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# CATALYSIS IN THE TRACE CLASS AND WEAK TRACE CLASS IDEALS

GUILLAUME AUBRUN, FEDOR SUKOCHEV, AND DMITRIY ZANIN

ABSTRACT. Given operators  $A, B$  in some ideal  $\mathcal{I}$  in the algebra  $\mathcal{L}(H)$  of all bounded operators on a separable Hilbert space  $H$ , can we give conditions guaranteeing the existence of a trace-class operator  $C$  such that  $B \otimes C$  is submajorized (in the sense of Hardy–Littlewood) by  $A \otimes C$ ? In the case when  $\mathcal{I} = \mathcal{L}_1$ , a necessary and almost sufficient condition is that the inequalities  $\text{Tr}(B^p) \leq \text{Tr}(A^p)$  hold for every  $p \in [1, \infty]$ . We show that the analogous statement fails for  $\mathcal{I} = \mathcal{L}_{1, \infty}$  by connecting it with the study of Dixmier traces.

## 1. INTRODUCTION

Let  $H$  be an infinite-dimensional separable Hilbert space,  $\mathcal{L}(H)$  be the algebra of all bounded operators on  $H$  and  $\mathcal{C}_0 = \mathcal{C}_0(\mathcal{H})$  the set of compact operators.

Given  $A \in \mathcal{C}_0$ , we denote by  $\mu(A) := \{\mu(k, A)\}_{k \geq 0}$  the sequence of singular values of the operator  $A$  (that is, eigenvalues of the operator  $|A|$ ) arranged in the decreasing order and taken with multiplicities (if any). We say that  $B \in \mathcal{C}_0$  is submajorized by  $A \in \mathcal{C}_0$  in the sense of Hardy–Littlewood (written  $B \ll A$ ) if for every integer  $n$

$$\sum_{k=0}^n \mu(k, B) \leq \sum_{k=0}^n \mu(k, A).$$

If  $A, B \in \mathcal{C}_0$  are such that  $B \ll A$ , then  $B \otimes C \ll A \otimes C$  for every  $C \in \mathcal{C}_0$ .<sup>1</sup>The converse does not hold, even in the finite-dimensional setting: if  $A, B, C$  are such that  $\mu(A) = (0.5, 0.25, 0.25, 0, \dots)$ ,  $\mu(B) = (0.4, 0.4, 0.1, 0.1, 0, \dots)$  and  $\mu(C) = (0.6, 0.4, 0, \dots)$ , one checks easily that  $B \otimes C \ll A \otimes C$  while  $B$  is not submajorized by  $A$ . This example appears in [7] and is related to the phenomenon of *catalysis* in quantum information theory (the operator  $C$  being called a catalyst). This corresponds to the situation where the transformation of some quantum state (in that case,  $B$ ) into another quantum state (in that case,  $A$ ) is only possible in the presence of an extra quantum state (in that case,  $C$ ) although the latter is not consumed in the process. It is argued in [7] that this phenomenon can be used to improve the efficiency of entanglement concentration procedures.

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<sup>1</sup>Suppose first that  $C \geq 0$  has finite rank. That is,  $C = \sum_{k=0}^{n-1} \mu(k, C) p_k$ , where  $p_k$ ,  $0 \leq k < n$ , are pairwise orthogonal rank one projections. Set  $A_k = A \otimes \mu(k, C) p_k$  and  $B_k = B \otimes \mu(k, C) p_k$ . It is immediate that  $B_k \ll A_k$  for  $0 \leq k < n$ . It follows from Lemma 2.3 in [4] that  $\sum_{k=0}^{n-1} B_k \ll \sum_{k=0}^{n-1} A_k$  or, equivalently,  $B \otimes C \ll A \otimes C$ . For an arbitrary  $C$ , the assertion follows by approximation.

In the following we restrict ourselves to  $A, B$  being positive elements in  $\bigcap_{p>1} \mathcal{L}_p$  ( $\mathcal{L}_p$  denoting the Schatten–von Neumann ideal) and compare the following statements.

- (i) There exists a nonzero  $C \in \mathcal{L}_1$  such that  $B \otimes C \ll A \otimes C$ .
- (ii) For every  $p > 1$ , we have  $\text{Tr}(B^p) \leq \text{Tr}(A^p)$ .

One checks that (i) implies (ii). This follows from the monotonicity of  $A \mapsto \text{Tr}(A^p)$  with respect to submajorization and from the formula

$$\text{Tr}(S \otimes T) = \text{Tr}(S) \cdot \text{Tr}(T), \quad S, T \in \mathcal{L}_1.$$

There is some hope to reverse the implication (i)  $\Rightarrow$  (ii) if we allow closure of the set

$$\{B : \exists C \in \mathcal{L}_1 \text{ such that } B \otimes C \ll A \otimes C\}$$

with respect to some topology (for the finite-dimensional case, see [1, 9, 15]).

