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Approximation de la superposition de processus de renouvellement par modèle d'âge virtuel : propriétés analytiques et application

Approximation of the superposition of renewal processes by virtual age model: analytical properties and application

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Résumé

Dans le cadre d'ingénierie de modèle et de maintenance, nous présentons une modélisation méthode, noté WARA (Weibull arithmetic reduction of age), qui vise à approximer et simplifier la superposition des Weibull processus de renouvellement, ce qui est souvent rencontrée face au système réparable en série ou à une file d'attente. Ce modèle nous permet d'estimer la distribution de temps inter panne en fonction de l'histoire de maintenance.

Nous présentons d'abord les propriétés analytiques du modèle avant de montrer l'adéquation de son utilisation face aux systèmes réparables en série par la simulation. Une application en données réelles (les temps de panne de climatiseurs collectés à partir des avions Boeing) est également présentée.

Summary

In the context of model engineering and maintenance, we present a modelling method, WARA (Weibull arithmetic reduction of age). Based on virtual age model, it aims to approximate the superposition of Weibull renewal processes (SRP) which is often encountered when studying a repairable serial system or in queueing theory. Compared to other RP-based approximation method, this model allows us to calculate more precisely the interval distribution conditioned on maintenance history. Some analytical properties of the model are presented before evaluating the quality of WARA approximation using simulations. An application on a real data (failure time of air conditioning system of a fleet of Boeing airplanes) is also presented.

1. Introduction and context

We consider a repairable serial system composed by n independent components, whose breakdown shall cause a system failure. A corrective maintenance with a negligible operation time is carried out after the failure, in order to repair or replace the failed part. This will form a renewal process for the component and an SRP (superposition of renewal process) for the system (figure 1).

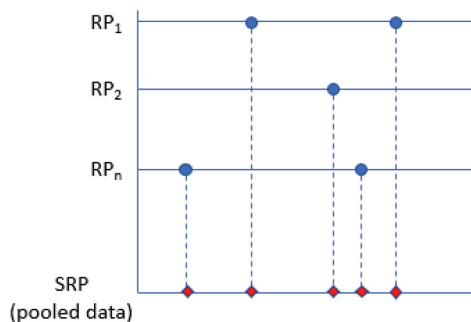


Figure 1. An example of SRP data

At a system level, the maintenance is between AGAN (as good as new) and ABAO (as bad as old) as only one constituent is repaired at one time. Therefore, an SRP can be viewed as an imperfect maintenance model.

It is well known that the superposition of a finite number of renewal processes are not renewal anymore. Although the inter failure times are statistically dependent, computing the interval's distribution is relatively easy. First found by (Cox et al., 1954), the survival function of the pooled data is given by:

$$R_s(t) = \frac{v_1 v_2 \dots v_n}{v_1 + v_2 + \dots + v_n} \sum_{i=1}^n \Psi_1 \dots \Psi_{i-1} R_i \Psi_{i+1} \dots \Psi_n \quad \{1\}$$

Where n the total number of RP superimposed, R_i the survival function of the interval of component i , $\Psi_i(t) = \int_t^\infty R_i(u) du$, and $v_i = \frac{1}{\Psi_i(0)}$ denote the average rate of i^{th} renewal source.

Analysing an SRP at a component level can be difficult considering the number of parameters involved and the efforts needed for ML estimations (Ye Tian, 2013). Thus, most of literature aims to simplify or approximate an SRP, and usually by a RP. (Whitt, 1982) has proposed 2 methods based on moment-matching: an approximation by hyper-exponential RP for high-variation case and a shifted-exponential model for low-variation case; (Kamen et al., 2001) has proved that a RP approximation can be rate-optimal if its interval distribution equals that of SRP ($\{1\}$); (Araghi et al., 2008) has proposed a 3 parameters generalised erlang RP model, matching the index of dispersion and the mean value of inter-failure time.

However, the main drawback of using RP approximation is that the dependency between the intervals is ignored. This dependency can be either positive or negative according to the nature of RP superposed. (Lawrence, 1973) has found the joint distribution of adjacent intervals and has proposed a formula for correlation calculation. He also noticed that a negative correlation between intervals might occur when 2 erlang RP are superimposed.

