Terao's conjecture for triangular arrangements

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Simone Marchesi and Jean Vallès

Abstract

In this work we study line arrangements consisting in lines passing through three non aligned points. We call them triangular arrangements. We prove that any of this arrangement is associated to another one with the same combinatorics, constructed by removing lines to a Ceva arrangement. We then characterize the freeness of such triangular arrangements, which will depend on the combinatorics of the deleted lines. We give two triangular arrangements having the same weak combinatorics (that means the same number t_i of points with multiplicity $i, i \geq 2$), such that one is free but the other one is not. Finally, we prove that Terao's conjecture holds for triangular arrangement.

1 Introduction

A line arrangement $\mathcal{A} = \{l_1, \ldots, l_n\}$ in $\mathbb{P}^2$ is a finite set of distinct lines. The union of these lines forms a divisor defined by an equation $f = \prod_i f_i = 0$ where $f_i = 0$ is the equation defining l_i. The cohomology ring of the complement $\mathbb{P}^2 \setminus \{f = 0\}$ was first studied by Arnold and Brieskorn who proved that it is generated by the logarithmic differential 1-forms $\frac{dl_i}{l_i}$. The sheaf $\Omega_{\mathcal{A}}$ of logarithmic 1-forms associated to a line arrangement $\mathcal{A}$ (more generally to a hyperplane arrangement), and its dual, the sheaf $\mathcal{T}_{\mathcal{A}}$ of vector fields tangent to this arrangement become of great interest and many important works appear concerning these objects (see for example [4], [7], [11], [12], [14]). This last sheaf can be heuristicaly unterstood as the tangent sheaf of the complement $\mathbb{P}^2 \setminus \{f = 0\}$ and can be defined as the kernel of the Jacobian map, which means (in $\mathbb{P}^2$):

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{\nabla f} \mathcal{J}_f(n-1) \longrightarrow 0,$$

where $\mathcal{J}_f$ is the ideal sheaf generated by the three partial derivatives $\nabla f = (\partial_x f, \partial_y f, \partial_z f)$. This ideal, called Jacobian ideal, defines the Jacobian scheme supported by the singular points of the arrangement; for instance when $\mathcal{A}$ is generic (i.e. it consists of n lines in general position) then $\mathcal{J}_f$ defines $\binom{n}{2}$ distinct points. These sheaves $\Omega_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{A}}$ are basic tools to study the link between the geometry, the topology and the combinatorics of $\mathcal{A}$. The combinatorics of $\mathcal{A}$ is determined by the intersection lattice $L(\mathcal{A})$ which is, roughly speaking, the set of all intersections of hyperplanes of the arrangement (see [11] for more details).

Notice that the sheaf $\mathcal{T}_{\mathcal{A}}$ is a reflexive sheaf over $\mathbb{P}^2$ and therefore it is a vector bundle. When $\mathcal{A}$ is generic, one can verify (see [7] for instance) that $\mathcal{T}_{\mathcal{A}}$ is a Steiner bundle (i.e. its resolution by free $\mathcal{O}_{\mathbb{P}^2}$-modules is given by a matrix of linear forms):

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{n-3}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{n-1} \longrightarrow \mathcal{T}_{\mathcal{A}}(n-2) \longrightarrow 0.$$

When $\mathcal{A}$ is not generic, the associated bundle $\mathcal{T}_{\mathcal{A}}$ can be of any kind, semi-stable, unstable and even decomposed as a sum of two line bundles. When $\mathcal{T}_{\mathcal{A}}$ is a sum of line arrangements, the arrangement $\mathcal{A}$ is called *free arrangement*; these free logarithmic sheaves were studied first by Saito in [12] for any reduced divisor and by Terao [13] for hyperplane arrangements. Let us define freeness precisely in our situation.

Definition 1.1. The arrangement $\mathcal{A}$ is free with exponents (a, b), where $0 \leq a \leq b$ are integers, if $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$.

In [11], the main reference about hyperplane arrangements, Terao conjectures that freeness depends only on the combinatorics of $\mathcal{A}$, where the combinatorics is described by the set $L(\mathcal{A})$ of all the intersections of lines in $\mathcal{A}$. More precisely, if two arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$ have the same combinatorics (a bijection between $L(\mathcal{A}_0)$ and $L(\mathcal{A}_1)$) and one of them is free then the other one is also free (of course with the same exponents). This conjecture, despite all the efforts, is proved, for line arrangements, only up to 13 lines (see [2]). Probably, one of the main difficulty is that few families of free arrangements are known.
A weaker problem concerns the weak combinatorics. The weak combinatorics of a given arrangement of n lines is defined the knowledge of the integers t_i, $i \geq 2$ of points with multiplicity exactly equal to i of the arrangement. Let us mention the following beautiful formula, found by Hirzebruch in [8], involving these numbers (when $t_n = t_{n-1} = t_{n-2} = 0$):

$$t_2 + t_3 \geq n + \sum_{i \geq 1} i t_{i+4}.$$

It is natural to ask if Terao's conjecture can be extended to the assumption of weak combinatorics, i.e.

Do there exist two arrangements with the same weak combinatorics with one free and the other one not?

In section 6, giving an explicit example, we prove that the answer is yes. To our knowledge, this example, is the first known example of two arrangements with the same weak combinatorics (but not the same combinatorics) such that one is free and the other is not.

In [5, Corollary 2.12], Elencjwag and Forster proved that a rank r vector bundle E on $\mathbb{P}^n$ with the same Chern classes than a sum of line bundles $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-a_i)$ and such that $E_l = \bigoplus_{i=1}^{r} \mathcal{O}_l(-a_i)$ for one line $l \subset \mathbb{P}^n$ is actually $E = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-a_i)$. In other words, if the Chern classes of $\mathcal{T}_{\mathcal{A}}$ are given (they are given by the knowledge of the number of triple points counted with multiplicities, which is weaker than the combinatorics, even weaker than the weak combinatorics), the freeness of $\mathcal{A}$ is completely determined by the splitting on one line!
So, the main difficulty is to determine the splitting type of the bundle $\mathcal{T}_{\mathcal{A}}$ along a line (of $\mathcal{A}$ or not). Wakefield and Yuzvinsky proved (see [14, Theorem 3.1]), using the notion of multiarrangements introduced by Ziegler (see [15]), that except for some special multiplicities, the splitting type of $\mathcal{T}_{\mathcal{A}}$ on one line of the arrangement does not depend only on the multiplicities of the restriction, but also on the positions of the restricted points.
Indeed, the cited result tells us the following. Take a line L belonging to an arrangement $\mathcal{A}$ of N lines and denote by n the number (without multiplicity) of intersection points on L and by $m_1 \geq \cdots \geq m_n$ their multiplicities. We have that:

- If $m_1 \geq \sum_{i=2}^{n} m_i$ then the splitting type of $\mathcal{T}_{\mathcal{A}}$ on L is $(\sum_{i=2}^{n} m_i, m_1)$.
- If $2n-1 \geq N$ then the splitting type is $(N-n, n-1)$.
- If $2n-1 \leq N$ then the splitting type is balanced when the n intersection points are in general position but can be unbalanced for special positions.

It means for instance that the splitting type of $\mathcal{T}_{\mathcal{A}}$ along a line of $\mathcal{A}$ containing 4 multiple points of $\mathcal{A}$ will depend on their cross-ratio. Since $\mathrm{PGL}(2,\mathbb{C})$ acts transitively on the set of three distinct points on $\mathbb{P}^1$, this implies that the combinatorics determine the splitting if there are no more than 3 multiple points on a line.

Let us recall that Terao's conjecture holds for specific configurations of line arrangements, i.e. when:

- All the lines of the arrangement $\mathcal{A}$ pass through two fixed points: such an arrangement is free if and only if the line joining the two fixed points belongs to $\mathcal{A}$;
- One line of the arrangement contains no more than 3 singular points;
- The singular points of $\mathcal{A}$ have multiplicities at most 3.

Let us mention also the infinite family of reflection arrangements $\mathcal{A}_3^0(n)$, $\mathcal{A}_3^1(n)$, $\mathcal{A}_3^2(n)$ and $\mathcal{A}_3^3(n)$ defined respectively by the equations $f_n = 0, xf_n = 0, xyf_n = 0$ and $xyzf_n = 0$ where $f_n = (x^n - y^n)(y^n - z^n)(x^n - z^n)$. They are free with exponents respectively $(n+1, 2n-2)$, $(n+1, 2n-1)$ and $(n+1, 2n)$ and $(n+1, 2n+1)$ (see [8] or Corollaries 2.9 and 2.10 of this text). Their number of multiple points $t_2, t_3, t_n, t_{n+1}, t_{n+2}$ (their *weak combinatorics*) are given in [11] and again in [8]. The last one, $\mathcal{A}_3^3(n)$, is often called Ceva arrangement.

