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Exponential mixing under controllability conditions for SDEs driven by a degenerate Poisson noise

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Abstract

We prove existence and uniqueness of the invariant measure and exponential mixing in the total-variation norm for a class of stochastic differential equations driven by degenerate compound Poisson processes. In addition to mild assumptions on the distribution of the jumps for the driving process, the hypotheses for our main result are that the corresponding control system is dissipative, approximately controllable and solidly controllable. The solid controllability assumption is weaker than the well-known parabolic Hörmander condition and is only required from a single point to which the system is approximately controllable. Our analysis applies to Galerkin projections of stochastically forced parabolic partial differential equations with asymptotically polynomial nonlinearities and to networks of quasi-harmonic oscillators connected to different Poissonian baths.

Keywords: stochastic differential equations, Poisson noise, exponential mixing, coupling, controllability, Hörmander condition

2010 MSC: 60H10, 37A25, 93B05

1. Introduction

Motivated by applications to thermally driven harmonic networks and to Galerkin approximations of partial differential equations (PDEs) randomly forced by degenerate noise, we consider a stochastic differential equation (SDE) of the form

$$dX_t = f(X_t) dt + B dY_t, \quad (1)$$

where $f : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a smooth vector field, $B : \mathbf{R}^n \rightarrow \mathbf{R}^d$ is a linear map, and $(Y_t)_{t \geq 0}$ is an n -dimensional *compound Poisson process* of the form

$$Y_t = \sum_{k=1}^{\infty} \eta_k \mathbf{1}_{[\tau_k, \infty)}(t). \quad (2)$$

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Throughout the paper, the jump displacements $\{\eta_k\}_{k \in \mathbf{N}}$ are independent and identically distributed random variables with law ℓ and the waiting times separating the jumps, defined as $t_1 = \tau_1$ and $t_k = \tau_k - \tau_{k-1}$ for $k \geq 2$, form a sequence $\{t_k\}_{k \in \mathbf{N}}$ of independent exponentially distributed random variables with common rate parameter $\lambda > 0$. Moreover, the sequences $\{\eta_k\}_{k \in \mathbf{N}}$ and $\{t_k\}_{k \in \mathbf{N}}$ are independent from one another. We are interested in the *noise-degenerate* case, that is when $\text{rank}(B) < d$.

The aim of this paper is to establish exponential mixing for the SDE (1) under some mild dissipativity and controllability conditions. The precise hypotheses are the following.

(C1) There are numbers $\alpha > 0$ and $\beta > 0$ such that

$$\langle f(y), y \rangle \leq -\alpha \|y\|^2 + \beta \quad (3)$$

for all $y \in \mathbf{R}^d$, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are a scalar product and the associated norm in \mathbf{R}^d .

Combined with the regularity of f and the fact that $\sum_{k=1}^{\infty} t_k = +\infty$ with probability 1, it ensures the global well-posedness of the SDE (1). It also strongly suggests the norm squared as a candidate Lyapunov function. The other two conditions are related to the controllability of the system: we ask that there exists a point $\hat{x} \in \mathbf{R}^d$ such that the system is *both* approximately controllable to \hat{x} and solidly controllable from \hat{x} . To formulate these conditions more precisely, we introduce the following (deterministic) mapping. For $T > 0$ a given time,

$$\begin{aligned} S_T : \mathbf{R}^d \times C([0, T]; \mathbf{R}^n) &\rightarrow \mathbf{R}^d, \\ (x, \zeta) &\mapsto y_T, \end{aligned} \quad (4)$$

where $(y_t)_{t \in [0, T]}$ is the solution of the controlled problem

$$\begin{cases} \dot{y}_t = f(y_t) + B\zeta_t, \\ y_0 = x. \end{cases} \quad (5)$$

Accordingly, we will refer to the first argument of $S_T(\cdot, \cdot)$ as an initial condition and to the second one as a control.

(C2) The system is *approximately controllable to $\hat{x} \in \mathbf{R}^d$* : for any number $\epsilon > 0$ and any radius $R > 0$, we can find a time $T > 0$ such that for any initial point $x \in \mathbf{R}^d$ with $\|x\| \leq R$, there exists a control $\zeta \in C([0, T]; \mathbf{R}^n)$ verifying

$$\|S_T(x, \zeta) - \hat{x}\| < \epsilon. \quad (6)$$

(C3) The system is *solidly controllable from \hat{x}* : there is a number $\epsilon_0 > 0$, a time $T_0 > 0$, a compact set \mathcal{K} in $C([0, T_0]; \mathbf{R}^n)$ and a non-degenerate ball G in \mathbf{R}^n such that, for any continuous function $\Phi : \mathcal{K} \rightarrow \mathbf{R}^d$ satisfying the relation

$$\sup_{\zeta \in \mathcal{K}} \|\Phi(\zeta) - S_{T_0}(\hat{x}, \zeta)\| \leq \epsilon_0,$$

we have $G \subset \Phi(\mathcal{K})$.

Condition (C2) is a well-known controllability property, and (C3) is an accessibility property that is weaker than the weak Hörmander condition at the point \hat{x} (see Section 4.1 for a discussion).

We denote by (X_t, \mathbb{P}_x) the Markov family associated with the SDE (1) parametrised by the time $t \geq 0$ and the initial condition $x \in \mathbf{R}^d$, by $P_t(x, \cdot)$ the corresponding transition function, and by \mathfrak{P}_t and \mathfrak{P}_t^* the Markov semigroups

$$\mathfrak{P}_t g(x) = \int_{\mathbf{R}^d} g(y) P_t(x, dy) \quad \text{and} \quad \mathfrak{P}_t^* \mu(\Gamma) = \int_{\mathbf{R}^d} P_t(y, \Gamma) \mu(dy),$$

where $g \in L^\infty(\mathbf{R}^d)$ and $\mu \in \mathcal{P}(\mathbf{R}^d)$. Recall that a measure $\mu^{\text{inv}} \in \mathcal{P}(\mathbf{R}^d)$ is said to be *invariant* if $\mathfrak{P}_t^* \mu^{\text{inv}} = \mu^{\text{inv}}$ for all $t \geq 0$.

Main Theorem. *Assume that Conditions (C1)–(C3) are satisfied and that the law of η_k has finite variance and possesses a continuous positive density with respect to the Lebesgue measure on \mathbf{R}^n . Then, the semigroup $(\mathfrak{P}_t^*)_{t \geq 0}$ admits a unique invariant measure $\mu^{\text{inv}} \in \mathcal{P}(\mathbf{R}^d)$. Moreover, there exist constants $C > 0$ and $c > 0$ such that*

$$\|\mathfrak{P}_t^* \mu - \mu^{\text{inv}}\|_{\text{var}} \leq C e^{-ct} \left(1 + \int_{\mathbf{R}^d} \|x\| \mu(dx) \right) \quad (7)$$

for any $\mu \in \mathcal{P}(\mathbf{R}^d)$ and $t \geq 0$.

In the literature, the problem of ergodicity for SDEs driven by a degenerate noise is mostly considered when the perturbation is a Brownian motion, the system admits a Lyapunov function, and the Hörmander condition is satisfied at all the points of the state space. Under these assumptions, the transition function of the underlying Markov process has a smooth density with respect to Lebesgue measure which is almost surely positive. This implies that the process is *strong Feller* and *irreducible*, so it has a unique invariant measure by Doob's theorem (see Theorem 4.2.1 in [9] and [23, 21] for related results).

Even with the assumption that the noise is Gaussian, there are only few papers that consider the problem of ergodicity for an SDE without the Hörmander condition being satisfied everywhere. In [4], the uniqueness property for invariant measures is proved for degenerate diffusions, under the assumption that the Hörmander condition holds at one point and that the process is irreducible. The proof relies heavily on the Gaussian nature of the noise. In the paper [34], an approach based on controllability and a coupling argument is given for a study of dynamical systems on compact metric spaces subject to a more general degenerate noise: under the controllability assumptions (C2) and (C3) and a *decomposability* assumption on the noise, exponential mixing in the total-variation metric is established. This approach can be carried to problems on a non-compact space, provided a dissipativity of the type of (C1) holds; see [30] for a study of networks of quasi-harmonic oscillators. The class of decomposable noises includes — but is not limited to — Gaussian measures.

The present paper falls under the continuity of the study carried out in these references. The main difficulty in our case comes from the fact that the Poisson noise we consider, in addition to being degenerate, does not have a decomposability structure; also see [27], where polynomial mixing is proved for the complex Ginzburg–Landau equation

driven by a non-degenerate compound Poisson process. Yet, the methods we use still stem from a control and coupling approach, which we outline in the following paragraphs; also see the beginning of Section 3. Indeed, the combination of coupling and controllability arguments has the advantage of yielding rather simple proofs of otherwise very technical results and also accommodates a wide variety of (non-Gaussian) noises for which other methods fail.

We hope that treating a relatively tractable problem in an essentially self-contained way will help interested readers in making their way to understanding technically more difficult problems for which methods of the same flavour are used.

For a discrete-time Markov family on a compact state space \mathcal{X} , existence of an invariant measure can be obtained from a Bogolyubov–Krylov argument and it is typical to derive uniqueness and mixing from a uniform upper bound on the total-variation distance between the transition functions from different points. One way to prove uniqueness using such a uniform *squeezing* estimate is through a so-called Doeblin coupling argument, where one constructs a Markov family on $\mathcal{X} \times \mathcal{X}$ whose projections to each copy of \mathcal{X} have the same distribution as the original Markov family, and with the property that it hits the diagonal $\{(x, x) : x \in \mathcal{X}\}$ soon enough, often enough. We refer the interested reader to the paper [14] and to Chapter 3 of the monograph [22] for an introduction to these ideas, which go back to Doeblin, Harris, and Vaserstein.

When the state space \mathcal{X} is not compact, existence of an invariant measure requires additional arguments and one can rarely hope to prove squeezing estimates which hold uniformly on the whole state space. The Bogolyubov–Krylov argument for existence can be adapted provided that one has a suitable Lyapunov structure. As for uniqueness and mixing, the coupling argument will go through with a squeezing estimate which only holds for points in a small ball, provided that one can obtain good enough estimates on the hitting time of that ball. Over the past years, it has become evident that control theory provides a good framework for formulating conditions that are sufficient for this endeavor when the noise is degenerate.

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Notation

For (\mathcal{X}, d) a Polish space, we shall use the following notation throughout the paper:

- $B_{\mathcal{X}}(x, \epsilon)$ for the closed ball in \mathcal{X} of radius ϵ centered at x (we shall simply write $B(x, \epsilon)$ in the special case $\mathcal{X} = \mathbf{R}^d$);
- $\mathcal{B}(\mathcal{X})$ for its Borel σ -algebra;
- $L^\infty(\mathcal{X})$ for the space of all bounded Borel-measurable functions $g : \mathcal{X} \rightarrow \mathbf{R}$, endowed with the norm $\|g\|_\infty = \sup_{y \in \mathcal{X}} |g(y)|$;

- $\mathcal{P}(\mathcal{X})$ for the set of Borel probability measures on \mathcal{X} , endowed with the total variation norm: for $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{X})$,

$$\begin{aligned}\|\mu_1 - \mu_2\|_{\text{var}} &:= \frac{1}{2} \sup_{\|g\|_{\infty} \leq 1} |\langle g, \mu_1 \rangle - \langle g, \mu_2 \rangle| \\ &= \sup_{\Gamma \in \mathcal{B}(\mathcal{X})} |\mu_1(\Gamma) - \mu_2(\Gamma)|,\end{aligned}$$

where $\langle g, \mu \rangle = \int_{\mathcal{X}} g(y) \mu(dy)$ for $g \in L^{\infty}(\mathcal{X})$ and $\mu \in \mathcal{P}(\mathcal{X})$.

Let (\mathcal{Y}, d') be another Polish space. The image of a measure $\mu \in \mathcal{P}(\mathcal{X})$ under a Borel-measurable mapping $F : \mathcal{X} \rightarrow \mathcal{Y}$ is denoted by $F_*\mu \in \mathcal{P}(\mathcal{Y})$.

On any space, $\mathbf{1}_{\Gamma}$ stands for the indicator function of the set Γ .

We use \mathbf{Z} for the set of integers and \mathbf{N} for the set of natural numbers (without 0). For any $m \in \mathbf{N}$, we set

$$\mathbf{N}_m := \{n \cdot m : n \in \mathbf{N}\} \quad \text{and} \quad \mathbf{N}_m^0 := \mathbf{N}_m \cup \{0\}. \quad (8)$$

We use $a \vee b$ [resp. $a \wedge b$] for the maximum [resp. minimum] of the numbers $a, b \in \mathbf{R}$.

2. Preliminaries and existence of an invariant measure

The SDE (1) has a unique càdlàg solution satisfying the initial condition $X_0 = x$. It is given by

$$X_t = \begin{cases} S_{t-\tau_k}(X_{\tau_k}) & \text{if } t \in [\tau_k, \tau_{k+1}), \\ S_{t_{k+1}}(X_{\tau_k}) + B\eta_{k+1} & \text{if } t = \tau_{k+1}, \end{cases} \quad (9)$$

where $\tau_0 = 0$ and $S_t(x) = S_t(x, 0)$ is the solution of the undriven equation. Relation (9) will allow us to reduce the study of the ergodicity of the full process $(X_t)_{t \geq 0}$ to that of the embedded process $(X_{\tau_k})_{k \in \mathbf{N}}$ obtained by considering its values at jump times τ_k . The strong Markov property implies that the latter is a Markov process with respect to the filtration generated by the random variables $\{t_j, \eta_j\}_{j=1}^k$. We denote by \hat{P}_k the corresponding transition function: for $x \in \mathbf{R}^d$ and $\Gamma \in \mathcal{B}(\mathbf{R}^d)$,

$$\hat{P}_k(x, \Gamma) := \mathbb{P}_x \{X_{\tau_k} \in \Gamma\}. \quad (10)$$

The key consequences of the dissipativity Condition (C1) are the moment estimates of the following lemma. They imply, in particular, existence of a suitable Lyapunov structure given by the norm squared.