In the following, we focus on a particular configuration of SRP, where the RP superposed are Weibull with a shape parameter greater than 1. To model this type of SRP, we propose to use the WARA method which conserves partially the negative dependence observed between Weibull SRP intervals.

In section 2, we recall the history and some notions of virtual age models. In section 3, several analytical

properties of WARA model are presented. We then compare an SRP to its WARA approximation in section 4. In section 5, we discuss the distribution of inter failure times conditioned on the maintenance history. In section 6, The WARA model is applied to model and explain the failure data of the air conditioning systems of Boeing airplanes. Conclusion are in section 7.

2. Virtual age models

Virtual age models (Kijima *et al.*, 1988) are the most frequently used imperfect repair models. The principle is that the wear-out does not depend on the chronological age of the system, but on a virtual age, commonly between zero and the elapsed time since the system was new. A virtual age model is entirely characterized by the failure rate of a new system and by the virtual age assumptions. (Kijima, 1989) has proposed two widespread classes of virtual age assumptions. He supposes that each repair efficiency is represented by a random variable supported on the interval $[0,1]$. A model under Kijima Type I assumption is such that a repair rejuvenates the virtual age of a proportional amount of the last inter-failure duration, whereas a model under Kijima Type II assumption supposes that the rejuvenated amount is proportional to the virtual age just before the repair. A special case is to consider that the repair efficiency is a constant $\rho \in [0,1]$, called restoration factor. The resulting models have been developed by (Malik, 1979) and by (Brown *et al.*, 1983) for the Kijima type I and II models, respectively. A unified version of the last two models has been presented by (Doyen *et al.*, 2004), called model of arithmetic reduction of age with memory m , and denoted ARA_m . The ARA_1 and ARA_∞ models are special cases of the Kijima Type I and II models, respectively.

In the following, we give a detailed introduction of the basic notions in an ARA_∞ model, which is then combined with the Weibull assumption in section 3.

A repairable system has been observed since it was new. The observations consist of the successive maintenance times $\{T_i\}$ $i \geq 0$. The corresponding inter-maintenance times are denoted $\{X_i\}$ $i \geq 1$ and the repair process can also be characterized by a counting process $\{N_t\}$ $t \geq 0$ where $N_t = \sum_{i=1}^{\infty} 1_{\{T_i \leq t\}}$. By convention, T_0 and X_0 are equal to zero and time can be either calendar or operational. The distributions are obtained from the failure intensity defined in {2}, where H_t^- is the history of the repair process at time t^- , commonly the failure times before t .

$$\forall t \geq 0, \lambda_t = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_t = 1 | H_t^-) \quad \{2\}$$

The failure rate of a new system, called initial failure intensity and denoted $\lambda(t)$, is assumed to be a deterministic and continuous function of time. It corresponds to the hazard rate of T_1 . The cumulative hazard rate function is denoted $\Lambda(t) = \int_0^t \lambda(u) du$. f , F and R are the probability density function, cumulative distribution function and survival function, respectively. As industrial systems are assumed to wear out, the initial failure intensity is increasing. Consequently, the two-parameter Weibull distribution has been chosen as in {3}. For wearing-out systems, the shape parameter β is greater than 1.

$$\forall t \geq 0, \lambda(t) = \alpha \beta t^{\beta-1} \quad \{3\}$$

A virtual age model (Kijima *et al.*, 1988) assumes that after the i^{th} repair, the system behaves as a new and unmaintained one of age A_i . This age is called effective age. The assumption is mathematically described in {4}, where Z is the time to failure of a new system and has the same distribution as X_1 .

$\forall i \geq 0, \forall t \geq 0, P(X_{i+1} > t | X_1, \dots, X_i) = P(Z > A_i + t | Z > A_i) \{4\}$
The conditional survival function of the $(i+1)^{\text{th}}$ inter-failure time in {4} is simply $R(A_i + t)/R(A_i)$. The age of the system A_0 at the beginning of the observation is zero if the system is as good as new, and greater than 0 otherwise. At a given time t , the virtual age of the system V_t is obtained from the latest effective age and the elapsed time since the last repair as in {5}.

$$V_t = A_{N_t^-} + t - T_{N_t^-} \quad \{5\}$$

The virtual age of the system just before the i^{th} repair is denoted A_i^- . The failure intensity can be derived from the initial failure intensity as in {6}.

