative coefficients and with only *real roots* is log-concave and that log-concavity implies the unimodality. The set of polynomials whose coefficients are symmetric with center of symmetry  $d/2$  is a vector space with a basis given by  $\{t^k(1+t)^{d-2k}\}_{k=0}^{\lfloor d/2 \rfloor}$ . If a polynomial  $p(t)$  has nonnegative coefficients when expanded in this basis, then the coefficients in the standard basis  $\{t^k\}_{k=0}^{\infty}$  form a unimodal sequence (being a nonnegative sum of symmetric and unimodal sequences with the same center of symmetry). Hence the expansion (1.2) implies both the symmetry and the unimodality of the Eulerian numbers.

Analogues of (1.2) for types  $B$  and  $D$  were given by Petersen [Pet07], Stembridge [Ste08] and Chow [Cho08]. In the recent years several  $q$ -analogues of this formula have been established in [SW10, SZ12, HJZ13, LZ14] by combining one of the Eulerian statistics in (1.1) and one of the two classical Mahonian statistics, i.e., inversion numbers and Major index. Recall that for  $\sigma \in \mathfrak{S}_n$  the inversion number  $\text{inv } \sigma$  is the number of pairs  $(i, j)$  such that  $i < j$  and  $\sigma(i) > \sigma(j)$  and the major index  $\text{maj } \sigma$  is the sum of the descent positions. In [SZ12] we have shown that some symmetric unimodal expansion formulas of Eulerian polynomials follow easily from the combinatorics of the continued fraction expansions of Jacobi type of their ordinary generating functions. In this paper, we shall prove an *INV*  $q$ -analogue of (1.2) as well as some expansions for the Eulerian polynomials of type  $B$  by

$n \setminus k$	0	1	2
1	1		
2	1		
3	1	$q(1+q)$	
4	1	$q(1+q)(q^2+q+2)$	
5	1	$q(1+q)(q^4+2q^3+3q^2+2q+3)$	$q^2(1+q)^2(q^4+q^3+q^2+1)$

TABLE 1. The values of  $\gamma_{n,k}(q)$  for  $1 \leq k \leq n \leq 5$

an analogue study of the correspondence between signed permutations and the weighted lattice paths.

For  $\sigma \in \mathfrak{S}_n$ , the statistic  $(31-2)\sigma$  (resp.  $(13-2)\sigma$ ) is the number of pairs  $(i, j)$  such that  $2 \leq i < j \leq n$  and  $\sigma(i-1) > \sigma(j) > \sigma(i)$  (resp.  $\sigma(i-1) < \sigma(j) < \sigma(i)$ ). Similarly, the statistic  $(2-13)\sigma$  (resp.  $(2-31)\sigma$ ) is the number of pairs  $(i, j)$  such that  $1 \leq i < j \leq n-1$  and  $\sigma(j+1) > \sigma(i) > \sigma(j)$  (resp.  $\sigma(j+1) < \sigma(i) < \sigma(j)$ ).

**Theorem 1.** *We have*

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma} q^{\text{inv } \sigma - \text{exc } \sigma} = \sum_{0 \leq k \leq (n-1)/2} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k}, \quad (1.3)$$

where the coefficient  $\gamma_{n,k}(q)$  has the following combinatorial interpretation

$$\gamma_{n,k}(q) = \sum_{\sigma \in \text{DD}_{n,k}} q^{2(2-13)\sigma + (31-2)\sigma}$$

and is divisible by  $q^k(1+q)^k$  for  $0 \leq k \leq (n-1)/2$ .

*Remark.* The first values of  $\gamma_{n,k}(q)$  in Table 1 are obtained by replacing  $p = q^2$  in [SZ12, Table of  $a_{n,k}(p, q)$  in Appendix].

There is a derangement analogue of (1.3). An element  $\sigma \in \mathfrak{S}_n$  is called a *derangement* if it has no fixed point. Denote by  $\mathfrak{D}_n$  the set of all derangements in  $\mathfrak{S}_n$  and let

$$\text{DE}_{n,k} := \{ \sigma \in \mathfrak{D}_n : \text{exc } \sigma = k \text{ and } \text{cda}(\sigma) = 0 \},$$

where  $\text{cda}(\sigma)$  is the number of *cyclic double ascents* (or *double excedances*) of  $\sigma$ , i.e., indices  $i \in [n]$  such that  $i < \sigma(i) < \sigma^2(i)$ . For example, we have  $\text{DE}_{4,1} = \{4123\} = \{(4321)\}$  and

$$\text{DE}_{4,2} = \{2143, 3412, 4321, 4312, 3421\} = \{(21)(43), (31)(42), (41)(32), (4231), (4132)\}.$$

**Theorem 2.** *For  $n \geq 1$  we have*

$$\sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv } \sigma} t^{\text{exc } \sigma} = \sum_{0 \leq k \leq n/2} \left( \sum_{\sigma \in \text{DE}_{n,k}} q^{\text{inv } \sigma} \right) t^k (1+t)^{n-2k}. \quad (1.4)$$

*Remark.* The  $t = -1$  case of Theorems 1 and 2 were first proved in [SZ10, Theorem 3] as a  $q$ -analogue of two well-known results connecting Eulerian polynomials (resp. derangement polynomials) to tangent (resp. secant) numbers.

n \ k	0	1	2
0	1		
1	0		
2	0	$q$	
3	0	$q^2$	
4	0	$q^3$	$q^2 + q^4 + 2q^5 + q^6$

TABLE 2. The values of  $\sum q^{\text{inv}\sigma}$  for  $\sigma \in \text{DE}_{n,k}$  and  $1 \leq k \leq n \leq 4$ 

Note that Athanasiadis and Savvidou [AS12] used the  $q = 1$  case of (1.4) to prove the  $\gamma$ -positivity of the local  $h$ -polynomials of certain simplicial divisions. More recently, Athanasiadis and Savvidou [AS13] and Athanasiadis [Ath13] have extended their results to hyperoctahedral group  $B_n$  and colored groups, while Mongelli [Mon13] studied the excedances in affine Weyl groups. Motivated by the aforementioned works, we shall study the similar expansion formula for two kinds of derangement Eulerian polynomials in  $\mathfrak{B}_n$  and wreath product  $\mathbb{Z}_r \wr \mathfrak{S}_n$ . We shall introduce the necessary definitions and present the main results for these derangement polynomials in the next section.

## 2. EXCEDANCES AND DERANGEMENTS IN COLORED PERMUTATIONS

**2.1. Excedances in  $\mathfrak{B}_n$ .** Let  $\mathfrak{B}_n$  be the set of permutations  $\sigma$  of  $\{\pm 1, \dots, \pm n\}$  such that  $\sigma(-i) = -\sigma(i)$  for every  $i \in [n]$ , that is,  $\mathfrak{B}_n = \mathbb{Z}_2 \wr \mathfrak{S}_n$  with  $-i = \bar{i}$ . From Steingrímsson [Ste94, Definition 3], an index  $i \in [n]$  is called an *excedance* of  $\sigma \in \mathfrak{B}_n$  if

$$i <_f \sigma(i)$$

in the *friends order*  $<_f$  of  $\{\pm 1, \dots, \pm n\}$ :

$$1 <_f -1 <_f 2 <_f -2 <_f \dots <_f n <_f -n$$

and  $\text{exc}(\sigma)$  is defined by the number of excedances of  $\sigma \in \mathfrak{B}_n$ . Following Brenti [Bre94, page 431], we say that  $i \in [n]$  is a *B-excedance* of  $\sigma$  if

$$\sigma(i) < \sigma(|\sigma(i)|) \quad \text{or} \quad \sigma(i) = -i$$

in the *natural order*  $<$  of  $\{\pm 1, \dots, \pm n\}$ :

$$-n < \dots < -2 < -1 < 1 < 2 < \dots < n.$$

and the number of *B-excedances* of  $\sigma$  will be denoted by  $\text{exc}_B(\sigma)$ . We also say that  $i \in [0, n-1]$  is a *B-descent* of  $\sigma$  if

$$\sigma(i) > \sigma(i-1)$$

in the natural order, where  $\sigma(0) = 0$ , and the number of *B-descents* of  $\sigma$  will be denoted by  $\text{des}_B(\sigma)$ . From [Ste94, Theorem 20] and [Bre94, eq.(14)] it follows that

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{B}_n} t^{\text{exc}(\sigma)} \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{\sigma \in \mathfrak{B}_n} t^{\text{des}_B(\sigma)} \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{\sigma \in \mathfrak{B}_n} t^{\text{exc}_B(\sigma)} \frac{z^n}{n!} = \frac{(1-t)e^{z(1-t)}}{1-te^{2z(1-t)}}, \quad (2.1)$$

i.e., the two statistics  $\text{exc}$  and  $\text{exc}_B$  are equidistributed on  $\mathfrak{B}_n$ . In 2004, Fire [Fir04, Definition 6.18] introduced the *flag excedance* statistic  $\text{fexc}$  of an element  $\sigma \in \mathfrak{B}_n$  by

$$\begin{aligned} \text{fexc}(\sigma) &= \#\{i \in \{\pm 1, \dots, \pm n\} : i <_c \sigma(i)\} \\ &= 2 \cdot \#\{i \in [n] : i < \sigma(i)\} + \#\{i \in [n] : \sigma(i) < 0\}, \end{aligned}$$

where we use the *color order*  $<_c$  of  $\{\pm 1, \dots, \pm n\}$ :

$$-1 <_c -2 <_c \dots <_c -n <_c 1 <_c 2 <_c \dots <_c n.$$

**2.2. Excedances in wreath product  $\mathbb{Z}_r \wr \mathfrak{S}_n$ .** The wreath product  $\mathbb{Z}_r \wr \mathfrak{S}_n$  is the set of *colored permutations*  $\binom{\pi}{\mathbf{z}}$ , where  $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$  and  $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_n) \in [0, r-1]^n$ . The number  $z_i$  will be considered as the color assigned to  $\pi_i$ . In this generalized symmetric group  $\mathbb{Z}_r \wr \mathfrak{S}_n$ , the product is defined by

$$\binom{\tau}{\mathbf{w}} \binom{\pi}{\mathbf{z}} = \binom{\tau\pi}{\pi(\mathbf{w}) + \mathbf{z}},$$

where the composition  $\tau\pi = \tau \circ \pi$  in  $\mathfrak{S}_n$  is composed from right to left,

$$\pi(\mathbf{w}) = (\mathbf{w}_{\pi_1}, \mathbf{w}_{\pi_2}, \dots, \mathbf{w}_{\pi_n}),$$

and the addition is summed modulo  $r$  in individual coordinate.