Let us introduce now the family of *triangular arrangements*. They are line arrangements consisting in lines passing through three non aligned fixed points. Reflection arrangements belong to this family. Our goal in this paper is to study these *triangular arrangements*. After describing some preliminary results (see the next section), the paper is organized as follows:

- First of all, we prove in Theorem 3.1, that we can associate, to any triangular arrangement, a further one having the same combinatorics and obtained by deleting particular lines from a reflection arrangement.
- We construct, for any possible exponent, a free arrangement and we describe its combinatorics. We show in Theorem 4.1 that these arrangements obtained from the reflection ones by deletion of lines are free if and only if the number of inner triple points in the arrangement consisting in the deleted lines is as small as possible.
- We characterize, in Section 5, every free uncomplete triangular arrangement, where uncomplete means a triangular arrangement in which at least one *side line* (i.e. line joining a couple of vertices among the three) is missing.
- We propose, in Section 6, two arrangements $\mathcal{A}$ and $\mathcal{B}$ having the same weak combinatorics (even quite the same combinatorics!) such that $\mathcal{A}$ is free and $\mathcal{B}$ is not.

- Finally, we prove Terao's conjecture for any triangular arrangement (Theorem 7.1).

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2 The inner triple points of a triangular arrangement

In this section we will explicit the importance of the set T of the triple points, defined by the triangular arrangement, which are not the vertices of the triangle. We will describe in particular the case where T is either empty or a complete intersection.

Let A, B, C be three points not aligned. A line arrangement such that any of its line passes through A, B or C is called *triangular arrangement*. If one of the three *side lines* is missing we will say that the triangular arrangement is uncomplete. The set of triangular arrangements consisting in $a+1$ lines through A, $b+1$ lines through B, $c+1$ lines through C and the three side lines is denoted by $\mathrm{Tr}(a, b, c)$; these arrangements possed $a + b + c$ lines.

Proposition 2.1. *Let $\mathcal{A} \in \mathrm{Tr}(a, b, c)$ then, there is an exact sequence*

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0,$$

where T is the smooth finite set of inner triple points (i.e. $A, B, C \notin T$).

Proof. Let $Z \subset \check{\mathbb{P}}^2$ the finite set of points corresponding by projective duality to the lines of $\mathcal{A}$. Since Z is contained in a triangle Δ. this induces the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{J}_{Z/\Delta}(1) \longrightarrow 0.$$

The hypothesis says that the vertices of Δ belong to Z, then it implies that $\mathcal{J}_{Z/\Delta}(1) = \mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b) \oplus \mathcal{O}_L(-c)$. Let us consider the incidence variety

$$\mathbb{F} = \{(x, l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \mid x \in l\}$$

and the projection maps $p : \mathbb{F} \to \mathbb{P}^2$ and $q : \mathbb{F} \to \check{\mathbb{P}}^2$. According to [6, Theorem 1.3] $\mathcal{T}_{\mathcal{A}} = p_*q^*(\mathcal{J}_Z(1))$ and the Fourier-Mukai transform p_*q^* applied to the above exact sequence gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow \mathcal{T}_{\mathcal{A}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \\ & & & & & \downarrow & \\ 0 & \longleftarrow \mathrm{R}^1p_*q^*\mathcal{O}_L(-a) \oplus \mathrm{R}^1p_*q^*\mathcal{O}_L(-b) \oplus \mathrm{R}^1p_*q^*\mathcal{O}_L(-c) & \longleftarrow & \mathrm{R}^1p_*q^*\mathcal{J}_Z(1), & & \end{array}$$

The sheaf $\mathrm{R}^1 p_* q^* \mathcal{J}_Z(1)$ is supported on the scheme of triple points defined by $\mathcal{A}$, while, the last sheaf of the sequence is supported on the vertices of the triangle (ABC). Therefore the kernel of the last map is the structural sheaf of the set of triple inner points T. This implies that we have the following exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0.$$

□

Remark 2.2. By the hypothesis on $\mathcal{A}$, the set T is smooth. Its length is related to the second Chern class of $\mathcal{T}_{\mathcal{A}}$, more precisely we have that $c_1(\mathcal{T}_{\mathcal{A}}) = 1 - a - b - c$ and

$$c_2(\mathcal{T}_{\mathcal{A}}) = \binom{a+b+c-1}{2} - \binom{a}{2} - \binom{b}{2} - \binom{c}{2} - \mid T \mid = (ab+bc+ac-a-b-c+1) - \mid T \mid . \tag{1}$$

First of all we prove that these arrangements, under conditions over a, b and c, lead to stable bundles.

Proposition 2.3. *Let us assume that $a \leq b \leq c$. Then, when $T = \emptyset$ we have*

$$\mathrm{H}^0(\mathcal{T}_{\mathcal{A}}(a+b-2)) = 0 \text{ and } \mathrm{H}^0(\mathcal{T}_{\mathcal{A}}(a+b-1)) \neq 0.$$

Proof. If $T = \emptyset$, then we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}(c) \longrightarrow \mathcal{T}_{\mathcal{A}}^{\vee} \longrightarrow 0.$$

Being $\mathcal{T}_{\mathcal{A}}^{\vee} = \mathcal{T}_{\mathcal{A}}(a+b+c-1)$, if we tensor by $\mathcal{O}_{\mathbb{P}^2}(-c)$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-c) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(a-c) \oplus \mathcal{O}_{\mathbb{P}^2}(b-c) \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{T}_{\mathcal{A}}(a+b-1) \longrightarrow 0,$$

which proves the proposition. □

Corollary 2.4. *Let us assume that $a \leq b \leq c$ and that $T = \emptyset$. Then, $\mathcal{T}_{\mathcal{A}}$ is stable if and only if $a + b > c + 1$.*

Proof. Under the hypothesis $T = \emptyset$, we have $\mathrm{H}^0(\mathcal{T}_{\mathcal{A}}(a+b-1)) \neq 0$ and $\mathrm{H}^0(\mathcal{T}_{\mathcal{A}}(a+b-2)) = 0$. Then $\mathcal{T}_{\mathcal{A}}$ is stable if and only if $c_1(\mathcal{T}_{\mathcal{A}}(a+b-1)) > 0$. Since $c_1(\mathcal{T}_{\mathcal{A}}(a+b-1)) = a+b-c-1$, this proves that $\mathcal{T}_{\mathcal{A}}$ is stable if and only if $a + b > c + 1$. □

Theorem 2.5. *The bundle $\mathcal{T}_{\mathcal{A}}$ is free with exponents $(a+b-1, c)$ if and only if T is a complete intersection $(a-1, b-1)$.*

Remark 2.6. This is the most unbalanced splitting that is allowed for $\mathcal{A} \in \mathrm{Tr}(a, b, c)$. Indeed, $|T|$ cannot be bigger than $(a-1)(b-1)$.

Proof. Assume that T is a complete intersection $(a-1, b-1)$. Since T is the locus of inner triple points, the curve defined by $(a-1)$ lines passing through A contains T and the curve defined by $(b-1)$ lines passing through B contains also T. These two curves generate the ideal defining T, which implies that the kernel of the last map of the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(-c) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0,$$

is $\mathcal{O}_{\mathbb{P}^2}(-a-b+1)\oplus\mathcal{O}_{\mathbb{P}^2}(-c)$. This proves that $\mathcal{T}_{\mathcal{A}}$ is free with exponents $(a+b-1,c)$. Reciprocally, if $\mathcal{T}_{\mathcal{A}}=\mathcal{O}_{\mathbb{P}^2}(-a-b+1)\oplus\mathcal{O}_{\mathbb{P}^2}(-c)$ then we have $\mathrm{H}^0(\mathcal{J}_T(a-1))\neq 0$ and $\mathrm{H}^0(\mathcal{J}_T(a-2))=0$. Moreover, the length of T, given by the numerical invariant of the above exact sequence, is $(a-1)(b-1)$ and this proves that T is a complete intersection $(a-1,b-1)$. □

Remark 2.7. If $c\geq a+b-1$ the splitting type of $\mathcal{T}_{\mathcal{A}}$ along the lines joining A to C or B to C is fixed and it is $\mathcal{O}_l(1-a-b)\oplus\mathcal{O}_l(-c)$; this is a consequence of [14, Theorem 3.1]. Therefore, under the condition $c\geq a+b-1$ the arrangement is free if and only if

$$\mathcal{T}_{\mathcal{A}}=\mathcal{O}_{\mathbb{P}^2}(1-a-b)\oplus\mathcal{O}_{\mathbb{P}^2}(-c).$$

That's why, if we want to describe all the possible splitting types of free triangular arrangements with $a+b+c$ lines ($a+1$ by A, $b+1$ by B and $c+1$ by C), we can assume that $c\leq a+b-2$. Then the biggest possible gap $|a+b-1-c|$ is realized by the complete intersection $(a-1)(b-1)$, in particular it could be described with a *Roots-of-Unity-Arrangement* (see the definition below): let ρ be a primitive $(c-1)$-root of unity, the arrangement

$$xyz\prod_{i=0}^{a-2}(x-\rho^i y)\prod_{j=0}^{b-2}(y-\rho^j z)\prod_{k=0}^{c-2}(x-\rho^k z)=0$$

belongs to $\mathrm{Tr}(a,b,c)$ and it is free with exponents $(a+b-1,c)$.

Definition 2.8. A triangular arrangement $\mathcal{A}$ of $a+b+c$ lines, defined by an equation

$$xyz\prod_{i=1}^{a-1}(x-\alpha_i y)\prod_{j=1}^{b-1}(y-\beta_j z)\prod_{k=1}^{c-1}(x-\gamma_k z)=0,$$

is called a *Roots-of-Unity-Arrangement* (RUA for short) if the coefficients α_i,β_j and γ_k can all be expressed as powers of a n-root of unity ρ.