$$\lambda_t = \lambda(V_t) = \lambda(A_{N_t^-} + t - T_{N_t^-}) \quad \{6\}$$

Many models have been developed from different assumptions on the virtual ages (Proschan *et al.*, 1983); (Dijoux *et al.*, 2013); (Doyen *et al.*, 2004); (Kijima, 1989). In particular, a Kijima type II model (Kijima, 1989) assumes that regarding the i^{th} repair, the effective age A_i is proportional to A_i^- . The model of arithmetic reduction of age with infinite memory ARA_∞ (Doyen *et al.*, 2004) assumes that this proportion is a constant ρ in the interval $[0,1]$ as in {7}. ρ is called the repair efficiency or the restoration factor.

$$A_i = (1 - \rho)A_i^- = (1 - \rho)(A_{i-1} + X_i) \quad \{7\}$$

In particular, a minimal and a perfect maintenance can be modelled with $\rho = 0$ and $\rho = 1$, respectively. The effective age in {7} can be explicitly expressed in terms of the inter-failure times and A_0 , as in {8}. One can find useful applications of the ARA_∞ model in different situations (Bartholomew-Biggs *et al.*, 2009); (Brown *et al.*, 1983); (Clavareau *et al.*, 2009); (Dijoux *et al.*, 2013); (Kahle, 2007); (Yun *et al.*, 1999).

$$A_i = (1 - \rho)^i A_0 + \sum_{j=1}^i (1 - \rho)^{i-j+1} X_j \quad \{8\}$$

Combining {6} and {8}, a model under the ARA_∞ assumption is defined by the initial age A_0 , the initial failure intensity $\lambda(\cdot)$ and the repair efficiency ρ .

3. Analytical properties of WARA model

The Weibull- ARA_∞ model, denoted WARA $_\infty$ or simply WARA, consists of the ARA_∞ assumption, the Weibull initial failure defined in {3} and a null initial age.

(Nguyen *et al.*, 2016) has found the closed forms of the distributions and expected values of age A_n and of interval X_n and has proved their convergence to a stationary regime. Some of the formulas are recalled below ({9}, {10}, {11}, {12}) while the proofs are omitted here.

$$\forall n \geq 1, R_{A_n}(t) = \sum_{k=1}^n \frac{1}{(q, q)_{n-k} \frac{(1, 1)_{k-1}}{(q, q)_{k-1}}} e^{-\frac{\alpha t^\beta}{q^k}} \quad \{9\}$$

Where $q = (1 - \rho)^\beta$ and $(a, q)_k = \prod_{j=0}^{k-1} (1 - aq^j)$ the q -Pochhammer symbol.

$$R_{X_{n+1}}(t) = \begin{cases} \sum_{k=1}^n \frac{\int_0^\infty \alpha \beta x^{\beta-1} e^{-\alpha(x+t)^\beta + \alpha(1-q^{-k})x^\beta} dx}{q^k (q, q)_{n-k} \frac{(1, 1)_{k-1}}{(q, q)_{k-1}}} & n \geq 1 \\ e^{-\alpha t^\beta} & n = 0 \end{cases} \quad \{10\}$$

In the asymptotic case where n is large, the limiting distributions of A_∞ and X_∞ are shown below:

$$R_{A_n}(t) = \sum_{k=1}^{\infty} \frac{1}{(q, q)_{\infty} \left(\frac{1}{q}\right)_{k-1}} e^{-\frac{\alpha t^{\beta}}{q^k}} \quad \{11\}$$

$$R_{X_n}(t) = \sum_{k=1}^{\infty} \frac{\int_0^{\infty} \alpha \beta x^{\beta-1} e^{-\alpha(x+t)^{\beta} + \alpha(1-q^{-k})x^{\beta}} dx}{q^k (q, q)_{\infty} \left(\frac{1}{q}\right)_{k-1}} \quad \{12\}$$

Based on these equations, we have further developed some properties of WARA model, including dependency between adjacent intervals, coefficient of variation, high order moments of A_n and X_n , as shown below. The proofs are also dropped for brevity.

Theorem 3.1 negative dependency between intervals

In an ARA model with parameters α, β, ρ where $\alpha > 0$, $\beta > 1$ and $0 < \rho < 1$, 2 successive inter-failure times X_n and X_{n+1} , $n \geq 1$, are negatively dependent.

