

Constructive Tensor Field Theory through an example

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Outline

Random tensors, random spaces

Loop Vertex Expansion

An example at work

Random tensors, random spaces

Random tensors, random spaces

Why?

How?

Loop Vertex Expansion

An example at work

Random surfaces

- $2D$ quantum gravity and matrix models:
 - Matrix models provide a theory of random discrete surfaces weighted by a discretized Einstein-Hilbert action.
 - Evidence for matrices being the right discretization of $2D$ quantum gravity.

- In the last 10 years, much progress from probabilists:
 - Random metric surfaces (e.g. Brownian map).
 - Universal limit of large planar maps.
 - Liouville quantum gravity sphere = Brownian map.

Why tensor fields?

1. Generalize matrix models to higher dimensions
 - w.r.t. their symmetry properties
 - provide a theory of random spaces

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- w.r.t. their symmetry properties
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2. LQG \Rightarrow GFT \Rightarrow T(G)FT

Invariant actions

Symmetry

Consider $T, \bar{T} : \mathbb{Z}^D \rightarrow \mathbb{C}$, complex rank D tensors with *no symmetry*.

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$$T_{n_1 n_2 \dots n_D} \longrightarrow \sum_m U_{n_1 m_1}^{(1)} U_{n_2 m_2}^{(2)} \dots U_{n_D m_D}^{(D)} T_{m_1 m_2 \dots m_D}$$

Invariant actions

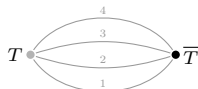
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- Invariants as D -coloured graphs



The diagram shows two vertices, a left one labeled T and a right one labeled \bar{T} . Between them are four curved edges, each labeled with a number from 1 to 4. Edge 1 is the bottom-most, edge 2 is the second from the bottom, edge 3 is the second from the top, and edge 4 is the top-most.

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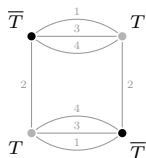
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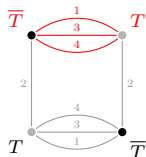
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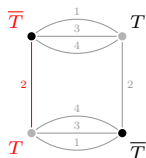
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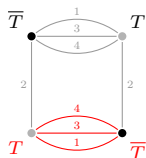
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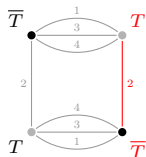
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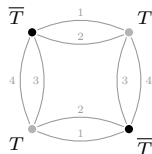
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Invariant actions

Feynman graphs

- Action of a tensor model

$$S(T, \bar{T}) = T \cdot \bar{T} + \sum_{B \in \mathfrak{J}} g_B \text{Tr}_B[T, \bar{T}],$$

$$\mathfrak{J} \subset \{D\text{-coloured graphs of order } \geq 4\}$$

interaction vertices

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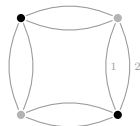
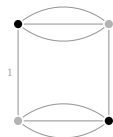
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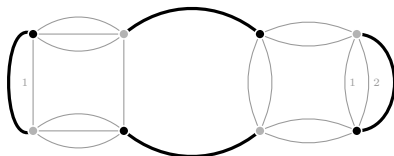
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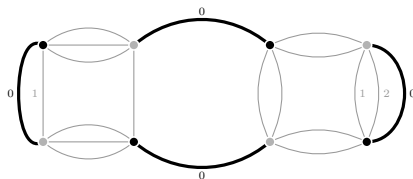
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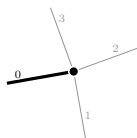
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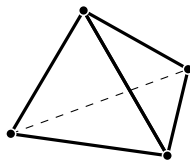
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D -simplex



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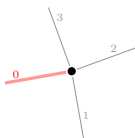
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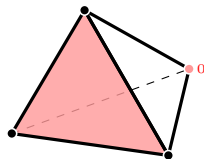
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half-edge