The following two results are well known (they are described in particular in [11]).

Corollary 2.9. *The arrangements $\mathcal{A}_3^3(n)$, defined by the equation $xyz(x^n-y^n)(y^n-z^n)(x^n-z^n)=0$, are free with exponents $(n+1,2n+1)$.*

Proof. The set of inner triple points T is a complete intersection of length n^2 defined by the ideal (x^n-y^n,y^n-z^n). □

Corollary 2.10. *The arrangements $\mathcal{A}_2^3(n),\mathcal{A}_1^3(n)$ and $\mathcal{A}_0^3(n)$ are obtained respectively from $\mathcal{A}_3^3(n)$, $\mathcal{A}_2^3(n)$ and $\mathcal{A}_1^3(n)$ by deleting one line between two vertices of the triangle. They are free with exponents respectively equal to $(n+1,2n)$, $(n+1,2n-1)$ and $(n+1,2n-2)$.*

Proof. Starting with $\mathcal{A}_3^3(n)$ we remove the line $l=\{x=0\}$. This gives the arrangement $\mathcal{A}_3^2(n)$. The line l contains $2n$ triple points (n at each vertex, 0 elsewhere). Then we have the following exact sequence

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^2}(-n-1)\oplus\mathcal{O}_{\mathbb{P}^2}(-2n-1)\longrightarrow\mathcal{T}_{\mathcal{A}_3^2(n)}\longrightarrow\mathcal{O}_l(-2n)\longrightarrow 0.$$

By the Addition-Deletion theorem (see [11, Theorem 4.51]) we obtain

$$\mathcal{T}_{\mathcal{A}_3^2(n)} = \mathcal{O}_{\mathbb{P}^2}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2n).$$

With the same arguments, removing the line $y = 0$ containing $2n-1$ triple points from $\mathcal{A}_2^3(n)$ we obtain

$$\mathcal{T}_{\mathcal{A}_3^1(n)} = \mathcal{O}_{\mathbb{P}^2}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2n+1),$$

and removing the line $z = 0$ containing $2n-2$ triple points from $\mathcal{A}_3^1(n)$ we obtain

$$\mathcal{T}_{\mathcal{A}_3^0(n)} = \mathcal{O}_{\mathbb{P}^2}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2n+2).$$

□

3 Roots-of-Unity-Arrangement

This section is dedicated to prove the following result.

Theorem 3.1. *Given a triangular arrangement, it is always possible to find a RUA with the same combinatorics.*

Proof. Let us consider the triangular arrangement defined by the following equations

$$\begin{cases} x = 0 \\ x = \alpha_i y \\ y = 0 \\ y = \beta_j z \\ z = 0 \\ z = \gamma_k x \end{cases}$$

where $x = y = z = 0$ are the lines which compose the triangle, $\alpha_i \neq 0$, $i = 1, \ldots, a-1$, and $\alpha_{i_1} \neq \alpha_{i_2}$ for $i_1 \neq i_2$, and the same properties hold for the β_j's and the γ_k's, $j = 1, \ldots, b-1$ and $k = 1, \ldots, c-1$.
Observe that the existence of an inner triple point, defined by three lines $x = \alpha_{\bar{\imath}} y, y = \beta_{\bar{\jmath}} z$ and $x = \gamma_{\bar{k}} z$ is given by a relation of the following type

$$\alpha_{\bar{\imath}} \beta_{\bar{\jmath}} \gamma_{\bar{k}} = 1.$$

Therefore, we can translate the combinatorics of the arrangement in a family of equalities

$$\alpha_{i_1} \beta_{j_1} \gamma_{k_1} = 1 \tag{2}$$

for each i_1, j_1, k_1 whose associated lines define an inner triple point of the arrangement, a family of inequalities

$$\alpha_{i_2} \beta_{j_2} \gamma_{k_2} \neq 1 \tag{3}$$

for each i_2, j_2, k_2 whose associated lines do not define an inner triple point of the arrangement, and finally, the inequalities

$$\alpha_{i_1} \neq \alpha_{i_2}, \alpha_{i_1} \neq 0, \beta_{j_1} \neq \beta_{j_2}, \beta_{j_1} \neq 0, \gamma_{k_1} \neq \gamma_{k_2}, \gamma_{k_1} \neq 0, \tag{4}$$

for each $i_1, i_2 = 1, \ldots, a-1$, with $i_1 \neq i_2$, $j_1, j_2 = 1, \ldots, b-1$, with $j_1 \neq j_2$, and $k_1, k_2 = 1, \ldots, c-1$, with $k_1 \neq k_2$.
Our goal is to find solutions, or at least prove their existence, which satisfy all the previous relations and that can be expressed as various powers of a n-th root of the unity, for a given n.
Let us consider a prime number p and one of its primitive roots ω, hence, working modulo p, we can translate all the relations of type (2) as

$$\omega^{v_{i_1}} \omega^{w_{j_1}} \omega^{t_{k_1}} \equiv 1 \pmod{p}$$

or equivalently, as a family of linear equations

$$v_{i_1} + w_{j_1} + t_{k_1} \equiv 0 \pmod{p-1}. \tag{5}$$

We claim that we always have solutions, for any choice of p, of the linear system defined by the family (5); indeed we have the following result.

Lemma 3.2. *If we consider the linear system given by the equations*

$$v_{i_1} + w_{j_1} + t_{k_1} = 0, \tag{6}$$

i.e. considering all the linear forms of (5) in $\mathbb{C}[v_{i_1}, w_{j_1}, t_{k_1}]$, we always have infinite solutions.

Proof. Let us consider the arrangement $\mathcal{A}_3^3(n)$. It leads to the maximal system of equations $v_{i_1} + w_{j_1} + t_{k_1} = 0$, in the sense that it contains all the possible equations involving the variables. Writing down the corresponding square matrix we verify that its determinant vanishes. The system of equations (6) can be seen as a subsystem of a maximal one associated to $\mathcal{A}_3^3(n)$, hence it will also have infinite solutions. □

Since the coefficients of the equations of (6) are integers we get in particular infinite integer solutions. Therefore, choosing well $p >> 0$, we get as many solutions of the system of linear congruences as we want.
Our next goal is to prove the following fact: consider the inequality

$$v + w + t \not\equiv 0 \pmod{p-1}$$

and add to our previous linear system, module p, the associated equality

$$v + w + t \equiv 0 \pmod{p-1}.$$

Then, for an infinite number of p's, either we have less solutions than before or the added condition is a consequence of the others.
Indeed, suppose that for a fixed p we have exactly the same solutions, this means that the added condition is a linear combination of the previous ones, i.e.

$$v + w + t \equiv \sum_s \lambda_{s,p}(v_{i_s} + w_{j_s} + t_{k_s}) \pmod{p-1}$$

which is equivalent to have

$$\omega^v \omega^w \omega^t \equiv \prod_s (\omega^{v_{i_s}} \omega^{w_{j_s}} \omega^{t_{k_s}})^{\lambda_{s,p}} \pmod{p}$$

and therefore

$$\alpha\beta\gamma \equiv \prod_s (\alpha_{i_s}\beta_{j_s}\gamma_{k_s})^{\lambda_{s,p}} \pmod{p}.$$

We conclude noticing that if we have the previous relation for an infinite set of prime numbers, then we must also have

$$\alpha\beta\gamma \equiv \prod_s (\alpha_{i_s}\beta_{j_s}\gamma_{k_s})^{\lambda_s},$$

which implies that the added condition is a consequence to the other equalities, which is a contradiction because of our hypothesis on the triple points.
Following the same reasoning as before for the relations expressed in the family (4), it is possible to find solutions that do also satisfy those inequalities.
Therefore, by what we have said, we can find a prime number p (big if necessary) such that we can find $\alpha_i = \rho^{\bar{\alpha}_i}$, $\beta_j = \rho^{\bar{\beta}_j}$ and $\gamma_k = \rho^{\bar{\gamma}_k}$, powers of the $(p-1)$-th unity root ρ, which satisfies all the conditions expressed in (2), (3) and (4). □

Example 3.3. Consider the line arrangement in $\mathbb{P}^2$ defined by the curve

$$xyz \prod_{i=-1}^{2} \left(x - \alpha^i y\right) \prod_{i=0}^{3} \left(\left(y - \alpha^j z\right)\left(x - \alpha^j z\right)\right) = 0$$

with α a generic complex number, which has 12 inner triple points, i.e. excluding the three vertices of the triangle.
Following the proof of he previous result, in order to find a root arrangement with the same combinatorics as the given one, we have to solve the following linear system of equalities

$$\begin{cases} x_1 + y_1 + z_1 = 0 \\ x_1 + y_2 + z_2 = 0 \\ x_1 + y_3 + z_3 = 0 \\ x_1 + y_4 + z_4 = 0 \\ x_2 + y_1 + z_2 = 0 \\ x_2 + y_2 + z_3 = 0 \\ x_2 + y_3 + z_4 = 0 \\ x_3 + y_1 + z_3 = 0 \\ x_3 + y_2 + z_4 = 0 \\ x_4 + y_2 + z_1 = 0 \\ x_4 + y_3 + z_2 = 0 \\ x_4 + y_4 + z_3 = 0 \end{cases}$$

and which, moreover, does not satisfy any other relation $x_i + y_j + z_k = 0$ which is not present in the previous system, and, finally, such that

$$x_p \neq x_q, \;\; y_p \neq y_q \;\; z_p \neq z_q, \quad \text{for} \quad p, q = 1, 2, 3, 4, \;\; p \neq q.$$

The set of integers

$$\begin{array}{lll} x_1 = 0 & y_1 = 0 & z_1 = 0 \\ x_2 = 1 & y_2 = 1 & z_2 = 5 \\ x_3 = 2 & y_3 = 2 & z_3 = 4 \\ x_4 = 5 & y_4 = 3 & z_4 = 3 \end{array}$$

satisfy all the required conditions, and therefore, we can consider ρ a 6-root of unity and the root arrangement with the same combinatorics as the starting one is given by

$$xyz\prod_{i=2}^{5}\left(\left(x-\rho^i y\right)\left(y-\rho^i z\right)\right)\prod_{j=0}^{3}\left(x-\rho^j z\right)=0$$

4 Free arrangements obtained by deletion from the Ceva's

In this section we will construct an explicit free arrangement in $\mathrm{Tr}(a,b,c)$ for each possible splitting.