Theorem 3.2 Low variability of interval X

In an ARA model with parameters α, β, ρ where $\alpha > 0$, $\beta > 1$ and $0 < \rho < 1$, the coefficient of variation of X is smaller than 1.

The m-order moment of A_n , $E(A_n^m)$, is given by the following equation:

$$E(A_n^m) = \alpha^{-\frac{m}{\beta}} \Gamma\left(1 + \frac{m}{\beta}\right) \sum_{k=1}^n \frac{q^{\frac{km}{\beta}}}{(q, q)_{n-k} \left(\frac{1}{q}\right)_{k-1}} \quad \{13\}$$

The 2nd, 3rd and 4th moments of X_n are used respectively to calculate the variance, skewness and kurtosis. The formula of the mth moment is as follows:

$$E(X_{n+1}^m) = \alpha^{-\frac{m}{\beta}} \Gamma\left(1 + \frac{m}{\beta}\right) \sum_{j=1}^n \frac{1}{(q, q)_{n-j} \left(\frac{1}{q}\right)_{j-1}} * \left\{ \frac{1-q^{\frac{jm}{\beta}}}{1-q} + m \sum_{k=1}^{m-1} \binom{m-1}{k} (-1)^k \frac{q^{\frac{jm}{\beta}}}{\beta+k} F \right\} \quad \{14\}$$

with $F = {}_2F_1\left(1, 1 + \frac{m}{\beta}; 2 + \frac{m}{\beta}; 1 - q\right)$ the hypergeometric function and $n \geq 1$, $m \geq 2$, $m, n \in \mathbb{Z}$

Like the first moment, the formula of the mth moment can also be generalized to the stationary regime: when n is large, the expected value of X and its moments will converge.

We have also obtained the expression of $E(X_n X_{n+k})$ which can be used for covariance calculation. It is not presented here for brevity reasons, but we will show in section 6 how it can be applied to a real data.

4. WARA approximation of SRP: theoretical goodness of fit evaluation

In this section, we show first how to deduce the WARA parameters (α , β and ρ) from an observed data; then we evaluate the approximation quality by comparing the estimated ARA moments, correlations and coefficient of variation of inter failure time X to the real values.

4.1. WARA parameter estimation

The most classic method for parameter estimation is the maximum likelihood estimation. (Nguyen *et al.*, 2016) used ML estimation by maximizing the WARA's likelihood function:

$$L(\alpha, \beta, \rho; t_1, t_2, \dots, t_n) = \prod_{i=1}^n \lambda(a_{i-1} + t_i - t_{i-1}) e^{-\sum_{i=1}^{n-1} [\Lambda(a_{i-1} + t_i - t_{i-1}) - \Lambda(a_{i-1})]} \quad \{15\}$$

Where $t_1, t_2, \dots, t_n$ represent the n failure times and a_i the virtual age. Three parameters (α, β, ρ) are then estimated. Even though explicit expression of the estimators is not available, they can be easily obtained by general op-

itemization methods, for instances the Nelder–Mead downhill simplex, interior point method etc.

4.2. Approximation quality evaluation

We need to answer the question how much our approximating process is like the approximated SRP. However, there are no classic criterion for comparing 2 stochastic processes and investigating simply the interval distributions is certainly not enough. We will compare a serial of statistics including moments, correlations and coefficients of variations. But first we need to define the SRP configurations to be studied.

4.2.1. SRP configurations

Here we investigate only the superposition of identical renewal processes, which is entirely determined by 3 parameters: ns , number of RP superimposed; a and b , scale and shape parameter of Weibull distribution. In our simulations, a is fixed to 1, b takes the following values: [1.5, 2, 2.5, 3, 3.5, 5, 10] as in industrial reliability analysis, a Weibull distribution of lifetime with shape parameter $b \geq 10$ are rarely encountered (Jiang *et al.*, 2011). And ns takes the values in [2, 3, 4, 5, 10, 20]. The interest of using ARA approximation lies mainly in the situations when ns are not too big, as it has been proven in (Cox *et al.*, 1954) that when ns becomes large, the interval between successive failure tends to be distributed exponentially.

