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Abstract

A geometric spatial reduction method is presented in this paper. It applies to port Hamiltonian models for open systems of balance equations. It is based on system projections which make use of the symmetries in the model and preserve the "natural" power pairing. Reductions from 3D to 2D and 1D domains are illustrated via two examples. The first one is a vibro-acoustic system with cylindrical symmetry where 3D-2D reduction is applied. The second one is the poloidal magnetic flux diffusion equation for tokamak reactors where the toroidal symmetry is used to perform a 3D-1D reduction.

Keywords: geometric reduction, distributed parameters systems, port Hamiltonian systems, tokamak plasma control, vibro-acoustic system

1. Introduction

A reduced model, reflecting the original one in some desired aspects, may be looked for in the purpose of system analysis, simulation or control synthesis. The term "geometric reduction" for Hamiltonian systems appeared in the 1980s in the works of J.E. Marsden ([15, 14]). For instance the method called momentum reductions (cf. [22, 14]) for Lagrangian or Hamiltonian systems was based on tangent and cotangent maps. The work of Blankenstein [3] presented the reduction of Dirac Structure including application to implicit Hamiltonian systems. In these early works, reduction was using symmetry Lie groups and intended to project the Hamiltonian in the reduced space. The system invariants, which can play a central role in control designs, sometimes disappeared after the reduction.

In the 1990s, the idea of geometric structure conservation was studied for the so-called multi-symplectic systems and used for the structure preserving total discretization of infinite dimensional systems. [6, 19] proposed the multi-symplectic formulation and corresponding discretization of the nonlinear Schrödinger equation and the water-wave problem. The multi-symplectic structure preservation in the numerical discretization of Hamiltonian dynamics was for instance investigated in [7], and detailed in [13, Chap. 6,7].

In this paper we focus on the spatial reduction problem for (open) port-Hamiltonian systems. More precisely we aim at reducing the dimension of the spatial domain: from a three dimensional (3D) model to a 2D or a 1D one. Our main motivation lies in the derivation of tractable continuous time infinite dimensional control model which may be used either for structure-based control design or further structure-preserving time discretization. In the port-Hamiltonian approach, a model is determined by a specific interconnection structure (Dirac structure) and its Hamiltonian function. The proposed reduction is based on the projection of the Dirac interconnection structure and therefore implies the preservation of the main geometrical properties of the original model such as conservativeness, dissipativeness, symplecticity, etc. The method applies mostly to open systems of balance equations written in the port- Hamiltonian formalism.

Two examples will be considered in this paper: a model of vibro-acoustic tubes (cf. [9, 10]) describing the wave transfer in cylindrical geometry and a 3D fluid-like model of the resistive diffusion of the poloidal magnetic flux in a tokamak reactor [26].

The paper is organized as follows. Section 2 details the system geometries and spatial symmetries that will be investigated in the paper. Section 3 describes the spatial geometric reduction general idea and the detailed methodology for 3D-2D and 3D-1D reductions. This methodology is applied on the two examples in section 4.

2. Considered geometries and spatial symmetries

We will make use of the *fiber bundle* concept which allows to define the reduction of the system variables using differential forms and their integration over a fiber.

2.1. Fiber bundle

The readers are referred to [8, 16] for detailed studies of fiber bundles or fibration, including topological aspects.

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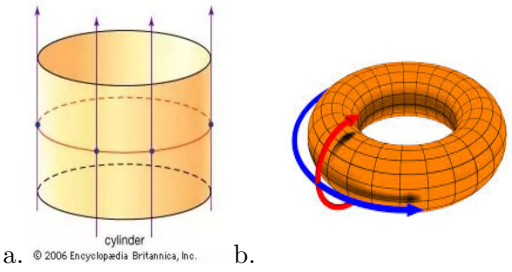
In this work, we restrict ourselves to some simple definitions and basic ideas sufficient to explain the reduction idea. A fiber bundle (E, π, B, F) will be defined using the continuous map:

$$\pi : E \to B \tag{2.1}$$

where E is the *total space,* B the *base space* (or *base),* F the fiber and π the submersion (or projection) map of the bundle, which projects every point $x \in E$ on the base B. Sometimes we also denote the fiber bundle π for the sake of simplicity.

In particular, the fiber bundle with $E = B \times F$ is called the *trivial bundle.* The simplest example of a non-trivial bundle is the Möbius Strip [8, 16]. One can consider a very simple example of the cylinder brush for the sake of comprehension. The cylinder brush is a trivial bundle E; the cylinder is the base B and the bristles, line segments are fibers F, π maps any point on the bristles to its root on the cylinder. Examples of trivial fiber bundles are the cylinder and the torus, which will be used in this paper. A cylinder bundle $E = S^1 \times Y$ (Fig. 2.1.a) is combined by the base S^1, the circle bundle, and the fiber is the line interval $Y = [0, 1]$. Note that we can also consider this cylinder bundle $E = Y \times S^1$ with the base Y and the fiber S^1. On the other hand, the torus is a $S^1 \times S^1$ bundle in which the base and the fiber are both S^1 circle bundle (Fig. 2.1.b).



a. b.

Figure 2.1: Cylinder bundle (a) and the Torus bundle (b)

An important notion associated with the fiber bundle is the cross section, defined as a continuous map s

$$s : B \to E \tag{2.2}$$

such that $\pi \circ s = identity$ or $\pi(s(x)) = x$ for all $x \in B$.