To explain why some closure is needed, we give an example of a pair  $A, B$  of positive operators satisfying (ii) but not (i). Consider positive operators with  $\mu(A) \neq \mu(B)$  and such that  $\text{Tr}(B^p) \leq \text{Tr}(A^p)$  for  $p \in (1, \infty)$ , while  $\text{Tr}(B^{p_0}) = \text{Tr}(A^{p_0})$  for some  $p_0 \in (1, \infty)$  (such an example exists among finite rank operators). Note that the norm in  $\mathcal{L}_{p_0}$  is strictly monotone with respect to submajorization (see Proposition 2.1 in [2]). That is, if  $K \in \mathcal{L}_{p_0}(H)$  and if  $L \ll K$ , then either  $\mu(L) = \mu(K)$  or  $\|L\|_{p_0} < \|K\|_{p_0}$ . Suppose that (i) holds, i.e. that  $B \otimes C \ll A \otimes C$  for some nonzero  $C \in \mathcal{L}_1$  (that is, no closure is taken). We then have  $\text{Tr}((B \otimes C)^{p_0}) = \text{Tr}((A \otimes C)^{p_0})$  and, by strict monotonicity,  $\mu(k, B \otimes C) = \mu(k, A \otimes C)$  for all  $k \geq 0$ . Now, taking into account that the sequences  $\mu(B \otimes C)$  and  $\mu(A \otimes C)$  coincide with decreasing rearrangements of sequences  $\mu(B) \otimes \mu(C)$  and  $\mu(A) \otimes \mu(C)$  respectively, we infer that  $\mu(A) = \mu(B)$ .

As we shall see, the choice of the topology plays a crucial role. Prior to stating the precise question, we recall a few definitions and relevant facts.

There is a remarkable correspondence between sequence spaces and two-sided ideals in  $\mathcal{L}(H)$  due to J.W. Calkin, [3]. Recall, that a linear subspace  $\mathcal{J}$  in  $\mathcal{L}(H)$  is a two-sided ideal if  $X \in \mathcal{J}$  and  $Y \in \mathcal{L}(H)$  imply  $YX, XY \in \mathcal{J}$ . Every non-trivial ideal necessarily consists of compact operators. A Calkin space  $J$  is a subspace of  $c_0$  (the space of all vanishing sequences) such that  $x \in J$  and  $\mu(y) \leq \mu(x)$  imply  $y \in J$ , where  $\mu(x)$  is the decreasing rearrangement of the sequence  $|x|$ . The Calkin correspondence may be explained as follows. If  $J$  is a Calkin space then associate to it the subset  $\mathcal{J}$  in  $\mathcal{L}(H)$

$$\mathcal{J} := \{X \in \mathcal{C}_0 : \mu(X) \in J\}.$$

Conversely, if  $\mathcal{J}$  is a two-sided ideal, then associate to it the sequence space

$$J := \{x \in c_0 : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{J}\}.$$

For the proof of the following theorem we refer to Calkin's original paper, [3], and to B. Simon's book, [13, Theorem 2.5].

**Theorem 1** (Calkin correspondence). *The correspondence  $J \leftrightarrow \mathcal{J}$  is a bijection between Calkin spaces and two-sided ideals in  $\mathcal{L}(H)$ .*

In the recent papers [8], [14] this correspondence has been specialised to quasi-normed symmetrically-normed ideals and quasi-normed symmetric sequence spaces [10]. We use the notation  $\|\cdot\|_\infty$  to denote the uniform norm on  $\mathcal{L}(H)$ .

**Definition 2.** (i) An ideal  $\mathcal{E}$  in  $\mathcal{L}(H)$  is said to be symmetrically (quasi)-normed if it is equipped with a Banach (quasi)-norm  $\|\cdot\|_{\mathcal{E}}$  such that

$$\|XY\|_{\mathcal{E}}, \|YX\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}\|Y\|_{\infty}, \quad X \in \mathcal{E}, Y \in \mathcal{L}(H).$$

(ii) A Calkin space  $E$  is a symmetric sequence space if it is equipped with a Banach (quasi)-norm  $\|\cdot\|_E$  such that  $\|y\|_E \leq \|x\|_E$  for every  $x \in E$  and  $y \in c_0$  such that  $\mu(y) \leq \mu(x)$ .

For convenience of the reader, we recall that a map  $\|\cdot\|$  from a linear space  $X$  to  $\mathbb{R}$  is a quasi-norm, if for all  $x, y \in X$  and scalars  $\alpha$  the following properties hold:

- (i)  $\|x\| \geq 0$ , and  $\|x\| = 0 \Leftrightarrow x = 0$ ;
- (ii)  $\|\alpha x\| = |\alpha|\|x\|$ ;
- (iii)  $\|x + y\| \leq C(\|x\| + \|y\|)$  for some  $C \geq 1$ .

The couple  $(X, \|\cdot\|)$  is a quasi-normed space and the least constant  $C$  satisfying the inequality (iii) above is called the modulus of concavity of the quasi-norm  $\|\cdot\|$  and denoted by  $C_X$ . A complete quasi-normed space is called quasi-Banach.