From now, for  $\sigma = \sigma(1) \dots \sigma(n) = \binom{\pi_1 \dots \pi_n}{\mathbf{z}_1 \dots \mathbf{z}_n} \in \mathbb{Z}_r \wr \mathfrak{S}_n$ , we write

$$\sigma(i) = \binom{\pi_i}{\mathbf{z}_i} \text{ as } \pi_i^{[\mathbf{z}_i]}, \quad (2.2)$$

while  $\pi_i$  (resp.  $\mathbf{z}_i$ ) is called the *number* (resp. the *color*) of  $\pi_i^{[\mathbf{z}_i]}$ , denote by

$$\pi_i = |\pi_i^{[\mathbf{z}_i]}| \text{ and } \mathbf{z}_i = \text{col}(\pi_i^{[\mathbf{z}_i]}).$$

Conventionally, we use also the window notation  $i = i^{[0]}$ ,  $\bar{i} = i^{[1]}$ , and  $\bar{\bar{i}} = i^{[2]}$ . For an example

$$\sigma = \binom{4 \ 7 \ 2 \ 5 \ 1 \ 6 \ 3}{0 \ 1 \ 0 \ 1 \ 2 \ 0 \ 0} = 4 \ \bar{7} \ 2 \ \bar{\bar{5}} \ \bar{\bar{1}} \ 6 \ 3 \in \mathbb{Z}_3 \wr \mathfrak{S}_7,$$

we have  $\binom{1}{2} = 1^{[2]} = \bar{\bar{1}}$ ,  $\binom{7}{1} = 7^{[1]} = \bar{7}$ ,  $|\bar{\bar{5}}| = 5$ , and  $\text{col}(\bar{\bar{5}}) = 1$ . So the wreath product  $\mathbb{Z}_r \wr \mathfrak{S}_n$  could be considered the set of  $r$ -colored permutations  $\sigma$  of the alphabet  $\Sigma$  of  $rn$  letters

$$\Sigma := \{1, \bar{1}, \bar{\bar{1}}, \dots, 1^{[r-1]}, 2, \bar{2}, \bar{\bar{2}}, \dots, 2^{[r-1]}, \dots, n, \bar{n}, \bar{\bar{n}}, \dots, n^{[r-1]}\},$$

such that  $\sigma(\bar{i}) = \overline{\sigma(i)}$  for all  $i \in \Sigma$ . From Steingrímsson [Ste94], an index  $i \in [n]$  is called an *excedance* of  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$  if  $i <_f \sigma(i)$  in the *friends order*  $<_f$  of  $\Sigma$ :

$$1 <_f \bar{1} <_f \dots <_f 1^{[r-1]} <_f 2 <_f \bar{2} <_f \dots <_f 2^{[r-1]} <_f \dots <_f n <_f \bar{n} <_f \dots <_f n^{[r-1]}.$$

Let  $\text{exc}(\sigma)$  be the number of excedances of  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ . Bagno and Garber [BG06, Definition 3.4] define the *flag excedance* statistic  $\text{fexc}$  of an element  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$  by

$$\begin{aligned} \text{fexc}(\sigma) &= \#\{i \in \Sigma : i <_c \sigma(i)\} \\ &= r \cdot \#\{i \in [n] : i <_c \sigma(i)\} + \sum_{i \in [n]} \text{col}(\sigma(i)), \end{aligned}$$

where we use the *color order*  $<_c$  of  $\Sigma$ :

$$1^{[r-1]} <_c 2^{[r-1]} <_c \dots <_c n^{[r-1]} <_c \dots <_c \bar{1} <_c \bar{2} <_c \dots <_c \bar{n} <_c 1 <_c 2 <_c \dots <_c n.$$

**2.3. Derangements in  $\mathbb{Z}_r \wr \mathfrak{S}_n$ .** An element  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$  is called a *derangement* if it has no fixed point, that is,  $\sigma(i) \neq i$  for all  $i \in \Sigma$ . Denote by  $\mathfrak{D}_n^{(r)}$  the set of all derangements in  $\mathbb{Z}_r \wr \mathfrak{S}_n$ . The two corresponding derangement Eulerian polynomials on  $\mathfrak{D}_n^{(r)}$  are defined by

$$D_n^{(r)}(t) = \sum_{\sigma \in \mathfrak{D}_n^{(r)}} t^{\text{fexc}(\sigma)} = \sum_{k \geq 0} D_{n,k}^{(r)} t^k \quad (2.3)$$

and

$$d_n^{(r)}(t) = \sum_{\sigma \in \mathfrak{D}_n^{(r)}} t^{\text{exc}(\sigma)} = \sum_{k \geq 0} d_{n,k}^{(r)} t^k. \quad (2.4)$$

By convention we set  $D_0^{(2)}(t) = d_0^{(2)}(t) = 1$ . The first values of  $D_n^{(2)}(t)$  and  $d_n^{(2)}(t)$  are as follows:

$$\begin{aligned} D_1^{(2)}(t) &= t, & d_1^{(2)}(t) &= t, \\ D_2^{(2)}(t) &= t + 3t^2 + t^3, & d_2^{(2)}(t) &= 4t + t^2, \\ D_3^{(2)}(t) &= t + 7t^2 + 13t^3 + 7t^4 + t^5, & d_3^{(2)}(t) &= 8t + 20t^2 + t^3, \\ D_4^{(2)}(t) &= t + 15t^2 + 57t^3 + 87t^4 + 57t^5 + 15t^6 + t^7, & d_4^{(2)}(t) &= 16t + 144t^2 + 72t^3 + t^4. \end{aligned}$$

The polynomials  $d_n^{(2)}(t)$  were first studied by Chen et al. [CTZ09] and Chow [Cho09]. In particular, the polynomials  $d_n^{(2)}(t)$  have only real roots, see [Cho09]. Recently Mongelli [Mon13] has proved the symmetry  $D_{n,j}^{(2)} = D_{n,2n-j}^{(2)}$  for all  $0 \leq j \leq n$  and conjectured its unimodality. Note that the polynomials  $D_3^{(2)}(t)$  has non-real complex roots. We will prove similar expansions for the derangement Eulerian polynomials  $D_n^{(r)}(t)$  and  $d_n^{(r)}(t)$  when  $r \geq 1$ . In particular, we prove the  $\gamma$ -positivity of  $D_n^{(2)}(t)$ , which implies both symmetry and unimodality of its coefficients, see (2.8).

**2.4. Main results in  $\mathbb{Z}_r \wr \mathfrak{S}_n$ .** For any integer  $n \geq 0$  define  $[n]_q = \frac{1-q^n}{1-q}$  as its  $q$ -analogue. Let  $\gamma_{n,i,j}$  be the number of permutations in  $\mathfrak{S}_n$  with exactly  $i$  fixed-points,  $j$  excedances and without double excedance, i.e.,

$$\gamma_{n,i,j} = |\{\sigma \in \mathfrak{S}_n : \text{fix } \sigma = i, \text{ exc } \sigma = j, \text{ cda } \sigma = 0\}|.$$

**Theorem 3.** For  $r \geq 1$ , we have  $\gamma_{n,i,j} > 0$  for  $1 \leq i + 2j \leq n$ ,

$$D_n^{(r)}(t) = \sum_{1 \leq i+2j \leq n} \gamma_{n,i,j} t^{i+j} (1+t)^{n-i-2j} ([r-1]_t)^i ([r]_t)^{n-i}, \quad (2.5)$$

and

$$d_n^{(r)}(t) = \sum_{1 \leq i+2j \leq n} \gamma_{n,i,j} t^{i+j} (1+t)^{n-i-2j} (r-1)^i r^{n-i}. \quad (2.6)$$

The special  $r = 1$  case of (2.5) corresponds to the  $q = 1$  case of (1.4). We derive immediately from Theorem 3 the following result.

**Corollary 4.** For all  $r \geq 1$  and  $n \geq 1$ , the polynomial  $D_n^{(r)}(t)$  is strictly unimodal and symmetric, namely, the coefficients  $D_{n,k}^{(r)}$  in (2.3) satisfy

$$0 \leq D_{n,0}^{(r)} < D_{n,1}^{(r)} < \cdots < D_{n,\lfloor rn/2 \rfloor}^{(r)}. \quad (2.7)$$

and  $D_{n,j}^{(r)} = D_{n,rn-j}^{(r)}$  for all  $0 \leq j \leq \lfloor rn/2 \rfloor$ .

*Proof.* Note that each summand at the right-hand side of (2.5) is unimodal and symmetric with the same center of symmetry at

$$(i+j) + \frac{n-i-2j}{2} + \frac{(r-2)i}{2} + \frac{(r-1)(n-i)}{2} = \frac{rn}{2}. \quad \square$$

Let  $\gamma_{n,k}^{(2)}$  be the number of permutations in  $\mathfrak{S}_n$  with exactly  $k$  weak excedances and without double excedance, i.e.,  $\gamma_{n,k}^{(2)} = \sum_{i+j=k} \gamma_{n,i,j}$ . For example, the permutations (written in product of disjoint cycles) in  $\mathfrak{S}_4$  with exactly 2 weak excedances and without double excedance are as follows:

$$(1)(432), (2)(431), (3)(421), (4)(432), (21)(43), (31)(42), (32)(41), (4132), (4231).$$

Thus  $\gamma_{4,2}^{(2)} = 9$ .

For  $r = 2$ , Theorem 3 reduces to the following result, of which (2.8) infers the  $\gamma$ -positivity of  $D_n^{(2)}(t)$ .

**Corollary 5.** We have  $\gamma_{n,k}^{(2)} > 0$  for  $1 \leq k \leq n$ ,

$$D_n^{(2)}(t) = \sum_{k=1}^n \gamma_{n,k}^{(2)} t^k (1+t)^{2n-2k}, \quad (2.8)$$

and

$$d_n^{(2)}(t) = \sum_{1 \leq i+2j \leq n} \gamma_{n,i,j} 2^{n-i} t^{i+j} (1+t)^{n-i-2j}. \quad (2.9)$$



$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	9	6	1		
5	0	1	23	35	10	1	
6	0	1	53	184	95	15	1

TABLE 3. The values of  $\gamma_{n,k}^{(2)}$  for  $0 \leq k \leq n \leq 6$

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	3				
3	0	1	7	11			
4	0	1	15	54	57		
5	0	1	31	197	458	361	
6	0	1	63	648	2551	4379	2763

TABLE 4. The values of  $\hat{\gamma}_{n,k}^{(2)}$  for  $0 \leq k \leq n \leq 6$

The first values of  $\gamma_{n,k}^{(2)}$  are given in Table 4. For example, we have

$$D_4^{(2)}(t) = t(1+t)^6 + 9t^2(1+t)^4 + 6t^3(1+t)^2 + t^4.$$

The following result is equivalent to [Ath13, Proposition 2.3], while we will give an alternative proof again using the generating functions.

**Theorem 6.** *For all  $r \geq 1$ , the coefficients  $D_{n,k}^{(r)}$  and  $d_{n,k}^{(r)}$  of the polynomials  $D_n^{(r)}(t)$  and  $d_n^{(r)}(t)$  (cf. (2.3) and (2.4)) are connected by  $d_{n,0}^{(r)} = D_{n,0}^{(r)}$  and*

$$d_{n,k}^{(r)} = \sum_{j=0}^{r-1} D_{n,rk-j}^{(r)} \quad \text{for all } 1 \leq k \leq n. \quad (2.10)$$

For example, when  $n = 4$  and  $r = k = 2$  we have  $144 = 57 + 87$ . In [CTZ09, Theorem 4.6], the so-called *spiral property* of the sequence  $\{d_{n,k}^{(2)}\}_{0 \leq k \leq n}$  was proven, namely,

$$0 \leq d_{n,0}^{(2)} < d_{n,n}^{(2)} < d_{n,1}^{(2)} < d_{n,n-1}^{(2)} < d_{n,2}^{(2)} < d_{n,n-2}^{(2)} < \cdots < d_{n,\lceil n/2 \rceil}^{(2)}.$$

The following theorem generalizes the above spiral property to the sequence  $\{d_{n,k}^{(r)}\}_{0 \leq k \leq n}$  for all  $r \geq 2$ .