$(D - 1)$ -face



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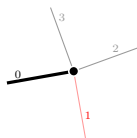
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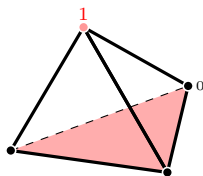
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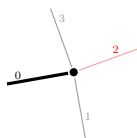
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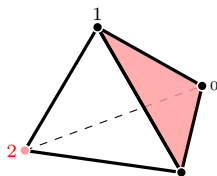
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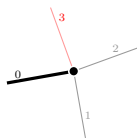
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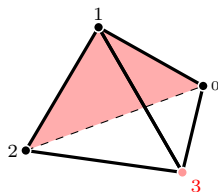
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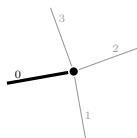
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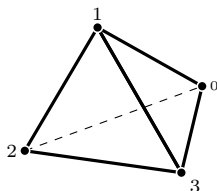
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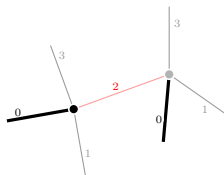
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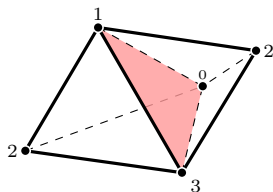
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gluing



Loop Vertex Expansion

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 - Initial goals:
 1. Constructive ϕ_4^{*4} ,
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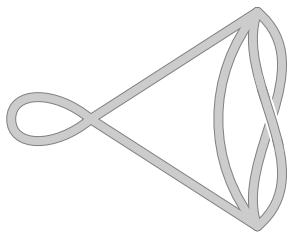
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$n + 2$ faces at order n



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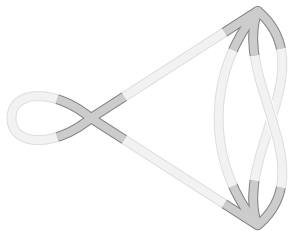
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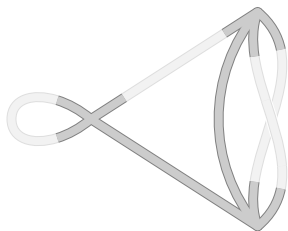
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$2n + 2$ free indices



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$$\begin{aligned} S(\lambda) &= \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} A_G = \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} \sum_{T \subset G} w(G, T) A_G \\ &= \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{T, n(T)=n} A_T, \quad A_T = \sum_{G \supset T} w(G, T) A_G \end{aligned}$$

The BKAR forest formula

- Fix an integer $n \geq 2$.
- f a function of $\frac{n(n-1)}{2}$ variables x_ℓ , sufficiently differentiable.
- K_n , complete graph on $\{1, 2, \dots, n\}$.

$$\#E(K_n) = \frac{n(n-1)}{2}$$

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- K_n , complete graph on $\{1, 2, \dots, n\}$. $\#E(K_n) = \frac{n(n-1)}{2}$

Then,

$$f(1, 1, \dots, 1) = \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \partial_{\mathcal{F}} f(X^{\mathcal{F}}(w_{\mathcal{F}}))$$

where

- the sum is over spanning forests of K_n ,
- $\int dw_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \int_0^1 dw_\ell$,
- $\partial_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial x_\ell}$,
- $X^{\mathcal{F}} = (x_\ell^{\mathcal{F}})_{\ell \in E(K_n)}$ with $x_\ell^{\mathcal{F}}$ the infimum of the $w_{\ell'}$ for ℓ' running over the unique path (if it exists) in \mathcal{F} joining the two ends of ℓ if it exists, 0 otherwise.

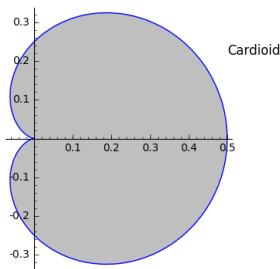
Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$Z(\lambda) = \int_{\mathbb{R}} e^{-\frac{\lambda}{2}\phi^4} d\mu(\phi), \quad d\mu(\phi) = \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

Theorem

$\log Z$ is analytic in the cardioid domain $\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2} \cos^2(\frac{1}{2} \arg \lambda)\}$.