It easily follows from Definition 2 that if  $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$  is a quasi-Banach ideal,  $X \in \mathcal{E}$  and  $Y \in \mathcal{L}(H)$  are such that  $\mu(Y) \leq \mu(X)$ , then  $Y \in \mathcal{E}$  and  $\|Y\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}$ . In particular, it is easy to see that if  $E$  is Calkin space corresponding to  $\mathcal{E}$ , then setting  $\|x\|_E := \|X\|_{\mathcal{E}}$  (where  $X \in \mathcal{E}$  is such that  $\mu(x) = \mu(X)$ ) we obtain that  $(E, \|\cdot\|_E)$  is a quasi-Banach symmetric sequence space. The converse implication is much harder and follows from main results of [8, 14].

With these preliminaries out of the way, we are now in a position to formulate the main question.

**Question 3.** Let  $\mathcal{I}$  be a (quasi-)Banach ideal such that  $\mathcal{I} \subset \bigcap_{p>1} \mathcal{L}_p$ . Let  $0 \leq A \in \mathcal{I}$ . Consider the sets

$$\text{PM}(A, \mathcal{I}) = \left\{ 0 \leq B \in \mathcal{I} : \text{Tr}(B^p) \leq \text{Tr}(A^p) \quad \forall p > 1 \right\}.$$

$$\text{Catal}(A, \mathcal{I}) = \left\{ 0 \leq B \in \mathcal{I} : \exists 0 \leq C \in \mathcal{L}_1 : C \neq 0, B \otimes C \ll A \otimes C \right\}.$$

Let also  $\overline{\text{Catal}}(A, \mathcal{I})$  denote the closure of  $\text{Catal}(A, \mathcal{I})$  with respect to the quasi-norm of  $\mathcal{I}$ . Is it true that  $\text{PM}(A, \mathcal{I}) = \overline{\text{Catal}}(A, \mathcal{I})$ ?

Note that  $\text{PM}(A, \mathcal{I})$  is a closed subset in  $\mathcal{I}$ . Indeed, let  $B_n \in \text{PM}(A, \mathcal{I})$  and let  $B_n \rightarrow B$  in  $\mathcal{I}$  as  $n \rightarrow \infty$ . Observe that it follows from Definition 2 that  $\mathcal{I}$  is continuously embedded into  $\mathcal{L}(H)$  and therefore, it follows from the Closed Graph Theorem that for every fixed  $p > 1$ , the identical embedding  $\mathcal{I} \subset \mathcal{L}_p$  is continuous, in particular, there exists a constant  $c(p, \mathcal{I})$  such that  $\|C\|_p \leq c(p, \mathcal{I})\|C\|_{\mathcal{I}}$ ,  $C \in \mathcal{I}$ . Thus,

$$\left| \|B_n\|_p - \|B\|_p \right| \leq \|B - B_n\|_p \leq c(p, \mathcal{I})\|B - B_n\|_{\mathcal{I}} \rightarrow 0.$$

Hence,

$$\text{Tr}(B^p) = \lim_{n \rightarrow \infty} \text{Tr}(B_n^p) \leq \text{Tr}(A^p), \quad p > 1.$$

We also have that  $\text{Catal}(A, \mathcal{I}) \subset \text{PM}(A, \mathcal{I})$ . Indeed, if  $B \otimes C \ll A \otimes C$ , then

$$\text{Tr}(B^p) = \frac{\text{Tr}((B \otimes C)^p)}{\text{Tr}(C^p)} \leq \frac{\text{Tr}((A \otimes C)^p)}{\text{Tr}(C^p)} = \text{Tr}(A^p), \quad p > 1.$$

Since  $\text{PM}(A, \mathcal{I})$  is closed, it follows that the inclusion  $\overline{\text{Catal}}(A, \mathcal{I}) \subset \text{PM}(A, \mathcal{I})$  always holds.

In this paper, we show that the answer to Question 3 is positive when  $\mathcal{I} = \mathcal{L}_1$  and negative when  $\mathcal{I} = \mathcal{L}_{1,\infty}$ . Recall that  $\mathcal{L}_{1,\infty}$  is the principal ideal generated by the element  $A_0 = \text{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \dots\})$ . Equivalently,

$$\mathcal{L}_{1,\infty} = \{A \in \mathcal{C}_0 : \sup_{k \geq 0} (k+1)\mu(k, A) < +\infty\}.$$

It becomes a quasi-Banach space (see e.g. [8, 14]) when equipped with the quasi-norm

$$\|A\|_{1,\infty} = \sup_{k \geq 0} (k+1)\mu(k, A), \quad A \in \mathcal{L}_{1,\infty}.$$

Here are our main results. We leave open the question of giving a complete description of the set  $\overline{\text{Catal}}(A, \mathcal{L}_{1,\infty})$ .