**Corollary 7.** *For all  $r \geq 2$  and  $n \geq 0$ , the polynomial  $d_n^{(r)}(t)$  has the spiral property, namely,*

$$d_{n,k}^{(r)} < d_{n,n-k}^{(r)} < d_{n,k+1}^{(r)}, \quad \text{for all } 0 \leq k < \lfloor n/2 \rfloor,$$

and  $d_{n,\lfloor n/2 \rfloor}^{(r)} < d_{n,\lceil n/2 \rceil}^{(r)}$  if  $n$  is odd, where  $d_{n,k}^{(r)}$  is the coefficient of  $t^k$  in the polynomial  $d_n^{(r)}(t)$ .

*Proof.* For  $r \geq 1$  and  $n \geq 1$ , letting  $a_{n,k}^{(r)} = D_{n,rk}^{(r)}$  and  $b_{n,k}^{(r)} = \sum_{j=1}^{r-1} D_{n,rk-j}^{(r)}$ , Corollary 4 gives that these two sequences  $\{a_{n,k}^{(r)}\}_{0 \leq k \leq n}$  and  $\{b_{n,k}^{(r)}\}_{1 \leq k \leq n}$  are also strict unimodal and symmetric, that is,

$$0 \leq a_{n,0}^{(r)} < a_{n,1}^{(r)} < \cdots < a_{n,\lfloor n/2 \rfloor}^{(r)}, \quad 0 < b_{n,1}^{(r)} < b_{n,2}^{(r)} < \cdots < b_{n,\lfloor n/2 \rfloor}^{(r)}, \quad (2.11)$$

and  $a_{n,k}^{(r)} = a_{n,n-k}^{(r)}$  for  $0 \leq k \leq \lfloor n/2 \rfloor$  and  $b_{n,k}^{(r)} = b_{n,n+1-k}^{(r)}$  for  $1 \leq k \leq \lceil n/2 \rceil$ . By Theorem 6, we have  $d_{n,0}^{(r)} = a_{n,0}^{(r)}$  and

$$d_{n,k}^{(r)} = a_{n,k}^{(r)} + b_{n,k}^{(r)} \quad \text{for all } 1 \leq k \leq n. \quad (2.12)$$

Combining (2.11) and (2.12), the sequence  $\{d_{n,k}^{(r)}\}_{1 \leq k \leq n}$  has a spiral property for all  $r \geq 2$  as follows. □

*Remark.* Let  $a_n^{(r)}(t) = \sum_{k=0}^n a_{n,k}^{(r)} t^k$  and  $b_n^{(r)}(t) = \sum_{k=1}^n b_{n,k}^{(r)} t^k$  such that

$$d_n^{(r)}(t) = a_n^{(r)}(t) + b_n^{(r)}(t).$$

Observing the equation (4.18) in proof of Theorem 6, we have

$$\sum_{n \geq 0} a_n^{(r)}(t) \frac{z^n}{n!} = \frac{e^{(r-1)tz} - te^{(r-1)z}}{e^{rtz} - te^{rz}}, \quad (2.13)$$

$$\sum_{n \geq 0} b_n^{(r)}(t) \frac{z^n}{n!} = \frac{te^{(r-1)z} - te^{(r-1)tz}}{e^{rtz} - te^{rz}}. \quad (2.14)$$

Athanasiadis and Savvidou [AS13] proved that there are nonnegative integers  $\xi_{n,k}^+$  and  $\xi_{n,k}^-$  satisfying

$$d_n^{(2)}(t) = \sum_{k \geq 0} \xi_{n,k}^+ t^k (1+t)^{n-2k} + \sum_{k \geq 1} \xi_{n,k}^- t^k (1+t)^{n+1-2k}. \quad (2.15)$$

We show that the polynomials  $D_n^{(2)}(t)$  have a similar expansion as (2.15) with the same coefficients  $\xi_{n,k}^+$  and  $\xi_{n,k}^-$  and also derive a combinatorial interpretation from Theorem 5. A permutation  $\sigma \in \mathcal{S}_n$  is called a *drop-colored permutation* if some of the drops of  $\sigma$  are colored by, say, *bar* or *tilde*. We denote by  $\text{cdrop}(\sigma)$  the number of colored drops in  $\sigma$ . Denote by  $\hat{\gamma}_{n,k}^{(2)}$  the number of permutations  $\sigma$  in  $\mathcal{Z}_n$  whose total number of weak excedances and colored drops is  $k$ , namely,  $\text{wex}(\sigma) + \text{cdrop}(\sigma) = k$ . Let  $\mathcal{Z}_n$  be the set of drop-colored permutations without double excedance in  $\mathfrak{S}_n$ . For example, the drop-colored permutations (written in product of disjoint cycles) in  $\mathcal{Z}_4$  whose total number of weak excedances and colored drops is 2 are:

$$(1)(4\bar{3}2), (2)(4\bar{3}1), (3)(4\bar{2}1), (4)(3\bar{2}1), (21)(4\bar{3}), (31)(4\bar{2}), (41)(3\bar{2}), \\ (4\bar{2}31), (41\bar{3}2), (\bar{4}3\bar{2}1), (\tilde{4}3\bar{2}1), (4\bar{3}\bar{2}1), (4\tilde{3}\bar{2}1), (4\bar{3}\bar{2}1), (4\bar{3}\tilde{2}1).$$

Thus  $\hat{\gamma}_{4,2}^{(2)} = 15$ . The first values of  $\hat{\gamma}_{n,k}^{(2)}$  are given in Table 4.

**Theorem 8.** *For  $0 \leq k \leq n$  we have*

$$\hat{\gamma}_{n,k}^{(2)} = \sum_{i=0}^k \gamma_{n,i}^{(2)} 2^{k-i} \binom{n-i}{k-i}. \quad (2.16)$$

Furthermore the following expansions hold true:

$$D_n^{(2)}(t) = \sum_{k \geq 0} \hat{\gamma}_{n,k}^{(2)} t^k (1+t^2)^{n-k}, \quad (2.17)$$

$$d_n^{(2)}(t) = \sum_{k \geq 0} \hat{\gamma}_{n,k}^{(2)} t^{\lceil k/2 \rceil} (1+t)^{n-k}, \quad (2.18)$$

where  $\lceil x \rceil$  denotes the least integer great than or equal to  $x$ .

For example, we have

$$\begin{aligned} D_4^{(2)}(t) &= t(1+t^2)^3 + 15t^2(1+t^2)^2 + 54t^3(1+t^2) + 57t^4, \\ d_4^{(2)}(t) &= t(1+t)^3 + 15t(1+t)^2 + 54t^2(1+t) + 57t^2. \end{aligned}$$

Some remarks are in order:

- (i) The two different polynomials  $D_n^{(2)}(t)$  and  $d_n^{(2)}(t)$  have the same coefficients  $\hat{\gamma}_{n,k}^{(2)}$ , when they are expanded in the basis  $\{t^k(1+t^2)^{n-k}\}$  and  $\{t^{\lceil k/2 \rceil}(1+t)^{n-k}\}$ , respectively. It would be interesting to have a combinatorial explanation of this coincidence.
- (ii) Comparing (2.15) and (2.18) we derive that

$$\hat{\gamma}_{n,2k}^{(2)} = \xi_{n,k}^+ \text{ for } k \geq 0 \quad \text{and} \quad \hat{\gamma}_{n,2k-1}^{(2)} = \xi_{n,k}^- \text{ for } k \geq 1. \quad (2.19)$$

- (iii) Athanasiadis [Ath13] has given an apparently different interpretation of the numbers in (2.19) using linear statistics on permutations. Is it easy to relate these two interpretations?

The rest of this paper is organized as follows. In Section 3, after recalling the necessary results in our previous paper [SZ12] we prove Theorems 1 and 2. In Section 4 we prove Theorems 3, 6, and 8. Finally, in Section 5, we prove Lemma 16 after recalling some background for the classical combinatorial theory of continued fractions.

### 3. PROOF OF THEOREMS 1 AND 2

**3.1. Preliminaries.** We need some more definitions. For  $\sigma \in \mathfrak{S}_n$ , let  $\sigma(0) = \sigma(n+1) = 0$ . Then any entry  $\sigma(i)$  ( $i \in [n]$ ) can be classified according to one of the four cases:

- a *peak* if  $\sigma(i-1) < \sigma(i)$  and  $\sigma(i) > \sigma(i+1)$ ;
- a *valley* if  $\sigma(i-1) > \sigma(i)$  and  $\sigma(i) < \sigma(i+1)$ ;
- a *double ascent* if  $\sigma(i-1) < \sigma(i)$  and  $\sigma(i) < \sigma(i+1)$ ;
- a *double descent* if  $\sigma(i-1) > \sigma(i)$  and  $\sigma(i) > \sigma(i+1)$ .

Let  $\text{peak}^* \sigma$  (resp.  $\text{valley}^* \sigma$ ,  $\text{da}^* \sigma$ ,  $\text{dd}^* \sigma$ ) denote the number of peaks (resp. valleys, double ascents, double descents) in  $\sigma$ . Clearly we have  $\text{peak}^* \sigma = \text{valley}^* \sigma + 1$ . Now, it is easy to see that

$$\text{DD}_{n,k} = \{\sigma \in \mathfrak{S}_n : \text{valley}^* \sigma = k \text{ and } \text{dd}^* \sigma = 0\}.$$

Define the generalized Eulerian polynomial by

$$A_n(p, q, t, u, v, w) = \sum_{\sigma \in \mathfrak{S}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des } \sigma} u^{\text{da}^* \sigma} v^{\text{dd}^* \sigma} w^{\text{valley}^* \sigma}. \quad (3.1)$$

We need the known formula for the so-called Jacobi-Rogers polynomials [GJ83].

**Lemma 9.** *If the ordinary generating function of the sequence  $\{\mu_n\}_n$  has the continued fraction expansion*

$$\sum_{n \geq 0} \mu_n z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{\dots}}},$$

then

$$\mu_n = \sum_{h \geq 0} \sum_{\substack{n_0, \dots, n_{h-1} \geq 1 \\ m_0, \dots, m_h \geq 0}} b_0^{m_0} \dots b_h^{m_h} \lambda_1^{n_0} \dots \lambda_h^{n_{h-1}} \rho(\mathbf{n}, \mathbf{m}), \quad (3.2)$$

where  $2(n_0 + \dots + n_{h-1}) + (m_0 + \dots + m_h) = n$  and

$$\rho(\mathbf{n}, \mathbf{m}) = \prod_{j=0}^{h-1} \binom{n_j + n_{j+1} - 1}{n_j - 1} \prod_{l=0}^h \binom{m_l + n_l + n_{l-1} - 1}{m_l}$$

with convention  $n_{-1} = 1$ ,  $n_h = 0$ ,  $\binom{p}{-1} = \delta_{p,-1}$ .

