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# A $q$ -Clausen-Orr type formula and its applications

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**Abstract.** We show that certain terminating  ${}_6\phi_5$  series can be factorized into a product of two  ${}_3\phi_2$  series. As applications we prove a summation formula for a product of two  $q$ -Delannoy numbers along with some congruences for sums involving  $q$ -Delannoy numbers. This confirms three recent conjectures of the second author.

*Keywords:*  $q$ -Delannoy numbers;  $q$ -analogue of Clausen's formula;  $q$ -binomial theorem;  $q$ -Chu-Vandermonde;  $q$ -Pfaff-Saalschütz.

*2000 Mathematics Subject Classifications:* Primary 11B65, Secondary 05A10, 05A30

## 1. Introduction

Clausen's formula

$$\left\{ {}_2F_1 \left[ \begin{matrix} a, b \\ a + b + 1/2 \end{matrix} ; z \right] \right\}^2 = {}_3F_2 \left[ \begin{matrix} 2a, 2b, a + b \\ a + b + 1/2, 2a + 2b \end{matrix} ; z \right]$$

plays a central role in Ramanujan's derivation for various series for  $1/\pi$ . See [1, 3] for some recent developments of this formula. More general formulas connecting products of two hypergeometric series as a single series were obtained by Orr in 1899 (see [15, p. 75]). This paper was motivated by a recent paper of the second author [7], where he proved some congruences of sums involving even powers of Delannoy numbers and raised some problems of finding the  $q$ -analogues. Recall that the Delannoy numbers count lattice paths from  $(0, 0)$  to  $(n, m)$  consisting of horizontal  $(1, 0)$ , vertical  $(0, 1)$ , and diagonal  $(1, 1)$  steps, and have the following explicit formulas in terms of binomial coefficients:

$$D(m, n) := \sum_{k=0}^n \binom{n}{k} \binom{n+m-k}{n} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} 2^k. \quad (1.1)$$

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The reader is referred to Dziemiańczuk [4] and the references therein for how to generalize Delannoy numbers via counting weighted lattice paths. Recall that the *basic hypergeometric series*  ${}_r\phi_s$  is defined as

$${}_r\phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1; q)_k (a_2; q)_k \cdots (a_r; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_s; q)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k,$$

where  $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$  for  $n = 1, 2, \dots$ , and  $(a; q)_0 = 1$ . Aiming to answer the  $q$ -problems in [7], we are led to prove the following  $q$ -Clausen-Orr type formula.

**Theorem 1.1.** *Let  $n$  be a non-negative integer. Then*

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, x \\ c, 0 \end{matrix} ; q, q \right] {}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, c/x \\ c, 0 \end{matrix} ; q, q \right] = a^n {}_6\phi_5 \left[ \begin{matrix} q^{-n}, cq^n, a, c/a, x, c/x \\ c, \sqrt{c}, -\sqrt{c}, \sqrt{cq}, -\sqrt{cq} \end{matrix} ; q, q \right]. \quad (1.2)$$

It is interesting to compare (1.2) with Jackson's  $q$ -analogue of Clausen's formula [9,10]:

$${}_2\phi_1 \left[ \begin{matrix} a, b \\ abq^{\frac{1}{2}} \end{matrix} ; q, z \right] {}_2\phi_1 \left[ \begin{matrix} a, b \\ abq^{\frac{1}{2}} \end{matrix} ; q, zq^{\frac{1}{2}} \right] = {}_4\phi_3 \left[ \begin{matrix} a, b, a^{\frac{1}{2}}b^{\frac{1}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}} \\ ab, a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{4}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{4}} \end{matrix} ; q^{\frac{1}{2}}, z \right], \quad |z| < 1,$$

and the following  $q$ -analogue of Clausen's formula [5, Appendix (III.22)]:

$$\left\{ {}_4\phi_3 \left[ \begin{matrix} a, b, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab \end{matrix} ; q, q \right] \right\}^2 = {}_5\phi_4 \left[ \begin{matrix} a^2, b^2, ab, abz, ab/z \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}, -ab, a^2b^2 \end{matrix} ; q, q \right], \quad (1.3)$$

where both series are supposed to be terminated. Indeed, letting  $c = x^2$ , the identity (1.2) reduces to the following formula, which seems to be new.

**Corollary 1.2.** *Let  $n$  be a non-negative integer. Then*

$$\left\{ {}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, x \\ x^2, 0 \end{matrix} ; q, q \right] \right\}^2 = a^n {}_5\phi_4 \left[ \begin{matrix} q^{-n}, x^2q^n, a, x^2/a, x \\ x^2, -x, xq^{\frac{1}{2}}, -xq^{\frac{1}{2}} \end{matrix} ; q, q \right]. \quad (1.4)$$

In particular, the right-hand side of (1.4) is non-negative for real  $a, x$ , and  $q$ . Furthermore, if  $n$  is even, then by (1.3), the right-hand side of (1.4) may be written as

$$a^n \left\{ {}_4\phi_3 \left[ \begin{matrix} q^{-\frac{n}{2}}, xq^{\frac{n}{2}}, a, x^2/a \\ xq^{\frac{1}{2}}, -xq^{\frac{1}{2}}, -x \end{matrix} ; q, q \right] \right\}^2.$$

Writing  $n = 2m$  in (1.4) and taking the square root we obtain an identity between two polynomials in  $a$  of degree  $2m$ , where the sign is determined by comparing the coefficients of  $a^{2m}$ . We record the resulting formula as the second corollary.