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Proof. LVE is done in 3 steps:

1. Intermediate field representation,
2. Replication of fields,
3. Forest formula.

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

1. Intermediate field representation:

$$e^{-\frac{\lambda}{2}\phi^4} = \int_{\mathbb{R}} e^{i\sigma\phi^2\sqrt{\lambda}} d\mu(\sigma)$$

$$\begin{aligned} Z(\lambda) &= \int_{\mathbb{R}} e^{-\frac{\lambda}{2}\phi^4} d\mu(\phi) \\ &= \int_{\mathbb{R}} e^{V(\sigma)} d\mu(\sigma), \quad V(\sigma) = -\frac{1}{2} \log(1 - i\sigma\sqrt{\lambda}). \end{aligned}$$

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2. Replication of fields:

$$Z(\lambda) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}} V(\sigma)^n d\mu(\sigma) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n V(\sigma_i) \right) d\mu_{\mathbb{1}_n}(\vec{\sigma}).$$

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3. Forest formula:

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{\mathcal{F} \subset K_n} \int dw_{\mathcal{F}} \int \left[\prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{F}}(w)}(\vec{\sigma})$$

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$$\log Z = \sum_{n \geq 1} \frac{1}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{\ell \in E(\mathcal{T})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$\begin{aligned}\log Z &= \sum_{n \geq 1} \frac{1}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{\ell \in E(\mathcal{T})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C\mathcal{T}(w)}(\vec{\sigma}) \\ &= \frac{1}{2} \sum_{n \geq 1} \frac{(-\lambda/2)^{n-1}}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \left[\prod_{i=1}^n \frac{(d_i - 1)!}{(1 - \nu \sigma_i \sqrt{\lambda})^{d_i}} \right] d\mu_{C\mathcal{T}(w)}(\vec{\sigma}).\end{aligned}$$

Loop Vertex Expansion

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Using $|1 - \nu \sigma \sqrt{\lambda}| \geq \cos(\frac{1}{2} \arg \lambda)$, we get

$$\begin{aligned}|\log Z| &\leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{|\lambda|}{2 \cos^2(\frac{1}{2} \arg \lambda)} \right)^{n-1} \sum_{\mathcal{T} \subset K_n} \prod_{i=1}^n (d_i - 1)! \\ &\leq 2 \sum_{n=1}^{\infty} \left(\frac{2|\lambda|}{\cos^2(\frac{1}{2} \arg \lambda)} \right)^{n-1}\end{aligned}$$

which is convergent for all $\lambda \in \mathbb{C}$ such that $|\lambda| < \frac{1}{2} \cos^2(\frac{1}{2} \arg \lambda)$. \square

An example at work

Random tensors, random spaces

Loop Vertex Expansion

An example at work

Presentation and main result

Sketch of the proof

The T_4^4 field theory

- Tensor fields:

$$T : \mathbb{Z}^4 \rightarrow \mathbb{C}, \quad T_{\mathbf{n}}, \overline{T_{\overline{\mathbf{n}}}} \text{ with } \mathbf{n}, \overline{\mathbf{n}} \in \mathbb{Z}^4.$$

The T_4^4 field theory

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$$T : \mathbb{Z}^4 \rightarrow \mathbb{C}, \quad T_{\mathbf{n}}, \bar{T}_{\bar{\mathbf{n}}} \text{ with } \mathbf{n}, \bar{\mathbf{n}} \in \mathbb{Z}^4.$$