2.2. Differential form on a fiber bundle

We present hereafter the projection of differential forms on a fiber bundle [17, 16]. A fiber bundle may be viewed as a particular coordinate system. When projected using this decomposition, a k-form ω may be decomposed using fiber and base coordinates.

Let us consider the trivial case where $E = B \times F$ with $dim\, B = n$ and $dim\, F = r$. Let ξ_i, $i = 1..n$ and ζ_j, $j = 1..r$ denote the coordinates of B and F, respectively. An example ($r = 1$) of k-form $\omega \in \Lambda^k$ on $E = \mathbb{R}^n \times [0, 1]$ with $r < k < n$ is

$$\begin{aligned} \omega \quad = \quad & \omega_B \, d\xi_{i_1} \wedge ... \wedge d\xi_{i_k} + \\ & \omega_F \, d\zeta \wedge d\zeta_{j_1} \wedge ... \wedge d\zeta_{j_{k-1}} \end{aligned} \tag{2.3}$$

where ω_B and ω_F are smooth function defined in the whole fiber bundle $E = B \times F$.

2.3. Reduction by integration over a fiber

We aim now at explaining our geometric reduction methodology. It allows to symplecticly reduce the spatial dimension of the considered system, by using the concept of integration over a fiber (cf. [5, 2]) which allows to lower the degree of a form by the fiber dimension.

Consider again the trivial fiber bundle (E, π, B, F) with $dim\, B = n$ and $dim\, F = r$. We define a canonical linear map $\pi_\star$, homogeneous of degree $-r$

$$\pi_\star : \Lambda^k(E) \to \Lambda^{k-r}(B) \tag{2.4}$$

which maps any k-form ω on E to a corresponding $(k-r)$-form $\pi_\star\omega$ on B such that**:**

$$\begin{cases} \pi_\star\omega = 0 & \textbf{if } k < r \\ \pi_\star\omega = \int_F \omega & \textbf{otherwise} \end{cases} \tag{2.5}$$

We call this map *integration over fiber* or *fiber integration.* Regarding the k-form in (2.3):,

$$\pi_\star\omega = \pi_\star\left(\beta d\zeta \wedge d\xi_{j_1} \wedge ...d\zeta_{j_{k-1}}\right) = \left(\int_0^1 \beta d\zeta\right) d\xi_{j_1} \wedge ...d\zeta_{j_{k-1}} \tag{2.6}$$

As the result, one can easily derive an integral of a k-form ω over the total space E as:

$$\int_{E=B\times F} \omega = \int_B \int_F \omega = \int_B \pi_\star\omega \tag{2.7}$$

The map $\pi_\star$ does symplecticly reduce the dimension of the integral space from $(n + r)$ to n . The term $\pi_\star\omega$ is named the *corresponding reduced form* of ω over the reduction $(n + r)$ to n.

In the sequel, we will consider two types of reduction: reduction by symmetry which appears quite naturally in the literature, and reduction by homogenization.

Reduction by symmetry . Model reduction using spatial symmetry is a classical approach to simplify a model. In order to guarantee the preservation of some power form in the reduction, we will make use of symmetry on fibers and fiber integration.

Let us consider a fiber bundle π with ζ_j, $j = 1, ..r$ the coordinates of the fiber F. A k-form $\alpha \in \Lambda^k(E)$ (for some $k \in \{0, \ldots, n\}$) is called symmetric with respect to the coordinate ζ_j if it satisfies

$$\partial_{\zeta_j}\alpha = 0 \tag{2.8}$$

The same k-form $\alpha \in \Lambda^k(E)$ is called symmetric with respect to the whole fiber F if $\partial_{\zeta_j}\alpha = 0\,\forall j \in 1,\ldots,r$. In this case, using fiber integration, one gets:

$$\pi_\star\omega = \int_F \omega = \omega_F d\xi_{j_1} \wedge ... \wedge d\xi_{j_{k-r}} \int d\zeta_1 \wedge ... \wedge d\zeta_r \in B \tag{2.9}$$

For instance, in the case of fiber bundles with a fiber of the S^1 bundle (such as in the cylinder and torus trivial bundle examples from Fig. 2.1):

$$\pi_\star\omega = \int_0^{2\pi} \omega d\zeta = 2\pi\omega \tag{2.10}$$

Reduction by homogenization. When there is no symmetry on the fiber space, fiber integration also allows to decrease the space dimension of the studied system. We will then perform a reduction by averaged value or by *homogenization*:

$$\pi_\star\omega = \int_F \omega = \langle\omega\rangle \int d\zeta_1 \wedge ... \wedge d\zeta_r \in B \tag{2.11}$$

where $\langle\omega\rangle$ is the averaged value of ω on the fiber F. Symmetry with respect to the fiber F, that is $\partial_{\zeta_j}\langle\omega\rangle = 0\;\forall j \in 1,\ldots,r$, results trivially from this fiber integration.

In the next section, we consider 3D systems and a special case where the k-form ω is chosen to be the power energy density (3-form) $\omega := \mathcal{H}^3 \in \Lambda^3(\Omega)$. In the port-Hamiltonian formulation, this density is defined as the power (wedge) product of efforts and flows differential forms, denoted respectively in the sequel α and β. The aim of the reduction is then to determine *the corresponding reduced efforts and flows*, respectively $\overline{\alpha}$ and $\overline{\beta}$, by making use of the fiber integration concept, when the space Ω is a fiber bundle. Depending on the way we define the base and the fiber, we may obtain 3D-2D reduction ($r = 1$, $n = 2$) or 3D-1D reduction ($r = 2$, $n = 1$), where r and n are the dimension of the fiber and of the base respectively.