**Corollary 1.3.** *Let  $m$  be a non-negative integer. Then*

$${}_3\phi_2 \left[ \begin{matrix} q^{-2m}, a, x \\ x^2, 0 \end{matrix} ; q, q \right] = a^m {}_4\phi_3 \left[ \begin{matrix} q^{-m}, xq^m, a, x^2/a \\ xq^{\frac{1}{2}}, -xq^{\frac{1}{2}}, -x \end{matrix} ; q, q \right].$$

For some other  $q$ -Clausen type formulas, the reader is referred to Gasper and Rahman [5, Exercise 8.17] and Schlosser [13]. On the other hand, in their study of some  $q$ -supercongruences for certain truncated basic hypergeometric series related to [16, 17], Guo and Zeng [8] stumbled on the following  $q$ -Clausen-Orr type formula:

$$\begin{aligned} & \left( \sum_{k=s}^n \frac{(q^{-2n}; q^2)_k (x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \left( \sum_{k=s}^n \frac{(q^{-2n}; q^2)_k (q/x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \\ &= \frac{(-1)^n (q^2; q^2)_n^2 q^{-n^2}}{(q^2; q^2)_{n-s} (q^2; q^2)_{n+s}} \sum_{k=s}^n \frac{(-1)^k (q^2; q^2)_{n+k} (x; q)_k (q/x; q)_k q^{k^2-2nk}}{(q^2; q^2)_{n-k} (q; q)_{k-s} (q; q)_{k+s} (q; q)_{2k}}. \end{aligned} \quad (1.5)$$

Noticing that  $(c; q)_{2k} = (\sqrt{c}; q)_k (-\sqrt{c}; q)_k (\sqrt{cq}; q)_k (-\sqrt{cq}; q)_k$ , we can rewrite (1.2) as

$$\begin{aligned} & \left( \sum_{k=0}^n \frac{(q^{-n}; q)_k (a; q)_k (x; q)_k q^k}{(q; q)_k (c; q)_k} \right) \left( \sum_{k=0}^n \frac{(q^{-n}; q)_k (a; q)_k (c/x; q)_k q^k}{(q; q)_k (c; q)_k} \right) \\ &= a^n \sum_{k=0}^n \frac{(q^{-n}; q)_k (cq^n; q)_k (a; q)_k (c/a; q)_k (x; q)_k (c/x; q)_k q^k}{(q; q)_k (c; q)_k (c; q)_{2k}}. \end{aligned} \quad (1.6)$$

Clearly (1.6) is an extension of (1.5). In the same vein we shall establish the following result.

**Theorem 1.4.** *Let  $n$  be a non-negative integer. Then*

$$\begin{aligned} & \left( \sum_{k=0}^n \frac{(q^{-n}; q)_k (x; q^2)_k q^k}{(q; q)_k (c; q)_k} \right) \left( \sum_{k=0}^n \frac{(q^{-n}; q)_k (x; q^2)_k c^k q^{nk - \binom{k}{2}}}{(q; q)_k (c; q)_k x^k} \right) \\ &= \sum_{k=0}^n \frac{(q^{-n}; q)_k (cq^n; q)_k (x; q^2)_k (c^2/x; q^2)_k q^k}{(q; q)_k (c; q)_k (c; q)_{2k}}. \end{aligned} \quad (1.7)$$

In this paper we shall consider two  $q$ -analogues of  $D(m, n)$ . We first recall some standard  $q$ -notation (see [5]). The  $q$ -binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} & \text{if } n \geq k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following two natural  $q$ -analogues of Delannoy numbers were introduced in [4, p. 30] and [12]:

$$D_q(m, n) := \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+m-k \\ n \end{bmatrix}, \quad (1.8)$$

$$D_q^*(m, n) := \sum_{k=0}^n q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+m-k \\ n \end{bmatrix}. \quad (1.9)$$

Note that  $D_q^*(m, n) := q^{mn} D_{q^{-1}}(m, n)$ . We first show that both  $D_q(m, n)$  and  $D_q^*(m, n)$  have a  $q$ -analogue of the second expression in (1.1) and provide a  $q$ -analogue of [7, (3.1)], which was asked in [7, Problem 5.2].