**Theorem 4.** *For every  $0 \leq A \in \mathcal{L}_1$ , the sets  $\text{PM}(A, \mathcal{L}_1)$  and  $\overline{\text{Catal}}(A, \mathcal{L}_1)$  coincide.*

**Theorem 5.** *There exists  $0 \leq A \in \mathcal{L}_{1,\infty}$  such that the set  $\text{PM}(A, \mathcal{L}_{1,\infty})$  strictly contains the set  $\overline{\text{Catal}}(A, \mathcal{L}_{1,\infty})$ .*

It is actually simple to deduce Theorem 4 from the finite-dimensional considerations from [1], as we explain in Section 2. This is in sharp contrast with Theorem 5, whose proof is infinite-dimensional in its nature and uses crucially fine properties of Dixmier traces, which we introduce in Section 3. The heart of the argument behind Theorem 5 appears in Section 4, while we relegate some needed computations to Section 5.

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## 2. THE CASE OF $\mathcal{L}_1$

We derive Theorem 4 from the following result which appears in [15] (see also Lemma 2 in [1]).

**Lemma 6.** *Let  $A, B$  be positive finite rank operators. Assume that for every  $1 \leq p \leq +\infty$ , we have the strict inequality  $\|B\|_p < \|A\|_p$ . Then there exists a nonzero finite rank operator  $C$  such that  $B \otimes C \ll A \otimes C$ .*

*Proof of Theorem 4.* Let us show the non-trivial inclusion, i.e. that every  $B \in \text{PM}(A, \mathcal{L}_1)$  belongs to  $\overline{\text{Catal}}(A, \mathcal{L}_1)$ .

Let  $p_k, k \geq 0$ , be a rank one eigenprojection of the operator  $A$  which corresponds to the eigenvalue  $\mu(k, A)$ . Similarly, let  $q_k, k \geq 0$ , be a rank one eigenprojection of the operator  $B$  which corresponds to the eigenvalue  $\mu(k, B)$ . We have

$$A = \sum_{k=0}^{\infty} \mu(k, A) p_k, \quad B = \sum_{k=0}^{\infty} \mu(k, B) q_k.$$

Without loss of generality,  $\mu(0, A) = 1$ . It follows that

$$(1 - (1 - \varepsilon)^p) \text{Tr}(A^p) \geq (1 - (1 - \varepsilon)^p) \mu(0, A)^p = 1 - (1 - \varepsilon)^p \geq \varepsilon^p.$$

The latter readily implies

$$\text{Tr}(A^p) - \varepsilon^p \geq (1 - \varepsilon)^p \text{Tr}(A^p), \quad p \geq 1, \quad \varepsilon \in (0, 1).$$

Now, fix  $\varepsilon \in (0, 1)$  and select  $n$  such that

$$\sum_{k=n}^{\infty} \mu(k, A) < \varepsilon, \quad \sum_{k=n}^{\infty} \mu(k, B) < \varepsilon.$$

Set

$$A_n = \sum_{k=0}^{n-1} \mu(k, A) p_k, \quad B_n = \sum_{k=0}^{n-1} \mu(k, B) q_k.$$

It is clear that

$$\begin{aligned} \mathrm{Tr}(A_n^p) &= \mathrm{Tr}(A^p) - \sum_{k=n}^{\infty} \mu(k, A)^p \geq \mathrm{Tr}(A^p) - \left( \sum_{k=n}^{\infty} \mu(k, A) \right)^p \\ &> \mathrm{Tr}(A^p) - \varepsilon^p \geq (1 - \varepsilon)^p \mathrm{Tr}(A^p), \quad p \geq 1. \end{aligned}$$

Therefore,

$$(1 - \varepsilon)^p \mathrm{Tr}(B_n^p) \leq (1 - \varepsilon)^p \mathrm{Tr}(B^p) \leq (1 - \varepsilon)^p \mathrm{Tr}(A^p) < \mathrm{Tr}(A_n^p), \quad p \geq 1.$$

Since both  $A_n$  and  $B_n$  are finite rank operators, it follows from Lemma 6 and the first footnote that there exists a finite rank operator  $C_n$  such that

$$(1 - \varepsilon)B_n \otimes C_n \ll A_n \otimes C_n \ll A \otimes C_n.$$

In particular, we have that  $(1 - \varepsilon)B_n \in \mathrm{Catal}(A, \mathcal{L}_1)$ . Observing that  $\|B - B_n\|_1 \leq 1$ , we further obtain

$$\|B - (1 - \varepsilon)B_n\|_1 \leq \varepsilon \|B\|_1 + (1 - \varepsilon) \|B - B_n\|_1 \leq \varepsilon (\|B\|_1 + 1).$$

Since  $\varepsilon$  is arbitrarily small, it follows that  $B \in \overline{\mathrm{Catal}}(A, \mathcal{L}_1)$ .  $\square$

### 3. DIXMIER TRACES

The crucial ingredient in the proof is the notion of a Dixmier trace on  $\mathcal{L}_{1,\infty}$ . Let  $\ell_\infty$  stand for the Banach space of all bounded sequences  $x = (x_n)_{n \geq 0}$  equipped with the usual norm  $\|x\|_\infty := \sup_{n \geq 0} |x_n|$ . A *generalized limit* is any positive linear functional on  $\ell_\infty$  which equals the ordinary limit on the subspace  $c$  of all convergent sequences.