Lemma 2.1. *Under Condition (C1), we have the following bounds:*

(i) *for any $\epsilon > 0$, there exists a constant $C_{\epsilon} > 0$ such that*

$$\|X_{\tau_k}\|^2 \leq (1 + \epsilon)^k e^{-2\alpha\tau_k} \|X_0\|^2 + C_{\epsilon} \sum_{j=1}^k e^{-2\alpha(\tau_k - \tau_j)} (1 + \epsilon)^{k-j} (1 + \|\eta_j\|^2) \quad (11)$$

for all $x \in \mathbf{R}^d$ and $k \in \mathbf{N}$;

(ii) there are numbers $\gamma \in (0, 1)$ and $C > 0$ such that

$$\mathbb{E}_x \|X_{\tau_k}\|^2 \leq \gamma^k \|x\|^2 + C(1 + \Lambda), \quad (12)$$

$$\mathbb{E}_x \|X_t\|^2 \leq (1 - \gamma)^{-1} \|x\|^2 + C(1 + \Lambda) \quad (13)$$

for all $x \in \mathbf{R}^d$, $k \in \mathbf{N}$, and $t \geq 0$, where $\Lambda := \mathbb{E}\|\eta_1\|^2$ and \mathbb{E}_x is the expectation with respect to \mathbb{P}_x .

Proof. First note that Condition (C1) implies the following estimate for the solution to the undriven equation:

$$\|S_t(x)\|^2 \leq e^{-2\alpha t} \|x\|^2 + \beta\alpha^{-1} \quad (14)$$

for all $x \in \mathbf{R}^d$ and $t \geq 0$. Let $\epsilon > 0$ be arbitrary. Combining (9) and (14), we find a positive constant C_ϵ such that

$$\|X_{\tau_k}\|^2 \leq (1 + \epsilon)e^{-2\alpha t_k} \|X_{\tau_{k-1}}\|^2 + C_\epsilon(1 + \|\eta_k\|^2).$$

Iterating this inequality, we get (11). Taking expectation in (11) and using the independence of the sequences $\{\eta_k\}$ and $\{\tau_k\}$, we obtain

$$\mathbb{E}_x \|X_{\tau_k}\|^2 \leq (1 + \epsilon)^k \left(\frac{\lambda}{\lambda + 2\alpha} \right)^k \|x\|^2 + C_\epsilon \sum_{j=1}^k \left(\frac{\lambda}{\lambda + 2\alpha} \right)^{k-j} (1 + \epsilon)^{k-j} (1 + \Lambda).$$

Choosing $\epsilon > 0$ so small that $\gamma := (1 + \epsilon)\frac{\lambda}{\lambda + 2\alpha} \in (0, 1)$ yields (12). To prove (13), we introduce the random variable

$$\mathcal{N}_t := \max\{k \geq 0 : \tau_k \leq t\}$$

and use (14):

$$\mathbb{E}_x \|X_t\|^2 \leq \mathbb{E}_x \|X_{\tau_{\mathcal{N}_t}}\|^2 + \beta\alpha^{-1} = \sum_{k=0}^{\infty} \mathbb{E}_x (\mathbf{1}_{\{\mathcal{N}_t=k\}} \|X_{\tau_k}\|^2) + \beta\alpha^{-1}. \quad (15)$$

Inequality (11) and the independence of $\{\eta_k\}$ and $\{\tau_k\}$ imply

$$\mathbb{E}_x (\mathbf{1}_{\{\mathcal{N}_t=k\}} \|X_{\tau_k}\|^2) \leq \gamma^k \|x\|^2 + C_\epsilon(1 + \Lambda) \sum_{j=1}^k (1 + \epsilon)^{k-j} \mathbb{E} (\mathbf{1}_{\{\mathcal{N}_t=k\}} e^{-2\alpha(\tau_k - \tau_j)}) \quad (16)$$

and

$$\sum_{k=1}^{\infty} \sum_{j=1}^k (1 + \epsilon)^{k-j} \mathbb{E} (\mathbf{1}_{\{\mathcal{N}_t=k\}} e^{-2\alpha(\tau_k - \tau_j)}) = \sum_{k=0}^{\infty} (1 + \epsilon)^k \mathbb{E} (e^{-2\alpha\tau_k}) = \sum_{k=0}^{\infty} (1 + \epsilon)^k \left(\frac{\lambda}{\lambda + 2\alpha} \right)^k,$$

which is finite by our choice of ϵ . Combining this with (15) and (16), we get (13) and complete the proof of the lemma. \square

As mentioned in the introduction, the dissipativity Condition (C1) guarantees the existence of an invariant measure. Indeed, the last lemma, combined with a Bogolyubov–Krylov argument and Fatou’s lemma yields the following result. We refer the reader to [22, §2.5.2] for more details.

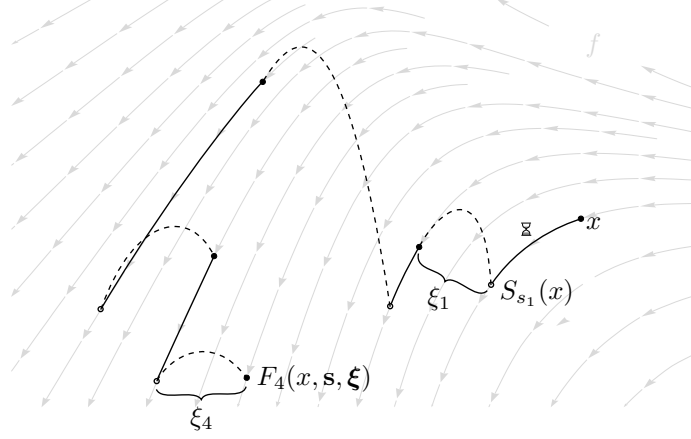


Figure 1: The map F_k takes as an input a point x , a sequence \mathbf{s} of times and a sequence $\boldsymbol{\xi}$ of displacement vectors and outputs the final position of a test particle which starts at x , follows the integral curves of f for a time s_1 , is immediately displaced by ξ_1 , follows the integral curves of f for a time s_2 , is immediately displaced by ξ_2 , and so on until it is finally displaced by ξ_k . We have sketched this for $k = 4$.

Lemma 2.2. *Under Condition (C1), the semigroup $(\mathfrak{P}_t^*)_{t \geq 0}$ admits at least one invariant measure $\mu^{\text{inv}} \in \mathcal{P}(\mathbf{R}^d)$. Moreover, any invariant measure $\mu \in \mathcal{P}(\mathbf{R}^d)$ has a finite second moment, that is*

$$\int_{\mathbf{R}^d} \|y\|^2 \mu(dy) < \infty. \quad (17)$$

We now turn to an important consequence of the solid controllability Condition (C3). The main ideas in its proof are borrowed from [34, §1] (also see the earlier works [1, §2] and [22, Ch. 3]). Such results are sometimes referred to as squeezing estimates, a concept to which we have referred in the introduction. This lemma is used to prove a key property of the coupling constructed in the next section.

We consider the family of maps $F_k : \mathbf{R}^d \times (\mathbf{R}_+)^{\mathbf{N}} \times (\mathbf{R}^n)^{\mathbf{N}} \rightarrow \mathbf{R}^d$ defined by

$$\begin{cases} F_0(x, \mathbf{s}, \boldsymbol{\xi}) = x, \\ F_k(x, \mathbf{s}, \boldsymbol{\xi}) = S_{s_k}(F_{k-1}(x, \mathbf{s}, \boldsymbol{\xi})) + B\xi_k \end{cases} \quad (18)$$

for $k \in \mathbf{N}$, $x \in \mathbf{R}^d$, $\mathbf{s} = (s_j)_{j \in \mathbf{N}} \in (\mathbf{R}_+)^{\mathbf{N}}$, and $\boldsymbol{\xi} = (\xi_j)_{j \in \mathbf{N}} \in (\mathbf{R}^n)^{\mathbf{N}}$; see Figure 1. Because F_k does not depend on $\{s_j, \xi_j\}_{j \geq k+1}$, i.e. the times and displacements for kicks that happen later than the k -th kick, we will often consider the domain of F_k to be $\mathbf{R}^d \times (\mathbf{R}_+)^m \times (\mathbf{R}^n)^m$ for some natural number $m \geq k$.

Lemma 2.3. *Suppose that \hat{x} is as in Condition (C3). Then, there exist numbers $m \in \mathbf{N}$, $r > 0$, and $p \in (0, 1)$ and a non-degenerate ball¹ Σ in $[0, T_0]^m$ such that*

$$\|F_m(x, \mathbf{s}, \cdot)_*(\ell^m) - F_m(x', \mathbf{s}, \cdot)_*(\ell^m)\|_{\text{var}} \leq p \quad (19)$$

for all $\mathbf{s} \in \Sigma$ and $x, x' \in B(\hat{x}, r)$, where $F_m(x, \mathbf{s}, \cdot)_*(\ell^m)$ is the image of ℓ^m (the m -fold product of the law ℓ with itself) under the mapping $F_m(x, \mathbf{s}, \cdot) : (\mathbf{R}^n)^m \rightarrow \mathbf{R}^d$.

¹Here $[0, T_0]^m$ is endowed with the metric inherited from \mathbf{R}^m .

Proof. Let us fix ϵ_0 , \mathcal{K} , and G as in Condition (C3). To simplify the presentation, we assume that $T_0 = 1$. For any $m \in \mathbf{N}$ and $\zeta \in C([0, 1]; \mathbf{R}^n)$, let $\iota_m(\zeta) : [0, 1] \rightarrow \mathbf{R}^n$ be the step function

$$\iota_m(\zeta) = \sum_{j=0}^{m-1} \mathbf{1}_{[\frac{j}{m}, \frac{j+1}{m})} \int_0^{\frac{j}{m}} \zeta(s) \, ds,$$

and let \mathcal{K}_m be the set $\iota_m(\mathcal{K})$. If ζ is a continuous function which allows the system to be controlled from \hat{x} to some target in time 1, then $\iota_m(\zeta)$ is a discretization in time of the antiderivative of ζ and we expect that feeding its jump discontinuities to F_m would result in a final position which is close to the target if m is large enough. With this in mind, we often identify the function $\iota_m(\zeta)$ with the m -tuple of vectors in \mathbf{R}^n consisting of its jumps at the times $\frac{1}{m}, \frac{2}{m}, \dots, \frac{m}{m}$.

We proceed in three steps. First, we show that Condition (C3) implies that the set $F_m(\hat{x}, \hat{s}, \mathcal{K}_m)$ contains a ball in \mathbf{R}^d . Then, combining this with Sard's theorem and some properties of images of measures under regular mappings, we show a uniform lower bound on $F_m(x, \mathbf{s}, \cdot)_*(\ell^m)$ for (x, \mathbf{s}) close enough to (\hat{x}, \hat{s}) where $\hat{s} := (\frac{1}{m}, \dots, \frac{1}{m}) \in [0, 1]^m$. Finally, from this uniform lower bound we derive the desired estimate in total variation.

Step 1: Solid controllability. Let S_T be the mapping defined by (4). By the compactness of \mathcal{K} , for any $\epsilon > 0$, there exists $m_0(\epsilon) \in \mathbf{N}$ such that

$$\sup_{\zeta \in \mathcal{K}} \left\| \iota_m \zeta - \int_0^\cdot \zeta(s) \, ds \right\|_{L^\infty([0, 1], \mathbf{R}^n)} \leq \epsilon$$

whenever $m \geq m_0(\epsilon)$. Hence, taking $m \geq m_0(\epsilon)$ for sufficiently small ϵ , we have

$$\sup_{\zeta \in \mathcal{K}} \|F_m(\hat{x}, \hat{s}, \iota_m \zeta) - S_1(\hat{x}, \zeta)\| \leq \epsilon_0,$$

where we use the aforementioned identification of functions in \mathcal{K}_m with m -tuples of displacement vectors in \mathbf{R}^n . Using the continuity of $F_m(\hat{x}, \hat{s}, \iota_m \cdot) : \mathcal{K} \rightarrow \mathbf{R}^d$ and Condition (C3), we conclude that $F_m(\hat{x}, \hat{s}, \mathcal{K}_m)$ contains a ball in \mathbf{R}^d . Until the end of the proof, we fix $m \geq m_0(\epsilon)$ for such a small ϵ .

Step 2: Uniform lower bound. We want to apply Lemma C.2 with $\mathcal{X} = B(\hat{x}, 1) \times [0, 1]^m$, $\mathcal{Y} = \mathbf{R}^d$, and $\mathcal{U} = (\mathbf{R}^n)^m$ and the map $F_m : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$ as before. As $F_m(\hat{x}, \hat{s}, \mathcal{K}_m)$ contains a ball in \mathbf{R}^d , Sard's theorem yields the existence of a point $\hat{u} \in \mathcal{K}_m \subset \mathcal{U}$ in which the derivative $D_{\xi} F_m(\hat{x}, \hat{s}, \cdot)$ has full rank. Hence, by Lemma C.2, there exists a continuous function $\psi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbf{R}_+$ and a radius $r_m > 0$ such that

$$\psi((\hat{x}, \hat{s}), F_m(\hat{x}, \hat{s}, \hat{u})) > 0$$

and

$$(F_m(x, \mathbf{s}, \cdot)_*(\ell^m))(\,dy) \geq \psi((x, \mathbf{s}), y) \, dy$$

(as measures, with y ranging over \mathbf{R}^d) whenever $x \in B(\hat{x}, r_m)$ and $\mathbf{s} \in B_{\mathbf{R}^m}(\hat{s}, r_m)$.