**Theorem 1.5.** *Let  $m$  and  $n$  be non-negative integers. Then*

$$D_q(m, n) = \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-1; q)_k, \quad (1.10)$$

$$D_q^*(n, m) = \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-q; q)_k. \quad (1.11)$$

Moreover,

$$D_q(m, n) D_q^*(m, n) = \sum_{k=0}^n q^{(m-k)(n-k)} \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} (-1; q)_k (-q; q)_k. \quad (1.12)$$

Applying the formula (1.12), we shall prove the following two results originally conjectured by the second author [7, Conjectures 5.3 and 5.4].

**Theorem 1.6.** *Let  $p$  be an odd prime and  $m$  a positive integer. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{1 - q^{2k+1}}{1 - q} D_q(m, k) D_{q^{-1}}(m, k) q^{-k} \\ & \equiv \begin{cases} \frac{1 - q^{-2m}}{1 - q^2} q \pmod{[p]^2} & \text{if } m \equiv 0 \pmod{p}, \\ \frac{1 - q^{2m+2}}{1 - q^2} q \pmod{[p]^2} & \text{if } m \equiv -1 \pmod{p}, \\ 0 \pmod{[p]^2} & \text{otherwise,} \end{cases} \end{aligned} \quad (1.13)$$

where  $[p] = 1 + q + \cdots + q^{p-1}$ , and the congruences are understood in the polynomial ring  $\mathbb{Z}[q]$ .

**Theorem 1.7.** *Let  $m$ ,  $n$ , and  $r$  be positive integers. Then all of*

$$\sum_{k=0}^{n-1} \frac{(1 - q^m)(1 - q^{m+1})(1 - q^{2k+1})}{(1 - q^2)(1 - q^n)^2} D_q(m, k) D_{q^{-1}}(m, k) q^{-k}, \quad (1.14)$$

$$\sum_{k=0}^{n-1} \frac{1 - q^{2k+1}}{1 - q^n} D_q(m, k)^r D_{q^{-1}}(m, k)^r q^{-k}, \quad (1.15)$$

$$\sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{1 - q^{2k+1}}{1 - q^n} D_q(m, k)^r D_{q^{-1}}(m, k)^r q^{\binom{k}{2}} \quad (1.16)$$

are Laurent polynomials in  $q$  with non-negative integer coefficients.

Note that Theorem 1.7 is a  $q$ -analogue of [7, Theorem 1.1] for the first three polynomials. The rest of the paper is organized as follows. We shall give three lemmas in Section 2 and prove Theorem 1.1 in Section 3. In Sections 4 and 5 we prove Theorems 1.4 and 1.5. In Sections 6 and 7, by using Theorem 1.5 we give proofs of Theorems 1.6 and 1.7 respectively.

## 2. Three lemmas

The following three lemmas are crucial ingredients of our proof of Theorem 1.1.

**Lemma 2.1.** *Let  $n$  and  $h$  be positive integers and let  $m$  be a non-negative integer with  $h \leq n - m$ . Then*

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (x; q)_j (x; q)_k (q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}}{(q; q)_j (q; q)_k (c; q)_j (c; q)_k} \\ &= \frac{(q; q)_n (q; q)_{h-1} (x; q)_{m+h} (c/x; q)_{n-h} x^{n-h} q^{\frac{m^2+3m}{2} - mn - mh - h^2 + h}}{(-1)^{m-1} (q; q)_m (c; q)_m (c; q)_n (q; q)_{n-m-h}}. \end{aligned} \quad (2.1)$$

*Proof.* Note that both sides of (2.1) are polynomials in  $x$  of degree  $m + n$  with the same leading coefficient. Therefore, to prove (2.1), it suffices to prove that both sides have the same roots as polynomials in  $x$ . Denote the left-hand side of (2.1) by  $L_{m,n}(x)$ . We first assert that

$$\begin{aligned} L_{m,n}(x) &= \sum_{j=0}^m \sum_{k=m+1}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (x; q)_j (x; q)_k}{(q; q)_j (q; q)_k (c; q)_j (c; q)_k} \\ &\quad \times (q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}. \end{aligned} \quad (2.2)$$

In fact, since  $(1 - q^{j-k}) q^{2k+j} = -(1 - q^{k-j}) q^{k+2j}$ , we have  $\sum_{j=0}^m \sum_{k=0}^m = 0$  for the summands in  $L_{m,n}(x)$ . We now consider the following two cases.

- For  $x = q^{-r}$  with  $0 \leq r \leq m + h - 1$ , we have

$$\begin{aligned} L_{m,n}(q^{-r}) &= \sum_{j=0}^m \sum_{k=0}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (q^{-r}; q)_j (q^{-r}; q)_k}{(q; q)_j (q; q)_k (c; q)_j (c; q)_k} \\ &\quad \times (q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}. \end{aligned}$$

If  $r \leq m$ , then  $L_{m,n}(q^{-r}) = 0$  by the antisymmetry of  $j$  and  $k$  in  $L_{m,n}(q^{-r})$ . If  $r \geq m + 1$ , then  $h \geq r - m + 1$ , i.e.,  $r - m - h + 1 \leq 0$ , and so  $(q^{k-m-h+1}; q)_{h-1} = 0$  for  $m + 1 \leq k \leq r$ . Hence, by (2.2), we again get  $L_{m,n}(q^{-r}) = 0$ .