3. Geometric reduction for 3D systems

In this section, we consider a 3D spatial domain Ω with an associated volume 3-form dV, a 2D domain denoted Σ with the corresponding volume (surface) 2-form dS and a 1D domain is denoted Π with its volume (length) 1-form $d\mathcal{C}$. The 3D model is stated in covariant form, that is the state and port variables are not defined as vector fields but rather as differential k-forms [12].

3.1. Methodology

It is assumed that the total energy in the 3D spatial domain Ω may be written $\mathbb{H} = \int_\Omega \mathcal{H}^3$ where the energy density $\mathcal{H}^3 \in \Lambda^3(\Omega)$ is the external product of two k-forms[1], either $\mathcal{H}^3 = \alpha^1 \wedge \beta^2$ or $\mathcal{H}^3 = \alpha^0 \wedge \beta^3$. The idea is then to partially integrate $\mathcal{H}$ on 2D or on 1D coordinate surfaces or contours in such a way that the total energy reads:

$$\begin{array}{lll} \mathbb{H} = \int_\Omega \mathcal{H}^3 & = \int_\Sigma \mathcal{H}^2 & = \int_\Pi \mathcal{H}^1 \\ \text{with} & \mathcal{H}^2 = \overline{\alpha}^1 \wedge \overline{\beta}^1 & \text{or } \mathcal{H}^2 = \overline{\alpha}^0 \wedge \overline{\beta}^2 \\ \text{and} & \mathcal{H}^1 = \overline{\alpha}^0 \wedge \overline{\beta}^1 & \text{or } \mathcal{H}^1 = \overline{\alpha}^1 \wedge \overline{\beta}^0 \end{array} \tag{3.1}$$

This leads to the definition of 2D reduced variables ($\overline{\alpha}^1$,$\overline{\beta}^1$) or ($\overline{\alpha}^0$,$\overline{\beta}^2$) which are power-conjugated in the 2D domain Σ (their wedge product in a power density 2-form in Σ), as well as to the definition of the 1D reduced variables ($\overline{\alpha}^0$,$\overline{\beta}^1$) in the 1D domain Π. We discuss hereafter how to determine these variables. In applications (see section \ref{4}), it will be necessary to project the considered models into chosen specific coordinate systems (according to the "natural symmetries"), such as cartesian, cylindrical, spherical or toroidal coordinate systems. Therefore, in the sequel, we will consider a general 3D curvilinear coordinate systems (ξ_1, ξ_2, ξ_3) and detail the reduction method in these coordinates.

3.2. 3D-2D

We assume that the system is symmetric with respect to the fiber ξ_3 coordinate (i.e. $\partial_{\xi_3}. = 0$ for all system variables) in the coordinate systems (ξ_1, ξ_2, ξ_3) . Then the power energy reads:

$$\int_\Omega \mathcal{H}^3 = \int_\Omega \alpha.\beta\, dV = \int_\Sigma \overline{\alpha}.\overline{\beta}\, dS = \int_\Sigma \mathcal{H}^2 \tag{3.2}$$

where the variables (α, β) (without upper-index) stand for the tensors in the considered 3D coordinate system, while ($\overline{\alpha}$,$\overline{\beta}$) are the reduced variables in the reduced 2D model. The inner product $\alpha.\beta$ associate to the volume form dV. Let us denote $(\lambda_1, \lambda_2, \lambda_3)$ the Jacobian coefficients, such that the volume element reads

$$dV = \sqrt{\lambda_1}d\xi_1\sqrt{\lambda_2}d\xi_2\sqrt{\lambda_3}d\xi_3 = \sqrt{\lambda}d\xi_1 d\xi_2 d\xi_3 \tag{3.3}$$

Then, with the previous symmetry assumption:

$$\begin{array}{rcl} \int_\Omega \alpha.\beta\, dV & = & \int_\Omega \alpha.\beta\sqrt{\lambda}d\xi_1 d\xi_2 d\xi_3 \\ & = & \underbrace{\int_{\xi_3\in F} d\xi_3}_{\Theta} \int_\Sigma \alpha.\beta\sqrt{\lambda}d\xi_1 d\xi_2 \end{array} \tag{3.4}$$

where Θ is a constant. The 2-form $\Theta\alpha.\beta\sqrt{\lambda}\, d\xi_1 d\xi_2$ is then identified with $\overline{\alpha}.\overline{\beta}\, dS$. Generally speaking, the obtained expression in local coordinates for $\alpha.\beta\sqrt{\lambda}$ depends on the existing symmetries and the corresponding chosen coordinate surfaces/contours for fiber integration. As it can been seen from (3.1), different choices exist for the reduction of the differential forms α and β which depends on their respective degrees. These choices are summarized in the table 3.1.