- Free action:

$$d\mu_C(T, \bar{T}) = \left(\prod_{\mathbf{n}, \bar{\mathbf{n}}} \frac{dT_{\mathbf{n}} d\bar{T}_{\bar{\mathbf{n}}}}{2i\pi} \right) \det(C^{-1}) e^{-\sum_{\mathbf{n}, \bar{\mathbf{n}}} T_{\mathbf{n}} C_{\bar{\mathbf{n}}\mathbf{n}}^{-1} \bar{T}_{\bar{\mathbf{n}}}},$$

$$C_{\mathbf{n}, \bar{\mathbf{n}}} = \frac{\mathbf{1}_{\leq j_{\max}} \mathbf{n} \bar{\mathbf{n}}}{\mathbf{n}^2 + 1} \delta_{\mathbf{n}, \bar{\mathbf{n}}}, \quad \mathbf{n}^2 := n_1^2 + n_2^2 + n_3^2 + n_4^2,$$

$$\mathbf{1}_{\leq j_{\max}} = \mathbf{1}_{\mathbf{n}^2 + 1 \leq M^{2j_{\max}}} \delta_{\mathbf{n}, \bar{\mathbf{n}}}.$$

The T_4^4 field theory

- Tensor fields:

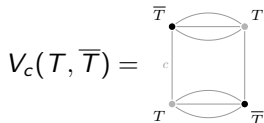
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- Interactions:

$$V(T, \bar{T}) = \frac{g}{2} \sum_{c=1}^4 V_c(T, \bar{T}),$$



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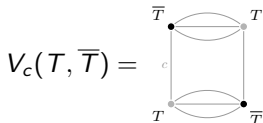
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$$Z_0(g) = \int e^{-\frac{g}{2} \sum_c V_c(T, \bar{T})} d\mu_C(T, \bar{T}).$$

The T_4^4 field theory

Perturbative renormalisation

Proposition

T_4^4 is renormalisable to all orders of perturbation.

The T_4^4 field theory

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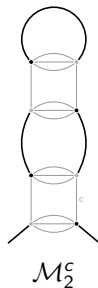
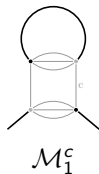
The T_4^4 field theory

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The T_4^4 field theory

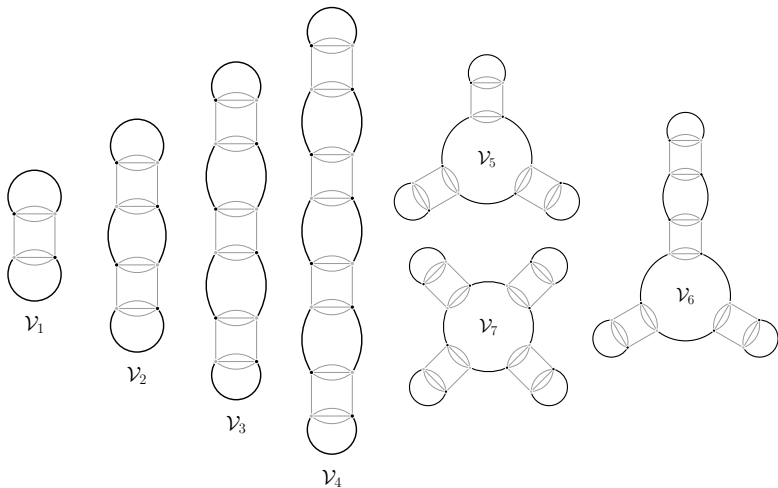
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- 7 divergent melonic vacuum graphs

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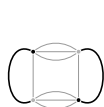
The T_4^4 field theory

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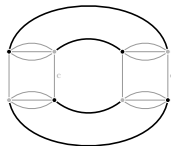
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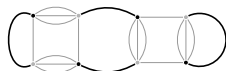
- Power counting similar to the one of ϕ_3^4 .
- 2 divergent 2-point graphs: $\mathcal{M}_1^c, \mathcal{M}_2^c$
- 7 divergent melonic vacuum graphs: $\mathcal{V}_i, i = 1, 2, \dots, 7$
- 3 divergent non melonic vacuum graphs:



\mathfrak{N}_1



\mathfrak{N}_2



\mathfrak{N}_3

The T_4^4 field theory

Analyticity

- Renormalised partition function:

$$Z_{j_{\max}}(\mathbf{g}) = \mathcal{N} \int e^{-\frac{\mathbf{g}}{2} \sum_c v_c(T, \bar{T}) + T \cdot \bar{T} \left(\sum_{G \in \mathcal{M}} \frac{(-\mathbf{g})^{|G|}}{s_G} \delta_G \right)} d\mu_C(T, \bar{T}),$$

$$\mathcal{M} = \{\mathcal{M}_1^c, \mathcal{M}_2^c, c = 1, 2, 3, 4\},$$

$\log \mathcal{N}$ = (finite) sum of the counterterms of the divergent vacuum connected graphs.

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$\log \mathcal{N}$ = (finite) sum of the counterterms of the divergent vacuum connected graphs.

Theorem (Rivasseau, V.-T. 2017)

There exists $\rho > 0$ such that $\lim_{j_{\max} \rightarrow \infty} \log Z_{j_{\max}}(g)$ is analytic in the cardioid domain defined by $|\arg g| < \pi$ and $|g| < \rho \cos^2(\frac{1}{2} \arg g)$.

The general strategy

0. Renormalised partition function:

$$Z_{j_{\max}}(\mathbf{g}) = \mathcal{N} \int e^{-\frac{\mathbf{g}}{2} \sum_c V_c(T, \bar{T}) + T \cdot \bar{T} \left(\sum_{G \in \mathcal{M}} \frac{(-\mathbf{g})^{|G|}}{s_G} \delta_G \right)} d\mu_C(T, \bar{T}).$$

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1. Intermediate field representation: $\sigma_c \in \text{Herm}_{M_{j_{\max}}}$, $c = 1, 2, 3, 4$

$$Z_{j_{\max}}(g) = \mathcal{N} e^{\delta_t} \int e^{-\text{Tr} \log(\mathbb{I} - \Sigma) - \imath \lambda \sum_c \delta_m^c \text{Tr}_c \sigma_c} d\nu_{\mathbb{I}}(\vec{\sigma})$$

$$\lambda = g^{1/2}$$

$$\Sigma = \imath \lambda C^{1/2} \sigma C^{1/2}$$

$$\sigma = \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \mathbb{I}_1 \otimes \sigma_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \dots$$

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2. Renormalised action:

$$Z_{j_{\max}}(\mathbf{g}) = \int e^{-\text{Tr} \log_3(\mathbb{I} - U) - \text{Tr}(D_1 \Sigma^2) - \frac{\lambda^2}{2} : \vec{\sigma} \cdot \mathbf{Q} \vec{\sigma} :} d\nu_{\mathbb{I}}(\vec{\sigma})$$

$$U = \Sigma + D_1 + D_2$$

$$D_1 = -\lambda^2 C^{1/2} A_{\mathcal{M}_1}^r C^{1/2}, \quad D_2 = \lambda^4 C^{1/2} A_{\mathcal{M}_2}^r C^{1/2}$$

$$\log_3(\mathbb{I} - U) = \log(\mathbb{I} - U) + U + \frac{U^2}{2}$$

The general strategy

3. Multiscale decomposition:

$$C_{\leq j} := \delta_{n, \bar{n}} \frac{(\mathbf{1}_{\leq j})_{n\bar{n}}}{n^2 + 1}$$

$$V_{\leq j} = \text{Tr} \log_3[\mathbb{I} - U_{\leq j}] + \text{Tr}[D_{1, \leq j} \Sigma_{\leq j}^2] + \frac{\lambda^2}{2} : \vec{\sigma} \cdot Q_{\leq j} \vec{\sigma} :$$