The following result was proved in [SZ12, Theorem 2]. We provide below a new and neat proof by using the Jacobi-Rogers formula.

**Lemma 10.** *We have the expansion formula*

$$A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q) (tw)^k (u + vt)^{n-1-2k}, \quad (3.3)$$

where

$$a_{n,k}(p, q) = \sum_{\sigma \in \text{DD}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}. \quad (3.4)$$

and, for all  $0 \leq k \leq \lfloor (n-1)/2 \rfloor$ , the polynomial  $a_{n,k}(p, q)$  is divisible by  $(p+q)^k$ .

*Proof.* By [SZ12, Eq. (28)] we have

$$\sum_{n \geq 1} A_n(p, q, t, u, v, w) x^{n-1} = \frac{1}{1 - (u + tv)[1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}twx^2}{1 - (u + tv)[2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}twx^2}{\dots}}},$$

where  $[n]_{p,q} = \frac{p^n - q^n}{p - q}$ . By Lemma 9, there are  $a_{n,k}(p, q) \in \mathbb{N}[p, q]$  such that

$$A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q)(u + tv)^{n-1-2k}(tw)^k. \quad (3.5)$$

Thus

$$A_n(p, q, 1, 1, 0, w) = \sum_{k \geq 0} a_{n,k}(p, q)w^k,$$

and we derive the combinatorial interpretation (3.4) for  $a_{n,k}(p, q)$  from (3.1). Finally, as

$$(p + q) \mid [n]_{p,q}[n + 1]_{p,q} \quad \text{for all } n \geq 1,$$

each  $w$  appears with a factor  $(p + q)$  in the expansion of the continued fraction, so the polynomial  $a_{n,k}(p, q)$  is divisible by  $(p + q)^k$ .  $\square$

**Definition 11.** For  $\sigma \in \mathfrak{S}_n$ , a value  $x = \sigma(i)$  ( $i \in [n]$ ) is called

- a cyclic peak if  $i = \sigma^{-1}(x) < x$  and  $x > \sigma(x)$ ;
- a cyclic valley if  $i = \sigma^{-1}(x) > x$  and  $x < \sigma(x)$ ;
- a double excedance if  $i = \sigma^{-1}(x) < x$  and  $x < \sigma(x)$ ;
- a double drop if  $i = \sigma^{-1}(x) > x$  and  $x > \sigma(x)$ ;
- a fixed point if  $x = \sigma(x)$ .

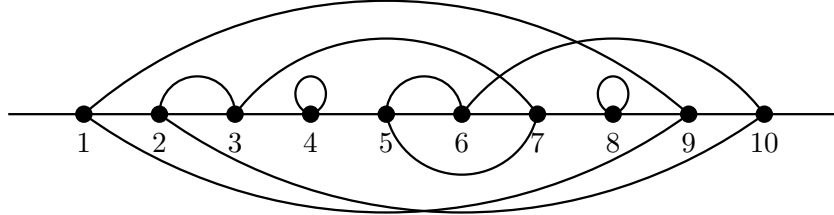
Let  $\text{cpeak } \sigma$  (resp.  $\text{cvalley } \sigma$ ,  $\text{cda } \sigma$ ,  $\text{cdd } \sigma$ ,  $\text{fix } \sigma$ ) be the number of cyclic peaks (resp. valleys, double excedances, double drops, fixed points) in  $\sigma$ .

For a permutation  $\sigma \in \mathfrak{S}_n$  the crossing and nesting numbers are defined by

$$\text{cros } \sigma = \# \{(i, j) \in [n] \times [n] : (i < j \leq \sigma(i) < \sigma(j)) \vee (i > j > \sigma(i) > \sigma(j))\}, \quad (3.6)$$

$$\text{nest } \sigma = \# \{(i, j) \in [n] \times [n] : (i < j \leq \sigma(j) < \sigma(i)) \vee (i > j > \sigma(j) > \sigma(i))\}. \quad (3.7)$$

For example, the diagram of  $\sigma = 9 \ 3 \ 7 \ 4 \ 6 \ 10 \ 5 \ 8 \ 1 \ 2$  is as follows:



It is easy to see that  $\text{cros } \sigma = 5$  and  $\text{nest } \sigma = 10$ . Indeed, 5 crossings are  $(1, 6)$ ,  $(2, 3)$ ,  $(3, 6)$ ,  $(5, 6)$ , and  $(10, 9)$  and 10 nestings are  $(1, 2)$ ,  $(1, 3)$ ,  $(1, 4)$ ,  $(1, 5)$ ,  $(1, 8)$ ,  $(3, 4)$ ,  $(3, 5)$ ,  $(6, 8)$ ,  $(9, 7)$ , and  $(10, 7)$ .

**Definition 12.** For  $\sigma \in \mathfrak{S}_n$ , let  $\sigma(0) = 0$  and  $\sigma(n+1) = n+1$ . The corresponding numbers of peaks, valleys, double ascents, and double descents of permutation  $\sigma$  are denoted by

$$\text{peak } \sigma, \quad \text{valley } \sigma, \quad \text{da } \sigma, \quad \text{dd } \sigma.$$

Moreover, a double ascent  $\sigma(i)$  of  $\sigma$  ( $i \in [n]$ ) is said to be a foremaximum if  $\sigma(i)$  is a left-to-right maximum of  $\sigma$ , i.e.,  $\sigma(j) < \sigma(i)$  for all  $1 \leq i < j$ . Let  $\text{fmax } \sigma$  denote the number of foremaxima of  $\sigma$ .

For instance,  $\text{fmax}(42157368) = 2$ . Note that  $\text{peak } \sigma = \text{valley } \sigma$  for any  $\sigma \in \mathfrak{S}_n$ . Recall the following result in [SZ12, Theorem 5].

**Lemma 13.** *There is a bijection  $\Phi$  on  $\mathfrak{S}_n$  such that for all  $\sigma \in \mathfrak{S}_n$  we have*

$$\begin{aligned} & (\text{nest}, \text{cros}, \text{drop}, \text{cda}, \text{cdd}, \text{cvalley}, \text{fix})\sigma \\ &= (2\text{-}31, 31\text{-}2, \text{des}, \text{da} - \text{fmax}, \text{dd}, \text{valley}, \text{fmax})\Phi(\sigma). \end{aligned}$$

Consider the enumerative polynomial

$$B_n(p, q, t, u, v, w, y) = \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{drop } \sigma} u^{\text{cda } \sigma} v^{\text{cdd } \sigma} w^{\text{cvalley } \sigma} y^{\text{fix } \sigma}. \quad (3.8)$$

By [SZ12, (34)] the ordinary generating function of  $B_n(p, q, t, u, v, w, y)$  has the following continued fraction expansion:

$$\sum_{n \geq 0} B_n(p, q, t, u, v, w, y) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\dots}}}, \quad (3.9)$$

where  $a_h = tw[h+1]_{p,q}$ ,  $b_h = yp^h + (qu + tv)[h]_{p,q}$ , and  $c_h = [h]_{p,q}$ .

Let  $\mathfrak{S}_n^*$  be the subset of  $\mathfrak{S}_n$  consisting of permutations of which each double ascent is also a foremaxima, and

$$\mathfrak{S}_{n,k,j} = \{\sigma \in \mathfrak{S}_n : \text{cvalley } \sigma = k, \text{fix } \sigma = j, \text{cda } \sigma = 0\}, \quad (3.10)$$

$$\mathfrak{S}_{n,k,j}^* = \{\sigma \in \mathfrak{S}_n^* : \text{valley } \sigma = k, \text{da } \sigma = j\}. \quad (3.11)$$

The following result is derived by combining Corollary 7 and Theorem 8 in [SZ12].

**Lemma 14.** *We have*

$$B_n(p, q, t, u, v, w, y) = \sum_{j=0}^n y^j \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} b_{n,k,j}(p, q) (tw)^k (qu + tv)^{n-j-2k},$$

where the coefficient  $b_{n,k,j}(p, q)$  is given by

$$b_{n,k,j}(p, q) = \sum_{\sigma \in \mathfrak{S}_{n,k,j}} p^{\text{nest } \sigma} q^{\text{cros } \sigma} = \sum_{\sigma \in \mathfrak{S}_{n,k,j}^*} p^{(2\text{-}31)\sigma} q^{(31\text{-}2)\sigma}.$$

A permutation  $\sigma \in \mathfrak{S}_n$  is a *coderangement* if  $\text{fmax } \sigma = 0$ . Let

$$\mathfrak{D}_n^* = \{\sigma \in \mathfrak{S}_n : \text{fmax } \sigma = 0\}.$$

For example, we have  $\mathfrak{D}_4^* = \{2143, 3142, 3241, 4123, 4132, 4213, 4231, 4312, 4321\}$ . Thus,  $\mathfrak{S}_{n,k,0}$  is the subset of derangements in  $\mathfrak{D}_n$  with exactly  $k$  cyclic valleys, and without double excedance, and  $\mathfrak{S}_{n,k,0}^*$  is the subset of coderangements in  $\mathfrak{D}_n^*$  with exactly  $k$  valleys and without double ascents.

**3.2. Proof of Theorem 1.** Let

$$\tilde{A}_n(p, q, t) = \sum_{\pi \in \mathfrak{S}_n} p^{(13-2)\pi} q^{(2-31)\pi} t^{\text{des } \pi}.$$

Since the triple statistics  $((2-31), (31-2), \text{des})$  and  $((2-13), (31-2), \text{des})$  are equidistributed on  $\mathfrak{S}_n$ , see [SZ12, Eq. (39)], Lemma 13 infers that

$$\tilde{A}_n(p, q, t) = \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{drop } \sigma}.$$

As  $\text{inv } \sigma^{-1} = \text{inv } \sigma$  and  $\text{drop } \sigma = \text{exc } \sigma^{-1}$  for any  $\sigma \in \mathfrak{S}_n$ , we have

$$A_n(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv } \sigma} t^{\text{exc } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv } \sigma} t^{\text{drop } \sigma}.$$

Invoking the known result (see [SZ10, Eq. (40)])

$$\text{inv} = \text{drop} + \text{cros} + 2 \text{nest}, \quad (3.12)$$

we have  $\text{inv} - \text{drop} = \text{cros} + 2 \text{nest}$  and

$$A_n(q, t/q) = \tilde{A}_n(q^2, q, t).$$

The result follows then from Lemma 10 with  $\gamma_{n,k}(q) := a_{n,k}(q^2, q)$ .  $\square$

**3.3. Proof of Theorem 2.** Let

$$D_n(q, t) := \sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv } \sigma} t^{\text{exc } \sigma} = \sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv } \sigma} t^{\text{drop } \sigma}. \quad (3.13)$$

By (3.8) we have  $D_n(q, t) = B_n(q^2, q, qt, 1, 1, 1, 0)$ . Hence, setting  $y = 0$ ,  $p = q^2$  and  $u = v = w = 1$  in Lemma 14 and then replacing  $t$  by  $qt$  we obtain

$$D_n(q, t) = \sum_{\sigma \in \mathfrak{D}_n} q^{\text{inv } \sigma} t^{\text{drop } \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k,0}(q^2, q) q^{n-k} t^k (1+t)^{n-2k},$$

where

$$b_{n,k,0}(q^2, q) q^{n-k} = \sum_{\sigma \in \mathfrak{S}_{n,k,0}} q^{2 \text{nest } \sigma + \text{cros } \sigma + n - \text{cvalley } \sigma}.$$