**Remark 7.** *Given a sequence  $(x_n)_{n \geq 0} \in \ell_\infty$ , there is a generalized limit  $\omega$  such that  $\omega((x_n)) = \limsup_{n \rightarrow \infty} x_n$ .*

*Proof.* Fix  $x = (x_n) \in \ell_\infty$  and let the sequence  $(n_k)_{k \geq 0}$  be such that  $\lim_{k \rightarrow \infty} x_{n_k} = \limsup_{n \rightarrow \infty} x_n$ . Consider the set of functionals  $(\varphi_{n_k})_{k \geq 0}$  on  $\ell_\infty$  defined by  $\varphi_{n_k}(y_n) := y_{n_k}$ ,  $y = (y_n) \in \ell_\infty$ ,  $k \geq 0$ . The set  $(\varphi_{n_k})_{k \geq 0}$  belongs to the unit ball  $B$  of the Banach dual  $\ell_\infty^*$ . The set  $B$  is compact in the weak\* topology  $\sigma(\ell_\infty^*, \ell_\infty)$  and therefore the set  $(\varphi_{n_k})_{k \geq 0}$  possesses a cluster point  $\omega \in \ell_\infty^*$  in that topology. The fact that  $\omega$  is a generalized limit on  $\ell_\infty$  such that  $\omega((x_n)) = \limsup_{n \rightarrow \infty} x_n$  follows immediately from the definition of the weak\* topology.  $\square$

The Dixmier traces are defined as follows.

**Theorem 8.** *Let  $\omega$  be a generalized limit. The mapping  $\mathrm{Tr}_\omega : \mathcal{L}_{1,\infty}^+ \rightarrow \mathbb{R}^+$  defined for  $0 \leq A \in \mathcal{L}_{1,\infty}$  by setting*

$$\mathrm{Tr}_\omega(A) := \omega \left( \left\{ \frac{1}{\log(N+2)} \sum_{k=0}^N \mu(k, A) \right\}_{N=0}^\infty \right)$$

*is additive and, therefore, extends to a positive unitarily invariant linear functional on  $\mathcal{L}_{1,\infty}$  called a Dixmier trace.*

Note that the positivity of generalized limits implies that

$$(1) \quad |\mathrm{Tr}_\omega(A)| \leq \|A\|_{1,\infty}$$

for every Dixmier trace  $\mathrm{Tr}_\omega$  and  $A \in \mathcal{L}_{1,\infty}$ .

Let us comment on how additivity is proved in Theorem 8. This is usually achieved under the extra assumption that  $\omega$  is scale invariant (see Theorem 1.3.1 in [11]), i.e. that  $\omega \circ \sigma_k = \omega$  for all positive integer  $k$ , where  $\sigma_k : \ell_\infty \rightarrow \ell_\infty$  is defined as

$$\sigma_k(x_1, x_2, \dots, x_n, \dots) = (\underbrace{x_1, \dots, x_1}_{k \text{ times}}, \underbrace{x_2, \dots, x_2}_{k \text{ times}}, \dots, \underbrace{x_n, \dots, x_n}_{k \text{ times}}, \dots).$$

Under this extra assumption the map  $\mathrm{Tr}_\omega$  is actually additive on the larger ideal  $\mathcal{M}_{1,\infty}$  (we refer to [11, Example 1.2.9] for the definition of the latter ideal and to [11, Section 6.8] for historical background). In the form presented here, Theorem 8 follows from Theorem 17 in [12]. For the reader's convenience we reproduce the argument here.

*Proof of Theorem 8.* Given  $A \in \mathcal{L}_{1,\infty}$ , consider the sequence  $(x_N(A))_{N=0}^\infty$  defined by

$$x_N(A) = \frac{1}{\log(N+2)} \sum_{j=0}^N \mu(j, A).$$

It is not hard to check that for all all positive integer  $k$ ,

$$(2) \quad \lim_{N \rightarrow \infty} x_N(A) - x_{kN}(A) = 0.$$

Let  $E \subset \ell_\infty$  be the subspace

$$E = \mathrm{span} \{ \sigma_k(\{x_N(A)\}) : k \geq 1, A \in \mathcal{L}_{1,\infty} \}.$$

It follows from (2) that the equation  $\omega \circ \sigma_k(x) = \omega(x)$  is satisfied for  $x \in E$ . By a version of the Hahn–Banach theorem (see [6], Theorem 3.3.1), the linear functional  $\omega|_E$  can be extended to a generalized limit  $\omega' : \ell_\infty \rightarrow \mathbf{R}$  which is scale-invariant. The usual argument ([11], Theorem 1.3.1) implies that  $\mathrm{Tr}_{\omega'}$  (which coincides with  $\mathrm{Tr}_\omega$  on  $\mathcal{L}_{1,\infty}$ ) is additive on  $\mathcal{L}_{1,\infty}$ .  $\square$

We also need a version of Fubini's theorem for Dixmier traces.