Let us begin by describing the case of the maximal number of inner triple points which corresponds to the more unbalanced splitting. The maximal number of inner triple points is evidently $|T| = (a-1)(b-1)$ and we already know that this complete intersection corresponds to a free arrangement with exponents $(c, a+b-1)$ (see Theorem 2.5). As shown in Remark 2.7, such an arrangement can be obtained by considering a $c-1$ root of unity ζ, which generates the multiplicative group of $(c-1)$-roots of unity, and then choosing $\alpha_i = \zeta^i$ for $i = 1,\ldots,a-1$ $\beta_j = \zeta^j$ for $j = 1,\ldots,b-1$ and $\gamma_k = \zeta^k$ for $k = 1,\ldots,c-1$.

According to Remark 2.7, we can assume that $c \leq a+b-2$ (on the contrary the splitting is known on the side lines by [14, Theorem 3.1]).

— If $c = a+b-2$ there is no other splitting, since with these values, the most unbalanced possible splitting type is actually balanced. In this case an arrangement is free if and only if $|T|$ is a complete intersection $(a-1, b-1)$.

— Assume that $c = a+b-3$, then another splitting is possible, that is

$$\mathcal{O}_{\mathbb{P}^2}(-c-1)\oplus\mathcal{O}_{\mathbb{P}^2}(2-a-b).$$

In order to obtain it, we begin with an arrangement $\mathcal{A}_3^3(c)$, which is free with exponents $(c+1, 2(c+1)-1)$. We remove one inner line from each vertex such that the inner triple points removed are $c+1$, then c then $c-1$. By the Addition-Deletion theorem, the new arrangement is free with exponents $(c+1, 2(c+1)-4) = (c+1, 2(c-1))$. We then remove $c-b$ lines from B and $c-a$ lines from A. Since we have removed the appropriate number of triple points, thanks to the Addition-Deletion theorem, the arrangement that we obtain is still free with exponents $(c+1, a+b-2)$.

— The same process gives the splitting $\mathcal{O}_{\mathbb{P}^2}(-c-i)\oplus\mathcal{O}_{\mathbb{P}^2}(1+i-a-b)$ for any $1 \leq i \leq [\frac{a+b-c}{2}]$: begin considering the arrangement $\mathcal{A}_3^3(c+i-1)$, remove i lines from each vertex in order to keep the arrangement free and then remove $c-b$ and $c-a$ lines from B and A. The way to keep the arrangement free is explained in the following theorem.

Theorem 4.1. *Let $\mathcal{A}$ be a roots of unity arrangement, then $\mathcal{A}$ is free if and only if is obtained from a Ceva one, deleting in each step a line such that the corresponding complementary arrangement has the miminum possible of triple points.*

Proof. Removing inner lines from a Ceva arrangement $\mathcal{A}_3^3(N)$ we find two triangular arrangements, $\mathcal{A}$ consisting in the remaining lines and $\mathcal{A}^c$ consisting in the deleted lines. More precisely, when we remove $a' = N-a+1$, $b' = N-b+1$ and $c' = N-c+1$ lines,

with equations $f_A = 0, f_B = 0, f_C = 0$, respectively from the vertices A, B and C of a Ceva arrangement $\mathcal{A}_3^3(N)$, these lines form a subarrangement $\mathcal{A}^c \in \mathrm{Tr}(a^{'}, b^{'}, c^{'})$ (defined by the equation $f_A f_B f_C = 0$) and this arrangement possesses, outside the vertices, a set T_{rem} of triple inner points (set that can be empty). This operation induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{O}_{\mathbb{P}^2}(-1-N) & \xrightarrow{\simeq} & \mathcal{O}_{\mathbb{P}^2}(-1-N) & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle (f_A,f_B,f_C)} & & & & \\
0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^3(-a)\oplus\mathcal{O}_{\mathbb{P}^2}^3(-b)\oplus\mathcal{O}_{\mathbb{P}^2}^3(-c) & \longrightarrow & \mathcal{J}_T(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_\Gamma(N+2-a-b-c) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J}_T(-1) & \longrightarrow & 0,
\end{array}$$

where $\mathcal{T}_{\mathcal{A}}$ is the logarithmic bundle associated to the triangular arrangement $\mathcal{A} \in \mathrm{Tr}(a, b, c)$, Γ is the zero locus of the section of $\mathcal{T}_{\mathcal{A}}$ induced by the syzygies (f_A, f_B, f_C), T is the set of triple inner points of $\mathcal{A}$ and $\mathrm{Sing}(\mathcal{F}) = T_{\mathrm{rem}}$. Since $\mathcal{T}_{\mathcal{A}}$ comes from the Ceva arrangement $\mathcal{A}_3^3(N)$ we have of course

$$\Phi = [\frac{x^N - y^N}{f_A}, \frac{y^N - z^N}{f_B}, \frac{x^N - z^N}{f_C}].$$

Moreover, since T_{rem} and T are disjoint sets of simple points, we conclude that $\Gamma = T_{\mathrm{rem}}$. This gives $|T_{\mathrm{rem}}| = c_2(\mathcal{T}_{\mathcal{A}}(N+1))$. Recall that, from Remark 2.2, we have that

$$\mid T \mid = (ab + bc + ac - a - b - c + 1) - c_2(\mathcal{T}_{\mathcal{A}}). \tag{7}$$

Then we obtain the following formula relating $|T|$ and $|T_{\mathrm{rem}}|$:

$$|T| = N^2 - (N+2)(a+b+c-3) - 3 + ab + ac + bc - |T_{\mathrm{rem}}|. \tag{8}$$

Observe that the equation (7) implies that we will reach the minimum possible number of inner triple points when the second Chern class of the bundle is as big as it can be (it means in particular that if the number of inner triple points of $\mathcal{A}$ is smaller than this minimum then $\mathcal{A}$ cannot be free). This happens exactly when the bundle $\mathcal{T}_{\mathcal{A}}$, when supposed to be free, is *as balanced as possible*, which means that, depending on the relations among a, b and c, the distance between the two exponents is as little as it can be. More precisely, *balanced* means that

$$\mathcal{T}_{\mathcal{A}} \simeq \mathcal{O}_{\mathbb{P}^2}\left(\frac{-a-b-c+1}{2}\right) \oplus \mathcal{O}_{\mathbb{P}^2}\left(\frac{-a-b-c+1}{2}\right)$$

when $-a-b-c+1$ is even, and

$$\mathcal{T}_{\mathcal{A}} \simeq \mathcal{O}_{\mathbb{P}^2}\left(\frac{-a-b-c}{2}\right) \oplus \mathcal{O}_{\mathbb{P}^2}\left(\frac{-a-b-c}{2}+1\right)$$

when $-a-b-c+1$ is odd.
On the other hand, by formula (8), we have that $\mid T \mid$ is as big as possible, when the complementary arrangement as the fewest triple points. This happens when we consider

a Ceva arrangement with N^2 inner triple points, we eliminate the first $N+1-a$ and $N+1-b$ lines respectively to two families, and we choose the other $N+1-c$ lines, to be eliminated from the third family, in order to get the least possible number of triple points in the complementary arrangement.

— If $N-a^{'}-b'-c' \geq 0$ (which is equivalent to the condition $2N \leq a+b+c-3$), then it is possible to eliminate this quantity of lines such that no triple points arise in the complementary arrangement. In such a case the minimal number of triple point in the complementary arrangement is 0, that means $T_{\rm rem}=\emptyset$ implying $\mathrm{H}^1_*(\mathcal{T}_{\mathcal{A}})=0$ and by Horrocks' criterion [9], $\mathcal{A}$ is free with exponents $(N+1, a+b+c-N-2)$. On the contrary, if $T_{\rm rem} \neq \emptyset$ (i.e. if we delete a line containing a triple point in the complementary when it was possible to delete a line with no triple point in the complementary) then the splitting type of $\mathcal{T}_{\mathcal{A}}$ on any line through Γ is different from the generic one, proving that $\mathcal{A}$ is not free.

The theorem is therefore proven for $2N \leq a+b+c-3$.