- For  $x = cq^r$  with  $0 \leq r \leq n - h - 1$ , we shall prove that

$$\sum_{k=0}^n \frac{(q^{-n}; q)_k (cq^r; q)_k (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}}{(q; q)_k (c; q)_k} = 0. \quad (2.3)$$

In fact, we can rewrite the left-hand side of (2.3) as

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{-nk + \binom{k}{2}} R_k, \quad (2.4)$$

where

$$R_k = \frac{(cq^r; q)_k (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}}{(c; q)_k}.$$

Since

$$\frac{(cq^r; q)_k}{(c; q)_k} = \frac{(cq^k; q)_r}{(c; q)_r},$$

we see that  $R_k$  is a polynomial in  $q^k$  of degree  $r + h - 1 + 2 \leq n$  with no constant term. By the  $q$ -binomial theorem (see, for example, [2, Theorem 3.3])

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k = (x; q)_n, \quad (2.5)$$

we have

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} q^{-ik} = 0 \quad \text{for } 0 \leq i \leq n-1.$$

It follows that the expression (2.4) is equal to 0. Namely, the identity (2.3) holds.

Hence, we see that all the  $m + n$  roots of  $L_{m,n}(x)$  are the same as those of the right-hand side of (2.1).  $\square$

**Lemma 2.2.** *Let  $n$  and  $h$  be positive integers and let  $m$  be a non-negative integer with  $h \leq n - m$ . Then*

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=m+h}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (a; q)_j (a; q)_k (1 - q^{k-j}) q^{j+k+jh} \begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix}}{(q; q)_j (q; q)_k (c; q)_j (c; q)_k} \\ &= \frac{(q; q)_n (a; q)_{m+h} (c/a; q)_{n-h} a^{n-h} q^{\frac{m^2+m-h^2+h}{2}-mn}}{(-1)^{m-h} (q; q)_m (c; q)_m (c; q)_n (q; q)_{h-1} (q; q)_{n-m-h}}. \end{aligned} \quad (2.6)$$

*Proof.* It is easy to see that  $\begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} = 0$  for  $m+1 \leq k < m+h$ . Therefore, the left-hand side of (2.6) remains unchanged when we replace  $\sum_{k=m+h}^n$  by  $\sum_{k=m+1}^n$ . Moreover,

$$\begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix} = \frac{(q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} q^{(m-j)(h-1) + \binom{h}{2}}}{(-1)^{h-1} (q; q)_{h-1}^2}.$$

The proof then follows from (2.1) and (2.2) with  $x = a$ .  $\square$

The following result has been proved in [8, (3.5)].

**Lemma 2.3** ([8]). *Let  $n$  be a positive integer. Then*

$$(x; q)_n + (a/x; q)_n = (x; q)_n(a/x; q)_n + (a; q)_n + \sum_{k=1}^{n-1} (x; q)_k(a/x; q)_k B_{n,k}(a), \quad (2.7)$$

where

$$B_{n,k}(a) := (1 - q^n) \sum_{h=1}^{n-k} (-1)^h \begin{bmatrix} n-k-1 \\ h-1 \end{bmatrix} \begin{bmatrix} k+h-1 \\ h-1 \end{bmatrix} \frac{q^{\binom{h}{2} + kh} a^h}{1 - q^h}.$$

### 3. Proof of Theorem 1.1

As

$$\left( \sum_{k=0}^n a_k \right) \left( \sum_{j=0}^n b_j \right) = \sum_{k=0}^n a_k b_k + \sum_{0 \leq j < k \leq n} (a_k b_j + a_j b_k),$$

the left-hand side of (1.6) is equal to

$$\begin{aligned} & \sum_{k=0}^n \frac{(q^{-n}; q)_k^2 (a; q)_k^2 q^{2k}}{(q; q)_k^2 (c; q)_k^2} (x; q)_k (c/x; q)_k \\ & + \sum_{0 \leq j < k \leq n} \frac{(q^{-n}; q)_j (q^{-n}; q)_k (a; q)_j (a; q)_k q^{j+k} ((x; q)_j (c/x; q)_k + (x; q)_k (c/x; q)_j)}{(q; q)_j (q; q)_k (c; q)_j (c; q)_k}. \end{aligned} \quad (3.1)$$