Step 3: Estimate in total variation. Shrinking r_m if necessary, Step 2 yields positive numbers $\epsilon_{m,1}$ and $\epsilon_{m,2}$ and a non-degenerate ball $\Sigma \subset [0, 1]^m$ such that

$$F_m(x, \mathbf{s}, \cdot)_*(\ell^m) \wedge F_m(x', \mathbf{s}, \cdot)_*(\ell^m) \geq \epsilon_{m,1} \, \text{Vol}_{\mathbf{R}^d}(\cdot \cap B(F_m(\hat{x}, \hat{s}, \hat{u}), \epsilon_{m,2}))$$

whenever $x, x' \in B(\hat{x}, r_m)$ and $\mathbf{s} \in \Sigma$. Therefore,

$$\|F_m(x, \mathbf{s}, \cdot)_*(\ell^m) - F_m(x', \mathbf{s}, \cdot)_*(\ell^m)\|_{\text{var}} \leq 1 - \epsilon_{m,1} \epsilon_{m,2}^d \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} =: p_m$$

whenever $x, x' \in B(\hat{x}, r_m)$ and $\mathbf{s} \in \Sigma$. This proves (19) with $r = r_m$ and $p = p_m$. \square

3. Coupling argument and exponential mixing

In this section, we shall always assume that Conditions (C1)–(C3) are satisfied. The Main Theorem is established by using the *coupling method*, which consists in proving uniqueness and convergence to an invariant measure for a Markov family by using the inequality

$$\|P_t(x, \cdot) - P_t(x', \cdot)\|_{\text{var}} \leq \mathbb{P}\{\mathcal{T} > t\},$$

where \mathcal{T} is a random time given by

$$\mathcal{T} := \inf \{s \geq 0 : Z_u = Z'_u \text{ for all } u \geq s\} \quad (20)$$

and $(Z_t, Z'_t)_{t \geq 0}$ is any $(\mathbf{R}^d \times \mathbf{R}^d)$ -valued random process defined on a space $(\Omega, \mathcal{F}, \mathbb{P}_{(x,x')})$ with $\mathbb{P}_{(x,x')}(Z_t \in \Gamma) = P_t(x, \Gamma)$ and $\mathbb{P}_{(x,x')}(Z'_t \in \Gamma) = P_t(x', \Gamma)$ for all $t \geq 0$ and all measurable $\Gamma \subseteq \mathbf{R}^d$. This inequality is of course most useful when the process $(Z_t, Z'_t)_{t \geq 0}$, called a *coupling*, is constructed in a such a way that $\mathbb{P}\{\mathcal{T} > t\}$ decays as fast as possible as $t \rightarrow \infty$, with a reasonable dependence on x and x' . To do so, one usually uses at some point a general result of the type of Lemma C.1 on the existence of so-called *maximal couplings* (see [22, Chapter 3]).

We first proceed to construct a coupling of two embedded discrete-time processes as introduced at the beginning of Section 2, but with different initial conditions: given x and x' in \mathbf{R}^d , we define a sequence $(z_k, z'_k)_{k \in \mathbf{N}}$ of $(\mathbf{R}^d \times \mathbf{R}^d)$ -valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_{(x,x')})$ with $\mathbb{P}_{(x,x')}(z_k \in \Gamma) = \hat{P}_k(x, \Gamma)$ and $\mathbb{P}_{(x,x')}(z'_k \in \Gamma) = \hat{P}_k(x', \Gamma)$ for all $k \in \mathbf{N}$ and measurable $\Gamma \subseteq \mathbf{R}^d$. In this context, we call $(z_k)_{k \in \mathbf{N}}$ [resp. $(z'_k)_{k \in \mathbf{N}}$] the first [resp. second] *component* of the coupling $(z_k, z'_k)_{k \in \mathbf{N}}$. The structure of the waiting times and the relation (9) then allow us to recover estimates for the original continuous-time process. The construction of this coupling is inductive and relies on the numbers $m \in \mathbf{N}$ and $r > 0$ in Lemma 2.3 and correlates the two components in a different way according to three cases: for $j \in \mathbf{N}_m^0$,

- if $z_j = z'_j$, then $z_k = z'_k$ for all $k \in \mathbf{N}$ with $k \geq j$;
- if z_j and z'_j are different but both in $B(\hat{x}, r)$, then the next m jumps are synchronous and, given the times of these jumps, z_{j+m} and z'_{j+m} are maximally coupled in the sense of Lemma C.1;
- if z_j and z'_j are different and not both in $B(\hat{x}, r)$, then the next m jumps are synchronous, but the respective jump displacements are independent.

In essence, the worst-case scenario is when the initial conditions x and x' are different and very far from the origin, but the number

$$I := \min\{i \in \mathbf{N}_m^0 : (z_i, z'_i) \in B(0, R) \times B(0, R)\} \quad (21)$$

of jumps needed for both components to enter a large² compact set around the origin is controlled by the Lyapunov structure inherited from (C1). Then, the approximate controllability assumption (C2) allows us to prove an estimate for an exponential moment of the number

$$J := \min\{j \in \mathbf{N}_m^0 : (z_j, z'_j) \in B(\hat{x}, r) \times B(\hat{x}, r)\} \quad (22)$$

of jumps needed for both components to simultaneously enter $B(\hat{x}, r)$. Finally, combining this with the solid controllability assumption (C3), we control the probability distribution of the number

$$\begin{aligned} K &:= \min\{k \in \mathbf{N}_m^0 : z_k = z'_k\} \\ &= \min\{k \in \mathbf{N}_m^0 : z_\ell = z'_\ell \text{ for all } \ell \in \mathbf{N} \text{ with } \ell \geq k\} \end{aligned} \quad (23)$$

of jumps after which the two components coincide.

Alternatively, in a language which avoids the particularities of the coupling method, one could rephrase the above strategy by saying that combining (C2) and the consequence of (C3) expressed in Lemma 2.3 gives a *local Doeblin condition* in $B(0, R)$ which, when combined with the Lyapunov structure conferred by (C1), yields exponential mixing by Meyn–Tweedie-type arguments [24].

3.1. Coupling for the embedded discrete-time process

In this section, we construct a coupling $(z_k, z'_k)_{k \in \mathbf{N}}$ for the embedded discrete-time process in such a way that the random time after which the two components coincide has an exponential moment which can we estimate in terms of the initial conditions (see Proposition 3.2).

Let us fix the numbers m , r , and p as in Lemma 2.3. The coupling is constructed by blocks of m steps as follows. Let $\mathcal{X} = \mathbf{R}^d \times (\mathbf{R}_+)^m \times (\mathbf{R}^n)^m$, $\mathcal{Y} = \mathbf{R}^d$, and $\mathcal{U} = \mathbf{R}^d \times \mathbf{R}^d \times (\mathbf{R}_+)^m$. Recall that the functions $F_i : \mathcal{X} \rightarrow \mathcal{Y}$ are defined by (18) for $i = 1, \dots, m$. We consider two random probability measures $u \in \mathcal{U} \mapsto \mu(u, \cdot)$, $\mu'(u, \cdot)$ on \mathcal{X} given by

$$\mu(u, \cdot) := \delta_z \times \delta_{\mathbf{s}} \times \ell^m \quad \text{and} \quad \mu'(u, \cdot) := \delta_{z'} \times \delta_{\mathbf{s}} \times \ell^m$$

for $u = (z, z', \mathbf{s}) \in \mathcal{U}$, where δ_z is the Dirac measure at $z \in \mathbf{R}^d$ and $\delta_{\mathbf{s}}$ is the Dirac measure at $\mathbf{s} \in (\mathbf{R}_+)^m$. By Lemma C.1 applied to F_m , there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and measurable mappings $\xi, \xi' : \mathcal{U} \times \tilde{\Omega} \rightarrow \mathcal{X}$ such that

$$\xi(u, \cdot)_* \tilde{\mathbb{P}} = \delta_z \times \delta_{\mathbf{s}} \times \ell^m, \quad \xi'(u, \cdot)_* \tilde{\mathbb{P}} = \delta_{z'} \times \delta_{\mathbf{s}} \times \ell^m,$$

and

$$\tilde{\mathbb{P}}\{\tilde{\omega} : F_m(\xi(u, \tilde{\omega})) \neq F_m(\xi'(u, \tilde{\omega}))\} = \|F_m(z, \mathbf{s}, \cdot)_*(\ell^m) - F_m(z', \mathbf{s}, \cdot)_*(\ell^m)\|_{\text{var}} \quad (24)$$

for each $u = (z, z', \mathbf{s}) \in \mathcal{U}$. Replacing $\tilde{\Omega}$ with a bigger space (still referred to as $\tilde{\Omega}$) if necessary, we may find a third measurable mapping $\xi'' : \mathcal{U} \times \tilde{\Omega} \rightarrow \mathcal{X}$ with the same

²The radius R of this compact set will be chosen to suitably fit the Lyapunov structure; cf. Corollary A.2.

distribution as ξ' , but independent from ξ .³ We set

$$\mathcal{R}_i(z, z', \mathbf{s}, \tilde{\omega}) := F_i(\xi(z, z', \mathbf{s}, \tilde{\omega}))$$

and

$$\mathcal{R}'_i(z, z', \mathbf{s}, \tilde{\omega}) := \begin{cases} F_i(\xi(z, z', \mathbf{s}, \tilde{\omega})) & \text{if } z = z', \\ F_i(\xi'(z, z', \mathbf{s}, \tilde{\omega})) & \text{if } z \neq z' \text{ both in } B(\hat{x}, r), \\ F_i(\xi''(z, z', \mathbf{s}, \tilde{\omega})) & \text{if } z \neq z' \text{ not both in } B(\hat{x}, r) \end{cases}$$

for each $(z, z', \mathbf{s}, \tilde{\omega}) \in \mathbf{R}^d \times \mathbf{R}^d \times (\mathbf{R}_+)^m \times \tilde{\Omega}$ and $i = 1, \dots, m$. Now, let \mathcal{E}_λ^m be the m -fold direct product of exponential laws with rate parameter λ . We denote by $(\Omega, \mathcal{F}, \mathbb{P}_{(x, x')})$ the direct product of the probability space $(\mathbf{R}^d \times \mathbf{R}^d, \mathcal{B}(\mathbf{R}^d) \times \mathcal{B}(\mathbf{R}^d), \delta_x \times \delta_{x'})$ with countably many copies of the probability space

$$((\mathbf{R}_+)^m \times \tilde{\Omega}, \mathcal{B}((\mathbf{R}_+)^m) \times \tilde{\mathcal{F}}, \mathcal{E}_\lambda^m \times \tilde{\mathbb{P}}),$$

and define the process $(z_k(\omega), z'_k(\omega))_{k \in \mathbf{N}}$ inductively. First, set $(z_0(\omega), z'_0(\omega)) = (y, y')$ where $\omega = (y, y', \omega_0, \omega_1, \dots) \in \Omega$ with $\omega_j = (\mathbf{s}_j, \tilde{\omega}_j) \in (\mathbf{R}_+)^m \times \tilde{\Omega}$, $j = 0, 1, 2, \dots$, and $i = 1, \dots, m$. Then,

$$\begin{aligned} z_{jm+i}(\omega) &:= \mathcal{R}_i(z_{jm}(\omega), z'_{jm}(\omega), \mathbf{s}_j, \tilde{\omega}_j), \\ z'_{jm+i}(\omega) &:= \mathcal{R}'_i(z_{jm}(\omega), z'_{jm}(\omega), \mathbf{s}_j, \tilde{\omega}_j). \end{aligned}$$

By construction, the pair (z_k, z'_k) , $k \in \mathbf{N}$ is a coupling for the embedded process:

$$\mathbb{P}_{(x, x')} \{\omega \in \Omega : z_k \in \Gamma\} = \hat{P}(x, \Gamma) \quad \text{and} \quad \mathbb{P}_{(x, x')} \{\omega \in \Omega : z'_k \in \Gamma\} = \hat{P}(x', \Gamma) \quad (25)$$

for all measurable $\Gamma \subseteq \mathbf{R}^d$.

We now state and prove two important properties of the constructed coupling. The first one relies on (C3) and elucidates the choice of a construction by blocks of m steps with m as in Lemma 2.3. The second combines this first property and some technical consequences of Conditions (C1) and (C2) proved in Appendix A to establish an estimate on the time K needed for the coupling to hit the diagonal, i.e. for the two coupled components to coincide; see (23). This will be crucial in the proof of the Main Theorem.

Proposition 3.1. *There is a number $\hat{p} \in (0, 1)$ such that*

$$\mathbb{P}_{(x, x')} \{z_m \neq z'_m\} < \hat{p} \quad (26)$$

for all $x, x' \in B(\hat{x}, r)$.