[1] The upper indexes denote the degree of the corresponding differential form

	0-form	1-form	2-form	3-form
3D variables	α^0	α^1	β^2	β^3
2D variables	$\overline{\alpha}^0$	$\overline{\alpha}^0$ (resp.$\overline{\alpha}^1$)	$\overline{\beta}^1$ (resp. $\overline{\beta}^2$)	$\overline{\beta}^2$

Table 3.1: Possible choices for the degrees of reduced power conjugated variables in the 3D-2D reduction

3.3. 3D-1D

Similarly we assume that the system is symmetric with respect to theξ_2, ξ_3 fiber coordinates (i.e. $\partial_{\xi_2} = \partial_{\xi_3} = 0$, for all system variables) in the coordinate systems (ξ_1, ξ_2, ξ_3) . Then the power energy reads:

$$\int_\Omega \mathcal{H}^3 = \int_\Omega \alpha.\beta \, dV = \int_\Pi \overline{\alpha}.\overline{\beta} \, d\mathcal{C} = \int_\diamond \mathcal{H}^1 \tag{3.5}$$

with

$$\begin{aligned} \textstyle\int_\Omega \alpha.\beta \, dV &= \textstyle\int_\Omega \alpha.\beta \sqrt{\lambda} d\xi_1 d\xi_2 d\xi_3 \\ &= \underbrace{\textstyle\int_{\xi_{2min}}^{\xi_{2max}} d\xi_2 \int_{\xi_{3min}}^{\xi_{3max}} d\xi_3}_{\Theta} \textstyle\int_\Pi \alpha.\beta \sqrt{\lambda} d\xi_1 \end{aligned} \tag{3.6}$$

The degrees of the reduced differential forms $\overline{\alpha}$ and $\overline{\beta}$ are determined by the corresponding degrees of the conjugated variables α and β according to the table 3.2.

	0-form	1-form	2-form	3-form
3D variables	α^0	α^1	β^2	β^3
2D variables	$\overline{\alpha}^0$	$\overline{\alpha}^0$ (resp. $\overline{\alpha}^1$)	$\overline{\beta}^1$ (resp. $\overline{\beta}^0$)	$\overline{\beta}^1$

Table 3.2: Degrees of reduced power conjugated variables in the 3D-1D reduction

4. Examples

The first example (subsection 4.1) in this section deals with 3D-2D reduction in cylindrical coordinates using axial symmetry. The second example (subsection 4.2) makes use of a 3D-1D reduction in toric coordinates with toroidal symmetries.

4.1. Vibro-acoustic tube with axial symmetry

Models of vibro-acoustic tubes are used in many engineering applications related for instance to the safety, passenger comfort and noise reduction in airplanes. They describe an acoustic wave traveling - without energy loss - in a tube (the cabin) equipped with a network of microphones/ loudspeakers [9]. The tube geometry will be depicted using cylindrical coordinates (x, r, ϕ) (see figure 4.1). The assumption of axis-symmetry [9] around the horizontal axis x (which means $\partial_\phi \cdot = 0$) leads to the 3D-2D reduced model with the 2D coordinate system (x, r) where $x \in [0, L]$ and $r \in [0, R]$ in which L and R denotes respectively the length and the radius of the tube.

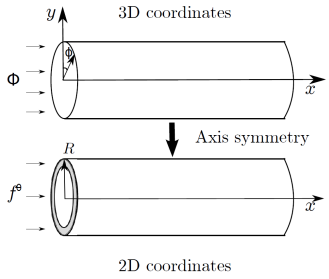


Figure 4.1: Cylindrical coordinate for the vibro-acoustic system and the considered axis symmetry. The 2D flux density f^θ corresponds to the 3D flux density Φ in the original domain.

4.1.1. The port-Hamiltonian 3D model

Reformulating the vibro-acoustic model [9, 10] in the Hamiltonian formalism may be done using the pair of power conjugated variables (Φ, v) and (Γ, P) and the total acoustic energy power (Hamiltonian)[2]:

$$\mathbb{H} = \frac{1}{2} \int_\Omega \Phi^1 \wedge \star v^1 + \Gamma^3 \wedge P^0 \tag{4.1}$$

where Φ is the 1-form kinetic momentum, v is the 1-form velocity, Γ is the 3-form volumetric expansion and P is the 0-form pressure. The Port Hamiltonian representation of this system consists in two conservation laws: momentum conservation and mass conservation:

$$\begin{pmatrix} \partial_t \Phi \\ -\partial_t \Gamma \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \star v \\ P \end{pmatrix} \tag{4.2}$$

In this model, we make use of the usual Stokes-Dirac structure for the 3D wave equation [21]

$$\begin{pmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -grad \\ div & 0 \end{pmatrix} \tag{4.3}$$

together with the two constitutive relations:

$$\begin{cases} v^1 = & \Phi^1/\mu_0 \\ P^0 = & \star\Gamma^3/\chi_s \end{cases} \tag{4.4}$$

where μ_0 and χ_s are the air mass density and the adiabatic compressibility coefficient respectively. The Hamiltonian (power energy) in (4.1) reduces to the quadratic form:

$$\mathbb{H} = \frac{1}{2} \int_\Omega \Phi^2 \wedge \frac{\star\Phi^2}{\mu_0} + \Gamma^3 \wedge \frac{\star\Gamma^3}{\chi_s} \tag{4.5}$$

In the sequel, we apply the proposed reduction method in section 3 to this system in order to determine the corresponding reduced variables and associated 2D model.