For  $0 \leq j < k$ , from (2.7) we deduce that

$$\begin{aligned} & (x; q)_j (c/x; q)_k + (x; q)_k (c/x; q)_j \\ & = (x; q)_j (c/x; q)_j ((xq^j; q)_{k-j} + (cq^j/x; q)_{k-j}) \\ & = (x; q)_k (c/x; q)_k + (x; q)_j (c/x; q)_j (cq^{2j}; q)_{k-j} + \sum_{i=1}^{k-j-1} (x; q)_{j+i} (c/x; q)_{j+i} B_{k-j,i}(cq^{2j}) \\ & = (x; q)_k (c/x; q)_k + (x; q)_j (c/x; q)_j + \sum_{i=0}^{k-j-1} (x; q)_{j+i} (c/x; q)_{j+i} B_{k-j,i}(cq^{2j}), \end{aligned}$$

where we have used the  $q$ -binomial theorem (2.5) in the last step:

$$(cq^{2j}; q)_{k-j} = 1 + \sum_{h=1}^{k-j} (-1)^h \begin{bmatrix} k-j \\ h \end{bmatrix} q^{\binom{h}{2} + 2jh} c^h.$$

It follows that (3.1) can be written as  $\sum_{m=0}^n \alpha_m (x; q)_m (c/x; q)_m$ , where

$$\begin{aligned} \alpha_m &= \sum_{j=0}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_m (a; q)_j (a; q)_m q^{j+m}}{(q; q)_j (q; q)_m (c; q)_j (c; q)_m} \\ & + \sum_{j=0}^m \sum_{k=m+1}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (a; q)_j (a; q)_k q^{j+k}}{(q; q)_j (q; q)_k (c; q)_j (c; q)_k} B_{k-j,m-j}(cq^{2j}). \end{aligned} \quad (3.2)$$



By the  $q$ -Chu-Vandermonde summation formula [5, Appendix (II.6)]:

$${}_2\phi_1 \left[ \begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q \right] = \frac{(c/a; q)_n}{(c; q)_n} a^n, \quad (3.3)$$

we have

$$\begin{aligned} & \sum_{j=0}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_m (a; q)_j (a; q)_m q^{j+m}}{(q; q)_j (q; q)_m (c; q)_j (c; q)_m} \\ &= (-1)^m \frac{(q; q)_n (a; q)_m (c/a; q)_n q^{\frac{m^2+m}{2}-mn}}{(q; q)_m (c; q)_m (c; q)_n (q; q)_{n-m}} a^n. \end{aligned} \quad (3.4)$$

Substituting (3.4) and (2.6) into (3.2), we obtain

$$\alpha_m = \frac{(-1)^m (q; q)_n q^{\frac{m^2+m}{2}-mn}}{(q; q)_m (c; q)_m (c; q)_n} \sum_{h=0}^{n-m} \frac{(a; q)_{m+h} (c/a; q)_{n-h} a^{n-h} q^{mh} c^h}{(q; q)_h (q; q)_{n-m-h}}. \quad (3.5)$$

The last sum can be summed again by the  $q$ -Chu-Vandermonde formula (3.3) and is equal to

$$\frac{(a; q)_m (c/a; q)_m (cq^{2m}; q)_{n-m} a^n}{(q; q)_{n-m}}. \quad (3.6)$$

It follows from (3.5) and (3.6) that  $\alpha_m$  is just the coefficient of  $(x; q)_m (c/x; q)_m$  on the right-hand side of (1.6).

*Remark.* Letting  $c = q^{2s+1}$ ,  $a = aq^s$ ,  $x = xq^s$ , and replacing  $n$  by  $n-s$  in (1.6) ( $0 \leq s \leq n$ ), we get the following result:

$$\begin{aligned} & \left( \sum_{k=s}^n \frac{(q^{-n}; q)_k (a; q)_k (x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \left( \sum_{k=s}^n \frac{(q^{-n}; q)_k (a; q)_k (q/x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \\ &= \frac{(q^{-n}; q)_s (a; q)_s a^{n-s} q^{(n+1)s-s^2}}{(q^{n+1}; q)_s (q/a; q)_s} \sum_{k=s}^n \frac{(q^{-n}; q)_k (q^{n+1}; q)_k (a; q)_k (q/a; q)_k (x; q)_k (q/x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s} (q; q)_{2k}}. \end{aligned} \quad (3.7)$$

It is clear that the  $a = -q^{-n}$  case of (3.7) reduces to (1.5).

## 4. Proof of Theorem 1.4

We need a special case of Theorem 1.1. Letting  $a = -x$  in (1.6), we are led to

$$\begin{aligned} & \left( \sum_{k=0}^n \frac{(q^{-n}; q)_k (x^2; q^2)_k q^k}{(q; q)_k (c; q)_k} \right) \left( \sum_{k=0}^n \frac{(q^{-n}; q)_k (-x; q)_k (c/x; q)_k q^k}{(q; q)_k (c; q)_k} \right) \\ &= (-x)^n \sum_{k=0}^n \frac{(q^{-n}; q)_k (cq^n; q)_k (x^2; q^2)_k (c^2/x^2; q^2)_k q^k}{(q; q)_k (c; q)_k (c; q)_{2k}}. \end{aligned} \quad (4.1)$$

We also need the following result.