Proof. With Σ as in and Lemma 2.3, the equality (24) gives

$$\begin{aligned} &(\mathcal{E}_\lambda^m \times \tilde{\mathbb{P}}) \{(\mathbf{s}, \tilde{\omega}) : F_m(\xi(x, x', \mathbf{s}, \tilde{\omega})) \neq F_m(\xi'(x, x', \mathbf{s}, \tilde{\omega}))\} \\ &\leq \mathcal{E}_\lambda^m(\Sigma) \sup_{\mathbf{s} \in \Sigma} \tilde{\mathbb{P}} \{\tilde{\omega} : F_m(\xi(x, x', \mathbf{s}, \tilde{\omega})) \neq F_m(\xi'(x, x', \mathbf{s}, \tilde{\omega}))\} + (1 - \mathcal{E}_\lambda^m(\Sigma)) \\ &= \mathcal{E}_\lambda^m(\Sigma) \sup_{\mathbf{s} \in \Sigma} \|F_m(x, \mathbf{s}, \cdot)_*(\ell^m) - F_m(x', \mathbf{s}, \cdot)_*(\ell^m)\|_{\text{var}} + (1 - \mathcal{E}_\lambda^m(\Sigma)) \end{aligned}$$

³For example, one can take as a new $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ the product of the old $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with itself and set $\xi_{\text{new}}(u, \tilde{\omega}_1, \tilde{\omega}_2) = \xi_{\text{old}}(u, \tilde{\omega}_1)$, $\xi'_{\text{new}}(u, \tilde{\omega}_1, \tilde{\omega}_2) = \xi'_{\text{old}}(u, \tilde{\omega}_1)$ and $\xi''_{\text{new}}(u, \tilde{\omega}_1, \tilde{\omega}_2) = \xi'_{\text{old}}(u, \tilde{\omega}_2)$ where $(\tilde{\omega}_1, \tilde{\omega}_2)$ is a generic element of the product of the old space with itself.

whenever x and x' are in the ball $B(\hat{x}, r)$. Therefore,

$$\mathbb{P}_{(x, x')} \{z_m \neq z'_m\} \leq 1 - \mathcal{E}_\lambda^m(\Sigma)(1 - p) =: \hat{p}$$

by Lemma 2.3. \square

Proposition 3.2. *There are positive constants θ_1 and A_1 such that*

$$\mathbb{E}_{(x, x')} e^{\theta_1 K} \leq A_1 (1 + \|x\| + \|x'\|) \quad (27)$$

for all $x, x' \in \mathbf{R}^d$.

Proof. Under Condition (C1), $(x, x') \mapsto 1 + \|x\|^2 + \|x'\|^2$ is a Lyapunov function for the coupling $(z_k, z'_k)_{k \in \mathbf{N}}$. As a consequence of this, we control an exponential moment of the number I of jumps needed to enter a ball of large radius R around the origin (see Corollary A.2). On the other hand, Condition (C2) guarantees the existence of a number $M \in \mathbf{N}_m$ of jumps in which transition probabilities from points in $B(0, R)$ to the ball $B(\hat{x}, r)$ are uniformly bounded from below (see Lemma A.5).

Combining these results, we get the following bound on an exponential moment of the first simultaneous hitting time of the ball $B(\hat{x}, r)$: there exist positive constants θ_2 and A_2 such that

$$\mathbb{E}_{(x, x')} e^{\theta_2 J} \leq A_2 (1 + \|x\|^2 + \|x'\|^2). \quad (28)$$

This is stated and proved as Proposition A.6 in the first appendix. Then, we introduce a sequence of random times defined inductively by $J_0 := 0$ and

$$J_i := \min \{j \in \mathbf{N}_m : z_j, z'_j \in B(\hat{x}, r) \text{ and } j > J_{i-1}\}$$

for $i \geq 1$. Using the strong Markov property and applying the inequality (28) repeatedly gives

$$\mathbb{E}_{(x, x')} e^{\theta_2 J_i} \leq \mathbb{E} \left(e^{\theta_2 J_{i-1}} \mathbb{E}_{(z_{J_{i-1}}, z'_{J_{i-1}})} e^{\theta_2 J_1} \right) \leq \hat{C}^i (1 + \|x\|^2 + \|x'\|^2) \quad (29)$$

for some positive constant \hat{C} .

Note that Proposition 3.1 implies that K is almost surely finite for all $x, x' \in \mathbf{R}^d$. Indeed,

$$\begin{aligned} \mathbb{P}_{(x, x')} \{K > J_i\} &\leq \mathbb{P}_{(x, x')} \{z_{J_i+m} \neq z'_{J_i+m}\} \\ &= \mathbb{P}_{(x, x')} (\{z_{J_i+m} \neq z'_{J_i+m}\} \mid \{z_{J_i} \neq z'_{J_i}\}) \mathbb{P}_{(x, x')} \{z_{J_i} \neq z'_{J_i}\} \\ &\leq \hat{p} \mathbb{P}_{(x, x')} \{z_{J_i} \neq z'_{J_i}\} \\ &\leq \hat{p} \mathbb{P}_{(x, x')} \{z_{J_{i-1}+m} \neq z'_{J_{i-1}+m}\} \\ &\leq \hat{p}^i \end{aligned} \quad (30)$$

and almost-sure finiteness follows from the Borel–Cantelli lemma. Now, by Hölder’s inequality,

$$\begin{aligned} \mathbb{E}_{(x, x')} e^{\theta_1 K} &\leq 1 + \sum_{i=0}^{\infty} \mathbb{E}_{(x, x')} (\mathbf{1}_{\{J_i < K \leq J_{i+1}\}} e^{\theta_1 J_{i+1}}) \\ &\leq 1 + \sum_{i=0}^{\infty} (\mathbb{P}_{(x, x')} \{K > J_i\})^{1-\frac{1}{q}} (\mathbb{E}_{(x, x')} e^{q\theta_1 J_{i+1}})^{\frac{1}{q}} \end{aligned}$$

for any $q \geq 1$. In each summand, the first term is controlled by the inequality (30) and the second one by (29), provided that $\theta_1 \leq \theta_2/q$:

$$\mathbb{E}_{(x,x')} e^{\theta_1 K} \leq 1 + \hat{C}^{\frac{1}{q}} \hat{p}^{\frac{1}{q}-1} (1 + \|x\|^2 + \|x'\|^2)^{\frac{1}{q}} \sum_{i=0}^{\infty} \left(\hat{C}^{\frac{1}{q}} \hat{p}^{1-\frac{1}{q}} \right)^i.$$

The proposition follows by taking $q \geq 2$ large enough that $\hat{C}^{\frac{1}{q}} \hat{p}^{1-\frac{1}{q}} < 1$. \square

3.2. Coupling for the original continuous-time process

Let the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{(x,x')})$ and the process (z_k, z'_k) be as in the previous subsection. Recall that an element ω of Ω consists in an initial condition in $\mathbf{R}^d \times \mathbf{R}^d$ and a sequence $(\mathbf{s}_j, \tilde{\omega}_j)_{j \in \mathbf{N}}$ of elements in $(\mathbf{R}_+)^m \times \tilde{\Omega}$ for some other probability space $\tilde{\Omega}$ we have constructed. Let $\tau_{jm+i}(\omega)$ be the positive real obtained by summing all the entries of $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_j$ and the first i entries of \mathbf{s}_{j+1} . Then, it follows from the construction of $\mathbb{P}_{(x,x')}$ that the sequence $(\tau_k)_{k \in \mathbf{N}}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P}_{(x,x')})$ has independent increments distributed according to an exponential distribution with rate parameter λ .

We define

$$Z_t(\omega) := \begin{cases} z_k(\omega) & \text{if } t = \tau_k(\omega), \\ S_{t-\tau_k(\omega)}(z_k(\omega)) & \text{if } t \in (\tau_k(\omega), \tau_{k+1}(\omega)) \end{cases}$$

and

$$Z'_t(\omega) := \begin{cases} z'_k(\omega) & \text{if } t = \tau_k(\omega), \\ S_{t-\tau_k(\omega)}(z'_k(\omega)) & \text{if } t \in (\tau_k(\omega), \tau_{k+1}(\omega)). \end{cases}$$

Then, (9), (10) and (25) imply that (Z_t, Z'_t) is a coupling of X_t and X'_t .

Proposition 3.3. *Under Conditions (C1)–(C3), there exist positive constants C and c such that*

$$\mathbb{P}_{(x,x')} \{\mathcal{T} > t\} \leq C(1 + \|x\| + \|x'\|)e^{-ct} \quad (31)$$

for any $x, x' \in \mathbf{R}^d$ and $t \geq 0$.

Proof. Let K be defined by (23). As τ_k is a sum of k independent exponentially distributed random variables with parameter λ , the expectation of $e^{2c\tau_k}$ can be computed explicitly for c in the interval $(0, \frac{1}{2}\lambda)$, and τ_K is also almost-surely finite. For such a number c , the Cauchy–Schwarz inequality yields

$$\mathbb{E}_{(x,x')} e^{c\tau_K} = \sum_{k=0}^{\infty} \mathbb{E}_{(x,x')} (e^{c\tau_k} \mathbf{1}_{\{K=k\}}) \leq \sum_{k=0}^{\infty} (\mathbb{E}_{(x,x')} e^{2c\tau_k})^{\frac{1}{2}} (\mathbb{P}_{(x,x')} \{K=k\})^{\frac{1}{2}}.$$

On the other hand, we control $\mathbb{P}_{(x,x')} \{K \geq k\}$ by Proposition 3.2 and Chebyshev's inequality. Therefore,

$$\begin{aligned} \mathbb{E}_{(x,x')} e^{c\tau_K} &\leq \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda - 2c} \right)^{\frac{k}{2}} (e^{-\theta_1 k} A_1 (1 + \|x\| + \|x'\|))^{\frac{1}{2}} \\ &\leq A_1^{\frac{1}{2}} (1 + \|x\| + \|x'\|) \sum_{k=0}^{\infty} \left(\frac{\lambda e^{-\theta_1}}{\lambda - 2c} \right)^{\frac{k}{2}}, \end{aligned}$$

where θ_1 and A_1 are as in Proposition 3.2. The series will converge for $c > 0$ small enough; fix such a value of c . By Chebyshev's inequality, we find $C > 0$ such that

$$\mathbb{P}_{(x,x')}\{\tau_K > t\} \leq C(1 + \|x\| + \|x'\|)e^{-ct}$$

for all $x, x' \in \mathbf{R}^d$. By construction, we have $\mathcal{T} \leq \tau_K$ almost surely and therefore

$$\mathbb{P}_{(x,x')}\{\mathcal{T} > t\} \leq C(1 + \|x\| + \|x'\|)e^{-ct}.$$

This completes the proof of the proposition. \square

3.3. Concluding the proof of the Main Theorem

In view of Lemma 2.2, if we can find constants $C > 0$ and $c > 0$ such that

$$\|\mathfrak{P}_t^* \delta_x - \mathfrak{P}_t^* \delta_{x'}\|_{\text{var}} \leq C(1 + \|x\| + \|x'\|)e^{-ct}$$

for all $x, x' \in \mathbf{R}^d$ and all $t \geq 0$, then integrating in x against μ and in x' against μ^{inv} gives the desired bound (7) with a different constant C . By construction of the coupling $(Z_t, Z'_t)_{t \geq 0}$, we have

$$(\mathfrak{P}_t g)(x) - (\mathfrak{P}_t g)(x') = \mathbb{E}_{(x,x')} (g(Z_t) - g(Z'_t))$$

for all $g \in L^\infty(\mathbf{R}^d)$. Therefore,

$$\begin{aligned} \|\mathfrak{P}_t^* \delta_x - \mathfrak{P}_t^* \delta_{x'}\|_{\text{var}} &= \frac{1}{2} \sup_{\|g\|_\infty \leq 1} |(\mathfrak{P}_t g)(x) - (\mathfrak{P}_t g)(x')| \\ &\leq \frac{1}{2} \sup_{\|g\|_\infty \leq 1} \mathbb{E}_{(x,x')} |g(Z_t) - g(Z'_t)| \\ &= \frac{1}{2} \sup_{\|g\|_\infty \leq 1} \mathbb{E}_{(x,x')} \{ \mathbf{1}_{\{Z_t \neq Z'_t\}} |g(Z_t) - g(Z'_t)| \} \\ &\leq \mathbb{P}_{(x,x')}\{Z_t \neq Z'_t\} \leq \mathbb{P}_{(x,x')}\{\mathcal{T} > t\} \end{aligned}$$

for all $x, x' \in \mathbf{R}^d$ and $t \geq 0$, and the result follows from Proposition 3.3.

4. Applications

In this section, we apply the Main Theorem to the Galerkin approximations of PDEs and to stochastically driven quasi-harmonic networks. For the Galerkin approximations we give a detailed derivation of the controllability conditions and in the case of the networks we appeal to the results obtained in [30]. Before we do so, we briefly discuss the solid controllability assumption (C3).

4.1. Criteria for solid controllability

The notion of solid controllability was introduced by Agrachev and Sarychev in [2] (see also the survey [3]) in the context of the controllability of the 2D Navier–Stokes and Euler systems. It has been used in [1] to prove the existence of density for finite-dimensional projections of the laws of the solutions of randomly forced PDEs. In [34], solid controllability is used to establish exponential mixing for some random dynamical systems in a compact space, and in [30], for some classes of quasi-harmonic networks of oscillators driven by a degenerate Brownian motion. It is the degeneracy allowed by this condition which sets our work apart from previous works on SDEs driven by compound Poisson processes (that are too numerous to be cited here).

We compare it to two related well-known properties, which might be more straightforward to check in some applications.

- (C3') *Continuous exact controllability from \hat{x}* : there exists a nondegenerate closed ball $D \subset \mathbf{R}^d$, a time $T_0 > 0$, and a continuous function $\Psi : D \rightarrow C([0, T_0]; \mathbf{R}^n)$ such that $S_{T_0}(\hat{x}, \Psi(x)) = x$ for all $x \in D$.
- (C3'') *Weak Hörmander condition at \hat{x}* : the vector space spanned by the family of vector fields

$$\{V_0, [V_1, V_2], [V_1, [V_2, V_3]], \dots : V_0 \in \mathbb{B}, V_1, V_2, \dots \in \mathbb{B} \cup \{f\}\} \quad (32)$$

at the point \hat{x} coincides with \mathbf{R}^d , where \mathbb{B} is the set of constant vector fields formed by the columns of the matrix B and $[U, V](x)$ is the *Lie bracket* of the vector fields U and V in the point x :

$$[U, V](x) = DV(x)U(x) - DU(x)V(x).$$

Here, $DU(x)$ is the Jacobian matrix of U at x .

It is shown in [34, §2.2] that (C3'') implies (C3') with arbitrary T_0 , and that (C3') in turn implies (C3) with the same T_0 ; see also [30, §3.2]. The first implication appeals to some ideas from geometric control theory. The second implication can be seen from a degree theory argument (or alternatively from an application of Brouwer's fixed point theorem).