— If $N-a^{'}-b'-c' < 0$ (which is equivalent to the condition $2N > a+b+c-3$) then necessarily $|T_{\rm rem}| \neq 0$. In order to minimize this number, we must minimize $r := a^{'}+b'+c'-N = 2N+3-a-b-c$. The way to do it is the following: the first $a^{'}$ and $b^{'}$ lines being deleted, we delete $c^{'}-r$ lines such that no triple point arise in the complementary arrangement. The last $r-1$ lines are chosen such that the number of triple points created in the complementary are

$$1+1+2+2+\cdots+(s-1)+(s-1)+s=\frac{r^2}{4}$$

when $r=2s$, and

$$1+1+2+2+\cdots+s+s=\frac{r^2-1}{4}$$

when $r=2s+1$.

Remark that $r=2N+3-a-b-c$ and $c_1(\mathcal{T}_{\mathcal{A}})=1-a-b-c$ have the same parity.

Consider the case when r, or identically, when $c_1(\mathcal{T}_{\mathcal{A}})=1-a-b-c$ is even.

Then, $|T_{\rm rem}|=\frac{r^2}{4}=\frac{(2N+3-a-b-c)^2}{4}$. From (7) we have that

$$\mid T \mid_{min} = (ab+bc+ac-a-b-c+1)-\frac{1}{4}\left(-a-b-c+1\right)^2,$$

while from (8), we get that

$$\mid T \mid_{max} = N^2-(N+2)(a+b+c-3)-3+ab+ac+bc-\frac{(2N+3-a-b-c)^2}{4}.$$

Through direct computation, we get that $\mid T \mid_{max} = \mid T \mid_{min}$ and this implies that $\mathcal{T}_{\mathcal{A}}$ is free if and only if it is constructed eliminating, from a Ceva arrangement, lines minimizing the triple points in the complementary arrangement.

Analogously, we get the same equality when $-a-b-c+1$ is odd.

The theorem is therefore proven for $2N > a+b+c-3$.

□

Let us illustrate the above theorem on an example of triangular arrangement in $\mathrm{Tr}(5,5,5)$.

Example 4.2. Let us consider the case of 15 lines (6 by each vertex). Then the splitting types allowed are $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-5) \oplus \mathcal{O}_{\mathbb{P}^2}(-9)$, which correspond to 16 inner triple points, $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-8)$, which correspond to 13 inner triple points, and $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-7) \oplus \mathcal{O}_{\mathbb{P}^2}(-7)$, which correspond to 12 inner triple points. Since we have an exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}^3_{\mathbb{P}^2}(-5) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0$$

then $\mathrm{H}^0(\mathcal{T}_{\mathcal{A}}(4)) = 0$ and these three splittings are the only possible ones.

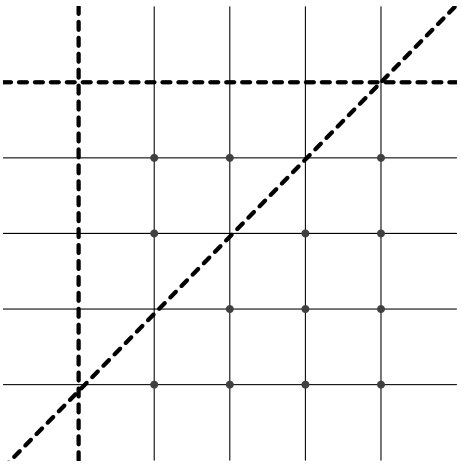
— To obtain the case $|T| = 16$ triple inner points and $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-5) \oplus \mathcal{O}_{\mathbb{P}^2}(-9)$ there is only one possibility. Indeed a syzygy of degree 0 means that the curve $\prod_{0 \leq i \leq 3}(x - \alpha_i y) = 0$ belongs to the pencil generated by $\prod_{0 \leq i \leq 3}(x - \gamma_i z) = 0$ and $\prod_{0 \leq i \leq 3}(y - \beta_i z) = 0$. This arises only when this system is equivalent by a linear change of coordinates to

$$[x^4 - y^4, x^4 - z^4, y^4 - z^4].$$

— To obtain the case $|T| = 13$ triple inner points and $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-8)$ we begin with the roots of unity of order 5. Then it is possible to remove three lines with the required number of triple points. The three associated partitions along the three directions are always $13 = 3 + 3 + 3 + 4$.
Another method to prove the freeness is also to remark that this inductive process gives a syzygy. In this case we find a syzygy (P, Q, R) of degree 1 verifying

$$(x - z)P - (y - z).Q + (x - \zeta^4 y).R = 0$$

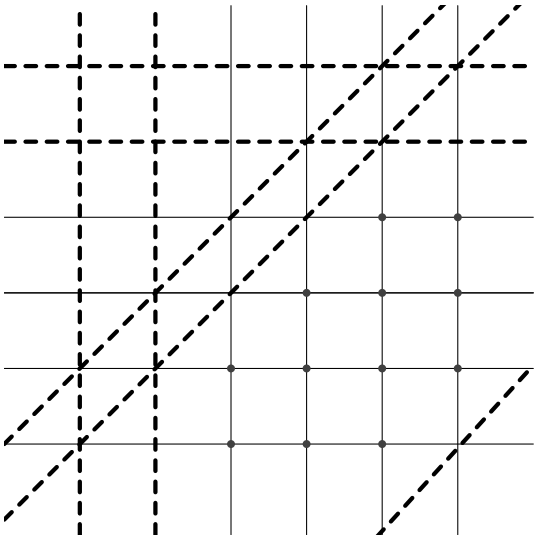
where $P = \frac{x^5 - z^5}{(x-z)}, Q = \frac{y^5 - z^5}{(y-z)}$ and $R = \frac{x^5 - z^5}{(x - \zeta^4 y)}$.



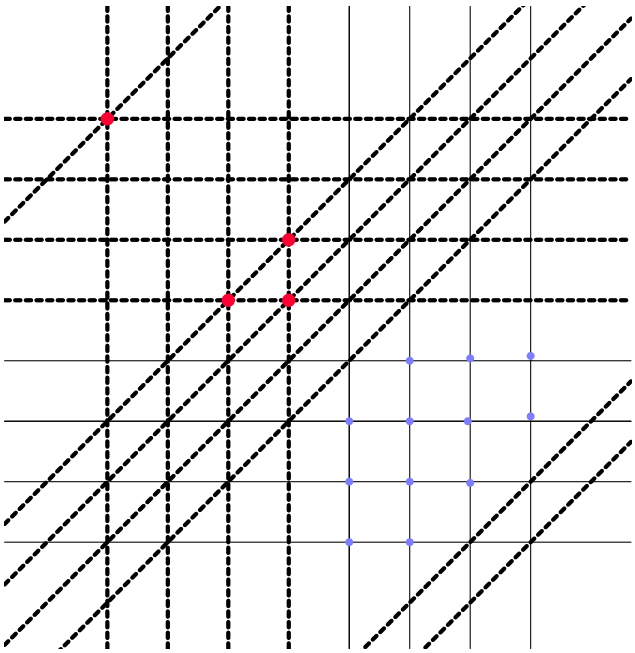
— To obtain the case $|T| = 12$ triple inner points and $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-7) \oplus \mathcal{O}_{\mathbb{P}^2}(-7)$ we begin with the roots of unity of order 6. Then we remove six lines with the required number of triple points. The three associated partitions along the three directions are always $12 = 2 + 3 + 4 + 3$. Observe that by [14, Theorem 3.1] the splitting type on the line l containing only 2 inner triple points is $\mathcal{O}_l(-7) \oplus \mathcal{O}_l(-7)$. Since its second Chern class is 49, $\mathcal{T}_{\mathcal{A}}$ is free by [5, Corollary 2.12]. Another method to prove the freeness is to remark that this inductive process gives a syzygy. In this case we find a syzygy (P, Q, R) of degree 2 verifying

$$(x - z)(x - \zeta z).P - (y - z)(y - \zeta z).Q + (x - \zeta^5 y)(x - \zeta^4 y).R = 0$$

where $P = \frac{x^6-z^6}{(x-z)(x-\zeta z)}, Q = \frac{y^6-z^6}{(y-z)(y-\zeta z)}$ and $R = \frac{x^6-z^6}{(x-\zeta^5 y)(x-\zeta^4 y)}$.



— As explained in Theorem 4.1 we can also find a free arrangement in $\mathrm{Tr}(5,5,5)$ with exponents $(7,7)$ (that is $|T| = 12$) deleting 4 lines from each vertex of $\mathcal{A}_3^3(8)$. The arrangement is free if and only if T_{rem} is minimal, which means here $|T_{\mathrm{rem}}| = 4$ as it can be seen on the picture below.