**Theorem 9.** *For every  $A \in \mathcal{L}_{1,\infty}$  and for every  $C \in \mathcal{L}_1$ , we have  $A \otimes C \in \mathcal{L}_{1,\infty}$  and*

$$(3) \quad \|A \otimes C\|_{1,\infty} \leq \|A\|_{1,\infty} \|C\|_1.$$

*Moreover, for every Dixmier trace  $\mathrm{Tr}_\omega$  on  $\mathcal{L}_{1,\infty}$ , we have*

$$(4) \quad \mathrm{Tr}_\omega(A \otimes C) = \mathrm{Tr}_\omega(A) \mathrm{Tr}(C).$$

*Proof.* We may assume  $\|A\|_{1,\infty} = 1$ . Recall that  $A_0 = \mathrm{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \dots\})$ . We have for all  $k \geq 0$ ,

$$\mu(k, A \otimes C) \leq \mu(k, A_0 \otimes C) \leq \frac{1}{k+1} \sum_{j=0}^k \mu(j, C) \leq \frac{\|C\|_1}{k+1}.$$

where the second inequality follows from Proposition 3.14 in [5]. This proves (3).

Observe that both sides of (4) depend linearly on  $A$  and  $C$  (thanks to Theorem 8). Thus, we can assume without loss of generality that  $A, C \geq 0$ . When  $C$  is a rank one projection, (4) follows from Theorem 8 since in that case  $\mu(k, A \otimes C) = \mu(k, A)$  for all  $k \geq 0$ . Again appealing to linearity of Dixmier traces, we infer the result for

the finite rank operator  $C$  and when  $A \in \mathcal{L}_{1,\infty}$  is arbitrary. Consider now a general  $C \in \mathcal{L}_1$  and let  $(C_n)$  be a sequence of finite rank operators such that  $\|C - C_n\|_1 \rightarrow 0$ . We have

$$|\mathrm{Tr}_\omega(A \otimes C_n) - \mathrm{Tr}_\omega(A \otimes C)| \leq \|A \otimes (C - C_n)\|_{1,\infty} \leq \|A\|_{1,\infty} \|C - C_n\|_1$$

and this quantity tends to 0 as  $n$  goes to infinity. Consequently,

$$\mathrm{Tr}_\omega(A \otimes C) = \lim_{n \rightarrow \infty} \mathrm{Tr}_\omega(A \otimes C_n) = \lim_{n \rightarrow \infty} \mathrm{Tr}_\omega(A) \mathrm{Tr}(C_n) = \mathrm{Tr}_\omega(A) \mathrm{Tr}(C). \quad \square$$

As a corollary, we obtain that Dixmier traces give necessary conditions for catalysis.

**Corollary 10.** *Let  $0 \leq A \in \mathcal{L}_{1,\infty}$  and  $0 \leq B \in \overline{\mathrm{Catal}}(A, \mathcal{L}_{1,\infty})$ . Then for every Dixmier trace  $\mathrm{Tr}_\omega$ , one has*

$$(5) \quad \mathrm{Tr}_\omega(B) \leq \mathrm{Tr}_\omega(A).$$

*Proof.* We know from (1) that Dixmier traces are continuous on  $\mathcal{L}_{1,\infty}$ , and therefore we may assume that  $B \in \mathrm{Catal}(A, \mathcal{L}_{1,\infty})$ . By definition of the latter set (see Question 3), there exists a nonzero positive  $C$  in  $\mathcal{L}_1$  with the property that  $B \otimes C \ll A \otimes C$ . Combining the definition of Hardy-Littlewood submajorization  $\ll$  and the definition of a Dixmier trace  $\mathrm{Tr}_\omega$  (see Theorem 8) we infer that the inequality  $\mathrm{Tr}_\omega(B \otimes C) \leq \mathrm{Tr}_\omega(A \otimes C)$  holds for every Dixmier trace  $\mathrm{Tr}_\omega$ . Inequality (5) now follows from (4) and from the fact that  $\mathrm{Tr}(C) > 0$ .  $\square$

#### 4. THE CASE OF $\mathcal{L}_{1,\infty}$ : THE MAIN ARGUMENT

Here is the main technical result used in the proof of Theorem 5. In the lemma below, we tacitly identify a sequence in the space  $\ell_\infty$  with the corresponding diagonal operator. For  $I \subset \mathbb{N}$ , we note by  $\chi_I$  the sequence defined by  $\chi_I(n) = 1$  if  $n \in I$  and  $\chi_I(n) = 0$  otherwise.