The weak Hörmander condition, also known as the parabolic Hörmander condition, has many important applications both in control theory (e.g., see [19, Ch. 5]) and stochastic analysis (e.g., see [29, §2.3 in Ch. 2] and [15]). It is often assumed to hold in *all* points of the state space. For finite-dimensional control systems, it ensures the global exact controllability; for Itô diffusions, it guarantees existence and smoothness of the density of solutions with respect to the Lebesgue measure — a major step towards proving important ergodic properties. We emphasize that we bypass the study of smoothing properties of the transition function of our Markov process and that the conditions stated need only hold in *one* point of the state space (where Condition (C2) is also satisfied).

Recall that a pair of matrices, $A : \mathbf{R}^d \rightarrow \mathbf{R}^d$ and $B : \mathbf{R}^n \rightarrow \mathbf{R}^d$, is said to satisfy the Kalman condition if any $x \in \mathbf{R}^d$ can be written as $x = By_0 + AB y_1 + \dots + A^{d-1} B y_{d-1}$ for some $y_0, \dots, y_{d-1} \in \mathbf{R}^n$. For a linear control system of the form $\dot{X} = AX + B\zeta$, the Kalman condition implies (C3'') in all points through a straightforward computation of the Lie brackets; see [7, §1.2–1.3] for other well-known implications. When f is a linear vector field $x \mapsto Ax$ plus a perturbation, Condition (C3'') can be deduced at a point \hat{x}

far from the origin by perturbing the Kalman condition on the pair (A, B) , provided that one has good control on the decay of derivatives of the perturbation along a sequence of points [30, §5].

4.2. Galerkin approximations of randomly forced PDEs

In this section, we apply the Main Theorem to the Galerkin approximations of the following parabolic PDE on the torus $\mathbb{T}^D := \mathbf{R}^D/2\pi\mathbf{Z}^D$:

$$\partial_t u(t, x) - \nu \Delta_x u(t, x) + F(u(t, x)) = h(x) + \zeta(t, x), \quad x \in \mathbb{T}^D, \quad (33)$$

where $\nu > 0$ is a constant, $h : \mathbb{T}^D \rightarrow \mathbf{R}$ is a given smooth function, and $F : \mathbf{R} \rightarrow \mathbf{R}$ is a function of the form

$$F(u) = au^p + g(u). \quad (34)$$

We assume that $a > 0$ is an arbitrary constant, $p \geq 3$ is an odd integer, and $g : \mathbf{R} \rightarrow \mathbf{R}$ is a smooth function satisfying the following two conditions⁴:

(i) there is a constant $C > 0$ such that

$$|g(u)| \leq C(1 + |u|)^{p-1}$$

for all $u \in \mathbf{R}$.

(ii) with $g^{(p)}$ the p -th derivative of g , the following limit holds

$$\lim_{u \rightarrow \pm\infty} g^{(p)}(u) = 0.$$

For any $N \in \mathbf{N}$, consider the following finite-dimensional subspace of $L^2(\mathbb{T}^D)$:

$$H_N := \text{span}\{s_k, c_k : k \in \mathbf{Z}^D, |k| \leq N\},$$

where $s_k(x) := \sin\langle x, k \rangle$, $c_k(x) := \cos\langle x, k \rangle$, $\langle x, k \rangle := x_1 k_1 + \dots + x_D k_D$ and $|k| := |k_1| + \dots + |k_D|$ for any multi-index $k = (k_1, \dots, k_D) \in \mathbf{Z}^D$ and any vector $x \in \mathbb{T}^D$. In particular, c_0 is the constant function 1. This subspace is endowed with the scalar product $\langle \cdot, \cdot \rangle_{L^2}$ and the norm $\|\cdot\|_{L^2}$ inherited from $L^2(\mathbb{T}^D)$. Let \mathbf{P}_N be the orthogonal projection onto H_N in $L^2(\mathbb{T}^D)$. The Galerkin approximations of (33) are given by

$$\dot{u}(t) - \nu \Delta u(t) + \mathbf{P}_N F(u(t)) = h + \zeta(t), \quad (35)$$

where u is an unknown H_N -valued function, h is an arbitrary vector in H_N and ζ is a continuous H_1 -valued function.

Let us emphasize that the space H_1 for the driving ζ is the same for any level $N \geq 1$ of approximation, any value of the constant ν and any function g satisfying (i) and (ii).

The main interest of the example considered in this section is that the perturbation term g in (34) is quite general. In particular, we may have $F(u) = 0$ in a large ball, so that the weak Hörmander condition is not necessarily satisfied at all the points of the state space.

⁴The results of this subsection remain true under weaker assumptions on the function g . This setting is chosen for the simplicity of presentation.

Theorem 4.1. Suppose that (i) and (ii) hold. Let $(Y_t)_{t \geq 0}$ be an H_1 -valued compound Poisson with jump distribution ℓ of finite variance and possessing a positive continuous density with respect to the Lebesgue measure on H_1 . Then, the semigroup $(\mathfrak{P}_t^*)_{t \geq 0}$ for the SDE

$$du - \nu \Delta u \, dt + P_N F(u) \, dt = h \, dt + dY$$

in H_N admits a unique invariant measure $\mu^{\text{inv}} \in \mathcal{P}(H_N)$. Moreover, it is exponentially mixing in the sense that (7) holds for some constants $C > 0$ and $c > 0$, any measure $\mu \in \mathcal{P}(H_N)$, and any time $t \geq 0$.

Proof. The SDE under consideration is of the form (1) with $d = \dim H_N$, $n = \dim H_1 = 2D + 1$, a smooth function $f_N : H_N \rightarrow H_N$ given by

$$f_N(u) = \nu \Delta u - P_N F(u) + h, \quad (36)$$

and $B : H_1 \rightarrow H_N$ the natural embedding operator. Let us show that Conditions (C1)–(C3) are verified. Using the assumption (i), the fact that s_k and c_k are eigenfunctions of the Laplacian, and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \langle f(u), u \rangle_{L^2} &= \langle \nu \Delta u - P_N F(u) + h, u \rangle_{L^2} \\ &\leq -\nu \int_{\mathbb{T}^D} |u(x)|^2 \, dx - C_1 \int_{\mathbb{T}^D} |u(x)|^{p+1} \, dx + C_2 \\ &\leq -\nu \|u\|_{L^2}^2 + C_2, \end{aligned}$$

where $C_1 > 0$ and $C_2 > 0$ are some constants and $u \in H_N$ is arbitrary. This implies Condition (C1).

Condition (C2) (to all points) is a consequence of the global approximate controllability property of Proposition 4.2 below, whose proof is given in Appendix B. Since it is proved in [34, §2.2] that the weak Hörmander condition implies solid controllability, Proposition 4.3 below yields Condition (C3).

Thus, Conditions (C1)–(C3) are satisfied and the proof of Theorem 4.1 is completed by applying our Main Theorem. \square

Proposition 4.2. Equation (35) is approximately controllable: for any number $\epsilon > 0$, any time $T > 0$, any initial condition $u_0 \in H_N$, and any target $\hat{u} \in H_N$, there exists a control $\zeta \in C([0, T]; H_1)$ such that the solution u of (35) with $u(0) = u_0$ satisfies

$$\|u(T) - \hat{u}\|_{L^2} < \epsilon.$$

Proposition 4.3. There is a number $R > 0$ such that the weak Hörmander Condition (C3'') is satisfied for equation (35) at any point $\hat{u} \in H_N$ with $\|\hat{u}\|_{L^2} \geq R$.

Proof of Proposition 4.3. In view of the weak Hörmander condition, we are interested in the nested subspaces $\{\mathcal{V}_i\}_{i \geq 0}$ of H_N defined by $\mathcal{V}_0 = H_1$ and

$$\mathcal{V}_{i+1}(\hat{u}) := \text{span}(\mathcal{V}_i \cup \{[V, f_N](\hat{u}) : V \in \mathcal{V}_i(\hat{u})\}),$$

where we at times identify the vector $V \in \mathcal{V}_i(\hat{u})$ with the corresponding constant vector field on H_N . Clearly, showing that $\mathcal{V}_i(\hat{u}) = H_N$ for some i large enough shows that

the weak Hörmander condition (C3'') holds in \hat{u} . We show in two steps that, indeed, $\mathcal{V}_{(N-1)p}(\hat{u}) = H_N$ if $\|\hat{u}\|_{L^2}$ is sufficiently large.

Step 1: Polynomial nonlinearity. In this step, we assume that $g \equiv 0$, so that

$$f_N(u) = \nu \Delta u - a P_N(u^p) + h. \quad (37)$$

In this case, Lie brackets with constant vector fields are especially straightforward to compute because Δ is a linear operator and h is a constant vector. In particular, for any constant vector fields $V_1, \dots, V_{p-2}, V_{p-1}$ and V_p ,

$$[V_1, \dots [V_{p-2}, [V_{p-1}, [V_p, f_N]]] \dots](\hat{u}) = -a p! P_N(V_1 \cdots V_{p-2} V_{p-1} V_p), \quad (38)$$

where the product $V_1 \cdots V_{p-2} V_{p-1} V_p$ is understood as a pointwise multiplication of functions.

We claim that, for each multi-index m with $0 < |m| \leq N$, the vectors c_m and s_m are in $\mathcal{V}_{(|m|-1)p}(\hat{u})$ for all $\hat{u} \in H_N$. To start, note that if $|l| \leq 1$, then c_l and s_l are in H_1 and thus in $\mathcal{V}_i(\hat{u})$ for each i .

Suppose now that c_m and s_m are in $\mathcal{V}_{(|m|-1)p}(\hat{u})$. As noted above, for all multi-indices l with $|l| \leq 1$, the vectors c_l and s_l are also in $\mathcal{V}_{(|m|-1)p}(\hat{u})$. Therefore, combining the computation (38) with trigonometric identities yields that

$$\begin{aligned} P_N c_{m \pm l} &= P_N(1 \cdots 1 c_l c_m) \mp P_N(1 \cdots 1 s_l s_m) \\ &= \frac{-1}{a p!} [c_0, \dots [c_0, [c_l, [c_m, f_N]]] \dots](\hat{u}) \pm \frac{1}{a p!} [c_0, \dots [c_0, [s_l, [s_m, f_N]]] \dots](\hat{u}) \end{aligned} \quad (39)$$

and

$$\begin{aligned} P_N s_{m \pm l} &= P_N(1 \cdots 1 s_l c_m) \pm P_N(1 \cdots 1 c_l s_m) \\ &= \frac{-1}{a p!} [c_0, \dots [c_0, [s_l, [c_m, f_N]]] \dots](\hat{u}) \pm \frac{-1}{a p!} [c_0, \dots [c_0, [c_l, [s_m, f_N]]] \dots](\hat{u}) \end{aligned} \quad (40)$$

are in $\mathcal{V}_{(|m|-1)p+p}(\hat{u})$. The result thus holds by induction on $|m|$.

Step 2: The General case. Let \tilde{f}_N be the vector field given by (37). If we consider the same Lie brackets as in Step 1, but now for the sum $\tilde{f}_N + P_N g$, the contribution of $P_N g$ will vanish as $\hat{u} \rightarrow \infty$, thanks to assumption (ii). Therefore, $\mathcal{V}_{(|N|-1)p}(\hat{u}) = H_N$, provided that $\|\hat{u}\|_{L^2}$ is sufficiently large. \square

4.3. Stochastically driven networks of quasi-harmonic oscillators

Stochastically driven networks of oscillators play an important role in the investigation of various aspects of nonequilibrium statistical mechanics. In its simplest form, the setup can be described as follows. Consider L unit masses, each labelled by an index $i \in \{1, \dots, L\}$ restricted to move in one dimension. Each of them is pinned by a spring of unit spring constant and, for $i \neq L$, the i th mass is connected to the $(i+1)$ th mass by a spring of unit spring constant. The equations of motion for the positions and momenta, $(q_i, p_i)_{i=1}^L$, are the Hamilton equations

$$\begin{cases} dq_i = p_i dt, & dp_i = -(3q_i - q_{i-1} - q_{i+1}) dt, & 1 < i < L, \\ dq_1 = p_1 dt, & dp_1 = -(2q_1 - q_2) dt, \\ dq_L = p_L dt, & dp_L = -(2q_L - q_{L-1}) dt. \end{cases}$$

Coupling the 1st [resp. the L th] oscillator to a fluctuating bath with dissipation constant γ_1 [resp. γ_L] leads to the SDE

$$\begin{cases} dq_i = p_i dt, & dp_i = -(3q_i - q_{i-1} - q_{i+1}) dt, & 1 < i < L, \\ dq_1 = p_1 dt, & dp_1 = -(2q_1 - q_2) dt - \gamma_1 p dt + dZ_{1,t}, \\ dq_L = p_L dt, & dp_L = -(2q_L - q_{L-1}) dt - \gamma_L p dt + dZ_{L,t}, \end{cases} \quad (41)$$

or variants thereof, where Z_1 and Z_2 are independent one-dimensional stochastic processes describing the fluctuations in the baths.

In the mathematical physics literature, many authors have considered nonlinear variants of this model where the thermal fluctuations—either acting on the momenta (the Langevin regime, as above) or on auxiliary degrees of freedom—are described by Gaussian white noise i.e. $Z_{j,t} = \sqrt{2\gamma_j\theta_j}W_{j,t}$, with $W_{j,t}$ a standard Wiener process. We refer the interested reader to [13, 38] for introductions to these models and discussions of their ergodic properties at thermal equilibrium; also see [16, 17] for a generalization to non-Markovian models. The existence and uniqueness of the invariant measure is much more problematic out of equilibrium; see [36, 12, 11, 10, 31, 8]. However, interesting phenomena pointed out in the physics literature for a single particle in a non-Gaussian bath [5, 37, 26, 25] motivate a rigorous study of the mixing properties of corresponding networks. While the methods used for most of the previously cited existence and uniqueness results are not suitable to deal with compound Poisson processes, most of the ideas of [34, 30] are. We develop the strategy to be followed in the present section.