5 Free arrangements in the non complete triangle

All the free triangular arrangements coming from Ceva's ones by deletion, as explained in the above construction, contain the three sides of the triangle, i.e. the three lines joining

the three vertices. In this section we will complete the study of triangular arrangement, proving the following result:

Theorem 5.1. *Let $\mathcal{A}_0 \in \mathrm{Tr}(a,b,c)$ and $(AB),(AC),(BC)$ its three sides.*

1. *If we remove one side, then*
   - *$\mathcal{A}_0 \setminus (AB)$ is free if and only if its set of inner triple points is a complete intersection $(a-1,b-1)$. Then its exponents are $(c, a+b-2)$.*
   - *$\mathcal{A}_0 \setminus (AC)$ is free if and only if $b=c$ and its set of inner triple points is a complete intersection $(a-1,b-1)$. Then its exponents are $(b, a+b-2)$.*
   - *$\mathcal{A}_0 \setminus (BC)$ is free if and only if $a=b=c$ and its set of inner triple points is a complete intersection $(a-1,a-1)$. Then its exponents are $(a, 2a-2)$.*
2. *If we remove two sides, then*
   - *$\mathcal{A}_0 \setminus [(AB)\cup(AC)]$ is free if and only if its set of inner triple points is a complete intersection $(a-1,b-1)$. Then its exponents are $(c, a+b-3)$.*
   - *$\mathcal{A}_0 \setminus [(AB)\cup(BC)]$ is free if and only if $b=c$ and its set of inner triple points is a complete intersection $(a-1,b-1)$. Then its exponents are $(b, a+b-3)$.*
   - *$\mathcal{A}_0 \setminus [(AC)\cup(BC)]$ is free if and only if $a=b=c$ and its set of inner triple points is a complete intersection $(a-1,a-1)$. Then its exponents are $(a, 2a-3)$.*
3. *If we remove three sides, then $\mathcal{A} \setminus [(AB)\cup(AC)\cup(BC)]$ is free if and only if its set of inner triple points is a complete intersection $(a-1,a-1)$. Then its exponents are $(a, 2a-4)$.*

Proof. We divide the proof in three parts: $\mathcal{A}$ is a triangular arrangement with one side removed (Part I), two sides removed (Part II) and three sides removed (Part III) from a complete triangular arrangement $\mathcal{A}_0 \in \mathrm{Tr}(a,b,c)$.

I) Let us assume first that $\mathcal{A}$ contains exactly two sides and let L be the missing one. Then $\mathcal{A}\cup L = \mathcal{A}_0 \in \mathrm{Tr}(a,b,c)$. It is clear that $\mathcal{A}$ and $\mathcal{A}\cup L$ have the same set T of inner triple points. Let us denote by $t_{\mathcal{A}}$ and $t_{\mathcal{A}\cup L}$ the number of triple points of $\mathcal{A}$ and $\mathcal{A}\cup L$ counted with multiplicities. Then we have

- $t_{\mathcal{A}\cup L} = |T| + \binom{a}{2} + \binom{b}{2} + \binom{c}{2}$.
- $t_{\mathcal{A}} = |T| + \binom{a}{2} + \binom{b-1}{2} + \binom{c-1}{2}$, if $L=(BC)$;
- $t_{\mathcal{A}} = |T| + \binom{a-1}{2} + \binom{b}{2} + \binom{c-1}{2}$, if $L=(AC)$;
- $t_{\mathcal{A}} = |T| + \binom{a-1}{2} + \binom{b-1}{2} + \binom{c}{2}$, if $L=(AB)$.

The difference $(t_{\mathcal{A}\cup L} - t_{\mathcal{A}})$ is $b+c-2$, $a+c-2$ or $a+b-2$ respectively when $L=(BC),(AC)$ or (AB).

$\circ$ Assume first that $L=(AB)$. By [7, Proposition 5.1] we have an exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_L(2-a-b) \longrightarrow 0.$$

– If $c \leq a+b-2$, the surjection of $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{O}_L(2-a-b)$ induces $(\mathcal{T}_{\mathcal{A}})_{|L} = \mathcal{O}_L(-c) \oplus \mathcal{O}_L(2-a-b)$ and, if we suppose $\mathcal{A}$ to be free, we have that $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-c) \oplus \mathcal{O}_{\mathbb{P}^2}(2-a-b)$. By Addition-Deletion, $\mathcal{T}_{\mathcal{A}\cup L} = \mathcal{O}_{\mathbb{P}^2}(-c) \oplus \mathcal{O}_{\mathbb{P}^2}(1-a-b)$ and, according to Theorem 2.5, the set of inner triple points is a complete intersection $(a-1, b-1)$.
– If $c \geq a+b-1$, we consider the multi-arrangement given by the restriction of $\mathcal{A}$ on the side (CB). The multiplicities of this multi-arrangement are $(c, 1, \ldots, 1, b-1)$ (with $a-1$ multiplicities of type "1"). This implies that the splitting type of $\mathcal{T}_{\mathcal{A}}$ on the side $l = (BC)$ is $\mathcal{O}_l(-c) \oplus \mathcal{O}_l(2-a-b)$ by [14, Theorem 3.1]. Then $\mathcal{A}$ is free if and only if $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-c) \oplus \mathcal{O}_{\mathbb{P}^2}(2-a-b)$. By Addition-Deletion again $\mathcal{T}_{\mathcal{A}\cup L} = \mathcal{O}_{\mathbb{P}^2}(-c) \oplus \mathcal{O}_{\mathbb{P}^2}(1-a-b)$ and the set of inner triple points is a complete intersection $(a-1, b-1)$.

∘ Assume that $L = (AC)$. By [7, Proposition 5.1] we have an exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_L(2-a-c) \longrightarrow 0.$$

– If $b \leq a+c-2$, the surjection of $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{O}_L(2-a-c)$ induces $(\mathcal{T}_{\mathcal{A}})_{|L} = \mathcal{O}_L(-b) \oplus \mathcal{O}_L(2-a-c)$ and $\mathcal{A}$ free implies that $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(2-a-c)$. Then by Addition-Deletion $\mathcal{T}_{\mathcal{A}\cup L} = \mathcal{O}_{\mathbb{P}^2}(-b) \oplus \mathcal{O}_{\mathbb{P}^2}(1-a-c)$, which is not possible if $b < c$ according to Remark 2.6. Indeed, in this case, the gap $|a+c-1-b|$ is strictly bigger than the gap $|a+b-1-c|$, implying that the number of inner triple points should be bigger than $(a-1)(b-1)$, which is impossible. This means that $b = c$ and $\mathcal{A}$ is free with exponents $(c, a+b-2)$.
– By hypothesis $a-1 > 0$ (we are considering triangular arrangements which have at least one line, for each vertex, aside the sides of the triangle) and, since $b \leq c$, we cannot have $b \geq a+c-1$.

∘ Assume now that $L = (BC)$. By [7, Proposition 5.1] we have an exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_L(2-b-c) \longrightarrow 0.$$

– Since $a \geq 2$ and $a \leq b \leq c$ we always have $a \leq b+c-2$. By consequence the surjection of $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{O}_L(2-b-c)$ induces $(\mathcal{T}_{\mathcal{A}})_{|L} = \mathcal{O}_L(-a) \oplus \mathcal{O}_L(2-b-c)$ and the freeness of $\mathcal{A}$ implies that $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(2-b-c)$. By Addition-Deletion $\mathcal{T}_{\mathcal{A}\cup L} = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(1-b-c)$. The gap $|b+c-1-a|$ is strictly bigger than the gap $|a+b-1-c|$ except when $a = b = c$; then, by Remark 2.6, this case cannot occur when $a \neq b$ or $b \neq c$. So $\mathcal{A} = (\mathcal{A} \cup (BC)) \setminus (BC)$ is free if and only if $a = b = c$ and T is a complete intersection $(a-1, a-1)$ (in this case $\mathcal{A} = \mathcal{A}_3^2(a-1)$).

II) Let us assume now that $\mathcal{A}$ contains only one side and let L and L' the missing sides. Then $\mathcal{A} \cup L \cup L' = \mathcal{A}_0 \in \mathrm{Tr}(a, b, c)$. Let $t_{\mathcal{A}}$, $t_{\mathcal{A}\cup L}$ and $t_{\mathcal{A}\cup L\cup L'}$ be the number of triple points of $\mathcal{A}$, $\mathcal{A} \cup L$ and $\mathcal{A} \cup L \cup L'$, respectively, counted with multiplicities. Then we have,

- $t_{\mathcal{A}\cup L\cup L'} = |T| + \binom{a}{2} + \binom{b}{2} + \binom{c}{2}$;
- $t_{\mathcal{A}} = |T| + \binom{a-2}{2} + \binom{b-1}{2} + \binom{c-1}{2}$, if $L = (AB)$ and $L' = (AC)$;
- $t_{\mathcal{A}} = |T| + \binom{a-1}{2} + \binom{b-2}{2} + \binom{c-1}{2}$, if $L = (BC)$ and $L' = (AB)$;
- $t_{\mathcal{A}} = |T| + \binom{a-1}{2} + \binom{b-1}{2} + \binom{c-2}{2}$, if $L = (AC)$ and $L' = (BC)$;

from which we get that the difference $(t_{\mathcal{A}\cup L} - t_{\mathcal{A}})$ is equal to $a+b-3$ when $L=(AB)$ and $L^{'}=(AC)$, $a+c-3$ when $L=(AC)$ and $L^{'}=(BC)$, and $b+c-3$ when $L=(BC)$ and $L^{'}=(AB)$.

$\circ$ Assume first that $L=(AB)$ and $L^{'}=(AC)$. By [7, Proposition 5.1] we have an exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_L(3-a-b) \longrightarrow 0.$$

– If $c \leq a+b-3$ the surjection of $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{O}_L(3-a-b)$ induces $(\mathcal{T}_{\mathcal{A}})_{|L} = \mathcal{O}_L(-c)\oplus\mathcal{O}_L(3-a-b)$ and if we suppose $\mathcal{A}$ to be free, it gives $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-c)\oplus\mathcal{O}_{\mathbb{P}^2}(3-a-b)$. By Addition-Deletion, $\mathcal{A}\cup L$ is free with exponents $(c, a+b-2)$ and, according to Part I), the set of inner triple points is a complete intersection of type $(a-1,b-1)$.
– If $c \geq a+b-2$ we consider the other exact sequence given by [7, Proposition 5.1], i.e.