[2]The Hodge star operator $\star$ (see for instance[12]) converts a k-form into $(n-k)$-form in a nD spatial domain. It is defined with respect to some metric which translates the geometric constitutive properties (e.g. anisotropy) of the considered spatial (material) domain. In the above example $\star v^1$ is thus a 2-form, while $\star\Phi^2$ is a 1-form and $\star\Gamma^3$ a 0-form. Unless stated otherwise, we will make use of the usual Euclidian metric

4.1.2. New variables in 2D domain and 2D model

The new variables in the reduced domain are defined from the energy conservation (4.1). The volume element written in cylindric coordinates is $dV = rdxdrd\phi$. The z-axis symmetry implies $\partial_\phi . = 0$ for all system variables and

$$\begin{array}{rcl} \mathbb{H} & = & \frac{1}{2}\int_V (\Phi v + \Gamma P)\, dV \\ & = & \frac{1}{2}\int_0^{2\pi} d\phi \int_\Sigma (\Phi v + \Gamma P)\, r dx dr \end{array} \tag{4.6}$$

The next step is to determine the reduced variables in the 2D domain. The integration domains are the circle section (r, ϕ) for the form Φ and the volume for the form Γ. One can derive:

$$\begin{array}{rcl} \mathbb{H} & = & \frac{1}{2}\int_S (\Phi dx)\,(2\pi r v dr) + (2\pi r \Gamma dx dr)\, P \\ & = & \frac{1}{2}\int_\Sigma \Phi^1 \wedge \overline{\star v}^1 + \overline{\Gamma}^2 \wedge P^0 \end{array} \tag{4.7}$$

where the1-form $\overline{\star v}^1 = 2\pi r v dr$ and the 2-form $\overline{\Gamma}^2 = 2\pi r \Gamma dx dr$ are the new variables in 2D domain. Consequently, the system (4.1) transforms into the 2D model:

$$\begin{pmatrix} \partial_t \Phi \\ -\partial_t \overline{\Gamma} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \overline{\star v} \\ P \end{pmatrix} \tag{4.8}$$

Formally, the spatial interconnexion operator

$$\begin{pmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -grad \\ div & 0 \end{pmatrix} \tag{4.9}$$

remains unchanged (although it is expressed now in the new reduced coordinate systems). The proposed reduction not only preserves the energy power but also the geometric structure of the model.

Remark 4.1. The reduced port-Hamiltonian model (4.8) with constitutive equations (4.4) are equivalent to the model in [20] where Euler's equation (momentum conservation) and mass balance are written in vectorial form, making use of the acoustic approximation assumption:

$$\begin{cases} \mu \dfrac{\partial v}{\partial t} & = & -gradP \\ \dfrac{\partial \mu}{\partial t} + div\,(\mu v) & = & 0 \end{cases} \tag{4.10}$$

The authors use the model to design a passivity-based controller which drives the system to a point of bounded energy with the minimum velocity of the wave at the boundaries. The same model is used in [29] where a structure-preserving spatial discretization scheme (mixed finite elements method) is developed.

4.2. Tokamak plasma with toroidal symmetry

Tokamak is a facility constructed with the shape of a torus (or dough-nut) in which a plasma is magnetically confined and heated in order to produce nuclear fusion reactions (see the classical Wesson's monograph [28]). Control problems of Tokamak plasma aim at many different objectives [18, 27, 1]. When trying to handle MHD (magneto-hydrodynamic) instabilities and control current, temperature and pressure density profiles, control models have been used quite successfully. A 3D model (TMHD) in port-Hamiltonian formulation has been proposed in [26] for the temperature, particle density, current density and magnetic flux profiles of a plasma gas in a tokamak toroidal chamber. We focus hereafter on the electromagnetic fields and entropy balance, respectively in sections 4.2.2 and 4.2.3. To derive a 1D model with the proposed approach, we will make use of two assumptions which have been already used to simplify the dynamics in tokamak reactors [4, Chap.6]:

- *axial symmetry:* the symmetric position of all the electric coils allows to carry out an axisymmetric magnetic field around the principle axis of the torus (a 2D reduced model could be deduced at this stage)
- *quasi-static equilibrium:* the plasma may be assumed to have reached a "mechanical" stationary profile at every instant t considered for the heat or magnetic flux diffusion phenomena. Therefore the dynamics in the mechanical domain may be considered at the equilibrium where the magnetic force balances the pressure forces:

$$J \times B = \nabla P \tag{4.11}$$

in which J, B, P are the plasma current, magnetic field and plasma pressure respectively.

With these two assumptions, it may be shown that the magnetic surfaces made with constant field lines of B (where also lies the plasma current density J) are also surfaces of constant pressure P. It may be proved that these surfaces form a set of nested toroids which are simultaneously isobaric, isothermal and iso-poloidal flux (cf. [28, sec. 3.2] and [4, chap. 1]). Therefore, after a continuous mapping, these surfaces may be matched into nested regular toroidal surfaces with circular cross-sections and a set of magnetic toric coordinates (ρ, θ, ϕ) (see figure 4.2) may be defined such that ρ denotes the index of the considered magnetic surface (the new "radial" coordinate) and such that all the system variables are independent of θ and ϕ. The model may be projected onto the 1D domain $\Pi = [0, a] \ni \rho$, $a = \rho_{max}$.