Allowing for different spring constants and different ways of connecting the masses while staying in the Langevin regime leads us to considering the following generalization of (41). Let I be a finite set and distinguish a nonempty subset $J \subset I$, where masses will be coupled to fluctuating baths. We use $\{\delta_i\}_{i \in I}$ [resp. $\{\delta_j\}_{j \in J}$] as the standard basis for \mathbf{R}^I [resp. \mathbf{R}^J]. Let $\omega : \mathbf{R}^I \rightarrow \mathbf{R}^I$ be a nonsingular linear map and let $\iota_j : \mathbf{R}^J \rightarrow \mathbf{R}^I$ be the rank-one map $\delta_j \langle \delta_j, \cdot \rangle$ for each $j \in J \subset I$. The SDE

$$d \begin{pmatrix} p \\ \omega q \end{pmatrix} = \begin{pmatrix} -\sum_{j \in J} \gamma_j \iota_j \iota_j^* & -\omega^* \\ \omega & 0 \end{pmatrix} \begin{pmatrix} p \\ \omega q \end{pmatrix} dt + \sum_{j \in J} \begin{pmatrix} \iota_j \\ 0 \end{pmatrix} dZ_j$$

in $\mathbf{R}^{2|I|}$ then describes the positions q and momenta p of $|I|$ masses connected to each other and pinned according to the matrix ω , with the j th oscillator being coupled to a Langevin bath with dissipation controlled by the constant $\gamma_j > 0$ and fluctuations described by the process Z_j .

In Proposition 4.4 and Corollary 4.5, we consider a nonlinear version of this SDE where the quadratic potential resulting from the springs is now perturbed by a potential $U : \mathbf{R}^d \rightarrow \mathbf{R}$. Their proofs are omitted since they are essentially the same as those of Proposition 4.6 and Corollary 4.7 respectively. We start with dissipativity and controllability properties of the control system.

Proposition 4.4. *Let I, J, ω and $(\gamma_j)_{j \in J}$ be as above. Then, the conditions*

- (K) *the pair $(\omega^* \omega, \sum_{j \in J} \gamma_j \iota_j \iota_j^*)$ satisfies the Kalman condition;*
- (G) *the gradient of U is a smooth globally Lipschitz vector field growing strictly slower than $q \mapsto 1 + |q|^{\frac{1}{4|I|}}$;*

(pH) there exists a sequence $\{q^{(n)}\}_{n \in \mathbf{N}}$ of points in \mathbf{R}^I , bounded away from 0, such that

$$\lim_{n \rightarrow \infty} |q^{(n)}|^k \|D^{k+1}U(q^{(n)})\| = 0$$

for each $k = 0, 1, \dots, d-1$;

imply that the control system

$$\begin{pmatrix} \dot{p} \\ \omega \dot{q} \end{pmatrix} = \begin{pmatrix} -\sum_{j \in J} \gamma_j \iota_j \iota_j^* & -\omega^* \\ \omega & 0 \end{pmatrix} \begin{pmatrix} p \\ \omega q \end{pmatrix} - \begin{pmatrix} \nabla U(q) \\ 0 \end{pmatrix} + \sum_{j \in J} \begin{pmatrix} \iota_j \\ 0 \end{pmatrix} \zeta$$

satisfies the conditions (C1), (C2) and (C3).

The exponent in the formulation of the growth condition is typically not optimal; see [30] for a formulation in terms of a power related to the Kalman condition. The following mixing result for the corresponding SDE with Poissonian noise essentially follows from our Main Theorem (see the proof of Corollary 4.7).

Corollary 4.5. *Under the same assumptions, if $(N_j)_{j \in J}$ is a collection of $|J|$ independent one-dimensional compound Poisson processes with jump distributions with finite variance and continuous positive densities with respect to the Lebesgue measure on \mathbf{R} , then the SDE*

$$d \begin{pmatrix} p \\ \omega q \end{pmatrix} = \begin{pmatrix} -\sum_{j \in J} \gamma_j \iota_j \iota_j^* & -\omega^* \\ \omega & 0 \end{pmatrix} \begin{pmatrix} p \\ \omega q \end{pmatrix} dt - \begin{pmatrix} \nabla U(q) \\ 0 \end{pmatrix} dt + \sum_{j \in J} \begin{pmatrix} \iota_j \\ 0 \end{pmatrix} \delta_j dN_j$$

admits a unique stationary measure $\mu^{\text{inv}} \in \mathcal{P}(\mathbf{R}^I \oplus \mathbf{R}^I)$. Moreover, it is exponentially mixing in the sense that (7) holds for some constants $C > 0$ and $c > 0$, any measure $\mu \in \mathcal{P}(\mathbf{R}^I \oplus \mathbf{R}^I)$, and any time $t \geq 0$.

In addition to the notation used so far, let $(\lambda_j)_{j \in J}$ be small positive numbers and let us use the shorthand $\gamma \iota^*$ for $\sum_j \gamma_j \iota_j \iota_j^*$, the shorthand $\lambda \iota^* \iota$ for $\sum_j \lambda_j \iota_j^* \iota_j$, and so on. The SDE

$$d \begin{pmatrix} r \\ p \\ \tilde{\omega} q \end{pmatrix} = \begin{pmatrix} -\gamma \iota^* & \lambda \iota^* & 0 \\ -\lambda \iota^* \iota & 0 & -\tilde{\omega}^* \\ 0 & \tilde{\omega} & 0 \end{pmatrix} \begin{pmatrix} r \\ p \\ \tilde{\omega} q \end{pmatrix} dt + \begin{pmatrix} \sqrt{2\gamma\theta} \iota^* \iota \\ 0 \\ 0 \end{pmatrix} dW$$

can be derived as the effective equation for the positions q and momenta p of a network of $|I|$ masses connected to each other and pinned according to the matrix ω , with the j th oscillator being coupled to a classical Gaussian field at temperature θ_j under some particular conditions on the coupling; see [12]. The $|J|$ auxiliary degrees of freedom $r \in \mathbf{R}^J$ are introduced to make the process Markovian. The parameters λ_j and γ_j describe the coupling and dissipation for the j th bath. Here, the matrix $\tilde{\omega}$ encodes an effective quadratic potential and is such that $\tilde{\omega}^* \tilde{\omega} = \omega^* \omega - \lambda^2 \iota^*$ (λ is small), where ω encodes the original quadratic potential.

Proposition 4.6. *Let I, J, ω and $(\gamma_j)_{j \in J}$ be as above. Then, for $(\lambda_j)_{j \in J}$ small enough, the conditions (K), (G) and (pH) as in the previous proposition imply that the control*

system

$$\begin{pmatrix} \dot{r} \\ \dot{p} \\ \dot{\tilde{\omega}q} \end{pmatrix} = \begin{pmatrix} -\gamma\iota^* & \lambda\iota^* & 0 \\ -\lambda\iota^*\iota & 0 & -\tilde{\omega}^* \\ 0 & \tilde{\omega} & 0 \end{pmatrix} \begin{pmatrix} r \\ p \\ \tilde{\omega}q \end{pmatrix} - \begin{pmatrix} 0 \\ \nabla U(q) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix} \zeta$$

satisfies the conditions (C1), (C2) and (C3).

Proof. The Kalman condition on the pair $(\omega^*\omega, \iota^*)$ implies the Kalman condition on the pair $(\tilde{\omega}^*\tilde{\omega}, \iota^*)$ if λ is small enough. This in turn implies that the pair

$$(A, B) := \left(\begin{pmatrix} -\gamma\iota^* & \lambda\iota^* & 0 \\ -\lambda\iota^*\iota & 0 & -\tilde{\omega}^* \\ 0 & \tilde{\omega} & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix} \right)$$

also satisfies the Kalman condition; see Proposition 4.1 in [30]. It follows by Lemma 5.1(2) in [18] that the eigenvalues of A then have strictly negative real part. Combined with the growth assumption (G), the negativity of the eigenvalues implies (C1) for a suitable inner product; see Lemma 3.1 in [30]. Proposition 3.3 in [30] says that the Kalman condition on (A, B) and the growth condition (G) on ∇U give (C2) everywhere. The fact that the Kalman condition on (A, B) and assumption (pH) give the weak Hörmander condition (C3'') in one point is the content of Proposition 5.1 in [30]. But, as previously mentioned, the weak Hörmander condition implies solid controllability. \square

Concerning the corresponding SDE with Poissonian noise, we have the following mixing result — which again parallels that of [30] — as a corollary of the controllability properties.

Corollary 4.7. *Under the same assumptions, if $(N_j)_{j \in J}$ is a collection of $|J|$ independent one-dimensional compound Poisson processes with jump distributions with finite variances and continuous positive densities with respect to the Lebesgue measure on \mathbf{R} , then the SDE*

$$d \begin{pmatrix} r \\ p \\ \tilde{\omega}q \end{pmatrix} = \begin{pmatrix} -\gamma\iota^* & \lambda\iota^* & 0 \\ -\lambda\iota^*\iota & 0 & -\tilde{\omega}^* \\ 0 & \tilde{\omega} & 0 \end{pmatrix} \begin{pmatrix} r \\ p \\ \tilde{\omega}q \end{pmatrix} dt - \begin{pmatrix} 0 \\ \nabla U(q) \\ 0 \end{pmatrix} dt + \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix} \sum_{j \in J} \delta_j dN_j.$$

admits a unique stationary measure $\mu^{\text{inv}} \in \mathcal{P}(\mathbf{R}^J \oplus \mathbf{R}^I \oplus \mathbf{R}^I)$. Moreover, it is exponentially mixing in the sense that (7) holds for some constants $C > 0$ and $c > 0$, any $\mu \in \mathcal{P}(\mathbf{R}^J \oplus \mathbf{R}^I \oplus \mathbf{R}^I)$, and any time $t \geq 0$.

Proof sketch. If the noise $\sum_{j \in J} \delta_j N_j$ were replaced by a single compound Poisson process whose jump distribution possesses a finite second moment and a positive continuous density with respect to the Lebesgue measure on \mathbf{R}^J , then our Main Theorem would apply.

Although the probability that jumps in the different baths occur simultaneously is zero by independence, there is a positive probability that they occur arbitrarily close to simultaneity. Since an independent sum of a jump from each distribution gives a random variable with a finite variance and a positive continuous density with respect to the Lebesgue measure on \mathbf{R}^J , our control arguments can be adapted by continuity. \square

Appendix A. Exponential estimates on hitting times

In this appendix, we present results on hitting times for the coupling (z_k, z'_k) constructed in Subsection 3.1. Loosely speaking, estimates on the hitting times of a small ball near \hat{x} are obtained by combining a lower bound on the hitting time of a (large) compact around the origin and a lower bound on the probability of making a transition from the aforementioned compact to the small ball. We shall assume that Conditions (C1)–(C3) are satisfied and fix the parameters m, r , and p as in Lemma 2.3.

We provide an estimate for the first simultaneous hitting time I of a ball of large radius R around the origin. To do this, we use the preliminary estimates of Lemma 2.1 to exhibit the existence of a suitable Lyapunov structure and conclude with a standard argument.

Lemma A.1. *The function V defined by $V(y, y') := 1 + \|y\|^2 + \|y'\|^2$ is a Lyapunov function in the sense that there exist positive constants R and C_* and a constant $0 < a < 1$ such that*

$$\mathbb{E}_{(x, x')} V(z_m, z'_m) \leq a V(x, x') \quad \text{for } \|x\| \vee \|x'\| \geq R, \quad (\text{A.1})$$

$$\mathbb{E}_{(x, x')} V(z_k, z'_k) \leq C_* \quad \text{for } \|x\| \vee \|x'\| < R, k \geq 0. \quad (\text{A.2})$$

Proof. By Lemma 2.1, there is $\gamma \in (0, 1)$ such that

$$\begin{aligned} \mathbb{E}_{(x, x')} (1 + \|z_k\|^2 + \|z'_k\|^2) &= 1 + \mathbb{E}_x \|X_{\tau_k}\|^2 + \mathbb{E}_{x'} \|X_{\tau_k}\|^2 \\ &\leq 1 + \gamma^k (\|x\|^2 + \|x'\|^2) + 2C(1 + \Lambda) \end{aligned} \quad (\text{A.3})$$

for all $k \in \mathbf{N}$ and $x, x' \in \mathbf{R}^d$. Taking $k = m$, any $a \in (0, \gamma^m)$, and any $x, x' \in \mathbf{R}^d$ such that

$$\|x\| \vee \|x'\| \geq (a - \gamma^m)^{-1/2} (1 - a + 2C(1 + \Lambda))^{1/2} =: R,$$

we get

$$\mathbb{E}_{(x, x')} (1 + \|z_m\|^2 + \|z'_m\|^2) \leq a (1 + \|x\|^2 + \|x'\|^2).$$

Thus, (A.1) holds. In the case $\|x\| \vee \|x'\| \leq R$, by (A.3), we have

$$\mathbb{E}_{(x, x')} (1 + \|z_k\|^2 + \|z'_k\|^2) \leq 1 + 2R^2 + 2C(1 + \Lambda) =: C_*.$$

This gives (A.2) and completes the proof of the lemma. \square

It is well known that the Lyapunov structure of the previous lemma implies a bound on an exponential moment for the time needed to reach a large enough level set of the Lyapunov function V . While arguments for this implication can be found in [24], we give a brief proof sketch and refer the reader to Proposition 3.1 in [33] for a statement and complete proof which more precisely reflects our approach.