Since  $n - \text{cvalley } \sigma = n - \text{cpeak } \sigma = \text{drop } \sigma$  for any  $\sigma \in \mathfrak{D}_n$  and

$$\mathfrak{S}_{n,k,0} = \{\sigma \in \mathfrak{D}_n : \text{cvalley}(\sigma) = k\} = \{\sigma \in \mathfrak{D}_n : \text{exc}(\sigma) = k\} = \text{DE}_{n,k},$$

the result follows from (3.12).  $\square$

4. PROOFS OF THEOREMS 3, 6 AND 8

4.1. **Preliminaries.** Recall that an integer  $i$  is called *fixed point* (resp. *A-excedance*, *A-weak excedance*) of  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$  if  $\sigma(i) = i$  (resp.  $\sigma(i) >_c i$ ,  $\sigma(i) \geq_c i$ ). Let

$$\begin{aligned} \text{fix } \sigma &= \# \{i \in [n] : i = \sigma(i)\}, & \text{exc}_A \sigma &= \# \{i \in [n] : i <_c \sigma(i)\}, \\ \text{wex}_A \sigma &= \# \{i \in [n] : i \leq_c \sigma(i)\} = \text{exc}_A \sigma + \text{fix } \sigma. \end{aligned}$$

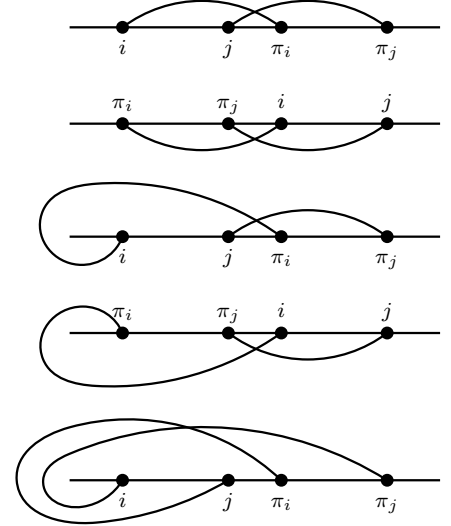
Also, we define two statistics about colors:

$$\begin{aligned} \text{wex}_c \sigma &= \# \{i \in [n] : i \leq |\sigma(i)| \text{ and } \text{col}(\sigma(i)) > 0\}, \\ \text{csum } \sigma &= \sum_{i \in [n]} \text{col}(\sigma(i)). \end{aligned}$$

We generalize the crossings of permutations in  $B_n$  (see [CKJV13]) to colored permutations.

**Definition 15** (Crossings). *For a permutation  $\sigma = \sigma(1) \dots \sigma(n) = \binom{\pi_1 \dots \pi_n}{\mathbf{z}_1 \dots \mathbf{z}_n} \in \mathbb{Z}_r \wr \mathfrak{S}_n$  a crossing is a pair  $(i, j)$  such that*

- $\mathbf{z}_i = \mathbf{z}_j = 0$  and  $i < j \leq \pi_i < \pi_j$ ; or
- $\mathbf{z}_i = \mathbf{z}_j = 0$  and  $\pi_i < \pi_j < i < j$ ; or
- $\mathbf{z}_i > 0$ ,  $\mathbf{z}_j = 0$ , and  $j \leq \pi_i < \pi_j$ ; or
- $\mathbf{z}_i > 0$ ,  $\mathbf{z}_j = 0$ , and  $\pi_j < i < j$ ; or
- $\mathbf{z}_i > 0$ ,  $\mathbf{z}_j > 0$ ,  $i < j$ , and  $\pi_j < \pi_i$ .



Let  $\text{cros } \sigma$  be the number of crossings in  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ .

**Lemma 16.** *We have*

$$\sum_{n \geq 0} \sum_{\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n} q^{\text{cros } \sigma} t^{\text{wex}_A \sigma} w^{\text{wex}_c \sigma} x^{\text{fix } \sigma} y^{\text{csum } \sigma} z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\dots}}}, \quad (4.1)$$

where the coefficients  $a_h$ ,  $b_h$  ( $h \geq 0$ ) and  $c_h$  ( $h \geq 1$ ) are given by

$$\begin{aligned} a_h &= (t + wy[r-1]_y q^h)(1 + y[r-1]_y q^{h+1}), \\ b_h &= (1 + y[r-1]_y q^h)[h]_q + t(x + q[h]_q) + wy[r-1]_y q^h[h+1]_q, \\ c_h &= [h]_q^2. \end{aligned}$$



*Remark.* Corteel et al [CKJV13] have proved (4.1) in the case of  $r = 2$  and  $x = 1$ . Actually we shall prove a refined version of (4.1). See (5.4) in Section 5.

**Lemma 17.** *We have*

$$\sum_{n \geq 0} D_n^{(r)}(t) z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{\dots}}}, \quad (4.2)$$

where  $b_h = t[r-1]_t + h(1+t)[r]_t$  ( $h \geq 0$ ) and  $\lambda_h = t[r]_t^2 h^2$  ( $h \geq 1$ ) and

$$\sum_{n \geq 0} d_n^{(r)}(t) z^n = \frac{1}{1 - b'_0 z - \frac{\lambda'_1 z^2}{1 - b'_1 z - \frac{\lambda'_2 z^2}{\dots}}}, \quad (4.3)$$

where  $b'_h = (r-1)t + rh(1+t)$  ( $h \geq 0$ ) and  $\lambda'_h = (rh)^2 t$  ( $h \geq 1$ ).

*Proof.* We can derive (4.2) and (4.3) from Lemma 16 as

$$\text{fexc} = r(\text{wex}_A - \text{fix}) + \text{csum}, \quad (4.4)$$

$$\text{exc} = (\text{wex}_A - \text{fix}) + \text{wex}_c. \quad (4.5)$$

Letting  $q = 1$ ,  $t \leftarrow t^r$ ,  $w = 1$ ,  $x = 0$  and  $y \leftarrow t$  in (4.1), we obtain (4.2). Also, letting  $q = 1$ ,  $t \leftarrow t$ ,  $w \leftarrow t$ ,  $x = 0$  and  $y = 1$  in (4.1), we obtain (4.3).  $\square$

*Remark.* It is also possible to derive (4.2) and (4.3) from the known generating functions.

- (i) Let  $B_n^{(r)}(t, x) = \sum_{\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n} t^{\text{fexc}} x^{\text{fix}} \sigma$ . Since  $D_n^{(r)}(t) = B_n^{(r)}(t, 0)$ , from the known generating function formula (see [Lin14, (2.23)] and [FH11, (1.13)] or [FH09, (1.9)] for  $r = 2$ )

$$\sum_{n \geq 0} D_n^{(r)}(t) \frac{z^n}{n!} = \frac{1-t}{e^{tz} - te^z}, \quad (4.6)$$

we derive that

$$\sum_{n \geq 0} \frac{D_n^{(r)}(t) z^n}{([r]_t)^n n!} = e^{-\frac{t^r}{[r]_t} z} \frac{1-t}{1 - te^{(1-t)z}}. \quad (4.7)$$

Also, it is known (see [CTZ09, Cho09]) that

$$\sum_{n \geq 0} \frac{d_n^{(r)}(t) z^n}{r^n n!} = e^{-tz/r} \frac{1-t}{1 - te^{(1-t)z}}. \quad (4.8)$$

Applying the addition formula of Rogers-Stieltjes (see [GJ83, Chap. 5] and [Zen93]) to (4.7) and (4.8) we also obtain the continued fraction expansions (4.2) and (4.3).

- (ii) Note that  $\frac{1-t}{1 - te^{(1-t)z}}$  is the exponential generating function of Eulerian polynomials for symmetric groups.

(iii) In (4.1) if we set  $r = 2$  and do the following substitution

$$q \leftarrow 1, \quad w \leftarrow t, \quad x \leftarrow t^{-1}, \quad \text{and} \quad y \leftarrow 1,$$

then we obtain  $a_h = 4t$ ,  $b_h = (2h + 1)(t + 1)$  and  $c_h = h^2$ . These are the same coefficients obtained when we specialize (3.9) as follows:

$$p = q = t = u = v = y = 1, \quad w \leftarrow \frac{4t}{(1+t)^2}, \quad \text{and} \quad z \leftarrow (1+t)z.$$

Combining Lemmas 14 and 16, we have

$$\sum_{\sigma \in \mathfrak{B}_n} t^{\text{exc } \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[ \sum_{j=0}^{n-2k} 2^{(n-2k-j)} b_{n,k,j}(1, 1) \right] (4t)^k (1+t)^{n-2k}, \quad (4.9)$$

where  $n \geq 2k + j$  and  $b_{n,k,j}(1, 1)$  is the cardinality of the set  $\mathfrak{S}_{n,k,j}$ , which is defined by (3.10). Note that this result is comparable with the  $\gamma$ -positivity result of Chow [Cho08, Theorem 4.7].

**4.2. Proof of Theorem 3.** Let

$$B_n(1, 1, 1, 0, 1, w, y) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{cda}(\sigma)=0}} w^{\text{exc } \sigma} y^{\text{fix } \sigma} = \sum_{i,j} \gamma_{n,i,j} w^j y^i, \quad (4.10)$$

where  $\gamma_{n,i,j} = |\{\sigma \in \mathfrak{S}_n : \text{cda } \sigma = 0, \text{ fix } \sigma = i, \text{ exc } \sigma = j\}|$ . Now, specializing the formula (3.9) yields

$$\sum_{n \geq 0} B_n(1, 1, 1, 0, 1, w, y) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\dots}}}, \quad (4.11)$$

where  $a_h = w(h + 1)$  ( $h \geq 0$ ),  $b_h = y + h$  ( $h \geq 0$ ), and  $c_h = h$  ( $h \geq 1$ ). Comparing the right-hand sides of (4.2) and (4.11), we derive

$$D_n^{(r)}(t) = (1+t)^n ([r]_t)^n B_n(1, 1, 1, 0, 1, w, y) \quad (4.12)$$

with

$$w = \frac{t}{(1+t)^2} \quad \text{and} \quad y = \frac{t[r-1]_t}{(1+t)[r]_t}.$$

Since  $\text{cvalley}(\sigma) = \text{exc}(\sigma)$  for any  $\sigma \in \mathfrak{S}_n$ , it follows from (4.10) and (4.12) that

$$D_n^{(r)}(t) = \sum_{1 \leq i+2j \leq n} \gamma_{n,i,j} t^{i+j} (1+t)^{n-i-2j} ([r-1]_t)^i ([r]_t)^{n-i}. \quad (4.13)$$

In the same vein, comparing the right-hand sides of (4.3) and (4.11), we derive

$$d_n^{(r)}(t) = (1+t)^n r^n B_n(1, 1, 1, 0, 1, w, y) \quad (4.14)$$

with

$$w = \frac{t}{r(1+t)^2} \quad \text{and} \quad y = \frac{t(r-1)}{r(1+t)}.$$

Therefore, by (4.10) and (4.14) we have

$$d_n^{(r)}(t) = \sum_{1 \leq i+2j \leq n} \gamma_{n,i,j} t^{i+j} (1+t)^{n-i-2j} (r-1)^i r^{n-i}. \quad (4.15)$$