4.2.1. Reduced variables in the 1D domain

Let g_ρ, g_θ, g_ϕ, and $g = g_\rho g_\theta g_\phi$[3] denote the transformation coefficients between geometric toric coordinates

[3] These coefficients have the form:
$\begin{cases} g_\rho = & (\partial_\rho r)^2 \\ g_\theta = & (\partial_\theta r)^2 + r^2 \\ g_\phi = & (R_0 + r cos\theta)^2 \end{cases}$; $g = g_\rho g_\theta g_\phi$

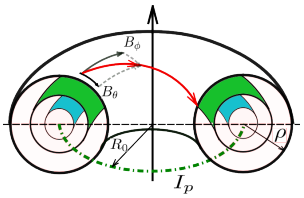


Figure 4.2: Magnetic toric coordinate: ρ denotes the magnetic surface index (corresponding to the small radius r), θ the polar angle and ϕ the azimuth angle. R_0 denotes the principal radius of the plasma, I_p the total plasma current and B_θ and B_ϕ the two components of the magnetic field

(r, θ, ϕ) and magnetic toric coordinates (ρ, θ, ϕ). Let us define the volume element as $dV = \sqrt{g} d\rho d\theta d\phi$. We will perform the integration of the energy conservation equation (3.1) along the magnetic toric curve coordinate (for 1-forms and 2-forms) and surface coordinate (for 3-forms). The obtained reduced power-conjugated variables α and β are summarized in table 4.1. Note that most of the variables "lie" on the magnetic surfaces, since the component in the radial direction ρ vanishes with the quasi-static equilibrium assumption.

k-form	new variables	corresponding values
1-form	$\alpha^0 = \left(\overline{\alpha_\theta}, \overline{\alpha_\phi}\right)$	$\left(\int_0^{2\pi} \sqrt{g_\theta}\alpha_\theta\, d\theta,\ \int_0^{2\pi} \sqrt{g_\phi}\alpha_\phi\, d\phi\right)$
2-form	$\beta^1 = \left(\overline{\beta_\theta}, \overline{\beta_\phi}\right) d\rho$	$\left(\int_0^{2\pi} \sqrt{g_\rho g_\phi}\beta_\theta\, d\phi,\ \int_0^{2\pi} \sqrt{g_\rho g_\theta}\beta_\phi\, d\theta\right) d\rho$
3-form	$\beta^1 = \overline{\beta} d\rho$	$\left(\int_0^{2\pi}\int_0^{2\pi} \sqrt{g}\beta d\theta d\phi\right) d\rho$

Table 4.1: Reduced variables definition in the 1D domain Π

This reduction is applied hereafter, first to Maxwell's equations in electromagnetic domain (section 4.2.2) and then to the entropy balance equation in the material domain in 3D (4.2.3). We will obtain respectively the so-called 1D resistive diffusion and thermal diffusion equations.

4.2.2. The resistive diffusion equation

The plasma electromagnetic 3D model was developed in port-Hamiltonian formulation in [26, Sec. 3]. It is defined from the covariant formulation of the Maxwell's equations (see for instance [12])

$$\begin{pmatrix} -\partial_t D \\ -\partial_t B \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} J \quad (4.12)$$

where the electric and magnetic field intensities $E, H \in \Lambda^1(\Omega)$ are the 1-form variables in a 3D domain Ω with volume V, while the field flows and total current density $D, B, J \in \Lambda^2(\Omega)$ are the 2-forms. Here d denotes the external spatial derivative (see e.g. [12]). The electromagnetic energy is:

$$\mathbb{H}_{EM} = \frac{1}{2}\int_\Omega [E^1 \wedge D^2 + H^1 \wedge B^2] \quad (4.13)$$

Let us now apply the geometric reduction described in the previous subsection to the magnetic domain to determine the corresponding 1D variables:

$$\begin{aligned} \mathbb{H}(B) &= \frac{1}{2}\int_\Omega H^1 \wedge B^2 = \frac{1}{2}\int_V H.B\mathrm{d}V \\ &= \frac{1}{2}\int_V \left(H_\rho B_\rho + H_\theta B_\theta + H_\phi B_\phi\right)\sqrt{g} d\rho d\theta d\phi \\ &= \frac{1}{2}\int_0^a d\rho[\int_0^{2\pi}\left(\sqrt{g_\theta}H_\theta\right)d\theta \int_0^{2\pi}\left(\sqrt{g_\rho g_\phi}B_\theta\right)d\phi \\ &\quad + \int_0^{2\pi}\left(\sqrt{g_\phi}H_\phi\right)d\phi \int_0^{2\pi}\left(\sqrt{g_\rho g_\theta}B_\phi\right)d\theta] \\ &= \frac{1}{2}\int_0^a d\rho\left[\left(\overline{H}_\theta\right)\left(\overline{B}_\theta\right) + \left(\overline{H}_\phi\right)\left(\overline{B}_\phi\right)\right] \\ &= \frac{1}{2}\int_\Pi \overline{H}^0 \wedge \overline{B}^1 \end{aligned} \quad (4.14)$$

Thus the 3D model (4.12) transforms into a 1D model with a similar power pairing product. The same reduction is applied to the electric domain with the energy density $\mathbb{H}(D) = \frac{1}{2}\int_\Omega E^1 \wedge D^2$ and leads to the definition of the reduced variables $\overline{E}^0$, $\overline{D}^1$ which are derived similarly. The Maxwell's equations (or EM Dirac structure) in *the 1D* domain Π are then simply written

$$\begin{bmatrix} -\dfrac{\partial}{\partial t}\overline{D}^1 \\ -\dfrac{\partial}{\partial t}\overline{B}^1 \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d}_\Pi \\ \mathrm{d}_\Pi & 0 \end{bmatrix} \begin{bmatrix} \overline{E}^0 \\ \overline{H}^0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overline{J}^1 \quad (4.15)$$

where the *exterior derivative* d_Π in the 1D reduced spatial domain Π is defined as

$$\mathrm{d}_\Pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \rho} \quad (4.16)$$