Corollary A.2. *There exist positive constants R , c_1 , and C_1 such that*

$$\mathbb{E}_{(x, x')} e^{c_1 I} \leq C_1 (1 + \|x\|^2 + \|x'\|^2)$$

for all $x, x' \in \mathbf{R}^d$, where

$$I := \min\{j \in \mathbf{N}_m^0 : z_j, z'_j \in B(0, R)\}$$

Proof sketch. One can show using the Markov property and (A.1) repeatedly that

$$\mathbf{E}_{(x,x')}[\mathbf{1}_{\{I > nm\}} V(z_{nm}, z'_{nm})] \leq a^n V(x, x')$$

and deduce using $V \geq 1$ that

$$\mathbf{P}_{(x,x')}[I > nm] \leq a^n V(x, x'). \quad (\text{A.4})$$

By (A.4) and the Borel–Cantelli lemma, I is almost surely finite. Therefore, one can use

$$\mathbf{E}_{(x,x')} e^{c_1 I} \leq 1 + \sum_{n=1}^{\infty} \mathbf{E}_{(x,x')}[\mathbf{1}_{\{I=nm\}} e^{c_1 I}]$$

and, for c_1 small enough, the right-hand side can be bounded using (A.4) in terms of $V(x, x')$ and a convergent geometric series. \square

In what follows R , c_1 and C_1 will be as in Corollary A.2. We continue with another estimate on an exponential moment.

Lemma A.3. *For any $M \in \mathbf{N}$, there is a constant $C_2 > 0$ such that*

$$\mathbb{E}_{(x,x')} e^{c_1 I_i} \leq C_2^i (1 + \|x\|^2 + \|x'\|^2) \quad (\text{A.5})$$

for all $x, x' \in \mathbf{R}^d$ and $i \in \mathbf{N}$, where $I_0 := 0$ and

$$I_i := \min \{j \in \mathbf{N}_m : j \geq I_{i-1} + M \text{ and } z_j, z'_j \in B(0, R)\}.$$

Remark A.4. The stopping time I_i depends on both M and R . The value of R was already fixed in Corollary A.2 and, in our application, M will be as in Lemma A.5. It is important that the constant C_2 does not depend on x and x' .

Proof. By our last corollary, the Markov property, and (12) in Lemma 2.1, we have

$$\begin{aligned} \mathbb{E}_{(x,x')} e^{c_1 I_1} &= e^{c_1 M} \mathbb{E}_{(x,x')} \left(\mathbb{E}_{(z_M, z'_M)} e^{c_1 I} \right) \\ &\leq C_1 e^{c_1 M} \mathbb{E}_{(x,x')} (1 + \|z_M\|^2 + \|z'_M\|^2) \\ &\leq C_1 e^{c_1 M} (1 + \gamma^M \|x\|^2 + \gamma^M \|x'\|^2 + 2C(1 + \Lambda)) \\ &\leq \tilde{C}_1 (1 + \|x\|^2 + \|x'\|^2) \end{aligned} \quad (\text{A.6})$$

for \tilde{C}_1 a combination of C , C_1 and Λ . In particular, for any $x, x' \in B(0, R)$,

$$\mathbb{E}_{(x,x')} e^{c_1 I_1} \leq \tilde{C}_1 (1 + R^2 + R^2) =: C_2.$$

Then $z_{I_{i-1}}, z'_{I_{i-1}} \in B(0, R)$ for any $i > 1$, and therefore

$$\mathbb{E}_{(x,x')} e^{c_1 I_i} = \mathbb{E}_{(x,x')} \left(e^{c_1 I_{i-1}} \mathbb{E}_{(z_{I_{i-1}}, z'_{I_{i-1}})} e^{c_1 I_1} \right) \leq C_2 \mathbb{E}_{(x,x')} e^{c_1 I_{i-1}} \leq C_2^{i-1} \mathbb{E}_{(x,x')} e^{c_1 I_1}.$$

Finally, using (A.6), we obtain (A.5). \square

Lemma A.5. *Consider the random variable*

$$J := \min \{j \in \mathbf{N}_m^0 : z_j, z'_j \in B(\hat{x}, r)\},$$

where \hat{x} is as in Condition (C2). There exists $M \in \mathbf{N}_m$ such that

$$0 < q := \inf_{x, x' \in B(0, R)} \mathbb{P}_{(x, x')} \{J \leq M\}. \quad (\text{A.7})$$

Proof. Let T be the time in Condition (C2) for $\epsilon = \frac{r}{2}$ and radius R . To simplify the presentation, we assume that $T = 1$.

Step 1: controlling a single trajectory of the SDE (1). First, let us show an inequality like (A.7) for a single trajectory of the SDE (1). Take an initial condition $x \in B(0, R)$. By Condition (C2), there exists a control $\zeta_x \in C([0, 1]; \mathbf{R}^n)$ such that

$$\|S(x, \zeta_x) - \hat{x}\| < \frac{r}{2}. \quad (\text{A.8})$$

By a standard continuity and compactness argument, we can find a finite set

$$Z := \{\zeta_i : i \in I\} \subset C([0, 1]; \mathbf{R}^n)$$

such that the control ζ_x in (A.8) can be chosen from Z for any $x \in B(0, R)$. For any integer $M \geq 1$, let the mapping $F_M : \mathbf{R}^d \times (\mathbf{R}_+)^M \times (\mathbf{R}^n)^M \rightarrow \mathbf{R}^d$ be defined by (18), let ι_M be as in Lemma 2.3, and consider the sets

$$\begin{aligned} \Delta &:= \left\{ \mathbf{s} = (s_j)_{j=1}^M \in (\mathbf{R}_+)^M : s_j \in \left(\frac{1-\delta}{M}, \frac{1}{M} \right), \quad j = 1, \dots, M \right\}, \\ \Xi_x &:= \left\{ \boldsymbol{\xi} = (\xi_j)_{j=1}^M \in (\mathbf{R}^n)^M : \|\iota_M(\zeta_x) - \boldsymbol{\xi}\|_{(\mathbf{R}^n)^M} < \delta, \quad j = 1, \dots, M \right\} \end{aligned}$$

for any $\delta > 0$. Again by a continuity and compactness argument, it is not hard to see that

$$\Delta \times \Xi_x \subset \{\mathbf{s} \in (\mathbf{R}_+)^M, \boldsymbol{\xi} \in (\mathbf{R}^n)^M : \|F_M(x, \mathbf{s}, \boldsymbol{\xi}) - \hat{x}\| < r\}$$

for sufficiently large $M \in \mathbf{N}_m$, small $\delta > 0$, and any $x \in B(0, R)$. Note that $F_M(x, \mathbf{s}, \boldsymbol{\xi}) = X_{\tau_M}$ when $\mathbf{s} = (t_j)_{j=1}^M$ and $\boldsymbol{\xi} = (\eta_j)_{j=1}^M$. By our assumptions on the laws of t_j and η_j , it is clear that⁵

$$\begin{aligned} \mathcal{E}_\lambda^M(\Delta) &= \prod_{j=1}^M \left(e^{-\lambda \frac{1-\delta}{M}} - e^{-\lambda \frac{1}{M}} \right) > 0, \\ \inf_{x \in B(0, R)} \ell^M(\Xi_x) &> 0, \end{aligned}$$

since there is only a finite number of sets Ξ_x for x in $B(0, R)$. We conclude that

$$0 < \inf_{x \in B(0, R)} \mathbb{P}_x \{\|X_{\tau_M} - \hat{x}\| < r\}. \quad (\text{A.9})$$

⁵Recall that \mathcal{E}_λ^M and ℓ^M stand for the M -fold products of the exponential distribution and ℓ , respectively.

Step 2: case of coupling trajectories. We consider three cases.

- If $x = x'$, then the trajectories z_j and z'_j coincide for all j and the result follows immediately from (A.9).
- If $x \neq x'$ with $x, x' \in B(\hat{x}, r)$, then

$$\mathbb{P}_{(x, x')} \{J = 0\} = 1.$$

• If $x \neq x'$ not both in $B(\hat{x}, r)$, consider $\mathbf{s} \in \Delta$, $\boldsymbol{\xi} \in \Xi_x$, and $\boldsymbol{\xi}' \in \Xi_{x'}$. By construction, both $F_M(x, \mathbf{s}, \boldsymbol{\xi})$ and $F_M(x', \mathbf{s}, \boldsymbol{\xi})$ lie in $B(\hat{x}, r)$. Then, there exists a minimal $k \in \mathbf{N}_m$ such that both $F_k(x, \mathbf{s}, \boldsymbol{\xi})$ and $F_k(x', \mathbf{s}, \boldsymbol{\xi})$ lie in $B(\hat{x}, r)$. Necessarily, k satisfies $k \leq M$. Therefore, the construction of the coupling⁶ implies that z_k, z'_k are guaranteed to be in $B(\hat{x}, r)$ for some $k \leq M$ for all $\omega = (x, x', (\mathbf{s}_j, \tilde{\omega}_j)_{j \in \mathbf{N}})$ such that $(\mathbf{s}_j)_{j=1}^{M/m}$ lies in Δ and such that $(\xi(x, x', \mathbf{s}_j, \tilde{\omega}_j))_{j=1}^{M/m}$ and $(\xi''(x, x', \mathbf{s}_j, \tilde{\omega}_j))_{j=1}^{M/m}$ lie respectively in Ξ_x and $\Xi_{x'}$. By construction,

$$\begin{aligned} \tilde{\mathbb{P}} \{\tilde{\omega}_j : \xi(x, x', \mathbf{s}_j, \tilde{\omega}_j) \in \Xi_x\} &= \ell^M(\Xi_x), \\ \tilde{\mathbb{P}} \{\tilde{\omega}_j : \xi''(x, x', \mathbf{s}_j, \tilde{\omega}_j) \in \Xi_{x'}\} &= \ell^M(\Xi_{x'}), \end{aligned}$$

and

$$\mathcal{E}_\lambda^M(\Delta) = \prod_{j=1}^M \left(e^{-\lambda \frac{1-\delta}{M}} - e^{-\lambda \frac{1}{M}} \right).$$

Then, independence gives

$$\mathbb{P}_{(x, x')} \{J \leq M\} \geq \ell^M(\Xi_x) \ell^M(\Xi_{x'}) \prod_{j=1}^M \left(e^{-\lambda \frac{1-\delta}{M}} - e^{-\lambda \frac{1}{M}} \right) > 0.$$

The uniformity in x and x' follows from the fact that there is only a finite number of sets Ξ_x and $\Xi_{x'}$ to consider as x and x' range over the set $B(0, R)$. \square

The main result of this appendix is the following exponential-moment bound on the random variable J . The argument used to deduce the proposition from the previous lemmas is well known and is for example discussed in depth in Section 3.3.2 in [22].

Proposition A.6. *There are constants $\theta_2 > 0$ and $A_2 > 0$ such that*

$$\mathbb{E}_{(x, x')} e^{\theta_2 J} \leq A_2 (1 + \|x\|^2 + \|x'\|^2) \quad (\text{A.10})$$

for all $x, x' \in \mathbf{R}^d$.

Proof. Let I_i be defined as in Lemma A.3 with constant $M \in \mathbf{N}_m$ as in Lemma A.5. Then

$$\mathbb{P}_{(x, x')} \{J > k\} \leq \mathbb{P}_{(x, x')} \{I_i < J\} + \mathbb{P}_{(x, x')} \{I_i \geq k\}$$

⁶When the coupling starts with $x \neq x'$ not both in $B(\hat{x}, r)$, the first m jumps are independent. The probability of $z_m = z'_m$ is zero by our assumptions on ℓ . Thus going by blocks of m steps, we see that the jumps are independent until both trajectories simultaneously hit $B(\hat{x}, r)$ at a time which is a multiple of m .

for any choice of integers $i, k \geq 1$. To control the first term, note that the Markov property and Lemma A.5 imply

$$\mathbb{P}_{(x,x')} \{I_i < J\} \leq (1-q) \mathbb{P}_{(x,x')} \{I_{i-1} < J\} \leq (1-q)^{i-1}.$$

For the second term, we have the bound

$$\mathbb{P}_{(x,x')} \{I_i \geq k\} \leq C_2^i e^{-c_1 k} (1 + \|x\|^2 + \|x'\|^2)$$

by Chebyshev's inequality and Lemma A.3. In particular, taking i scaling like ϵk for ϵ small enough, we find

$$\begin{aligned} \mathbb{P}_{(x,x')} \{J > k\} &\leq (1-q)^{\epsilon k-1} + C_2^{\epsilon k} e^{-c_1 k} (1 + \|x\|^2 + \|x'\|^2) \\ &\leq a^k (1 + \|x\|^2 + \|x'\|^2) \end{aligned}$$

for some $a \in (0, 1)$. This exponential decay of the probability yields the proposition for θ_2 small enough and A_2 large enough. \square

Appendix B. Controllability of ODEs with polynomially growing nonlinearities

When the perturbation term g in (34) is a polynomial, Proposition 4.2 follows from [20, Thm. 3] or [19, Thm. 11 in Ch. 5] and the system is even exactly controllable. In the general case, when g is an arbitrary smooth function satisfying (i) and (ii), these results cannot be applied since the Hörmander condition is not necessarily satisfied at all the points. We adapt an argument used in [28, Thm. 2.5] which is particularly simple in the case of ordinary differential equations. Let us consider the equation

$$\dot{u}(t) - \nu \Delta(u(t) + \xi(t)) + P_N F(u(t) + \xi(t)) = h + \zeta(t), \quad (\text{B.1})$$

with two controls ξ and ζ in $C([0, T]; H_N)$.⁷ We denote by $S_t(u_0, \xi, \zeta)$ the solution of (B.1) satisfying the initial condition $u(0) = u_0$. To simplify the presentation, we shall assume that $a = 1$ in (34). Let us define a sequence $\{\mathcal{H}_i\}_{i \geq 1}$ of subspaces of H_N as follows: $\mathcal{H}_1 = H_1$ and

$$\mathcal{H}_i = \text{span} \{P_N(\varphi_1 \cdots \varphi_p) : \varphi_j \in \mathcal{H}_{i-1}, j = 0, \dots, p\}$$

for $i \geq 2$. The trigonometric identities (39) and (40) give that $s_{l \pm m}, c_{l \pm m} \in \mathcal{H}_i$, provided that $s_l, s_m, c_l, c_m \in \mathcal{H}_{i-1}$. Recalling the definition of H_1 , it is easy to infer that

$$\mathcal{H}_i = H_N \quad \text{for sufficiently large } i \geq 1. \quad (\text{B.2})$$

We will also use another form of these subspaces:

$$\mathcal{H}_i = \text{span} \{\varphi_0, P_N \varphi^p : \varphi_0, \varphi \in \mathcal{H}_{i-1}\} \quad (\text{B.3})$$

for $i \geq 2$, which can be verified as in Lemma 4.2 in [28].