**Lemma 11.** *Let  $I$  be the subset of  $\mathbb{N}$  defined as*

$$I = \bigcup_{n \geq 0} [2^{2n}, 2^{2n+1}).$$

*Consider the operator<sup>2</sup>*

$$B = \bigoplus_{m \in I} 2^{-m} \chi_{[0, 2^m)}.$$

*Then  $B \in \mathcal{L}_{1,\infty}$ . Moreover,*

$$(6) \quad \limsup_{s \rightarrow 0+} s \mathrm{Tr}(B^{1+s}) \leq \frac{5}{9 \log 2} < \frac{2}{3 \log 2} \leq \limsup_{N \rightarrow \infty} \frac{1}{\log N} \sum_{k=0}^N \mu(k, B).$$

Let us postpone the proof of Lemma 11 and show how it implies the result stated in Theorem 5. Consider  $B$  as in Lemma 11 and fix a number  $\alpha$  such that  $\frac{5}{9 \log 2} < \alpha < \frac{2}{3 \log 2}$ . Recall that  $A_0 = \mathrm{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \dots\})$ . Since

$$\lim_{s \rightarrow 0+} s \mathrm{Tr}((\alpha A_0)^{1+s}) = \lim_{s \rightarrow 0+} s \zeta(s) \alpha^{1+s} = \alpha,$$

it follows from (6) that there exists  $\delta > 0$  such that the inequality

$$(7) \quad \mathrm{Tr}(B^{1+s}) \leq \mathrm{Tr}((\alpha A_0)^{1+s})$$

<sup>2</sup>In the subsequent formulas, the symbol  $\oplus$  stands for the direct sum of operators.



holds whenever  $0 < s \leq \delta$ . Define the operator  $A = \alpha A_0 \oplus \|B\|_{1+\delta} p$  where  $p$  is a rank one projection. We claim that  $B \in \text{PM}(A, \mathcal{L}_{1,\infty})$ : indeed, for  $s > \delta$  we may write

$$\text{Tr}(B^{1+s}) = \text{Tr}\left((B^{1+\delta})^{\frac{1+s}{1+\delta}}\right) \leq (\text{Tr}(B^{1+\delta}))^{\frac{1+s}{1+\delta}} = \|B\|_{1+\delta}^{1+s} \leq \text{Tr}(A^{1+s})$$

while for  $0 < s \leq \delta$  the inequality  $\text{Tr}(B^{1+s}) \leq \text{Tr}(A^{1+s})$  follows immediately from (7).

We now assume by contradiction that  $B$  belongs to the set  $\overline{\text{Catal}}(A, \mathcal{L}_{1,\infty})$ . We know from Corollary 10 that  $\text{Tr}_\omega(B) \leq \text{Tr}_\omega(A)$  for every Dixmier trace  $\text{Tr}_\omega$ . Observing that any such trace vanishes on finite rank operators, we see that the value  $\text{Tr}_\omega(A)$  coincides with  $\text{Tr}_\omega(\alpha A_0)$  and hence is equal to  $\alpha$  for every Dixmier trace  $\text{Tr}_\omega$  (see the definition given in Theorem 8). On the other hand, we may choose a generalized limit  $\omega$  such that

$$\text{Tr}_\omega(B) = \limsup_{N \rightarrow \infty} \frac{1}{\log N} \sum_{k=0}^N \mu(k, B)$$

and obtain from (6) that  $\frac{2}{3 \log 2} \leq \alpha$ , a contradiction.

We note that the Dixmier trace considered in the proof does not behave in a monotone way with respect to trace of powers: we have  $\text{Tr}(B^p) \leq \text{Tr}(A^p)$  for every  $p > 1$ , but  $\text{Tr}_\omega(B) > \text{Tr}_\omega(A)$ .

## 5. PROOF OF LEMMA 11

Let  $I$  and  $B$  be as defined in Lemma 11, and denote by  $E_B$  the spectral measure of  $B$ . First, note that for every integer  $m$ ,

$$(8) \quad \text{Tr}(E_B(2^{-m}, \infty)) \leq \sum_{l < m} 2^l \leq 2^m.$$

Hence, for every positive integer  $n$ , writing  $2^m \leq n < 2^{m+1}$ , we infer

$$(9) \quad \text{Tr}(E_B(\frac{1}{n}, \infty)) \leq \text{Tr}(E_B((2^{-m-1}, \infty)) \leq 2^{m+1} \leq 2n.$$

Recall also (see e.g. [11, Chapter 2, Section 2.3]) that  $\mu(k, B)$ ,  $k \geq 0$  can be computed via the formula

$$\mu(k, B) = \inf\{s \geq 0 : \text{Tr}(E_{|B|}((s, \infty)) \leq k\}$$

Hence, it follows from (9) that  $\mu(k, B) \leq \frac{2}{k+1}$  for every  $k \geq 0$  and, in particular,  $B \in \mathcal{L}_{1,\infty}$ . We now prove the right inequality in (6). For a given  $n$ , let  $N = \text{Tr}(E_B(2^{-2^{2n+1}}, \infty))$ . We know from (8) that  $N \leq 2^{2^{2n+1}}$ . Therefore,

$$\sum_{k=0}^{N-1} \mu(k, B) = \text{Tr}(B E_B(2^{-2^{2n+1}}, \infty)) = \text{card}(I \cap [0, 2^{2^{2n+1}}]) = \frac{2}{3} \cdot 2^{2^{2n+1}} - \frac{1}{3}.$$

Hence, for  $N$  as above, we have

$$\frac{1}{\log(N)} \sum_{k=0}^{N-1} \mu(k, B) \geq \frac{1}{\log(2^{2^{2n+1}})} \cdot \left(\frac{2}{3} \cdot 2^{2^{2n+1}} - \frac{1}{3}\right) = \frac{2}{3 \log(2)} + o(1).$$

as needed.