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L'} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_{L'}(3-a-c) \longrightarrow 0.$$

If $a=2$ considering the multiplicities of the multiarrangement obtained by restricting $\mathcal{A}$ onto the unique line through A we find $|T|$ double points and $c+b-|T|$ simple points. Since $2(c+b-|T|)-1 \geq b+c$ because $|T| \leq b-1$ we get a splitting type $(c+b-1-|T|,|T|)$ on this line. So if $\mathcal{A}$ is free its exponents are $(c+b-1-|T|,|T|)$. If $b<c$ then $|T|<c-1$ then the surjection $\mathcal{T}_{\mathcal{A}} \to \mathcal{O}_{L'}(3-a-c) = \mathcal{O}_{L'}(1-c)$ imposes $|T|-1-c-b = 1-c$ i.e. $|T|=b$ which is not possible. Then $c=b$ and $|T|=b-1$ i.e. T is a complete intersection $(a-1,b-1)=(1,b-1)$ and $\mathcal{A}$ is free with exponents $(b-1,c)$.
Let us assume now that $a \geq 3$.
Since $a \geq 3$ it is clear that $b \leq a+c-3$ and the surjection of $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{O}_{L'}(3-a-c)$ induces $(\mathcal{T}_{\mathcal{A}})_{|L'} = \mathcal{O}_{L'}(-b)\oplus\mathcal{O}_{L'}(3-a-c)$ and if we suppose $\mathcal{A}$ to be free, we obtain $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-b)\oplus\mathcal{O}_{\mathbb{P}^2}(3-a-c)$. Then, by Addition-Deletion, $\mathcal{A}\cup L^{'}$ is free with exponents $(b, a+c-2)$. According to Part I), this occurs if and only if $b=c$ and T is a complete intersection $(a-1)(b-1)$.

$\circ$ Assume now that $L=(BC)$ and $L^{'}=(AB)$ (or $L=(BC)$ and $L^{'}=(AC)$). By [7, Proposition 5.1] we have an exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_L(3-b-c) \longrightarrow 0.$$

– Besides the case when $a=b=c=2$ (in this case the four lines give a free arrangement with exponents $(1,2)$ if and only if $|T|=1$), we have $a \leq b+c-3$ and the surjection of $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{O}_L(3-b-c)$ induces $(\mathcal{T}_{\mathcal{A}})_{|L} = \mathcal{O}_L(-a)\oplus\mathcal{O}_L(3-b-c)$ and if we suppose $\mathcal{A}$ to be free, it gives $\mathcal{T}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-a)\oplus\mathcal{O}_{\mathbb{P}^2}(3-b-c)$. By Addition-Deletion, $\mathcal{A}\cup L$ is free with exponents $(a, b+c-2)$. The gap $|b+c-2-a|$ is strictly bigger than the gap $|a+b-2-c|$ except when $a=b=c$. Then by Remark 2.6 this case cannot occur when $a \neq b$ or $b \neq c$. So $\mathcal{A}$ is free if and only if $a=b=c$ and T is a complete intersection $(a-1,a-1)$ (in this case $\mathcal{A}=\mathcal{A}_3^1(a-1)$).

III) Let us assume now that $\mathcal{A}$ contains no side of the triangle and let $L=(AB)$, $L^{'}=(AC)$ and $L^{''}=(BC)$ denote the missing sides. Then $\mathcal{A}\cup L\cup L^{'}\cup L^{''} = \mathcal{A}_0 \in \mathrm{Tr}(a,b,c)$. Consider the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup L''} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{O}_{L''}(4-b-c) \longrightarrow 0.$$

If $a \leq b + c - 4$ then, by the same technique used before, $\mathcal{A}$ is free with exponents $(a, b + c - 4)$ implying that $\mathcal{A} \cup L''$ is free with exponents $(a, b + c - 3)$. Since the gap for this last splitting is bigger than the gap given by the exponents $(c, a + b - 3)$, this proves, according to the Part II), that $\mathcal{A}$ is free if and only if $a = b = c$ and T is a complete intersection $(a - 1, a - 1)$.

□

As a direct consequence of the previous result, we can explicit the following cases, which describe when we have the same number of lines passing through each vertex.

Corollary 5.2.
- *The only free triangular arrangement of* $3n-1$ *lines passing through three points, with* $n - 1$ *inner lines through each vertex, plus two sides, is* $\mathcal{A}_3^2(n)$.
- *The only free triangular arrangement of* $3n - 2$ *lines passing through three points, with* $n - 1$ *inner lines through each vertex, plus one side, is* $\mathcal{A}_3^1(n)$.
- *The only free arrangement of* $3n - 3$ *lines passing through three points, with* $n - 1$ *inner lines through each vertex, and no side, is* $\mathcal{A}_3^0(n)$.

Proof. According to the previous theorem, for each one of these free arrangements the set T is a complete intersection $(n - 1, n - 1)$. It means that they come by deletion of one, two or three sides from $\mathcal{A}_0 \in \mathrm{Tr}(n, n, n)$ with exponents $(n, 2n - 1)$. In other words we have

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}_0} = \mathcal{O}_{\mathbb{P}^2}(-n) \oplus \mathcal{O}_{\mathbb{P}^2}(1 - 2n) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3(-n) \xrightarrow{(f(x,y),g(y,z),h(x,z))} \mathcal{J}_T(-1) \longrightarrow 0,$$

where T is the set of inner triple points, and $(f(x, y), g(y, z), h(x, z))$ are three homogeneous polynomials of degree $(n - 1)$. According to the splitting of $\mathcal{T}_{\mathcal{A}}$ there is a relation $af + bg + ch = 0$ with $(a, b, c) \in \mathbb{C}^3$. This implies that after a diagonal change of coordinates, we can choose $f(x, y) = x^{n-1} - y^{n-1}, g(x, y) = y^{n-1} - z^{n-1}, h(x, y) = x^{n-1} - z^{n-1}$, and this proves that these arrangements are $\mathcal{A}_3^0(n)$, $\mathcal{A}_3^1(n)$ or $\mathcal{A}_3^2(n)$. □

6 Weak combinatorics

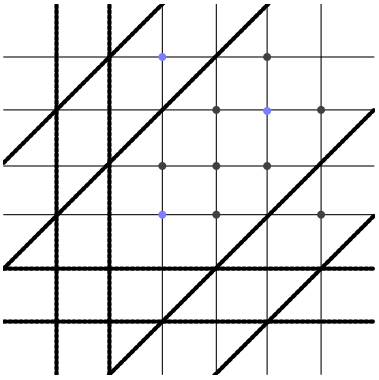
The combinatorics of $\mathcal{A}$ is determined by the set $L(\mathcal{A})$ of all the intersections of lines in $\mathcal{A}$. There is a partial order on this set corresponding to the inclusion of points ($L_1 \cap L_2 \subset L_1$ for instance) in lines. Two line arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$ have the same combinatorics if and only if there is a bijection between $L(\mathcal{A}_0)$ and $L(\mathcal{A}_1)$ preserving the partial order. In [11] Terao conjectures that if two arrangements have the same combinatorics and one of them is free then the other one is also free. This problem posed in any dimension and on any field is still open even on the projective plane and seems far from being proved, probably because few free arrangements are known.
In this section we will show that if we only suppose the weak combinatorics hypothesis, the conjecture does not hold. Indeed, we get the following result.

Theorem 6.1. *There exist pairs of arrangements possessing the same weak combinatorics such that one is free and the other is not.*

Proof. We prove it by describing an example. We will explain next how to produce a family of examples of the same kind.

We will construct two triangular arrangements of 15 lines $\mathcal{A}_0$, which will be free with exponents $(7,7)$, and $\mathcal{A}_1$, which won't be free, in $\mathrm{Tr}(5,5,5)$ with the same following numbers of multiple points $t_3 = 12, t_4 = t_5 = 0$, $t_6 = 3$ and $t_i = 0$ for $i > 6$ (the number of double points is a given by the combinatorial formula $\binom{15}{2} = t_2 + 3t_3 + \binom{6}{2}t_6$).