**Lemma 4.1.** *Let  $n$  be a non-negative integer. Then*

$$\sum_{k=0}^n \frac{(q^{-n}; q)_k (x; q)_k (y; q)_k q^k}{(q; q)_k (c; q)_k} = \sum_{k=0}^n (-1)^k \frac{(q^{-n}; q)_k (x; q)_k (c/y; q)_k x^{n-k} y^k q^{nk - \binom{k}{2}}}{(q; q)_k (c; q)_k}. \quad (4.2)$$

*Proof.* This follows from combining Jackson's two transformations of terminating  ${}_2\phi_1$  series [5, Appendix (III.7) and (III.8)]:

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c, 0 \end{matrix} ; q, q \right] = b^n {}_3\phi_1 \left[ \begin{matrix} q^{-n}, b, q/z \\ bq^{1-n}/c \end{matrix} ; q, \frac{z}{c} \right]$$

with  $b \rightarrow x$ ,  $c \rightarrow xq^{1-n}/c$  and  $z \rightarrow qy/c$ .  $\square$

Replacing  $x$  and  $y$  by  $-x$  and  $c/x$  respectively in (4.2), we are led to

$$\sum_{k=0}^n \frac{(q^{-n}; q)_k (-x; q)_k (c/x; q)_k q^k}{(q; q)_k (c; q)_k} = (-x)^n \sum_{k=0}^n \frac{(q^{-n}; q)_k (x^2; q^2)_k c^k q^{nk - \binom{k}{2}}}{(q; q)_k (c; q)_k x^{2k}}. \quad (4.3)$$

Combining (4.1) and (4.3) (also  $x \rightarrow \sqrt{x}$ ), we obtain Theorem 1.4.

*Remark.* Letting  $c = q^{2s+1}$ ,  $x \rightarrow xq^{2s}$ , and replacing  $n$  by  $n - s$  ( $0 \leq s \leq n$ ) in (1.7), we get the following identity

$$\begin{aligned} & \left( \sum_{k=s}^n \frac{(q^{-n}; q)_k (x; q^2)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \left( \sum_{k=s}^n \frac{(q^{-n}; q)_k (x; q^2)_k q^{(n+1)k - \binom{k}{2}}}{(q; q)_{k-s} (q; q)_{k+s} x^k} \right) \\ &= \frac{(-1)^s (q; q)_n^2 (x; q^2)_s q^s}{(q; q)_{n-s} (q; q)_{n+s} (q^2/x; q^2)_s x^s} \sum_{k=s}^n \frac{(q^{-n}; q)_k (q^{n+1}; q)_k (x; q^2)_k (q^2/x; q^2)_k q^k}{(q; q)_{k-s} (q; q)_{k+s} (q; q)_{2k}}, \end{aligned}$$

which was originally conjectured in a preliminary version (arXiv:1408.0512v1) of [8].

## 5. Proof of Theorem 1.5

If we replace  $n$  by  $n - i$  in the  $q$ -Chu-Vandermonde summation formula (3.3) with  $a = q^{-m+i}$  and  $c = q^{i+1}$  then

$$\sum_{k=i}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-i \\ k-i \end{bmatrix} = \begin{bmatrix} n+m-i \\ n \end{bmatrix}.$$

Hence, by the  $q$ -binomial theorem (2.5) we have

$$\begin{aligned} \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (x; q)_k &= \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{i=0}^k (-1)^i \begin{bmatrix} k \\ i \end{bmatrix} q^{\binom{i}{2}} x^i \\ &= \sum_{i=0}^m (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} q^{\binom{i}{2}} x^i \sum_{k=i}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \\ &= \sum_{i=0}^m q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+m-i \\ n \end{bmatrix} (-x)^i. \end{aligned}$$

When  $x = -1$  and  $x = -q$ , we obtain (1.10) and (1.11), respectively. Now, letting  $c = q$  and  $a = q^{-m}$  in (1.6), we get

$$\begin{aligned} & \left( \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (x; q)_k \right) \left( \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (q/x; q)_k \right) \\ &= \sum_{k=0}^m q^{(m-k)(n-k)} \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} (x; q)_k (q/x; q)_k. \end{aligned}$$

This entails the identity (1.12) by taking  $x = -1$  or  $-q$ .

## 6. Proof of Theorem 1.6

The following identity can be easily proved by induction.

$$\sum_{k=j}^{n-1} (1 - q^{2k+1}) \begin{bmatrix} k+j \\ 2j \end{bmatrix} q^{-(j+1)k} = \frac{(1 - q^n)(1 - q^{n-j})}{1 - q^{j+1}} \begin{bmatrix} n+j \\ 2j \end{bmatrix} q^{-(j+1)(n-1)}. \quad (6.1)$$

By (1.12) and (6.1), the left-hand side of (1.13) is equal to

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{1 - q^{2k+1}}{1 - q} \sum_{j=0}^k q^{j^2 - mj - (j+1)k} \begin{bmatrix} k+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} (-1; q)_j (-q; q)_j \\ &= \sum_{j=0}^{p-1} q^{j^2 - mj - (j+1)(p-1)} \frac{(1 - q^p)(1 - q^{p-j})}{(1 - q)(1 - q^{j+1})} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} (-1; q)_j (-q; q)_j. \end{aligned} \quad (6.2)$$