4.2.2. asymptotic interval distribution

The asymptotic inter-failure time distribution is an intuitive measure of the similarity between a point process and its approximation. The calculations (using {1} and {11}) show that ARA model has a density that is close to the SRP density when b is small (figure 2). It should also be noticed that the distance between the density functions increases as b increases.

4.2.3. 1st-4th moments

We consider the relative error E_r between the moments of ARA and that of SRP in their stationary regimes. ARA moments are calculated by {14} whereas SRP moments by {1}. Here are some basic observations:

- (1) E_r of the first moment is small. For all configurations considered, the maximal absolute value of relative error is 0.000682;
- (2) For 2nd, 3rd and 4th moments, E_r are negative, meaning that the moments of ARA are smaller than that of SRP;
- (3) For all combinations of ns and b , $|E_r(m_4)| \geq |E_r(m_3)| \geq |E_r(m_2)| \geq |E_r(m_1)|$ (figure 3);
- (4) For 2nd, 3rd and 4th moments, $|E_r|$ increases as b increases.

4.2.4. Coefficient of variation

Having calculated the first and second moment we can compare the coefficient of variation c between SRP and its ARA approximation. c is defined as σ/μ , usually used to characterise the variability of a random variable. $c > 1$ represent a high-variability case and $c < 1$ low variability, see (Whitt, 1982). Among the common distributions, the exponential distribution has a c that equals 1 and the Weibull distribution has a c smaller than 1. Our simulations show that the c of an SRP among the configurations considered above, is smaller than 1, so do that of their ARA approximations.

4.2.5. Correlation between 2 successive inter-failure time

One of the most attractive characteristic of ARA approximation, when compared to RP approximations, is that it conserves partially the dependencies between inter-failure times. It is known that this dependency is extremely complicated in SRP and some of the special cases have been studied in (Lawrence, 1973). According to our simulations, 2 successive inter-failure times of Weibull SRP are negatively correlated. As proved in the section 3, any 2 inter-failure times in ARA model has also a negative correlation.

4.2.6. Conclusion

Both ARA and pooled Weibull RP have negative interval dependencies and a coefficient of variation smaller than 1. In terms of approximation errors, using WARA model to approximate the SRP can lead to small errors in moments or correlation. For small values of b , these errors are usually acceptable. As b grows larger, however, the errors increase.

5. Conditional distribution of intervals

Imagine that we want to use a RP approximation to describe an observed SRP data (with known parameters). Under renewal assumption, the distribution of X_n given X_{n-1} is just the asymptotic interval distribution described by {1}: information contained in X_{n-1} is unused. Note that the dependency between SRP intervals makes sure that the conditional distribution is not the same as {1}.

Under ARA assumption, the information contained in X_{n-1} can be used to get a more accurate estimation of the distribution of X_n . In fact, in ARA model, the distribution of X_n depends directly on the virtual age A_{n-1} , whose density is then determined by X_{n-1} and A_{n-2} . This is demonstrated by the following simulation: an SRP data of length 1000000 is generated (by pooling Weibull point processes) and the triplet $(\alpha \beta \rho)$ is estimated thanks to {15}. The original data is noted $\{X^{srp}\}$, whose first and third quantile are noted respectively q_1 and q_3 . Using $(\alpha \beta \rho)$, we calculate the survival function of X_n conditioned on X_{n-1} . Consider the following 4 distributions:

- (1) $R_{q_1}^{srp}$: the empirical distribution of X in $\{X^{srp}\}$ whose previous value is smaller than q_1 ;
- (2) $R_{q_3}^{srp}$: the empirical distribution of X in $\{X^{srp}\}$ whose previous value is bigger than q_3 ;
- (3) $R_{q_1}^{ara}$: the theoretical distribution of X in ARA, knowing its previous value is smaller than q_1 ;
- (4) $R_{q_3}^{ara}$: the theoretical distribution of X in ARA, knowing its previous value is bigger than q_3 .

If the ARA model can indeed give a more accurate estimation of the conditional distribution, the difference between $R_{q_1}^{srp}$ and $R_{q_1}^{ara}$ (as well as $R_{q_3}^{srp}$ and $R_{q_3}^{ara}$) should be small, or at least less than that between {1} and $R_{q_1}^{srp}$.