This completes the proof of Theorem 3.  $\square$

**4.3. Proof of Theorem 6.** Since any primitive  $r$ th root  $\omega = e^{2\pi i/r}$  of unity satisfies

$$\sum_{i=0}^{r-1} (\omega^k)^i = \begin{cases} 0 & \text{if } r \nmid k, \\ r & \text{if } r \mid k, \end{cases}$$

for integer  $j = 0, \dots, r-1$ , we have

$$\frac{1}{r} \sum_{i=0}^{r-1} \frac{\omega^{ij}}{X - \omega^i Y} = \begin{cases} \frac{X^{r-1}}{X^r - Y^r} & \text{if } j = 0, \\ \frac{X^{j-1} Y^{r-j}}{X^r - Y^r} & \text{if } 1 \leq j < r. \end{cases} \quad (4.16)$$

It follows that for all  $0 \leq j < r$ ,

$$\begin{aligned} \sum_{i=0}^{r-1} \omega^{ij} D_n^{(r)}(\omega^i t) &= \sum_{k \geq 0} D_{n,k}^{(r)} \left( \sum_{i=0}^{r-1} \omega^{ij} (\omega^i t)^k \right) \\ &= \sum_{k \geq 0} D_{n,k}^{(r)} \left( \sum_{i=0}^{r-1} (\omega^{k+j})^i \right) t^k = r \sum_{k \geq \lceil j/r \rceil} D_{n,rk-j}^{(r)} t^{rk-j}. \end{aligned} \quad (4.17)$$

From (4.17) and (4.6), for all  $0 \leq j < r$ , we derive that

$$\begin{aligned} \sum_{n \geq 0} \sum_{k \geq \lceil j/r \rceil} D_{n,rk-j}^{(r)} t^{rk-j} \frac{x^n}{n!} &= \sum_{n \geq 0} \frac{1}{r} \left( \sum_{i=0}^{r-1} \omega^{ij} D_n^{(r)}(\omega^i t) \right) \frac{x^n}{n!} \\ &= \frac{1}{r} \sum_{i=0}^{r-1} \frac{\omega^{ij} - \omega^{i(j+1)}}{e^{t^r x} - \omega^i t e^x} \\ &= \begin{cases} \frac{e^{(r-1)t^r x} - t^r e^{(r-1)x}}{e^{r t^r x} - t^r e^{r x}} & \text{if } j = 0, \\ \frac{t^{r-j} e^{(j-1)t^r x + (r-j)x} (1 - e^{(t^r-1)x})}{e^{r t^r x} - t^r e^{r x}} & \text{if } 1 \leq j < r. \end{cases} \end{aligned} \quad (4.18)$$

Summing (4.18) multiplied by  $t^j$  over all  $0 \leq j < r$  yields

$$\sum_{n \geq 0} \sum_{k \geq 0} D_{n,k}^{(r)} t^{r \lceil k/r \rceil} \frac{x^n}{n!} = \frac{(1-t^r) e^{(r-1)t^r x}}{e^{r t^r x} - t^r e^{r x}}. \quad (4.19)$$

Comparing (4.19) with (4.8) we see that

$$d_n^{(r)}(t) = \sum_{\sigma \in \mathfrak{D}_n^{(r)}} t^{\lceil \text{fexc}(\sigma)/r \rceil}. \quad (4.20)$$

Since  $\lceil \text{fexc}(\sigma)/r \rceil = k$  if and only if  $\text{fexc}(\sigma) = rk - j$  for  $j = 0, \dots, r - 1$ , we obtain (2.10).  $\square$

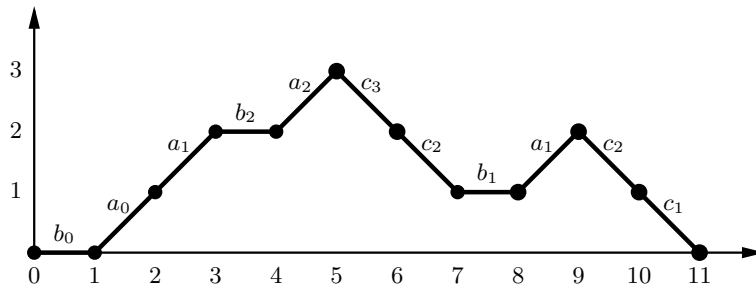
**4.4. Proof of Theorem 8.** By Theorem 5, there are  $\gamma_{n,i}^{(2)}$  permutations in  $\mathfrak{S}_n$  with  $i$  weak excedances with  $0 \leq i \leq k$ , for any such permutation, there are  $\binom{n-i}{k-i}$  ways to choose  $k - i$  drops among  $n - i$  drops and  $2^{k-i}$  ways to color the chosen drops by either *tilde* or *bar*. Summing over  $i$  we see that the number of permutations in  $\mathbb{Z}_n$  whose total number of weak excedances and colored drops is  $k$  is given by  $\sum_{i=0}^k \gamma_{n,i}^{(2)} 2^{k-i} \binom{n-i}{k-i}$ , which is the formula(2.16). Now, writing  $(1+t)^{2n-2k} = (1+2t+t^2)^{n-k} = \sum_{j \geq 0} \binom{n-k}{j} (2t)^j (1+t^2)^{n-k-j}$  and substituting this into (2.8) we get

$$\begin{aligned} D_n^{(2)}(t) &= \sum_{k=1}^n \sum_{j \geq 0} \gamma_{n,k}^{(2)} 2^j \binom{n-k}{j} t^{k+j} (1+t^2)^{n-k-j} \\ &= \sum_{k=1}^n \sum_{i \geq 0} \gamma_{n,i}^{(2)} 2^{k-i} \binom{n-k}{k-i} t^k (1+t^2)^{n-k}, \end{aligned}$$

which is (2.17) in view of (2.16). Finally, we derive (2.18) from Theorem 6.  $\square$

### 5. A CONTINUED FRACTION EXPANSION

**5.1.  $r$ -colored Laguerre histories.** A *Motzkin path* of length  $n$  is a sequence of points  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$  in the plan  $\mathbb{N} \times \mathbb{N}$  with  $\gamma_i = (x_i, y_i)$  ( $0 \leq i \leq n$ ) such that  $\gamma_0 = (0, 0)$  and  $\gamma_n = (n, 0)$ ; and each step  $(\gamma_{i-1}, \gamma_i)$  stisfies  $\gamma_i - \gamma_{i-1} = (1, 1), (1, 0)$  or  $(1, -1)$ , and is called East, North-East, and South-East, respectively. The *height*  $h_i$  of the  $i$ th step  $(\gamma_{i-1}, \gamma_i)$  is the ordinate of  $\gamma_{i-1}$ . We can depict a Motzkin path  $\gamma$  by drawing a line connecting  $\gamma_i$  and  $\gamma_{i+1}$  as follows:



Given a Motzkin path  $\gamma$ , we weight each step  $(x_{i+1}, y_{i+1}) - (x_i, y_i)$  of height  $i$  by  $a_i$  (resp.  $b_i$  and  $c_i$ ) and define the weight of  $\gamma$  as the product of its step weights. For example, the weight of the above Motzkin path is  $\mathbf{w}(\gamma) = a_0 a_1^2 a_2 b_0 b_1 b_2 c_1 c_2^2 c_3$ . Denote by  $\mathfrak{M}_n$  the set

of Motzkin paths of length  $n \geq 1$ . It is folklore (see [GJ83]) that

$$1 + \sum_{n \geq 1} \sum_{\gamma \in \mathfrak{M}_n} \mathbf{w}(\gamma) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\dots}}}. \quad (5.1)$$

Our starting point is the following  $r$ -dilatation of Laguerre's continued fraction:

$$\sum_{n=0}^{\infty} (n! r^n) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\dots}}}, \quad (5.2)$$

where  $a_h = r^2$ ,  $b_h = (2h + 1)r$  ( $h \geq 0$ ),  $c_h = h^2$  ( $h \geq 1$ ). Indeed, this formula is related to the moment sequence of the simple orthogonal Laguerre polynomials when  $r = 1$ . Clearly  $r^n n!$  is the cardinality of  $\mathbb{Z}_r \wr \mathfrak{S}_n$ . In order to make the counterpart of  $\mathbb{Z}_r \wr \mathfrak{S}_n$  in terms of Motzkin paths, we need to generalize the notion of *Laguerre history* [dMV94].

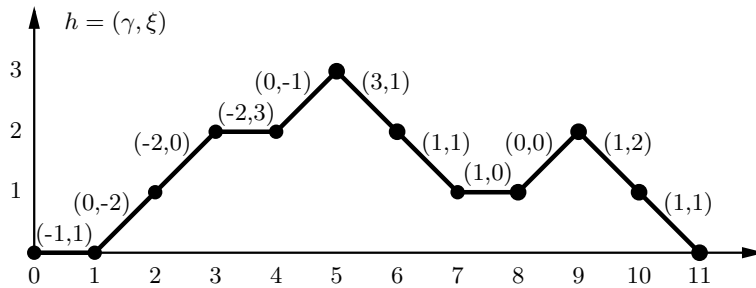
**Definition 18** ( $r$ -colored Laguerre history). *An  $r$ -colored Laguerre history of length  $n$  is a couple  $h = (\gamma, \xi)$ , where  $\gamma$  is a Motzkin path of length  $n$  and  $\xi = ((p_1, q_1), \dots, (p_n, q_n))$  is an integer-pair sequence (as labels of steps) satisfying*

- (i) if the  $i$ th step is North-East, then  $(p_i, q_i) \in [-(r - 1), 0]^2$ ;
- (ii) if the  $i$ th step is East, then  $(p_i, q_i) \in [1, h_i] \times [-(r - 1), 0] \cup [-(r - 1), 0] \times [1, h_i + 1]$ ;
- (iii) if the  $i$ th step is South-East, then  $(p_i, q_i) \in [1, h_i]^2$ ;

where  $h_i$  is the height of the  $i$ th step of  $\gamma$ .

Counting the number of possible pairs  $(p, q)$  for each step of height  $h$ , the number of  $r$ -colored Laguerre histories associated to a Motzkin path  $\gamma$  is equal to the weight  $\mathbf{w}(\gamma)$ . Hence, according to (5.2), the number of  $r$ -colored Laguerre histories of length  $n$  is  $n! r^n$ .

The following is an example of 3-colored Laguerre history of length 11, where the label on the  $i$ th step stands for  $(p_i, q_i)$ .

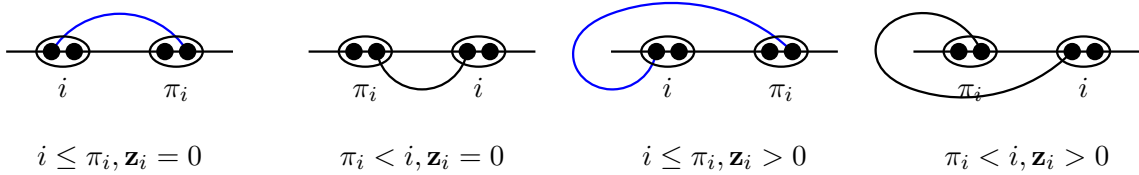


When  $r = 1$ , there are two well-known bijections between  $\mathfrak{S}_n$  and the Laguerre histories of length  $n$  due to Françon-Viennot and Foata-Zeilberger, which correspond to the two interpretations of Eulerian polynomials in  $\mathfrak{S}_n$ : one using linear statistics and the other

using cyclic statistics. The connection between these two bijections is explained in [CSZ97] (see also [dMV94, SZ12]).