The considered boundary in our system is the magnetic surface at the plasma external radius a . Since $E_\rho = H_\rho = 0$ on the magnetic surfaces, the energy flux $\int_{\partial\Omega} H^1 \wedge E^1$ which goes through the boundary is:

$$\begin{aligned} \int_{\partial\Omega} H^1 \wedge E^1 &= \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \sqrt{g_\theta g_\phi}\left(H_\theta E_\phi - H_\phi E_\theta\right)|_0^a \\ &= \left(\int_0^{2\pi}\sqrt{g_\phi}E_\phi d\phi\right)\left(\int_0^{2\pi} d\theta\sqrt{g_\theta}H_\theta\right)|_0^a \\ &\quad - \left(\int_0^{2\pi}\sqrt{g_\phi}H_\phi d\phi\right)\left(\int_0^{2\pi} d\theta\sqrt{g_\theta}E_\theta\right)|_0^a \\ &= \int_{\partial\Pi} \overline{H}_\theta \overline{E}_\phi - \overline{H}_\phi \overline{E}_\theta \\ &= \int_{\partial\Pi} \overline{H}^0 \wedge \overline{E}^0 \end{aligned} \quad (4.17)$$

Let us focus now on half of the model 4.15 related to the diffusion of poloidal flux ψ; with $-\partial_\partial \psi = \overline{B}_\theta$. Only components B_θ, H_θ, E_ϕ, J_ϕ) are concerned. This half models reads

$$\begin{pmatrix} f_{el} \\ f_{mg} \end{pmatrix} = \begin{pmatrix} 0 & -\partial_\rho \\ -\partial_\rho & 0 \end{pmatrix} \begin{pmatrix} e_{el} \\ e_{mg} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} f_d \quad (4.18)$$

where f_{el}, f_{mg}, e_{el}, e_{mg}, f_d are flow and effort variables respectively defined by $\partial_t\left(-\overline{D}_\phi\right)$, $\partial_t\left(-\overline{B}_\theta\right)$, $\overline{E}_\phi$, $\overline{H}_\theta$ and $\overline{J}_\phi$. The closure equations (written in toric coordinates) for the balance equations are:

$$\begin{cases} e_{el} = \frac{\eta}{C_3}\overline{J}_\Omega & \text{Ohm's law} \\ e_{mg} = \frac{C_2}{\mu_0}\overline{B}_\theta & \text{magnetic constitutive equation} \end{cases} \quad (4.19)$$

where $C_2 = \frac{\sqrt{g_\theta}}{\sqrt{g_\rho g_\phi}}$, $C_3 = \frac{\sqrt{g_\rho g_\theta}}{\sqrt{g_\phi}}$. $\overline{J}_\Omega$ is the 1D ohmic current equal to $\left(\overline{J}_\phi - \overline{J}_{ni}\right)$. The current $\overline{J}_{ni}$ is the corresponding 1D non-inductive current, it equals to the sum of the bootstrap current $\overline{J}_{bs}$ described in [28] (a magneto-hydrodynamics coupling effect which produces and extra current density) and external current source $\overline{J}_{ext}$ which is controlled through external heating sources [28, p. 238]. The magnetic permeability is considered to be the void permeability μ_0 since Tokamaks are operating at very low densities.

The balance equations (4.18) and constitutive equations (4.19) are equivalent to the so-called *resistive diffusion equation* for the poloidal magnetic flux (cf. [4, chap. 6]; [28, p. 152]):

$$\frac{\partial \psi}{\partial t} = \eta \frac{1}{C_3}\frac{\partial}{\partial \rho}\left(\frac{1}{\mu}C_2\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{C_3}\left(\eta \overline{J}_{ni}\right) \tag{4.20}$$

Remark 4.2. The plasma resistivity η, and the bootstrap current $\overline{J}_{bs}$ are significantly varying with the plasma temperature T (cf. [4, p.172]). However, in most existing control designs (for the poloidal flux control) these TMHD couplings have been neglected and the temperature T has been considered as an external parameter. Then $\eta := \eta(z,t)$ and $\overline{J}_{bs} := \overline{J}_{bs}(z,t)$ are considered as time and space dependent parameters. In section 4.2.3, an explicit dependence of these parameters with the temperature T has been considered by adding a diffusion model.

Remark 4.3. The structured model (4.18) and (4.19) has been used in [24] to design a symplectic geometric discretization scheme for the resistive diffusion equation of the poloidal magnetic flux. In [25] and IDA-PBC (Interconnection and Damping Assignment - Passivity Based Control) control design has been proposed, based on the same model.

4.2.3. The thermal diffusion equation

In [26, Sec. 4], the material domain balance equations for mass, momentum, energy and entropy are written firstly from the Boltzmann equation using the kinetic theory. The connection between the classical macroscopic transport equation and the port-based formulation is made by using the material derivative in covariant form. Then the irreversible entropy source term is derived from the Gibbs-Duhem relation (following the "port-based" approach in [23]). The irreversible entropy production contains terms which accounts for the heat conduction, the viscous dissipation, the Joule (ohmic) terms, and the external heating sources. It defines the constitutive Onsager relations for the heat balance or thermal diffusion equation.