It should be noticed that the theoretical calculation of $R_{q_1}^{srp}$ and of $R_{q_1}^{ara}$ are under the following assumptions:

- (a) A_{n-2} is asymptotically distributed as {11}
- (b) X_{n-1} is asymptotically distributed as {12}
- (c) $X_{n-1} \leq q_1$ or $X_{n-1} \geq q_3$

We can then calculate the conditional density of the virtual age A_{n-1} , before using {4} to deduce the distribution of X_n .

5.1. $R_{q_1}^{srp}$ and $R_{q_1}^{ara}$

The condition that the X_{n-1} is smaller than q_1 signify that 2 maintenances are performed in a short time. Thus, the system can be considered as 'young' if the number of RP superposed is small. Figure 4 shows an example of how ARA approximation provides a more accurate conditional density estimation compared to RP model, with $ns=2$, $a=1$,

$b=1.5$. Note that survival functions of both ARA and RP are under that of SRP.

5.2. $R_{q_3}^{srp}$ and $R_{q_3}^{ara}$

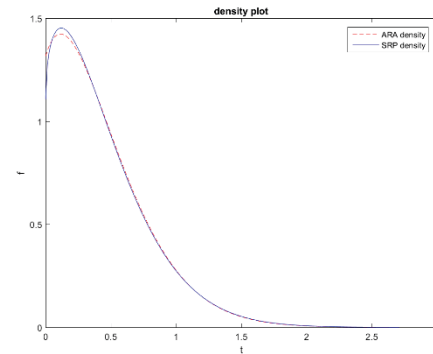


Figure 2. An example of density comparison where 2 identical RP with shape parameter $b=1.5$ are superposed.

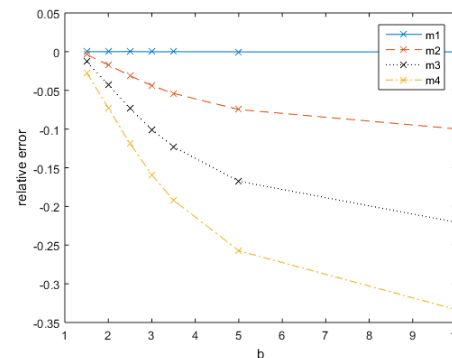


Figure 3. The relative errors of 1st – 4th moment augment with shape parameter b .

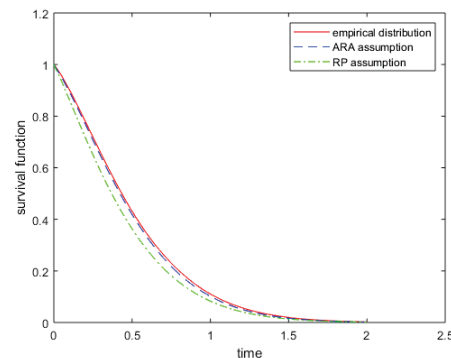


Figure 4. The survival functions conditioned on q_1 .

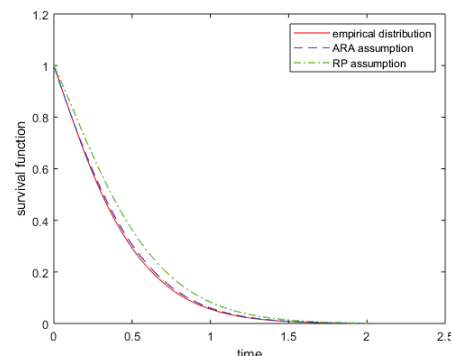


Figure 5. The survival functions conditioned on q_3 .

The condition that the previous inter failure time is larger than q_3 implies a long functional time since last repair and the system is considered as 'old' if ns is large. Figure 5

shows an example of how ARA approximation provides a more accurate conditional density estimation compared to RP model, with $ns=2$, $a=1$, $b=1.5$. Note that survival functions of both ARA and RP are above that of SRP.

As b becomes larger, the improvement of the accuracy of conditional density estimation brought by ARA is more significant; but the distance between ARA and empirical densities increases too.

6. Application of ARA model on air conditioning system of Boeing airplanes

In this section, we show how to use the WARA method to model and explain the failure data of the air conditioning systems of Boeing airplanes.