The following lemma will play an important role in the proof of Proposition 4.2. It is established at the end of this subsection.

⁷The idea of introducing the second control ξ comes from [2] and is nowadays extensively used in the control theory of PDEs with finite-dimensional controls (see the surveys [3, 35]).

Lemma B.1. *Under the conditions of Theorem 4.1, for any vectors $u_0, \varphi, \psi \in H_N$, we have*

$$S_\delta(u_0, \delta^{-1/p}\varphi, \delta^{-1}\psi) \rightarrow u_0 + \psi - P_N\varphi^p \quad \text{in } H_N \text{ as } \delta \rightarrow 0. \quad (\text{B.4})$$

Proof of Proposition 4.2. By a general argument (see for example Step 4 in the proof of Theorem 2.3 in [28]) approximate controllability in any fixed time $T > 0$ can be obtained from controllability in arbitrarily small time.

Lemma B.1 gives that for all $u_0 \in H_N$, $\psi \in H_1 = \mathcal{H}_1$, $\epsilon > 0$, and $T > 0$, there exists $\zeta \in C([0, \delta]; H_1)$ with $0 < \delta < T$ such that

$$\|S_\delta(u_0, \zeta) - (u_0 + \psi)\|_{L^2} < \epsilon. \quad (\text{B.5})$$

Because $H_N = \mathcal{H}_i$ for some i , we may proceed by induction on i : let us suppose that for all $u_0 \in H_N$, $\psi \in \mathcal{H}_{i-1}$, $\epsilon > 0$, and $T > 0$, there exists $\zeta \in C([0, \delta]; H_1)$ with $0 < \delta < T$ such that (B.5) holds; we will show that this property then also holds for i , and the proof of the proposition will be complete.

Fix $u_0 \in H_N$. By (B.3), any $\psi \in \mathcal{H}_i$ can be written as a linear combination of elements of the form $P_N\varphi^p$ with $\varphi \in \mathcal{H}_{i-1}$, plus a vector in \mathcal{H}_{i-1} . Hence, by an iteration argument, it suffices to consider vectors ψ of the form $-P_N\varphi^p$ for some $\varphi \in \mathcal{H}_{i-1}$. Let $\epsilon > 0$ and $T > 0$ be arbitrary. By Lemma B.1, there exists $\delta_2 \in (0, \frac{1}{3}T)$ such that

$$\|S_{\delta_2}(u_0, \delta_2^{-1/p}\varphi, 0) - (u_0 - P_N\varphi^p)\|_{L^2} < \frac{1}{4}\epsilon.$$

On the other hand, a change of variable shows

$$S_{\delta_2}(u_0, \delta_2^{-1/p}\varphi, 0) = S_{\delta_2}(u_0 + \delta_2^{-1/p}\varphi, 0) - \delta_2^{-1/p}\varphi$$

so that

$$\|S_{\delta_2}(u_0 + \delta_2^{-1/p}\varphi, 0) - (u_0 - P_N\varphi^p + \delta_2^{-1/p}\varphi)\|_{L^2} < \frac{1}{4}\epsilon.$$

By continuity, there exists a radius $\rho > 0$ such that

$$\|S_{\delta_2}(u, 0) - (u_0 - P_N\varphi^p + \delta_2^{-1/p}\varphi)\|_{L^2} < \frac{1}{2}\epsilon$$

for all u with

$$\|u - (u_0 + \delta_2^{-1/p}\varphi)\|_{L^2} < \rho.$$

By the induction hypothesis, there exists $\tilde{\zeta}_1 \in C([0, \delta_1]; H_1)$ with $0 < \delta_1 < \frac{1}{3}T$ such that $\|S_{\delta_1}(u_0, \tilde{\zeta}_1) - (u_0 + \delta_2^{-1/p}\varphi)\|_{L^2} < \rho$, and therefore such that

$$\|S_{\delta_2}(S_{\delta_1}(u_0, \tilde{\zeta}_1), 0) - (u_0 - P_N\varphi^p + \delta_2^{-1/p}\varphi)\|_{L^2} < \frac{1}{2}\epsilon.$$

Yet again by the induction hypothesis, there exists $\tilde{\zeta}_3 \in C([0, \delta_3]; H_1)$ with $0 < \delta_3 < \frac{1}{3}T$ such that

$$\|S_{\delta_3}(S_{\delta_2}(S_{\delta_1}(u_0, \tilde{\zeta}_1), 0), \tilde{\zeta}_3) - (S_{\delta_2}(S_{\delta_1}(u_0, \tilde{\zeta}_1), 0) - \delta_2^{-1/p}\varphi)\|_{L^2} < \frac{1}{4}\epsilon.$$

Therefore, by the triangle inequality,

$$\|S_{\delta_3}(S_{\delta_2}(S_{\delta_1}(u_0, \tilde{\zeta}_1), 0), \tilde{\zeta}_3) - (u_0 - P_N\varphi^p)\|_{L^2} < \frac{3}{4}\epsilon.$$

We conclude that (B.5) holds with $\zeta \in C([0, \delta_1 + \delta_2 + \delta_3]; H_1)$ a good enough continuous approximation of the function $\mathbf{1}_{[0, \delta_1)}\tilde{\zeta}_1 + \mathbf{1}_{[\delta_1 + \delta_2, \delta_1 + \delta_2 + \delta_3)}\tilde{\zeta}_3(\cdot - (\delta_1 + \delta_2))$. Note that $0 < \delta_1 + \delta_2 + \delta_3 < T$ by construction. \square

Proof of Lemma B.1. Fix $\varphi, \psi \in H_N$ and let $u(t) = S_t(u_0, \xi, \zeta)$ with the constant controls $\xi(t) \equiv \varphi$ and $\zeta(t) \equiv \psi$. Also let

$$w(t) := u_0 + t(\psi - \mathbf{P}_N \varphi^p) \quad \text{and} \quad v(t) := u(\delta t) - w(t).$$

Clearly, the fact that u solves (B.1) with $u(0) = u_0$ implies that v solves

$$\dot{v}(t) - \nu \delta \Delta(v(t) + w(t) + \delta^{-1/p} \varphi) + \delta \mathbf{P}_N F(v(t) + w(t) + \delta^{-1/p} \varphi) - \mathbf{P}_N \varphi^p = \delta h$$

with $v(0) = 0$. Taking the scalar product in L^2 of this equation with $v(t)$, applying the Cauchy–Schwarz inequality, and dropping the arguments (t) for notational simplicity, we get

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 \leq \left(\nu \delta \|\Delta w\|_{L^2} + \nu \delta^{1-1/p} \|\Delta \varphi\|_{L^2} + \delta \|h\|_{L^2} \right. \quad (\text{B.6})$$

$$\left. + \|\delta \mathbf{P}_N F(v + w + \delta^{-1/p} \varphi) - \mathbf{P}_N \varphi^p\|_{L^2} \right) \|v\|_{L^2} \\ \leq C_1 \left(\delta^{1-1/p} + \|\delta \mathbf{P}_N F(v + w + \delta^{-1/p} \varphi) - \mathbf{P}_N \varphi^p\|_{L^2} \right) \|v\|_{L^2} \quad (\text{B.7})$$

for any $t \leq 1$ and $\delta \leq 1$. Using the assumption (i) and the Young inequality, we obtain

$$\|\delta \mathbf{P}_N F(v + w + \delta^{-1/p} \varphi) - \mathbf{P}_N \varphi^p\|_{L^2} \leq C_2 \delta \left(\|v\|_{L^2}^p + \|w\|_{L^2}^p + \delta^{-(p-1)/p} \|\varphi\|_{L^2}^{p-1} + 1 \right) \\ \leq C_3 \delta \left(\|v\|_{L^2}^p + \delta^{-(p-1)/p} + 1 \right). \quad (\text{B.8})$$

Combining (B.7) and (B.8), we see that

$$\frac{d}{dt} \|v(t)\|_{L^2}^2 \leq C_4 \delta^{1/p} \left(\|v(t)\|_{L^2}^{p+1} + 1 \right). \quad (\text{B.9})$$

Let us set $A_\delta := C_4 \delta^{1/p}$ and

$$\Phi(t) := A_\delta + A_\delta \int_0^t \|v(s)\|_{L^2}^{p+1} ds. \quad (\text{B.10})$$

Then, (B.9) is equivalent to

$$(\dot{\Phi})^{2/(p+1)} \leq A_\delta^{2/(p+1)} \Phi,$$

and

$$\frac{\dot{\Phi}}{\Phi^{(p+1)/2}} \leq A_\delta.$$

Integrating this inequality, we derive

$$\Phi(t) \leq A_\delta \left(1 - \frac{p-1}{2} A_\delta^{(p+1)/2} t \right)^{-2/(p-1)}$$

for all $0 \leq t < 1 \wedge T_*(\delta)$, where

$$T_*(\delta) := \left(\frac{p-1}{2} A_\delta^{(p+1)/2} \right)^{-1}.$$

Because $T_*(\delta) \uparrow \infty$ monotonically as $\delta \downarrow 0$, there exists $\delta_0 > 0$ small enough that

$$\Phi(t) \leq 2A_\delta \quad (\text{B.11})$$

for all $0 \leq t \leq 1$, whenever $0 < \delta \leq \delta_0$. Then, combining (B.9)–(B.11), we obtain

$$\|v(1)\|_{L^2}^2 \leq C_5 \delta^{1/p}$$

for some constant C_5 independent of δ . Thus $v(1) \rightarrow 0$ as $\delta \rightarrow 0$, which implies (B.4). \square

Appendix C. Some results from measure theory

Appendix C.1. Maximal couplings

Let \mathcal{X}, \mathcal{Y} , and \mathcal{U} be Polish spaces endowed with their Borel σ -algebras, $u \in \mathcal{U} \mapsto \mu(u, \cdot), \mu'(u, \cdot)$ be two random probability measures on \mathcal{X} , and $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a measurable mapping. We denote by $F_*\mu(u, \cdot)$ the image of $\mu(u, \cdot)$ under F (similarly for μ'). The following lemma on the existence of maximal couplings is a particular case of Exercise 1.2.30.ii in [22] (see the last section of the book for a proof).

Lemma C.1. *There is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and measurable mappings $\xi, \xi' : \mathcal{U} \times \Omega \rightarrow \mathcal{X}$ such that the following two properties are satisfied:*

- *for all $u \in \mathcal{U}$, $(\xi(u, \cdot), \xi'(u, \cdot))$ is a coupling of $\mu(u, \cdot)$ and $\mu'(u, \cdot)$ in the sense that*

$$\xi(u, \cdot)_*\mathbb{P} = \mu(u, \cdot) \quad \text{and} \quad \xi'(u, \cdot)_*\mathbb{P} = \mu'(u, \cdot); \quad (\text{C.1})$$

- *for all $u \in \mathcal{U}$, $(F(\xi(u, \cdot)), F(\xi'(u, \cdot)))$ is a maximal coupling of $F_*\mu(u, \cdot)$ and $F_*\mu'(u, \cdot)$ in the sense that*

$$\mathbb{P}(\{\omega \in \Omega : F(\xi(u, \omega)) \neq F(\xi'(u, \omega))\}) = \|F_*\mu(u, \cdot) - F_*\mu'(u, \cdot)\|_{\text{var}} \quad (\text{C.2})$$

and the random variables $F(\xi(u, \cdot))$ and $F(\xi'(u, \cdot))$ conditioned on the event

$$\{\omega \in \Omega : F(\xi(u, \omega)) \neq F(\xi'(u, \omega))\}$$

are independent.

Appendix C.2. Images of measures under regular mappings

Let \mathcal{X} be a compact metric space, \mathcal{Y} and \mathcal{U} be finite-dimensional spaces, and $F : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$ be a continuous mapping. The following is a consequence of a more general result proved in Theorem 2.4 in [32] (see also Chapter 9 of [6]). In this simplified context in finite dimension, it can be proven directly from the implicit function theorem and a change of variable.

Lemma C.2. *Assume that the mapping $F(x, \cdot) : \mathcal{U} \rightarrow \mathcal{Y}$ is differentiable for any $x \in \mathcal{X}$, the derivative $D_u F$ is continuous on $\mathcal{X} \times \mathcal{U}$, the image of the linear operator $(D_u F)(\hat{x}, \hat{u})$ has full rank for some $(\hat{x}, \hat{u}) \in \mathcal{X} \times \mathcal{U}$, and $\varrho \in \mathcal{P}(\mathcal{U})$ is a measure possessing a positive continuous density with respect to the Lebesgue measure on \mathcal{U} . Then there is a continuous function $\psi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbf{R}_+$ and a number $r > 0$ such that*

$$\psi(\hat{x}, F(\hat{x}, \hat{u})) > 0,$$

and

$$(F_*(x, \cdot)\varrho)(dy) \geq \psi(x, y) dy$$

(as measures on \mathcal{Y}) for all $x \in B_{\mathcal{X}}(\hat{x}, r)$.

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