The *pignose diagram* representation for a permutation was introduced in [dMV94, CKJV13]. This is a useful device to illustrate the crossings of a permutation by the intersecting points of two arcs. Actually, we can associate a pignose diagram to any  $r$ -colored permutation  $\sigma(= \pi^{\mathbf{z}}) = \begin{pmatrix} \pi_1 \pi_2 \dots \pi_n \\ \mathbf{z}_1 \mathbf{z}_2 \dots \mathbf{z}_n \end{pmatrix} \in \mathbb{Z}_r \wr \mathfrak{S}_n$  (see § 2.2) as follows: A pair of two vertices positioned side by side enclosed by an ellipse is called a *pignose*. We arrange  $n$  pignoses labeled with  $1, 2, \dots, n$  from left to right on an horizontal line. For each  $i \in [n]$ , we connect the *left* vertex of  $i$ th pignose and the *right* vertex of  $\pi_i$ th pignose with an arc:

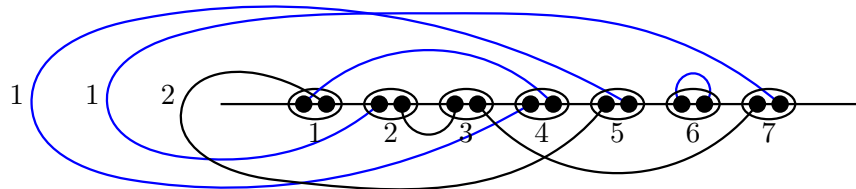
- if  $i \leq \pi_i$  and  $\mathbf{z}_i = 0$ , draw an arc above the horizontal line;
- if  $\pi_i < i$  and  $\mathbf{z}_i = 0$ , draw an arc below the horizontal line;
- if  $\mathbf{z}_i > 0$ , draw an arc starting from the left vertex of  $i$ th pignose below the horizontal line to the right vertex of  $\pi_i$ th pignose above the horizontal line clockwise (like a spiral) and label this spiral arc with  $\mathbf{z}_i$ ;
- optionally, if  $i \leq \pi_i$ , use a blue color in drawing arcs.



Note that  $\text{cros } \sigma$  is equal to the number of crossing points of two arcs in the pignose diagram of  $\sigma$ . For example, the pignose diagram of

$$\sigma = \begin{pmatrix} 4 & 7 & 2 & 5 & 1 & 6 & 3 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_3 \wr \mathfrak{S}_7$$

is depicted as follows:



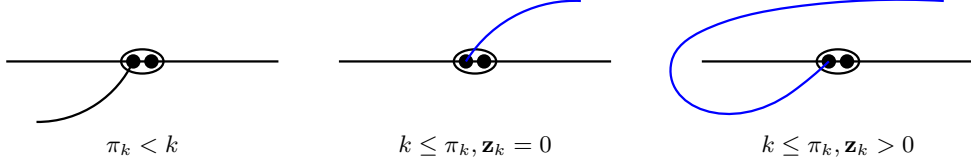
and  $\text{cros}(\sigma) = 6$ .

**5.2. Bijection  $\Phi$ .** We construct a bijection  $\Phi$  between  $\mathbb{Z}_r \wr \mathfrak{S}_n$  and the  $r$ -colored Laguerre histories of size  $n$  by generalizing the Foata-Zeilberger bijection. Given an  $r$ -colored permutation  $\sigma = \sigma(1) \dots \sigma(n) \in \mathbb{Z}_r \wr \mathfrak{S}_n$ , for  $k = 1, \dots, n$ , we build successively the partial pignose diagrams of the restrictions of  $\sigma$  on  $[i]$  by drawing the pignoses labeled with  $1, \dots, k$  along with a half-arc connecting with each vertex in these pignoses. Denote by  $P^{(k)}$  the partial pignose diagram on  $[k]$ . Let  $P^{(0)}$  denote the  $n$  pignoses on the horizontal line. Also  $\gamma^{(0)} = ((0, 0))$  and  $\xi^{(0)} = \emptyset$ . For  $1 \leq k \leq n$ , assume that the mappings

$P^{(k-1)} \mapsto (\gamma^{(k-1)}, \xi^{(k-1)})$ , where

$$\gamma^{(k-1)} = (\gamma_0, \dots, \gamma_{k-1}) \text{ and } \xi^{(k-1)} = (\xi_1, \dots, \xi_{k-1}),$$

are already defined from  $\sigma$ . We show how to extend this to  $k$ . First we draw a half-arc starting from the left vertex of the  $k$ th pignose in  $P^{(k-1)}$  in one of the three possible ways:

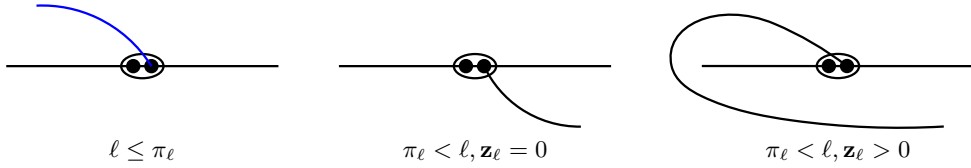


For the first case let

$$p_k = \#\{j : \mathbf{z}_k = \mathbf{z}_j = 0, \pi_k < \pi_j < k < j\} + \#\{j : \mathbf{z}_k > 0, \mathbf{z}_j = 0, \pi_j < k < j\} \\ + \#\{j : \mathbf{z}_k > 0, \mathbf{z}_j > 0, \pi_j < \pi_k < k < j\} + 1,$$

which is positive, for the second case let  $p_k = 0$ , and for the third case let  $p_k = -\mathbf{z}_k$ , which is negative. If  $p_k$  is positive, then  $(p_k - 1)$  means the number of new crossing points by connecting two vertices by a new arc into a partial pignose diagram, otherwise  $-p_k$  means the color of new arc drawn into a partial pignose diagram.

Then we get the partial diagram  $P^{(k)}$  by drawing a half-arc connecting the right vertex in  $k$ th pignose in the above diagram, for the index  $\ell$  such that  $\pi_\ell = k$ , as one of the followings.



For the first case let

$$q_k = \#\{j : \mathbf{z}_\ell = \mathbf{z}_j = 0, \ell < j \leq \pi_\ell < \pi_j\} + \#\{j : \mathbf{z}_\ell > 0, \mathbf{z}_j = 0, j \leq \pi_\ell < \pi_j\} \\ + \#\{i : \mathbf{z}_i > 0, \mathbf{z}_\ell > 0, i < \ell \leq \pi_\ell < \pi_i\} + 1,$$

which is positive, for the second case let  $q_k = 0$ , and for the third case let  $q_k = -\mathbf{z}_\ell$ , which is negative. Similarly, if  $q_k$  is positive,  $(q_k - 1)$  means the number of new crossing points by connecting two vertices by a new arc into a partial pignose diagram, otherwise  $-q_k$  means the color of new arc drawn into a partial pignose diagram.

Lastly, when  $\gamma_{k-1} = (k-1, h_{k-1})$ , we get  $\gamma_k = (k, h_k)$  by defining

- (i)  $h_k = h_{k-1} - 1$  (South-East) if both of  $p_i$  and  $q_i$  are positive,
- (ii)  $h_k = h_{k-1}$  (East) if exactly one of  $p_i$  and  $q_i$  is positive, and
- (iii)  $h_k = h_{k-1} + 1$  (North-East) if none of  $p_i$  and  $q_i$  are positive.

Actually,  $h_k$  means half of the number of open half-arcs, which do not become complete-arcs, in the partial pignose diagram  $P^{(k)}$ .

By taking the above process inductively for  $k = 1, \dots, n$ , we obtain the pignose diagram  $P = P^{(n)}$  of  $\sigma$ . Since there is no half-arcs in  $P^{(n)}$ , we have  $h_n = 0$  and  $\gamma^{(n)}$  is a Motzkin path of length  $n$ . As the  $r$ -colored Laguerre history corresponding to an  $r$ -colored permutation  $\sigma$ , let  $\Phi(\sigma) = (\gamma^{(n)}, \xi^{(n)})$ .

Note that for  $r = 1$ , the above bijection is the same as the Foata-Zeilberger bijection. In Figure 1, we run the above algorithm for the permutation  $\sigma = 4 \bar{7} 2 \bar{5} \bar{1} 6 3 \in \mathbb{Z}_3 \wr \mathfrak{S}_7$  in the previous subsection.

**5.3. Proof of Lemma 16.** Introduce some more statistics on an  $r$ -colored permutation  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ :

$$\begin{aligned} \text{fix } \sigma &= \# \{i \in [n] : i = \pi_i \text{ and } \mathbf{z}_i = 0\}, \\ \text{fix}_c \sigma &= \# \{i \in [n] : i = \pi_i \text{ and } \mathbf{z}_i > 0\}, \\ \text{wex}_A \sigma &= \# \{i \in [n] : i \leq \pi_i \text{ and } \mathbf{z}_i = 0\}, \\ \text{wex}_c \sigma &= \# \{i \in [n] : i \leq \pi_i \text{ and } \mathbf{z}_i > 0\}, \\ \text{drop}_A \sigma &= \# \{i \in [n] : \pi_i < i \text{ and } \mathbf{z}_i = 0\}, \\ \text{drop}_c \sigma &= \# \{i \in [n] : \pi_i < i \text{ and } \mathbf{z}_i > 0\}, \\ \text{csum}_w \sigma &= \sum_{1 \leq i \leq \pi_i \leq n} \mathbf{z}_i, \quad \text{csum}_d \sigma = \sum_{1 \leq \pi_i < i \leq n} \mathbf{z}_i. \end{aligned}$$

For the above example, we have

$$\begin{array}{lll} \text{fix } \sigma = 1, & \text{wex}_A \sigma = 2, & \text{drop}_A \sigma = 2, \\ \text{fix}_c \sigma = 0, & \text{wex}_c \sigma = 2, & \text{drop}_c \sigma = 1, \end{array}$$

also  $\text{csum}_w \sigma = 2$  and  $\text{csum}_d \sigma = 2$ .