6.1. Description of data and related work

The data first appeared in (Proschan, 1963), where the author tried to explain the observed DFR (decreasing failure rate) of the life distribution of a system. He showed that the mixture of independent exponential failure times with different failure rate could lead to a pooled failure data with DFR.

The data arose in the following way. Records were kept for the time of successive failures of the air conditioning system of each member of a fleet of Boeing 720 jet airplanes. 13 planes are recorded in total. After roughly 2000 hours of service the planes received major overhaul. In (Proschan 1963), the failure interval containing major overhaul is omitted from the listing since the length of that failure interval may have been affected.

6.2. Summary of ongoing analysis

In this article, we focus on a subset of the original data: instead of using all 13 airplanes' failure times, we study the system formed by 3 planes: plane 7909, 7911 and 8045. Their failure times are grouped in the following table (only failure times before major overhaul are considered).

Plane number	Successive failure times
7909	90,10,60,186,61,49,14,24,56,20,79,84,44,59,29,118,25,156,310,76,26,44,23,62
7911	55,320,56,104,220,239,47,246,176,182,33
8045	102,209,14,57,54,32,67,59,134,152,27,14,230,66,61,34

Table 1. The failure times of 3 Boeing planes

The reason why we choose these particular aircrafts are as follows: First, the gain of using ARA model lies mainly in situations where a small number of point process are pooled. Secondly, when a Weibull model is fitted to these data, the shape parameters is relatively significant compared to others.

In the following, we prove first the necessary conditions of using ARA model; then the ARA parameters are deduced from the pooled data before finally showing the variation of virtual age and failure rate under ARA model.

6.3. Necessary conditions of using ARA model

2 conditions are indispensable if we want to fit a WARA model to a SRP data: first, we need to verify if the Weibull assumption holds for a single failure time data, and specifically, the shape parameter needs to be larger than 1; second, the pooled data should have an increasing failure rate (IFR).

6.3.1. Weibull assumption

Anderson-Darling test is performed respectively for the 3 failure time data, with null hypothesis that the observed

successive failure times come from a Weibull distribution. Their p-values and estimated parameters are grouped in table 2.

Plane number	p-value	Weibull scale parameter	Weibull shape parameter
7909	0.1935	76.8222	1.2418
7911	0.1671	170.7914	1.6485
8045	0.4156	89.8924	1.3461

Table 2. The results of AD goodness of fit test

Setting the significance level $\alpha = 0.05$, it seems safe to accept the hypothesis that these failure data come from the Weibull distribution, as all p-values are larger than α .

6.3.2. Increasing failure rate

In WARA model, the failure rate defined by $\{3\}$ is an increasing function. If the pooled data is observed to be DFR, using ARA is not a good choice.

In (Proschan 1963), the author proved that the inter failure times of the pooled data (13 planes) is DFR. We use the same method to prove that for the subset of plane 7909, 7911 and 8045, the inter failure times of pooled data is IFR.

Figure 6 shows the empirical survival function and the fitted exponential survival function. One can easily notice the fact that the empirical survival function lies consistently above the exponential for $0 \leq t \leq 23.5$ and then lies consistently below the exponential for $t \geq 23.5$. The fact that the survival curve seems to cross the exponential once only, and from above, suggests strongly the possibility that the failure distribution for the pooled failure intervals has an increasing failure rate, as supported by the inverse of Theorem 1 of (Proschan 1963):

Theorem 1 inverse: Let F be a distribution with density f , satisfying (a) $F(t) = 0$ for $t < 0$, (b) the failure rate $r(t) = f(t)/\bar{F}(t)$ is increasing for $t \geq 0$, (c) $F(\xi_p) = p$; i.e., ξ_p is the p^{th} percentile. Then

$$\bar{F}(t) \begin{cases} \geq e^{-\alpha t} & \forall t \leq \xi_p \\ \leq e^{-\alpha t} & \forall t \geq \xi_p \end{cases}$$

Where $\alpha = -\log(1 - p)/\xi_p$. The proof is omitted here.

6.4. ARA model fit

Using ML estimation, we deduce the ARA parameters from the pooled data before investigating whether the model fits well the observed inter-failure times or not.

	α	β	ρ
ARA parameters	0.0141	1.1685	0.7358

Table 3. The parameters of fitted ARA model

The value of ρ seems to suggest a rather acceptable maintenance efficiency, while a β close to 1 suggests a low degrading (aging) rate.