— Let us construct $\mathcal{A}_0$: it is obtained by removing the six lines $x = z$, $x = \zeta z$, $y = z$, $y = \zeta z$, $x = \zeta^2 y$ and $x = \zeta^4 y$ from the Ceva arrangement $xyz(x^6 - y^6)(y^6 - z^6)(x^6 - z^6) = 0$ as represented in the following picture:



This arrangement is free because the syzygy of degree 2, that is

$$\psi = [(x - z)(x - \zeta z), (y - z)(y - \zeta z), (x - \zeta^2 y)(x - \zeta^4 y)],$$

has no zero. Indeed this syzygy gives for

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}_0} \longrightarrow \mathcal{O}^3_{\mathbb{P}^2}(-5) \xrightarrow{\phi} \mathcal{J}_T(-1) \longrightarrow 0$$

a non zero section $\mathcal{O}_{\mathbb{P}^2}(-7) \longrightarrow \mathcal{T}_{\mathcal{A}_0}$, being

$$\phi = [\frac{x^6 - z^6}{(x - z)(x - \zeta z)}, \frac{y^6 - z^6}{(y - z)(y - \zeta z)}, \frac{x^6 - y^6}{(x - \zeta^2 y)(x - \zeta^4 y)}].$$

This induces a commutative diagram:

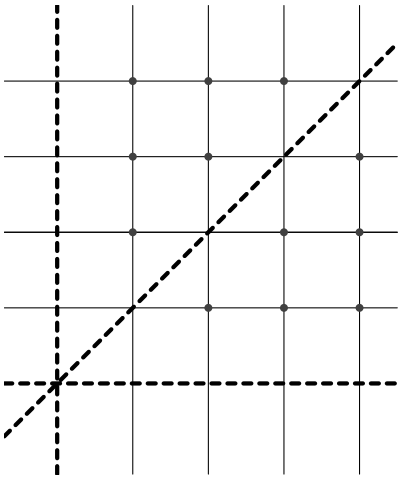
$$\begin{array}{ccccccccc}
 & & \mathcal{O}_{\mathbb{P}^2}(-7) & \xrightarrow{\simeq} & \mathcal{O}_{\mathbb{P}^2}(-7) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{T}_{\mathcal{A}_0} & \longrightarrow & \mathcal{O}^3_{\mathbb{P}^2}(-5) & \longrightarrow & \mathcal{J}_T(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_\Gamma(-7) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J}_T(-1) & \longrightarrow & 0,
\end{array}$$

where the singular locus of the rank two sheaf $\mathcal{F}$ is the zero set of ϕ. Since this zero set is empty, $\mathcal{F}$ is a vector bundle. That proves $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}) = 0$ and then $\Gamma = \emptyset$.

Another argument can be used to establish the freeness: the 12 inner triple points are distributed as a partition $3 + 3 + 3 + 3$ along the vertical lines, $3 + 3 + 3 + 3$ along the

horizontal lines but $2+3+4+3$ along the diagonal (this means that this example is very closed to have the same combinatorics than the non free case: indeed in the following example we'll see that the partition along vertical, horizontal and diagonal lines is always $3+3+3+3$); thanks to [14, Theorem 3.1], the bundle restricted to the line containing only 2 inner triple point has the splitting $(7,7)$ which proves the freeness according to [5, Corollary 2.12].

— Let us construct now $\mathcal{A}_1$: it is obtained by removing the three lines $x=z$, $y=z$, $x=y$ from the Ceva arrangement $xyz(x^5-y^5)(y^5-z^5)(x^5-z^5)=0$ as it appears on the picture:



Beginning with $\mathcal{A}_3^3(5)$ which is free with exponents $(6,11)$ and removing the first line we obtain, by Addition-Deletion theorem, a free bundle with exponents $(6,10)$. Removing the second line we find again a free arrangement with exponents $(6,9)$. Removing the third line, we don't find a free bundle (with splitting $(7,7)$) but a nearly free bundle with generic splitting $(6,8)$. The jumping point is the intersection point of the three removed lines. The three partitions appearing along the horizontal, vertical and diagonal lines are $12=3+3+3+3$.

Let us make it more explicit. We found a syzygy of degree 1, which is

$$\psi=[x-z,y-z,x-y],$$

and which induces a non zero section $\mathcal{O}_{\mathbb{P}^2}(-6)\longrightarrow\mathcal{T}$ where

$$0\longrightarrow\mathcal{T}_{\mathcal{A}_1}\longrightarrow\mathcal{O}_{\mathbb{P}^2}^3(-5)\xrightarrow{\phi}\mathcal{J}_T(-1)\longrightarrow 0$$

and

$$\phi=[\frac{x^5-z^5}{x-z},\frac{y^5-z^5}{y-z},\frac{x^5-y^5}{x-y}].$$

This syzygy admits a common zero $p = (1, 1, 1)$ and induces a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathcal{O}_{\mathbb{P}^2}(-6) & \xrightarrow{\simeq} & \mathcal{O}_{\mathbb{P}^2}(-6) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{T}_{\mathcal{A}_1} & \longrightarrow & \mathcal{O}^3_{\mathbb{P}^2}(-5) & \longrightarrow & \mathcal{J}_T(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_\Gamma(-8) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J}_T(-1) & \longrightarrow & 0,
\end{array}$$

where the singular locus of the rank two sheaf $\mathcal{F}$ is the zero set of p. Since $p \notin T$ then $p \in \Gamma$ (actually $p = \Gamma$) and $\mathcal{T}_{\mathcal{A}_1}$ cannot be free.

This example proves that the arrangement consisting in these 15 lines is Nearly free (defined in [3] and studied by the authors in [10]) with the same weak-combinatorics (same numbers $t_2, t_3, \ldots$) and quite the same combinatorics (only one partition along the diagonals differs) than the one described just before. This shows that we cannot replace the term combinatorics by weak-combinatorics in the hypothesis of Terao's conjecture. □

Remark 6.2. In the famous Ziegler's example of two arrangements (9 lines with 6 triple points) with the same combinatorics but with different free resolutions, the situation was explained by the existence of a smooth conic containing the 6 triple points. Here the situation can be geometrically explained by the existence of a cubic containing the 12 inner triple points. Indeed, since the bundle $\mathcal{T}_{\mathcal{A}_1}$ described in the previous example is the kernel of the following exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}_1} \longrightarrow \mathcal{O}^3_{\mathbb{P}^2}(-5) \longrightarrow \mathcal{J}_T(-1) \longrightarrow 0,$$

(where $|T| = 12$) it gives $\mathrm{H}^0(\mathcal{J}_T(3)) = \mathrm{H}^1(\mathcal{T}_{\mathcal{A}_1}(4))$. Moreover, the following non zero global section

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-6) \longrightarrow \mathcal{T}_{\mathcal{A}_1} \longrightarrow \mathcal{J}_p(-8) \longrightarrow 0,$$

where p is the jumping point associated to the Nearly Free arrangement, proves that

$$\mathrm{h}^1(\mathcal{T}_{\mathcal{A}_1}(4)) = \mathrm{h}^1(\mathcal{J}_p(-4)) = \mathrm{h}^0(\mathcal{O}_p) = 1.$$

Remark 6.3. It is possible to generalize the described examples, and find a family of them in the following way: consider triangular arrangements in the family $\mathrm{Tr}(n, n, n)$. The multiplicity of each vertex is $n + 1$. Assume that $n = 2k + 1$. For arrangements of this family, the maximal possible number for the inner triple points is $|T| = 4k^2$ (then the arrangement is $\mathcal{A}^3_3(n-1)$). For a general triangular arrangement, the set of inner triple points T is empty but if we want to consider free arrangements, T must contain at least $3k^2$ points. This minimal number corresponds to the balanced free arrangement with exponents $(3k + 1, 3k + 1)$. We construct a nearly free arrangement with generic splitting $\mathcal{O}_l(-3k) \oplus \mathcal{O}_l(-2-3k)$ and $3k^2$ triple inner points in the same family $\mathrm{Tr}(n, n, n)$. This generic splitting is also the one of the free arrangement with $3k^2 + 1$ triple inner points constructed by removing $(k-1)$ (inner) lines from each vertex of the arrangement $xyz(x^{3k-1} - y^{3k-1})(y^{3k-1} - z^{3k-1})(x^{3k-1} - z^{3k-1}) = 0$. In the last step, instead of removing a line with $k - 1$ triple inner points we remove a line with k triple points. It is always

possible by choosing a line of the third direction passing through a intersection point $\{p\}$ of the previous removing lines in the two other directions. This construction induces and exact sequence (see section **??**):

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3k) \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{J}_p(-2-3k) \to 0.$$

7 On Terao's conjecture

In this final section we use the characterization given in theorem 4.1 of free arrangements obtained from the Ceva to prove that Terao's conjecture holds for triangular arrangements.

Theorem 7.1. *Terao's conjecture holds for triangular arrangements.*

Proof. Consider an arrangement $\mathcal{A}$ and the associated one $\mathcal{B}$ which comes from a Ceva arrangement and which have the same combinatorics. Our goal is to prove that $\mathcal{A}$ is free if and only if $\mathcal{B}$ is free. Indeed, having characterized the freeness of the arrangements that comes from Ceva one in Theorem 4.1, this implies the conjecture.
— Suppose first that $\mathcal{B}$ is free.
We will prove that $\mathcal{A}$ is free through induction on the number of lines.
It is shown in [2] that the conjecture holds up to 13 lines for any arrangement, and in particular it holds for triangular ones with such number of lines.
Suppose that it holds for triangular arrangements constructed with $n-1$ lines and let us consider a triangular arrangement $\mathcal{A}$ that has n lines. Being $\mathcal{B}$ free, then, thanks again to Theorem 4.1, we can delete a line in order to get $\mathcal{B}'$ again free, and we delete the associate line to $\mathcal{A}$, in order to get $\mathcal{A}'$. This arrangement is free because of the induction hypothesis, and due to the Addition-Deletion Theorem, $\mathcal{A}$ must be free as well.
— Suppose now that $\mathcal{A}$ is free.
Having $\mathcal{B}$ the same Chern classes as $\mathcal{A}$, and in particular the same inner triple points, the only way to construct it is eliminating lines from the Ceva, minimizing the number of triple points in the complementary arrangement. This implies, due to Theorem 4.1, that $\mathcal{B}$ is free.

□

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