By [7, Theorem 2.1], we know that

$$\begin{aligned} & \frac{(1 - q^m)(1 - q^{m+1})(1 - q^{p-j})}{(1 - q)(1 - q^p)(1 - q^{j+1})} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} \\ &= \frac{(1 - q^{p-j})(1 - q^{j+1})}{(1 - q)(1 - q^p)} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix} \begin{bmatrix} m+j \\ j+1 \end{bmatrix} \end{aligned}$$

is a polynomial in  $q$  with non-negative integer coefficients. Since  $[p] = (1 - q^p)/(1 - q)$  is an irreducible polynomial in  $q$  for any prime  $p$  and  $\gcd(q^m - 1, q^n - 1) = (q^{\gcd(m,n)} - 1)$ , we conclude that

$$\frac{1 - q^{p-j}}{1 - q^{j+1}} \begin{bmatrix} p+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} \equiv 0 \pmod{[p]} \quad \text{if } m \not\equiv 0, -1 \pmod{p},$$

and so the right-hand side of (6.2) is congruent to 0 modulo  $[p]^2$  in this case.

On the other hand, if  $m \equiv 0, -1 \pmod{p}$ , then

$$\frac{1 - q^{p-j}}{1 - q^{j+1}} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} \equiv 0 \pmod{[p]} \quad \text{if } j = 0, 1, \dots, p-2.$$

Therefore, if  $m \equiv 0 \pmod{p}$ , then the right-hand side of (6.2) is congruent to

$$\begin{aligned} q \begin{bmatrix} 2p-1 \\ 2p-2 \end{bmatrix} \begin{bmatrix} m \\ p-1 \end{bmatrix} \begin{bmatrix} m+p-1 \\ p-1 \end{bmatrix} (-1; q)_{p-1} (-q; q)_{p-1} &\equiv q \frac{1-q^{2p-1}}{1-q} \cdot \frac{1-q^m}{1-q^{p-1}} (-q^{-1}) \cdot \frac{2q}{1+q} \\ &\equiv -\frac{2q(1-q^m)}{1-q^2} \\ &\equiv \frac{1-q^{-2m}}{1-q^2} q \pmod{[p]^2}, \end{aligned}$$

where we have used the congruence  $(-q; q)_{p-1} \equiv 1 \pmod{[p]}$  (see [6, (1.6)] or [11]); while if  $m \equiv -1 \pmod{p}$ , then the right-hand side of (6.2) is congruent to

$$\begin{aligned} \frac{1-q^{2p-1}}{1-q} \cdot \frac{1-q^{m+1}}{1-q^{p-1}} \cdot \frac{2q}{1+q} &\equiv \frac{2q(1-q^{m+1})}{1-q^2} \\ &\equiv \frac{1-q^{2m+2}}{1-q^2} q \pmod{[p]^2}. \end{aligned}$$

## 7. Proof of Theorem 1.7

Similarly to (6.2), the left-hand side of (1.14) is equal to

$$\sum_{j=0}^{n-1} q^{j^2-mj-(j+1)(n-1)} \frac{(1-q^m)(1-q^{m+1})(1-q^{n-j})}{(1-q^2)(1-q^n)(1-q^{j+1})} \begin{bmatrix} n+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} (-1; q)_j (-q; q)_j.$$

It is easy to see that

$$\begin{aligned} &\frac{(1-q^m)(1-q^{m+1})(1-q^{n-j})}{(1-q^2)(1-q^n)(1-q^{j+1})} \begin{bmatrix} n+j \\ 2j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} m+j \\ j \end{bmatrix} (-q; q)_j \\ &= \begin{cases} \frac{(1-q^m)(1-q^{m+1})}{(1-q^2)(1-q)} & \text{if } j = 0, \\ \frac{(1-q^{n-j})(1-q^{j+1})}{(1-q)(1-q^n)} \begin{bmatrix} n+j \\ 2j \end{bmatrix} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix} \begin{bmatrix} m+j \\ j+1 \end{bmatrix} (-q^2; q)_{j-1} & \text{if } j \geq 1 \end{cases} \end{aligned}$$

is a polynomial in  $q$  with non-negative integer coefficients by [7, Theorem 2.1]. We conclude that (1.14) is the desired Laurent polynomial in  $q$ .

Let

$$S_n(x_0, \dots, x_n; q) = \sum_{k=0}^n \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix} q^{-nk} x_k.$$

To prove that (1.15) and (1.16) also have the same properties, we first establish the following result.