We then compare some statistics calculated by ARA model to the empirical ones as in section 4. Table 4 shows that the errors caused by ARA are acceptable. The correlation used is Pearson's rho, whose value is calculated theoretically by a formula that is not exhibited in this article. Compared to RP model which supposes that the pooled inter failure times are independent, and which uses $\{1\}$ and the estimated Weibull parameters in table 2, the ARA approximation is evidently more accurate. This is mainly because the observed data are not enough to support firmly the Weibull assumption, nor to deduce an accurate estimation of the Weibull parameters.

	mean	Moment order 2	Correlation between successive intervals
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Empirical (observed)	33.4510	2184.8	-0.0687
ARA (calculated)	33.4167	2033.6	-0.0387
RP	30.6491	1654.5	0

Table 4. ARA model captures more accurately the moments and correlations.

6.5. Virtual age and failure rate under ARA assumption

One of the central concept of ARA model is the virtual age. Once a repair is carried out, ARA model assume that the system's age is reduced; meanwhile, the change of age leads to the change of failure intensity (failure rate). Here we show how the virtual age and failure rate change under ARA model.

Figure 7 shows how the virtual age of system changes with time. One may notice that the when the VA grows, the slope is not constant. This is because the VA is a linear combination of all past inter failure times. The significant growth of VA usually appears when the system is not maintained for a long time. For example, the 5th repair takes place at $t=160$ followed by a virtual age $va=15.88$; while the 6th repair is carried out at $t=311$ followed by a virtual age $va=44.0932$.

Having calculated the virtual age, we can easily draw the variation of failure intensity using {3} and {6} (figure 8). Note that the failure rate is reduced to 0 right after a maintenance and then grows rapidly: the intensity can reach 0.02 in just 3 hours.

Conclusion

We study the statistical properties of a WARA model, which is a combination of the ARA virtual age model and the Weibull RP assumptions. We then evaluate the adequacy of using it to approximate or simplify the superposition of Weibull renewal processes, by comparing their moments, coefficient of variation and correlation between intervals.

The WARA model performs well in estimating the interval's conditional distribution, due to the fact that it preserves partially the negative dependency between intervals of an SRP. It can therefore be used to evaluate a system's health state (virtual age) and to forecast the RUL based on the maintenance history. Compared to traditional failure data analysis, the estimation of WARA model can be easily done and it leads to more accurate estimation.

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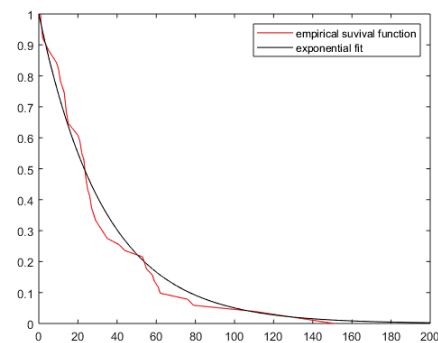


Figure 6. The empirical survival function and its exponential fit

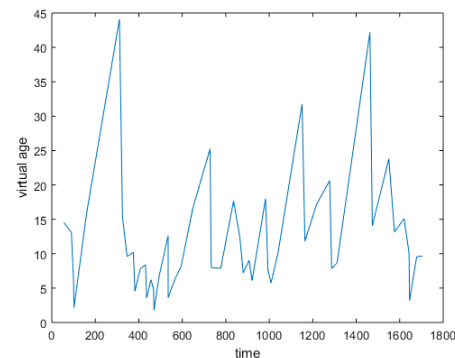


Figure 7. The variation of virtual age

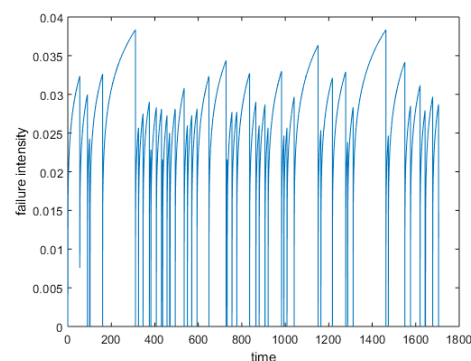


Figure 7. The variation of failure intensity

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