We now focus on the left inequality in (6) and use the following summation formula, whose proof we postpone. For a given sequence  $(x_n) \in \ell_\infty$  and for a given  $s > 0$ , we have that

$$(10) \quad \sum_{m=0}^{\infty} \left( \sum_{k=0}^m \sum_{l=0}^k x_l \right) 2^{-ms} = (1 - 2^{-s})^{-2} \sum_{l=0}^{\infty} x_l 2^{-ls}.$$

Note that  $\text{Tr}(B^{1+s}) = \sum_{m \in I} 2^{-ms} = \sum_{m \geq 0} \chi_I(m) 2^{-ms}$  (here,  $\chi_I(0) = 0$ ). Applying (10) to  $x = \chi_I$ , we obtain, for every  $M > 0$

$$\begin{aligned} \limsup_{s \rightarrow 0^+} s \sum_{l \geq 0} \chi_I(l) 2^{-ls} &= \limsup_{s \rightarrow 0^+} s (1 - 2^{-s})^2 \sum_{m \geq 0} \left( \sum_{k=0}^m \sum_{l=0}^k \chi_I(l) \right) 2^{-ms} \\ &= \limsup_{s \rightarrow 0^+} s (1 - 2^{-s})^2 \sum_{m \geq M} \left( \frac{1}{(m+1)^2} \sum_{k=0}^m \sum_{l=0}^k \chi_I(l) \right) \cdot (m+1)^2 2^{-ms} \\ &\leq \left( \sup_{m \geq M} \frac{1}{(m+1)^2} \sum_{k=0}^m \sum_{l=0}^k \chi_I(l) \right) \cdot \left( \limsup_{s \rightarrow 0^+} s (1 - 2^{-s})^2 \sum_{m \geq M} (m+1)^2 2^{-ms} \right). \end{aligned}$$

Passing  $M \rightarrow \infty$ , we infer that

$$\limsup_{s \rightarrow 0^+} s \sum_{l \geq 0} \chi_I(l) 2^{-ls} \leq C \limsup_{s \rightarrow 0^+} s (1 - 2^{-s})^2 \sum_{m \geq 0} (m+1)^2 2^{-ms},$$

where

$$C := \limsup_{m \rightarrow \infty} \frac{1}{(m+1)^2} \sum_{k=0}^m \sum_{l=0}^k \chi_I(l).$$

An elementary computation gives

$$\sum_{m=0}^{\infty} (m+1)^2 2^{-ms} = \frac{1 + 2^{-s}}{(1 - 2^{-s})^3}.$$

It follows that

$$\limsup_{s \rightarrow 0^+} s \text{Tr}(B^{1+s}) \leq \frac{2C}{\log 2}.$$

It remains to show that  $C \leq 5/18$  (we actually show  $C = 5/18$ ). To that end, we think of  $\chi_I$  as an element of  $L_\infty(0, \infty)$  and define  $z \in L_\infty(0, \infty)$  by setting  $z = \chi_{\bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1})}$ . Observe that  $\chi_I \leq z$ . Therefore,

$$C \leq \limsup_{t \rightarrow \infty} \frac{1}{t^2} \int_0^t \int_0^s z(u) \, du \, ds.$$

Since  $z(4t) = z(t)$  for every  $t > 0$ , applying Fubini's theorem we have

$$C \leq \sup_{t \in (1, 4)} \frac{1}{t^2} \int_0^t z(u)(t-u) \, du.$$

However,

$$\frac{1}{t^2} \int_0^t z(u)(t-u) \, du = \begin{cases} \frac{1}{2} - \frac{2}{3t} + \frac{2}{5t^2}, & 1 \leq t \leq 2 \\ \frac{4}{3t} - \frac{8}{5t^2}, & 2 \leq t \leq 4 \end{cases}$$

Hence, the latter supremum is, in fact, a maximum which is attained at  $t = \frac{12}{5}$  and equal to  $\frac{5}{18}$ .

*Proof of (10).* Write

$$\begin{aligned}
\sum_{m \geq 0} \left( \sum_{k=0}^m \sum_{l=0}^k x_l \right) 2^{-ms} &= \sum_{m \geq k \geq 0} \left( \sum_{l=0}^k x_l \right) 2^{-ms} = \sum_{k=0}^{\infty} \left( \sum_{l=0}^k x_l \right) \sum_{m=k}^{\infty} 2^{-ms} \\
&= (1 - 2^{-s})^{-1} \sum_{k=0}^{\infty} \left( \sum_{l=0}^k x_l \right) 2^{-ks} = (1 - 2^{-s})^{-1} \sum_{k \geq l \geq 0} x_l 2^{-ks} \\
&= (1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} x_l \sum_{k=l}^{\infty} 2^{-ks} = (1 - 2^{-s})^{-2} \sum_{l=0}^{\infty} x_l 2^{-ls}. \quad \square
\end{aligned}$$

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