Let σ_s denote the 3-form entropy source term, s the 3-form entropy density and T the 0-form temperature. They are all defined in the moving material domain $\mathcal{M}$. The entropy balance equation reads:

$$\begin{pmatrix} T\dfrac{ds}{dt} \\ F \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ -\mathrm{d} & 0 \end{pmatrix}\begin{pmatrix} T \\ f_q \end{pmatrix} + \begin{pmatrix} \sigma_s \\ 0 \end{pmatrix} \tag{4.21}$$

with the 2-form heat flux f_q and the 1-form thermal force F. In the case where there is no fusion reaction, average variables may be used (instead of species temperatures and densities) and only one energy balance equation may be considered. This energy balance equation may be written using magnetic toric coordinates:

$$\begin{aligned} \mathbb{H}_T &= \int_{\mathcal{M}} T^0 \wedge S^3 = \int_\Omega T^0 \wedge n\sigma_s^3 \\ &= \int_V T n \sigma_s \sqrt{g} d\rho d\theta d\phi \\ &= \int_0^a d\rho \left[T \int_0^{2\pi}\int_0^{2\pi} \left(n\sqrt{g}\sigma_s\right) d\theta d\phi\right] \\ &= \int_0^a T\overline{\sigma_s} d\rho = \int_\Pi T^0 \wedge \overline{\sigma}_s^1 \end{aligned} \tag{4.22}$$

where we have applied the proposed reduction scheme and defined the corresponding 1D reduced port-conjugated variables in the thermal domain. Let us point out that (4.21) is a material domain balance equation (see details in [26, Sec. 4]). Therefore we have used a transformation from this moving material domain$\mathcal{M}$ into the fixed volume domain Ω (see the first line in equation (4.22)). This simple transformation uses the average particle density n (see also remark 4.4). Thus, the 3D thermal model in (4.21) transforms into the 1D port-Hamiltonian model:

$$\begin{pmatrix} f_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -\partial_\rho \\ -\partial_\rho & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} \overline{\sigma_s} \\ 0 \end{pmatrix} \tag{4.23}$$

where f_1, f_2, e_1, e_2 are the flows and efforts which are respectively defined by $\overline{nT\left(D_t s\right)}$, $\overline{nf_q}$, T, and $\overline{F}$. One of the associated closure relations is the Fourier's law:

$$f_2 = n\chi \frac{\sqrt{g_\theta g_\phi}}{\sqrt{g_\rho}} e_2 \tag{4.24}$$

where χ is the diffusion coefficient. The ideal gas law is used as the second constitutive equation, relating e_1 and f_1, by considering no particle source injection (i.e. $\frac{dn}{dt} = 0$):

$$s = \ln\left(\frac{T^{3/2}}{n}\right) \Rightarrow \frac{ds}{dt} = \frac{\partial s}{\partial T}\frac{dT}{dt} = \frac{3}{2}\frac{1}{T}\frac{dT}{dt} \tag{4.25}$$

The derived reduced port-Hamiltonian model (made of the balance equations (4.23) and constitutive equations (4.24) and (4.25)) is formally equivalent to the usual thermal diffusion equation:

$$\sqrt{g}\frac{3}{2}\frac{\partial nT}{\partial t} = \partial_\rho\left(\frac{\sqrt{g_\theta g_\phi}}{\sqrt{g_\rho}}\left(n\chi\partial_\rho T\right)\right) + \overline{\sigma_s} \tag{4.26}$$

For instance, equation (4.26) is equivalent to the electronic heat transport equation which may be written [11]:

$$V'\partial_t\left(n_e T_e\right) = \partial_\rho\left(G_1 V' n_e \chi_e \partial_\rho T_e\right) + V' P_e \tag{4.27}$$

where the terms V', $G_1 = \langle\nabla\rho\rangle^2$, and P_e are equivalent respectively to our parameters $\sqrt{g}$, g_ρ^{-1}, and $n\sigma_{\mathbf{s}}$ which are the magnetic toric coordinate coefficients and the source terms in (4.26).

Remark 4.4. In the plasma magneto-hydrodynamic (MHD) couplings, one must consider not only the Lorentz forces-but also the transformation from the fix volumetric frame Ω into the moving massic frame $\mathcal{M}$, named Eulerian-Lagrangian transformation. The power product of (e_2, f_2) in $\mathcal{M}$ domain is preserved as (e_1, f_1) in Ω domain:

$$\int_{\mathcal{M}=\phi_t(\Omega)} e_2 \wedge f_2 = \int_{\Omega} \phi^* (e_2 \wedge f_2) = \int_{\Omega} \phi^* (e_2) \wedge f_1 \tag{4.28}$$

with the pullback $\phi^* (e_2) = n e_2 \circ \phi^{-1}$ and where n denotes the average particles density (which is assumed constant by the quasi-static assumption).

5. Conclusion

A structure preserving geometric spatial reduction methodology is proposed in this paper. It is based on the projection map of the fiber bundle structure used to describe the spatial symmetry. The reduction is based on the integration along fibers of the differential k-forms. It gives rise to port-Hamiltonian models described by Stokes-Dirac structures in the lower dimensional spaces which are the projection of the original ones. Constitutive equations are reduces applying the same ideas. Two examples have been used to illustrate all the possible cases for the reduction of power pairings in 3D domains (see table 4.1) to 2D or 1D spatial domains. The reduced models obtained with thess examples have been proven to be equivalent to existing models in the literature and have been used in recent works on discretization and control.

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