$$V_{\leq j_{\max}} = \sum_{j=1}^{j_{\max}} (V_{\leq j} - V_{\leq j-1}) =: \sum_{j=1}^{j_{\max}} V_j$$

$$Z_{j_{\max}}(g) = \int \prod_j e^{-V_j} d\nu_{\mathbb{I}}(\vec{\sigma}) = \int e^{-\sum_j \bar{\chi}_j W_j(\sigma) \chi_j} d\nu_{\mathbb{I}}(\vec{\sigma}) d\mu(\bar{\chi}, \chi)$$

$$W_j = e^{-V_j} - 1$$

The general strategy

0. Renormalised partition function:

$$Z_{j_{\max}}(\mathbf{g}) = \mathcal{N} \int e^{-\frac{\mathbf{g}}{2} \sum_c V_c(T, \bar{T}) + T \cdot \bar{T} \left(\sum_{G \in \mathcal{M}} \frac{(-\mathbf{g})^{|G|}}{5_G} \delta_G \right)} d\mu_C(T, \bar{T}).$$

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The general strategy

4. Multiscale Loop Vertex Expansion:

- 2 forest formulas on top of each other
- First, a Bosonic forest then a Fermionic one

[Gurau, Rivasseau 2014]

(2-jungle formula)

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4. Multiscale Loop Vertex Expansion:

[Gurau, Rivasseau 2014]

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(2-jungle formula)

$$\log Z_{j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (-\bar{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\vec{\sigma}^a) \chi_{j_a}^{\mathcal{B}}) \right]$$

- $\mathbf{w}_{\mathcal{J}}$ = weakening parameters w_{ℓ} , $\ell \in E(\mathcal{J})$
- $\nu_{\mathcal{J}}$ = interpolated Gaussian Bosonic and Fermionic measures
- $\partial_{\mathcal{J}}$ = derivatives with respect to the σ -, χ - and $\bar{\chi}$ -fields

The general strategy

4. Multiscale Loop Vertex Expansion:

[Gurau, Rivasseau 2014]

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$$\begin{aligned}\log Z_{j_{\max}}(g) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} (-\bar{\chi}_{j_a}^{\mathcal{B}} W_{j_a}(\vec{\sigma}^a) \chi_{j_a}^{\mathcal{B}}) \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\mathbf{w}_{\mathcal{J}} \prod_{\mathcal{B}} \left(\prod_{\substack{a, b \in \mathcal{B} \\ a \neq b}} (1 - \delta_{j_a j_b}) \right) I_{\mathcal{B}} \\ I_{\mathcal{B}} &= \int \partial_{\mathcal{B}} \prod_{a \in \mathcal{B}} W_{j_a}(\vec{\sigma}^a) d\nu_{\mathcal{B}} = \sum_{\mathcal{G}} \int \left(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \right) A_{\mathcal{G}}(\vec{\sigma}) d\nu_{\mathcal{B}}(\vec{\sigma})\end{aligned}$$

- Graphs \mathcal{G} are plane forests.

The general strategy

Bosonic bounds

$$I_{\mathcal{B}} = \sum_{\mathbf{G}} \int \left(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \right) A_{\mathbf{G}}(\vec{\sigma}) d\nu_{\mathcal{B}}(\vec{\sigma})$$
$$|I_{\mathcal{B}}| \leq \underbrace{\left(\int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|} d\nu_{\mathcal{B}} \right)^{1/2}}_{I_{\mathcal{B}}^{NP}, \text{ non-perturbative}} \sum_{\mathbf{G}} \underbrace{\left(\int |A_{\mathbf{G}}(\vec{\sigma})|^2 d\nu_{\mathcal{B}} \right)^{1/2}}_{I_{\mathcal{B}}^P, \text{ perturbative}}$$

The general strategy

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5. Non-perturbative bound:

Theorem

For ρ small enough and for any value of the w interpolating parameters,

$$I_{\mathcal{B}}^{NP} = \int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|} d\nu_{\mathcal{B}} \leq O(1)^{|\mathcal{B}|}.$$