For  $n \geq 1$  define the weight of a permutation  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$  by

$$\mathbf{w}(\sigma) := q^{\text{cros } \sigma} t^{\text{wex}_A \sigma} \tilde{t}^{\text{drop}_A \sigma} w^{\text{wex}_c \sigma} \tilde{w}^{\text{drop}_c \sigma} x^{\text{fix } \sigma} \tilde{x}^{\text{fix}_c \sigma} y^{\text{csum}_w \sigma} \tilde{y}^{\text{csum}_d \sigma}. \quad (5.3)$$

Then, we have the following refined version of Lemma 16:

$$1 + \sum_{n \geq 1} \sum_{\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n} \mathbf{w}(\sigma) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\dots}}}, \quad (5.4)$$

where

$$\begin{cases} a_h = (t + wy[r-1]_y q^h)(\tilde{t} + \tilde{w}\tilde{y}[r-1]_{\tilde{y}} q^{h+1}), & c_h = [h]_q^2, \\ b_h = (\tilde{t} + \tilde{w}\tilde{y}[r-1]_{\tilde{y}} q^h)[h]_q + t(x + q[h]_q) + wy[r-1]_y q^h([h]_q + \tilde{x}q^h). \end{cases} \quad (5.5)$$

Clearly, the formula (5.4) becomes Lemma 16 for  $\tilde{y} = y$  and  $\tilde{t} = \tilde{w} = \tilde{x} = 1$ .

In order to interpret  $\mathbf{w}(\sigma)$  in (5.3) as a weight of  $r$ -colored Laguerre history  $\Phi(\sigma)$ , we define the weight of  $k$ th step  $(\gamma_{k-1}, \gamma_k)$  of height  $h_{k-1}$  with its label  $(p_k, q_k)$  of Laguerre



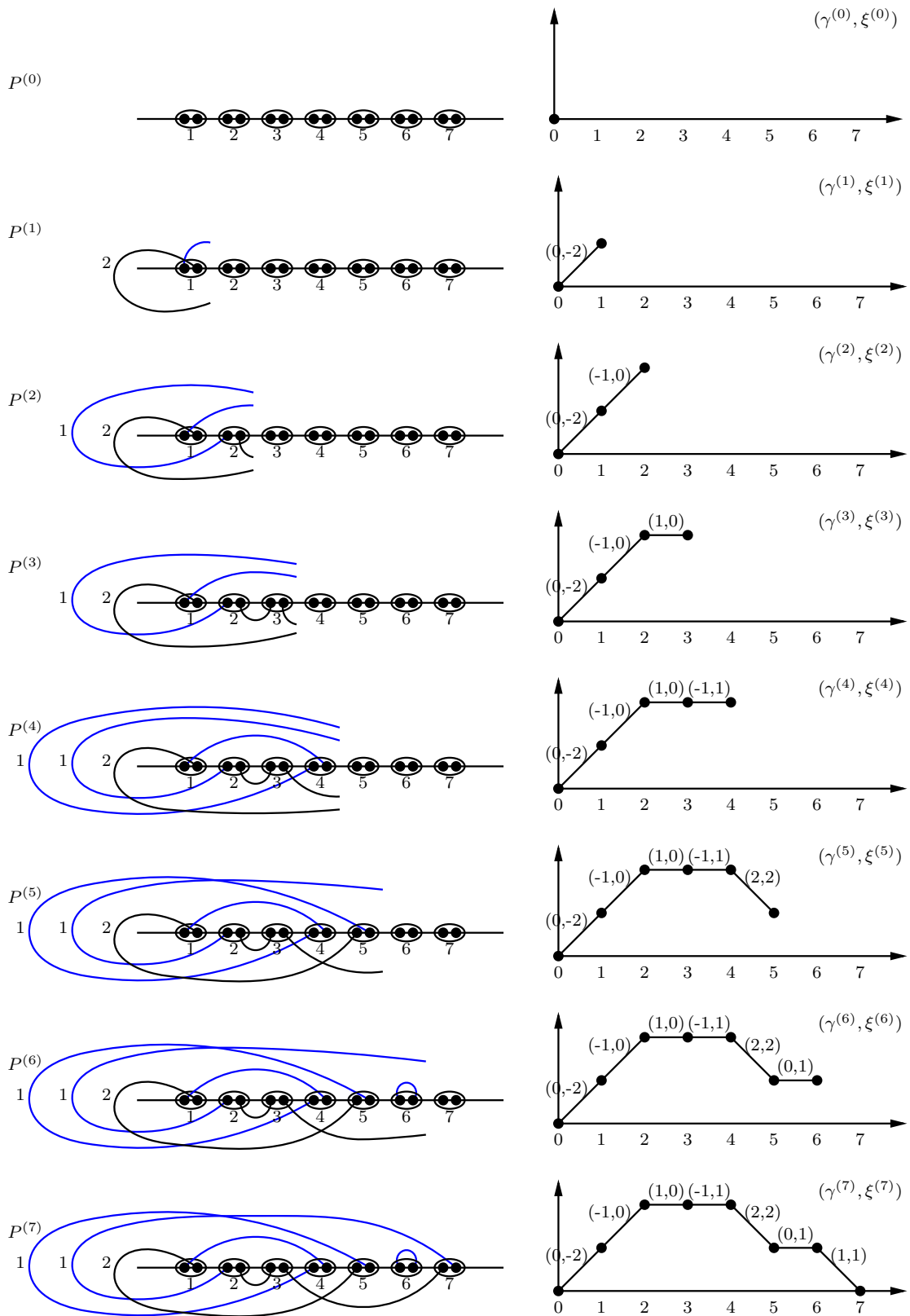


FIGURE 1. An illustration of the bijection  $\Phi$

history  $(\gamma, \xi)$  by

$$\mathbf{w}((\gamma_{k-1}, \gamma_k), (p_k, q_k)) = \begin{cases} \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1}) & \text{if } p_k > 0, \\ x \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1} + 1) & \text{if } p_k = 0 \text{ and } q_k = 1, \\ \tilde{x} \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1} + 1) & \text{if } p_k < 0 \text{ and } q_k = h_{k-1} + 1, \\ \mathbf{w}_l(p_k, h_{k-1}) \cdot \mathbf{w}_r(q_k, h_{k-1} + 1) & \text{otherwise,} \end{cases}$$

where

$$\mathbf{w}_l(z, h) = \begin{cases} q^{z-1} & \text{if } z > 0, \\ t & \text{if } z = 0, \\ wy^{-z}q^h & \text{if } z < 0, \end{cases} \quad \mathbf{w}_r(z, h) = \begin{cases} q^{z-1} & \text{if } z > 0, \\ \tilde{t} & \text{if } z = 0, \\ \tilde{w}\tilde{y}^{-z}q^h & \text{if } z < 0. \end{cases}$$

We find the weight of  $r$ -colored Laguerre history  $(\gamma, \xi)$  of length  $n$  as

$$\mathbf{w}(\gamma, \xi) = \prod_{k=1}^n \mathbf{w}((\gamma_{k-1}, \gamma_k), (p_k, q_k)).$$

It is easy to check the followings.

- (i)  $k = \pi_k, \mathbf{z}_k = 0$  if and only if  $(p_k, q_k) = (0, 1)$ . (for the statistic  $\text{fix}$ )
- (ii)  $k = \pi_k, \mathbf{z}_k > 0$  if and only if  $(p_k, q_k) = (-\mathbf{z}_k, h_{k-1} + 1)$ . (for the statistic  $\text{fix}_c$ )
- (iii)  $k > \pi_k$  if and only if  $p_k > 0$  and we make a full arc, by connecting two half-arcs, which meets  $(p_k - 1)$  open half-arcs. (for the statistic  $\text{cross}$ ).
- (iv)  $k \leq \pi_k, \mathbf{z}_k = 0$  if and only if  $p_k = 0$ . (for the statistic  $\text{wex}_A$ )
- (v)  $k \leq \pi_k, \mathbf{z}_k > 0$  if and only if  $p_k = -\mathbf{z}_k$ . (for two statistics  $\text{wex}_c$  and  $\text{csum}_w$ )
- (vi)  $k = \pi_\ell \geq \ell$  if and only if  $q_k > 0$  and we make a full arc, by connecting two half-arcs, which meets  $(q_k - 1)$  open half-arcs. (for the statistic  $\text{cross}$ )
- (vii)  $k = \pi_\ell < \ell, \mathbf{z}_\ell = 0$  if and only if  $q_k = 0$ . (for the statistic  $\text{drop}_A$ )
- (viii)  $k = \pi_\ell < \ell, \mathbf{z}_\ell > 0$  if and only if  $q_k = -\mathbf{z}_\ell$ . (for two statistics  $\text{drop}_c$  and  $\text{csum}_d$ )
- (ix) if  $\mathbf{z}_\ell > 0$ , we draw a spiral arc, which meets  $h$  open half-arcs. (for the statistic  $\text{cross}$ )

According to the above conditions, given an  $r$ -colored permutation  $\sigma \in \mathbb{Z}_r \wr \mathfrak{S}_n$ , we have  $\mathbf{w}(\sigma) = \mathbf{w}(\gamma, \xi)$ , where  $\Phi(\sigma) = (\gamma, \xi)$ .

Summing the weight of possible pairs  $(p, q)$  for each step of height  $h$ , the weight sum of  $r$ -colored Laguerre histories associated to a Motzkin path  $\gamma$  is equal to the weight  $\mathbf{w}(\gamma)$  with

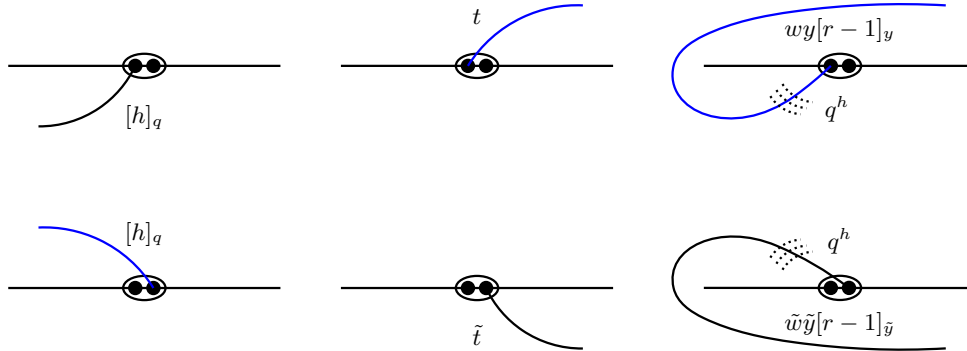
$$\begin{cases} a_h = \sum_{p=1-r}^0 \sum_{q=1-r}^0 \mathbf{w}(h, (p, q)), & c_h = \sum_{p=1}^h \sum_{q=1}^h \mathbf{w}(h, (p, q)), \\ b_n = \sum_{p=1}^h \sum_{q=1-r}^0 \mathbf{w}(h, (p, q)) + \sum_{p=1-r}^0 \sum_{q=1}^{h+1} \mathbf{w}(h, (p, q)), \end{cases} \quad (5.6)$$

where

$$\mathbf{w}(h, (p, q)) = \begin{cases} \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h) & \text{if } p > 0, \\ x \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h + 1) & \text{if } (p, q) = (0, 0), \\ \tilde{x} \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h + 1) & \text{if } (p, q) = (0, h + 1), \\ \mathbf{w}_l(p, h) \cdot \mathbf{w}_r(q, h + 1) & \text{otherwise.} \end{cases}$$

We derive (5.5) from (5.6). □

*Remark.* It is instructive to look at the weight of each step from the corresponding pignose diagram. For  $x = \tilde{x} = 1$ , every vertex is connected to only one half-arc with its weight in one of following six ways:



Here the parameter  $h$  stands for the number of open half-arcs, which do not become complete-arcs, above horizontal line in the partial pignose diagram.

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