**Lemma 7.1.** *Let  $n$  and  $r$  be positive integers. Then both*

$$\sum_{k=0}^{n-1} \frac{1 - q^{2k+1}}{1 - q^n} S_k(x_0, \dots, x_k)^r q^{-k} \text{ and } \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{1 - q^{2k+1}}{1 - q^n} S_k(x_0, \dots, x_k)^r q^{\binom{k}{2}}$$

*are polynomials in  $x_0, \dots, x_{n-1}, q$  and  $q^{-1}$  with non-negative integer coefficients.*

*Proof.* Recall the identity

$$\begin{bmatrix} k+i \\ 2i \end{bmatrix} \begin{bmatrix} 2i \\ i \end{bmatrix} \begin{bmatrix} k+j \\ 2j \end{bmatrix} \begin{bmatrix} 2j \\ j \end{bmatrix} = \sum_{s=i}^{i+j} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} j \\ s-i \end{bmatrix} \begin{bmatrix} s \\ j \end{bmatrix} \begin{bmatrix} k+s \\ 2s \end{bmatrix} \begin{bmatrix} 2s \\ s \end{bmatrix} q^{(i+j-s)(k-s)}.$$

which can be proved using the  $q$ -Pfaff-Saalschütz identity (see [14, Lemma 2.1]). It follows that

$$\begin{aligned} S_k(x_0, \dots, x_k)^r &= \sum_{0 \leq i_1, \dots, i_r \leq k} \prod_{j=1}^r \begin{bmatrix} k+i_j \\ 2i_j \end{bmatrix} \begin{bmatrix} 2i_j \\ i_j \end{bmatrix} q^{-ki_j} x_{i_j} \\ &= \sum_{0 \leq i_1, \dots, i_r \leq k} x_{i_1} \cdots x_{i_r} \sum_{s=i_1}^{i_1+\dots+i_r} P(i_1, \dots, i_r, s) \begin{bmatrix} k+s \\ 2s \end{bmatrix} \begin{bmatrix} 2s \\ s \end{bmatrix} q^{-ks}, \end{aligned} \quad (7.1)$$

where  $P(i_1, \dots, i_r, s)$  is a Laurent polynomial in  $q$  independent of  $k$  with non-negative integer coefficients. Therefore, by (6.1), we see that

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{1 - q^{2k+1}}{1 - q^n} S_k(x_0, \dots, x_k)^r q^{-k} \\ &= \sum_{0 \leq i_1, \dots, i_r \leq n-1} x_{i_1} \cdots x_{i_r} \sum_{s=i_1}^{i_1+\dots+i_r} P(i_1, \dots, i_r, s) \frac{1 - q^{n-s}}{1 - q^{s+1}} \begin{bmatrix} n+s \\ 2s \end{bmatrix} \begin{bmatrix} 2s \\ s \end{bmatrix} q^{-(s+1)(n-1)} \end{aligned}$$

is a polynomial in  $x_0, \dots, x_{n-1}, q$  and  $q^{-1}$  with non-negative integer coefficients since

$$\frac{1 - q^{n-s}}{1 - q^{s+1}} \begin{bmatrix} n+s \\ 2s \end{bmatrix} \begin{bmatrix} 2s \\ s \end{bmatrix} = \begin{bmatrix} n+s \\ s \end{bmatrix} \begin{bmatrix} n \\ s+1 \end{bmatrix}.$$

Similarly, since

$$\sum_{k=s}^{n-1} (-1)^{n-k-1} \frac{1 - q^{2k+1}}{1 - q^n} \begin{bmatrix} k+s \\ 2s \end{bmatrix} \begin{bmatrix} 2s \\ s \end{bmatrix} q^{\binom{k}{2}-sk} = \begin{bmatrix} n-1 \\ s \end{bmatrix} \begin{bmatrix} n+s \\ s \end{bmatrix} q^{\binom{n}{2}-sn},$$

we deduce from (7.1) that

$$\begin{aligned} &\sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{1 - q^{2k+1}}{1 - q^n} S_k(x_0, \dots, x_k)^r q^{\binom{k}{2}} \\ &= \sum_{0 \leq i_1, \dots, i_r \leq n-1} x_{i_1} \cdots x_{i_r} \sum_{s=i_1}^{i_1+\dots+i_r} P(i_1, \dots, i_r, s) \begin{bmatrix} n-1 \\ s \end{bmatrix} \begin{bmatrix} n+s \\ s \end{bmatrix} q^{\binom{n}{2}-sn} \end{aligned}$$

is a polynomial in  $x_0, \dots, x_{n-1}, q$  and  $q^{-1}$  with non-negative integer coefficients.  $\square$

For  $k = 0, \dots, n-1$ , let

$$x_k = \begin{bmatrix} m+k \\ 2k \end{bmatrix} (-1; q)_k (-q; q)_k q^{k^2-mk}.$$

Then the identity (1.12) may be rewritten as

$$D_q(m, n) D_{q^{-1}}(m, n) = \sum_{k=0}^n \begin{bmatrix} n+k \\ 2k \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix} q^{-nk} x_k.$$

It is clear that  $x_0, \dots, x_{n-1}$  are Laurent polynomials in  $q$  with non-negative integer coefficients. By Lemma 7.1, so are the expressions (1.15) and (1.16).

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