The general strategy

Bosonic bounds

$$|I_{\mathcal{B}}| \leq \underbrace{\left(\int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|} d\nu_{\mathcal{B}} \right)^{1/2}}_{I_{\mathcal{B}}^{NP}, \text{ non-perturbative}} \sum_G \underbrace{\left(\int |A_G(\vec{\sigma})|^2 d\nu_{\mathcal{B}} \right)^{1/2}}_{I_{\mathcal{B}}^P, \text{ perturbative}}$$

6. Perturbative bound:

Theorem

Let \mathcal{B} be a Bosonic block. Then there exists $K \in \mathbb{R}_+^*$ such that

$$I_{\mathcal{B}}^P(G) \leq K^{|\mathcal{B}|-1} ((|\mathcal{B}|-1)!)^{37/2} \rho^{e(G)} \prod_{a \in \mathcal{B}} M^{-\frac{1}{48}j_a}.$$

The general strategy

0. Renormalised partition function
1. Intermediate field representation
2. Renormalised action
3. Multiscale decomposition
4. Multiscale Loop Vertex Expansion
5. Non-perturbative Bosonic bound
6. Perturbative Bosonic bound

The general strategy

0. Renormalised partition function
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6. Perturbative Bosonic bound
7. Put everything together and cross your fingers. . . □

Conclusion and perspectives

- Tensor field theory provides a combinatorial theory of random D -spaces.
- In the last 8 years, a lot of results with tensors:
 $\frac{1}{N}$ -expansion, perturbative and constructive renormalisation, continuum limit of the dominant triangulations, double scaling limit, uniform random complexes etc.
- Regarding T_4^4 , one could also prove Borel summability and analyticity of the connected correlation functions.

Conclusion and perspectives

- Tensor field theory provides a combinatorial theory of random D -spaces.
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-
- T_5^4 (just renormalisable, asymptotically free)
 - New Loop Vertex Representation
 - Simplify Bosonic constructive theory?

[Rivasseau 2017]

Thank you for your attention

Non-perturbative bounds

Theorem

For ρ small enough and for any value of the w interpolating parameters,

$$I_B^{NP} = \int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|} d\nu_{\mathcal{B}} \leq O(1)^{|\mathcal{B}|} e^{O(1)\rho^{3/2}|\mathcal{B}|}.$$

Proof.

1. Expand each node:

$$e^{2|V_{j_a}|} = \sum_{k=0}^{P_a} \frac{(2|V_{j_a}|)^k}{k!} + \int_0^1 dt_{j_a} (1 - t_{j_a})^{P_a} \frac{(2|V_{j_a}|)^{P_a+1}}{P_a!} e^{2t_{j_a}|V_{j_a}|}.$$

2. Crude non-perturbative bound: (Quadratic bound)

$$\int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|} d\nu_{\mathcal{B}} \leq K^{|\mathcal{B}|} e^{K' \rho M^{j_1}}.$$

3. Power counting (via quartic bound) beats both combinatorics and the crude non-perturbative bound. □

Perturbative bound

Theorem

Let \mathcal{B} be a Bosonic block. Then there exists $K \in \mathbb{R}_+^*$ such that

$$I_{\mathcal{B}}^P(\mathbb{G}) = \int |A_{\mathbb{G}}(\vec{\sigma})|^2 d\nu_{\mathcal{B}} \leq K^{|\mathcal{B}|-1} ((|\mathcal{B}| - 1)!)^{37/2} \rho^{\epsilon(\mathbb{G})} \prod_{a \in \mathcal{B}} M^{-\frac{1}{48}j_a}.$$

- $A_{\mathbb{G}}(\vec{\sigma})$ depends on σ (essentially) through resolvents.
- If not for resolvents, $A_{\mathbb{G}}$ would be the amplitude of a convergent graph.
- Remove resolvents with iterated Cauchy-Schwarz estimates.